Gamma process model for reliability analysis and replacement of aging structural components[1]

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ABSTRACT: The time-dependent reliability analysis to model age-related deterioration is necessary to support strategic planning and optimization of civil engineering infrastructure systems. The paper presents a stochastic gamma process model to account for temporal variability associated with a deterioration process and its impact on the optimal age of replacement. The proposed method is more versatile than the random-variable deterioration rate model commonly used in the structural reliability literature. The paper presents an intuitive interpretation and a comparison of assumptions associated with the random variable and gamma process deterioration models. Numerical results are presented about the optimal age of replacement of structural components.

1 INTRODUCTION

The time-dependent reliability analysis to model age-related deterioration is necessary to support strategic planning and optimization of civil engineering infrastructure systems. In the modelling of deterioration, two types of uncertainties are encountered, namely, sampling and temporal uncertainties. The sampling uncertainty means that parameters of the deterioration process vary from sample to sample, and uncertainty associated with the evolution or progression of deterioration over time is referred to as temporal variability.

In engineering mechanics, the form of deterioration law, such as a linear or power law in time, is often known through experimental analysis and mathematical modelling. The sampling uncertainty can be captured by randomizing the parameters of the deterioration law, whereas its uncertain evolution over time should ideally be modelled as a stochastic process.

In structural reliability estimation, the deterioration of structures is usually modelled through random variables (such as the deterioration rate) and the computation is based on the First-Order Reliability Method (FORM). Such models, referred to as Random Variable (RV) deterioration models, do not account for temporal variability in a formal mathematical sense. The paper presents a general model of deterioration based on the gamma process (GP), which is a stochastic process with independent non-negative increments having a gamma distribution (Figure 1). In contrast with the Brownian motion with drift, also called the Wiener process, the gamma process is monotonically increasing and positive, which makes it ideal for modelling deterioration processes.

Although the use of stochastic processes for modelling deterioration has been advocated, an intuitive explanation to draw a clear distinction between the features of the random variable versus the random process approach is not articulated in the engineering literature. The paper presents a comparative assessment and interpretation of the random variable and gamma process models for time-dependent reliability analysis in a simple setting to emphasize the additional value of information gained through the stochastic modelling. The paper also presents results on optimal age of replacement

2 RANDOM VARIABLE DETERIORATION MODEL

2.1 *Model Definition*

For the sake of clarity, consider that the deterioration of resistance follows a linear model given as

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$$R(t) = r_0 - X(t) = r_0 - At \quad (1)$$

where r_0 is the initial resistance, A is the random deterioration rate, and t is the time interval (or age). A common approach in time-dependent structural reliability analysis is to randomize the deterioration rate, A, and assign it a probability distribution based on some available data and expert judgment.

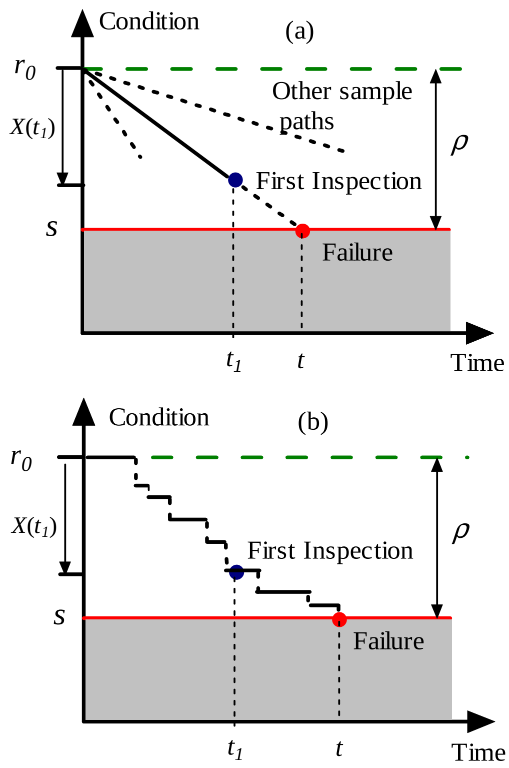

Figure 1: Models of deterioration: (a) RV model, and (b) GP model

The randomization of the degradation rate reflects its variability in a large population of similar components (in a similar manner as the variability in lifetimes). The failure of a component is defined as the down-crossing of resistance below an applied stress s, and the probability of failure can be estimated from the limit state function:

$$R(t) - s \le 0 \Leftrightarrow (r_0 - s) - X(t) \le 0 \Leftrightarrow \rho - At \le 0, \quad (2)$$

where $\rho = (r_0 - s)$ denotes the design margin or a deterioration threshold. Failure is defined as the event at which the cumulative amount of deterioration $X(t)$ exceeds the deterioration threshold ρ. The threshold ρ is taken as a constant for simplicity of discussion. We introduce the following notation to define the mean, variance and coefficient of variation (COV) of a random variable X:

$$E[X] = \mu_X, \ \mathrm{VAR}[X] = \sigma_X^2, \ \mathrm{COV}[X] = \nu_X = \frac{\sigma_X}{\mu_X} \quad (3)$$

The mean, variance and the COV of deterioration at time t are given as

$$\mu_{At} = \mu_A t, \ \sigma_{At}^2 = \sigma_A^2 t^2, \text{ and } \nu_{At} = \frac{\sigma_{At}}{\mu_{At}} = \nu_A \quad (4)$$

where μ_A, σ_A and ν_A denote the mean, standard deviation and COV of the deterioration rate, respectively. From Equation (2), the cumulative probability distribution function of the lifetime, T, can be simply written as

$$F_T(t) = P[T \le t] = P[(\rho / A) \le t] = P[A \ge (\rho / t)] \quad (5)$$

Depending on the probability distribution of A, the lifetime distribution can be derived analytically or computed numerically using FORM or simulation.

2.2 *Remarks*

Although the degradation model in Equation (1) can be considered as a stochastic process in a technical sense, a sample path of deterioration of a specific component remains fixed over its entire lifetime. In other words, the temporal variability associated with the evolution of the deterioration process of a sample path over time is ignored. It is also reflected from Equation (4) that the COV of deterioration is constant over time. In fact, the future sample path in the linear deterioration model becomes completely deterministic after a single inspection at a time t_1 quantifying the amount of the deterioration, as shown in Figure 1. It means that a single inspection can determine the remaining lifetime of the component without any uncertainty. In principle, if there are n random variables in the deterioration law, n number of inspections will determine the remaining lifetime of a component.

The RV deterioration model is implicit in several studies that apply FORM for time-dependent reliability analysis (Pandey 1998). In the condition assessment and rehabilitation planning of existing structures, the uncertainty associated with the evolution of degradation over time is an important consideration. Since the RV model is not adequate to model temporal variability associated with deterioration, we present a more formal stochastic gamma process model to overcome this limitation.

2.3 *Illustration*

For sake of a consistent comparison between the random variable and the gamma process model we assume that the degradation rate, A, is a gamma distributed random variable and the damage threshold ρ has a fixed deterministic value. The probability density function of the deterioration rate is given as

$$f_A(a) = \frac{\delta^\eta}{\Gamma(\eta)} a^{\eta-1} e^{-\delta a} = \mathrm{Ga}(a \mid \eta, \delta), \quad a > 0, \quad (6)$$

where η and δ are the shape and scale parameter of the gamma probability distribution, respectively.

Note that we adopt Ga($x|\eta,\delta$) as a compact notation for the density function of a gamma distributed random variable X with shape and scale parameters of η and δ, respectively. The cumulative gamma distribution function is denoted as

$$\mathrm{GA}(y|\eta,\delta) = \int_0^y \mathrm{Ga}(x \mid \eta,\delta)dx \tag{7}$$

The mean and variance of the degradation rate are given as

$$\mu_A = \frac{\eta}{\delta} \quad \text{and} \quad \sigma_A^2 = \frac{\eta}{\delta^2} \tag{8}$$

Conversely, the shape and scale parameters can be related to the mean and COV, $\nu_A = \sigma_A/\mu_A$, as follows:

$$\eta = \frac{1}{(\nu_A)^2} \quad \text{and} \quad \delta = \frac{\eta}{\mu_A} = \frac{1}{\mu_A \nu_A^2} \tag{9}$$

The cumulative damage in time interval (0, t], $X(t) = At$, is also a gamma distributed random variable with probability density function:

$$f_{X(t)}(x) = \mathrm{Ga}\left(x \middle| \eta, \frac{\delta}{t}\right) \tag{10}$$

In this example, we assume μ_A = 2 units/year with a COV of ν_A = 0.4, and the threshold ρ = 100 units. The distribution parameters are η = 6.25 and δ = 3.125. The probability density functions of deterioration in different time intervals ranging from 0 – 200 years are shown in Figure 2, which are calculated using Equation (10).

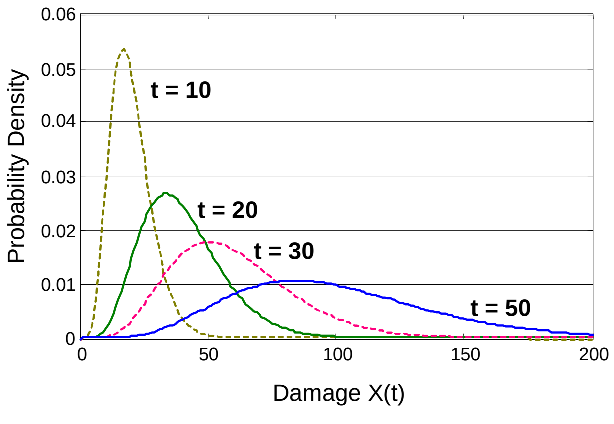

Figure 2: Probability density functions of deterioration $X(t)$ in RV model

Since the deterioration rate is gamma distributed, the lifetime ($T=\rho/A$) follows an inverted gamma distribution with the following density function:

$$f_T(t) = \frac{(\rho\delta)^\eta}{\Gamma(\eta)}\left(\frac{1}{t}\right)^{\eta+1} e^{-\rho\delta/t} \tag{11}$$

and the cumulative distribution function is written as

$$F_T(t) = 1 - \mathrm{GA}\left(\frac{1}{t}\middle|\eta,\rho\delta\right) \tag{12}$$

3 GAMMA PROCESS DETERIORATION MODEL

3.1 *Background*

A key issue in time-dependent reliability analysis is the modeling of the deterioration process that typically increases in an uncertain manner over the life of a structure. In structural engineering, a distinction is made between a structure's resistance (e.g. the crest-level of a dike) and its applied stress (e.g. the water level to be withstood). A failure may then be defined as the event in which the deteriorating resistance drops below the stress. Maintenance management mainly deals with condition failure rather than structural failure (collapse). The uncertain deterioration can be regarded as a stochastic process, and the associated uncertainty can be represented by the normal distribution. This probability distribution has been used for modelling the exchange value of shares and the movement of small particles in fluids and air. A characteristic feature of this model, also denoted by the Brownian motion with drift, is that a structure's resistance alternately increases and decreases, like the exchange value of a share. For this reason, the Brownian motion is inadequate for modelling deterioration. For example, a dike of which the height is subject to a Brownian deterioration can, according to the model, spontaneously rise up, which of course cannot occur in practice.

The gamma process is ideally suited to model gradual damage that monotonically accumulates over time, such as wear, corrosion, erosion, and creep of materials, which are common causes of failure of engineering components (Abdel-Hameed, 1975).

3.2 *Gamma Process Model for Deterioration*

The gamma process belongs to a general class of stochastic processes, referred to as the Markov process. In the gamma process, increments are independent and non-negative random variables (e.g. metal loss due to corrosion) having a gamma distribution with an identical scale parameter and a time dependent shape function. The other examples of processes with independent increments are the Brownian motion with drift having Gaussian increments and the compound Poisson process.

In the gamma process model, the cumulative deterioration at time t follows a gamma distribution with the shape function, $\lambda(t)$>0, and the scale parameter, β, is a constant. To reflect the monotonic nature of deterioration over time, the shape function,

$\lambda(t)$, must be an increasing function of time. The probability density function is given as

$$f_{X(t)}(x) = \frac{\beta^{\lambda(t)}}{\Gamma(\lambda(t))} x^{\lambda(t)-1} e^{-\beta x} = \text{Ga}(x \mid \lambda(t), \beta) \tag{13}$$

The damage increment from time t_1 to t_2, given as $[X(t_2) - X(t_1)]$, is a non-negative quantity that is independent of the cumulative deterioration at time t_1, $X(t_1)$. The increment is gamma distributed with shape parameter $[\lambda(t_2) - \lambda(t_1)]$ and scale parameter β. The fact that the damage increment from any state $X(t)$ is an independent random variable is a consequence of the properties of the gamma process. The gamma distribution is an infinitely divisible distribution, because of which the increments and their cumulative sum are gamma distributed. The sample paths of the gamma process are discontinuous and monotone, as shown in Figure 1.

The mean, variance and COV of the cumulative deterioration at time t are given as

$$\mu_{X(t)} = \frac{\lambda(t)}{\beta},\ \sigma^2_{X(t)} = \frac{\lambda(t)}{\beta^2},\ \text{and}\ \nu_{X(t)} = \frac{1}{\sqrt{\lambda(t)}} \tag{14}$$

A comparison of Equations (4) and (14) shows that COV of deterioration in the random variable model is time invariant, but it is a time-dependent function in the gamma process model. The cumulative distribution function of the lifetime can be derived as

$$F_T(t) = P[X(t) \geq \rho] = 1 - \text{GA}[\rho \mid \lambda(t), \beta] \tag{15}$$

It should be noted that the probability density function, $f(t) = dF(t)/dt$, has no closed form expression, though it can be computed numerically.

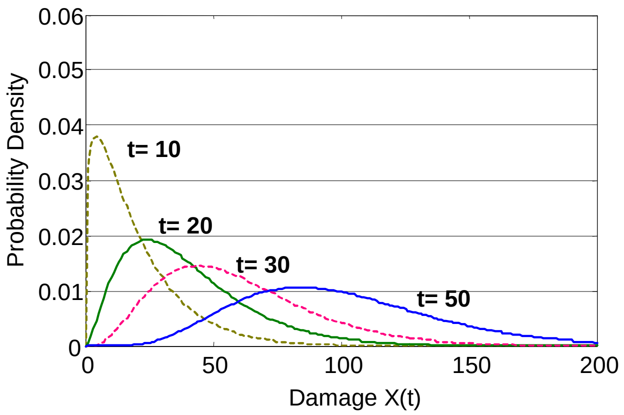

Figure 3: Probability density functions of deterioration $X(t)$ in the gamma process model

3.3 *Illustration*

For the sake of a consistent comparison with the RV model, we consider deterioration as a stationary gamma process with $\lambda(t) = \alpha t$. It implies that the average deterioration increases linearly with time, though the gamma process is not restricted to a linear model. As a matter of fact, any shape function $\lambda(t)$ suffices, as long as it is a non-decreasing, right continuous and real-valued function.

The data given in the example of Section 2.3 are used to calibrate the parameters of the gamma process. The COV of deterioration is time dependent in the GP model (Equation 14), and given as

$$\nu_{X(t)} = \frac{1}{\sqrt{\alpha t}} \tag{16}$$

To calibrate the gamma process, we take $\nu_{X(t)} = \nu_{At} = \nu_A = 0.4$ at the time at which the expected deterioration exceeds the failure threshold, that is, at $t_F = 50$ years. The shape parameter is calculated as $\alpha = \eta/t_F = 6.25/50 = 0.125$. The scale parameter is obtained by matching the mean damage rate in both the RV model and GP model, which leads to $\beta = \delta/t_F = 3.125/50 = 0.0625$. The probability density functions of cumulative damage at different time intervals are shown in Figure 3.

4 COMPARISON OF GP AND RV MODELS

4.1 *Lifetime Distributions*

As seen from Figure 1, the main difference between the GP model and the RV model is that the sample paths of the gamma process are discontinuous and monotone, whereas the sample paths of the latter approach are straight lines for a linear deterioration law. According to the theory of the gamma process, one inspection reveals only one observed increment which can be used to predict the probability distribution of future deterioration. According to the random-variable degradation model, however, one inspection already fixes the future deterioration beforehand. Although the RV model can be a good approximation, one should be careful as soon as inspections are involved. Details of inspection models based on the gamma process are presented elsewhere by van Noortwijk et al. (1995, 1997).

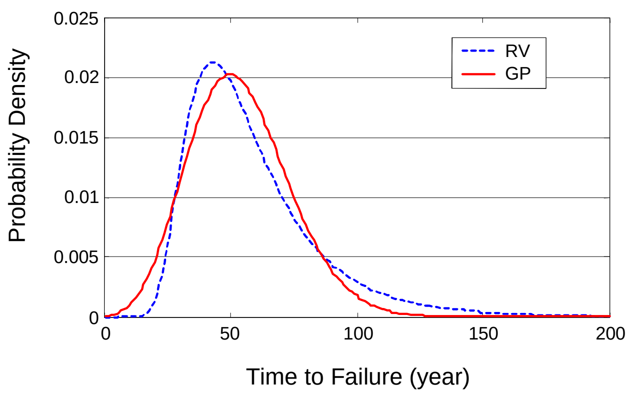

Figure 4: Comparison of lifetime probability density functions for RV and GP models

In Figure 4, the lifetime density of the RV model (Equation 11) is compared with that computed for the GP model by discretizing the cumulative distribution given in Equation (15). The lifetime cumulative distribution functions are compared in Figure 5(a), which shows that the tails of the two distributions are different. This difference is seen more clearly in Figure 5(b), where the survival curves for both models are plotted on a log scale. It is interesting to note that survival probabilities in advanced ages (t > 50) in the GP model are much less than those obtained from the RV model, in spite of the fact that the mean deterioration rate is the same in both models. The pessimistic prediction of the GP model is attributed to uncertainty associated with evolution of deterioration in form of independent gamma increments. In contrast, the deterioration rate in the RV model is fully correlated over the lifetime, which possibly results in the overestimation of survival probabilities.

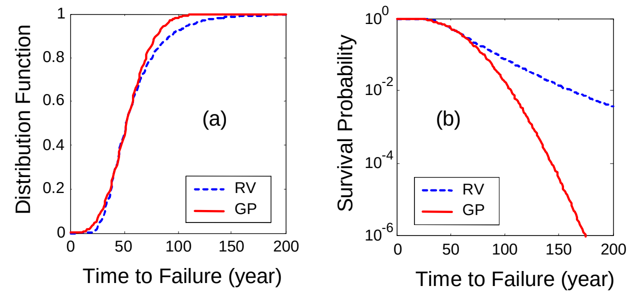

Figure 5: Comparison of lifetime distribution obtained from RV and GP models: (a) Cumulative distribution functions, and (b) Survival curves

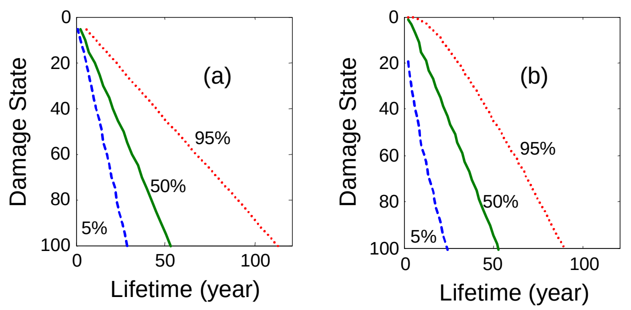

Figure 6: Probability of realizing different condition during the lifetime for ν_A =0.4: (a) RV model, and (b) GP model

Another way to compare the GP and RV models is to study the probability of realizing different levels of condition (or remaining resistance margin) during the lifetime, such as a case shown in Figure 6. In case of the GP model (ν_A = 0.4), at the origin (ρ =100, t = 0) the lifetime corresponding to the 5th, 50th and 95th percentiles (5%, 50% and 95% quantiles) are estimated as 22, 50 and 90 years, respectively (Figure 6a), whereas the RV model leads to lifetime estimates of 30, 50 and 110 years for the 5th, 50th and 95th percentiles, respectively. These bound are computed from Equations (12) and (15) by decreasing ρ from 100 to 0. It can also be computed by using Equations (10) and (13) and determining percentiles for different values of t. The discrepancy between the GP and the RV models is amplified substantially, when the COV of the deterioration rate is increased to ν_A =1 (Figure 7). The upper bound curve (95% percentile) in the RV model is highly skewed and it results in unreasonably large values of the lifetime (Figure 7b), whereas the GP model provides reasonable bounds on the lifetime. An arbitrary increase in the uncertainty associated with the deterioration process is handled by the gamma process model in a logical manner, whereas the RV model appears to provide unreasonable estimates of the lifetime.

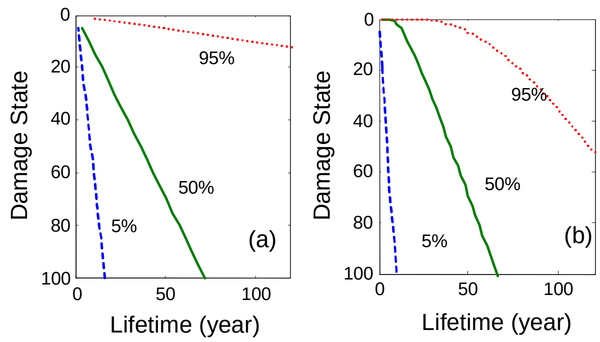

Figure 7: Probability of realizing different condition levels during the lifetime for ν_A =1: (a) RV model, and (b) GP model

5 OPTIMAL AGE OF REPLACEMENT

A key application of time-dependent reliability analysis is to determine the optimal age-based replacement policy. It is the simplest policy to be used in renewal of infrastructure. In this policy, a component is replaced when it reaches to a specific age (t_0) regardless of its condition. The component is also replaced, if failure occurs before the replacement time, t_0.

Consider the total cost associated with all the consequences of a structural failure are denoted as C_F, and the cost of a preventive replacement is C_P. According to the renewal theory (Cox 1962), the expected average cost per unit time in long term can be expressed as (Barlow and Proschan 1965):

$$CR(t_0) = \frac{F_T(t_0)C_F + (1 - F_T(t_0))C_P}{\int_0^{t_0} (1 - F_T(t))dt}, \qquad (17)$$

where $F_T(t)$ is the cumulative lifetime distribution, which can be calculated on the basis of the RV and GP models, i.e., Equation (12) and (15), respectively. The optimal age preventive replacement (t_0) should be chosen so as to minimize the average cost rate.

Since the calculation of the cost rate is sensitive to the lifetime distribution, it would be of interest to examine the impact of the RV and GP model on the replacement policy.

For the illustration purpose, assume C_P=1 and C_F=50. The basic data given in Section 2.3 and 3.3 are used in the calculation of lifetime distributions.

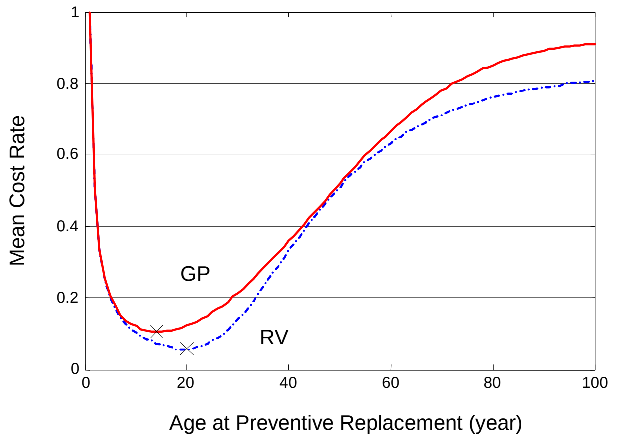

Figure 8: Expected cost per year as a function of replacement age for RV and GP models (COV = 0.4)

The first case corresponds to a COV of the deterioration rate of ν_A = 0.4. The comparison of the expected cost per year obtained from GP and RV models as a function of replacement age is illustrated in Figure 8. The optimal replacement ages can be read directly from the figure as 20 and 14 units of time for RV and GP model, respectively. The mean cost rate associated with GP model is always greater than that in RV. The reason is that the GP model reflects higher uncertainty, as seen from Figure 5 in which the lifetime distribution of GP model is above the distribution obtained from that of RV model. The implication is that in the GP model the numerator of Equation (17) is greater while the denominator is less than those computed in the RV model.

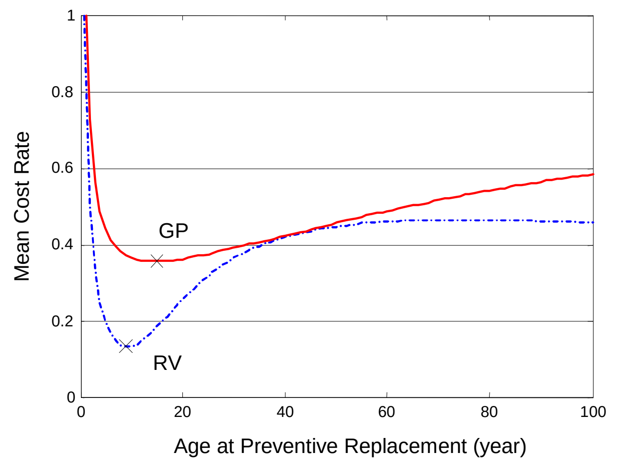

Figure 9: Expected cost per year as a function of replacement age for RV and GP models (COV = 1.0)

The second case corresponds to a COV of the deterioration rate of ν_A = 1. In this case the difference between RV and GP are further exaggerated (Figure 9). It is noteworthy that the optimal replacement age in RV model (9 units) is smaller than that obtained from GP model (15 unit). However, the RV model leads to a fairly conservative estimate of the minimum mean cost rate (0.132/year) in comparison to that obtained from the GP model (0.356/year).

6 CONCLUSIONS

The paper presents a stochastic gamma process model to account for both sampling and temporal variability associated with a deterioration process that increases the probability of failure with aging of the structure. The proposed gamma process model is more versatile than the random-variable damage rate model commonly used in the structural reliability literature. The reason being that the random rate model cannot capture temporal variability associated with the evolution of degradation. For example, the coefficient of variation of cumulative deterioration in the random variable model is time independent, whereas the gamma process has an explicit dependence over time. A consequence is that the random variable model overestimates the probability of survival over a long-term horizon.

In the context of optimization of an age-replacement policy, the illustrative examples presented in the paper show that the random variable (RV) model provides more conservative estimates of the average cost rate and optimistic estimates of the replacement age. In contrast, the gamma process (GP) model leads to higher mean cost estimates and lower replacement age to account for the temporal variability associated with the degradation process.

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