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On Nash equilibria of a competitive location problem

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On Nash equilibria of a competitive location problem *

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The location decision of a facility for two competing chains in a new planar market is described by a Huff-like attraction model. This means that the market share capture is given by a gravity model. The profit that can be reached depends on the actions of the competitor. The main questions are: what are the possible Nash equilibria in such a situation, how are they characterised and by which computational methods can they be determined?

Key words: Competitive location, continuous optimisation, Branch-and-Bound, Nash equilibrium, bi-matrix game

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1. Introduction

Many factors must be taken into account when locating a new facility which provides a new good or service to the customers of a given area. One of the most important points is the existence of a competitor that enters the market at the same time. In that case, the new facility will have to compete for the market.

Many competitive location models are available in the literature, see for instance the survey papers Eiselt and Laporte (1996), Eiselt et al. (1993), Plastria (2001) and the references therein. They vary in the ingredients which form the model. For instance, the *location space* may be the plane, a network or a discrete set. Demand is usually supposed to be concentrated in a discrete set of points, called *demand points* and in our case the demand quantities are assumed to be known.

The *patronising behaviour* of the customers must also be taken into account, since the *market share* captured by the facilities depends on it. In some models customers select among the facilities in a *deterministic* way, i.e, the full demand of the customer is served by the facility to which he/she is *attracted* most. In other cases, the customer splits his/her demand among more that one facility, leading to *probabilistic* patronising behaviour. In the model under study we will consider both types of behaviour. Usually attraction depends on distance to and quality of the facility. Even the simplest case of a new facility with fixed quality locating in continuous space, may lead to a hard to solve multi-extremal problems. Recent research even finds the global optimum where two firms are competing in a leader-follower Stackelberg model, see Sáiz et al. (2009).

Few research has been done on the kind of problems with simultaneous decisions on location and quality in continuous space. For a single competing facility, the problem has been studied under deterministic customer behaviour in Drezner (1994), Plastria (1997), using attraction functions of gravity type, and in Plastria and Carrizosa (2004), using different kinds of attraction functions. For probabilistic customer behaviour, the problem has been studied in Drezner and Drezner (1994),

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where the location problem is solved for a wide range of quality values. For the current study we will focus on the rational behaviour of suppliers with respect to their choice on the quality of the new facility to be located.

In this paper, we consider a facility location problem on the plane, with given demand. We look at both cases where behaviour of customers is either probabilistic or deterministic, based on an attraction function depending on both the location and the quality of the facility to be located. We study the situation where there are two competitors, firm 1 and firm 2 that decide on location and quality of a new facility in a new market. One can consider the models as a two stage game, where on the lower level one chooses the quality and on more strategic level suppliers choose the location. We describe the models in Section 2, where notation is introduced and both the deterministic and probabilistic model are given.

Our first question is under which circumstances the models have Nash equilibria on choice of quality. We deal with that in Section 3. As co-location (locating at the same place) is a natural way to compete, our second question is whether if equilibria on the quality level exist, do Nash equilibria occur apart from co-location? What characterises these equilibria and how can we determine them? We deal with this question in Section 4 for both models. Conclusions and future work are discussed in Section 5.

2. Description of the problem

The following notation is used to describe the models under study:

Indices

i index of demand points, $i=1,\ldots,n$

Variables

$x=(x_1,x_2)$	location of firm 1,	$y=(y_1,y_2)$	location of firm 2
q_1	quality facility firm 1,	q_2	quality facility firm 2

Data

p_i location of the i-th demand point (customer)

w_i demand (or buying power) at p_i

S location space where the leader and follower will locate the new facility

Miscellaneous

$d_i(z)$	distance between p_i and $z=x$ or $z=y$
$c_1()$, $c_2()$	cost functions for firm 1 and firm 2 with respect to quality. Usually they are convex.
$M_i(x,y,q_1,q_2)$	market share of customer i obtained by firm 1

In M_i we find the big difference in the probabilistic and deterministic model. In the probabilistic model M_i takes mostly a value between 0 and 1, as the customer will get part of the demanded service from firm 1 and part from firm 2. In the deterministic or binary model, the customer will choose between either of them, unless they have exactly the same attraction where they both supply half of the demand. The Huff based probabilistic model is described by

$$M_i(x,y,q_1,q_2)=\frac{q_1 d_i(y)}{q_1 d_i(y)+q_2 d_i(x)} \tag{1}$$

and capturing all demand $M_i(x,y,q_1,q_2)=1$ being the only one at the customer front door, i.e. $x=p_i$ and $y\neq p_i$. In the extreme case of co-location at the specific customer, $x=y=p_i$, then $M_i(x,y,q_1,q_2)=\frac{q_1}{q_1+q_2}$.

For the binary (deterministic) model, the customers usually chose one of the facilities; that one which is most attractive. This can be described as follows.

$$M_i(x,y,q_1,q_2)=1 \;\; if \;\; q_1 d_i(y)>q_2 d_i(x) \tag{2}$$

and 0 in most other cases. In the exceptional case that $q_1 d_i(y) = q_2 d_i(x)$ they take an equal share $M_i(x,y,q_1,q_2) = \frac{1}{2}$, called the braking tie rule. In the case of co-location at the specific customer, $x = y = p_i$, then we focus on the consequences of two variants. In one variant we stick to equation (2) and in the other variant we consider a more continuous behaviour with respect to the breaking tie rule taking $M_i(x,y,q_1,q_2) = \frac{q_1}{q_1+q_2}$.

The profit function of firm 1 is given by

$$\Pi_1(x,y,q_1,q_2) = \sum_{i=1}^{n} w_i M_i(x,y,q_1,q_2) - c_1(q_1). \tag{3}$$

As the market share sums to 1 for each customer, the profit function of firm 2 is given by

$$\Pi_2(x,y,q_1,q_2) = \sum_{i=1}^{n} w_i(1 - M_i(x,y,q_1,q_2)) - c_2(q_2). \tag{4}$$

The maximisation of the profit of one firm, when the decision of the other is given is typically a global optimisation problem as studied in Fernández et al. (2007). In this paper the research question is what are the possible Nash equilibria and how can we determine them. For this we first have to define the Nash equilibrium in this context. Decision vector (x^*, y^*, q_1^*, q_2^*) is a Nash equilibrium if

$$x^*, q_1^* \in \arg\max \Pi_1(x, y^*, q_1, q_2^*) \quad ; \quad y^*, q_2^* \in \arg\max \Pi_2(x^*, y, q_1^*, q_2). \tag{5}$$

In the symmetric case where both firms have the same cost function to determine the quality, an obvious equilibrium is to co-locate, such that market share as well as profit is equal for the firms. The question is what happens if the firms are not equal. We first focus on the question of Nash equilibria with respect to the choice of the quality of the facilities. Then in Section 4, the next question is whether equilibria exist on the decision of locating given the equilibria on the level of quality choice.

3. Optimum quality levels

The optimum values of investment in the quality q_j fulfil first order conditions. Putting the partial derivative to zero for firm 1, $\frac{\partial \Pi_1}{\partial q_1} = 0$ leads for the probabilistic model to the expression

$$\sum w_i \frac{q_2 d_i(x) d_i(y)}{(q_1 d_i(y) + q_2 d_i(x))^2} = \frac{dc_1(q_1)}{dq_1} \tag{6}$$

where the sum only applies for these i where $p_i \neq x, y$. It means that marginal cost in investment of quality should equal marginal gain in market share. For the deterministic model many discontinuities are involved when customers switch facility.

For the analysis we will consider two different cost functions: the basic linear case where

$$c_j(q_j) = \alpha_j q_j$$

and the quadratic case where

$$c_j(q_j) = \alpha_j q_j^2.$$

We analyse 4 different cases. First we condsider what are optimum quality investments if the two firms do not co-locate for the two models in Sections 3.1 and 3.2. This is followed by studying optimum choices in case of co-location for the two models in Sections 3.3 and 3.4.

3.1. No co-location, probabilistic model

The first order conditions (6) for the quality choice are usually not easy to solve analytically, but for the linear and quadratic cost models, expressions can be found. The linear cost functions gives as first order conditions

$$q_2 \sum_{p_i \neq x,y} w_i \frac{d_i(x)d_i(y)}{(q_1 d_i(y) + q_2 d_i(x))^2} = \alpha_1 \ ; \tag{7}$$

from which can be derived that $q_1^* \alpha_1 = q_2^* \alpha_2$, i.e. the cost of investment in quality is the same in the optimum. Elaboration of this equality in (7) gives as optimum levels.

$$\begin{aligned} q_1^* &= \alpha_2 \sum w_i \frac{d_i(x)d_i(y)}{(\alpha_2 d_i(y) + \alpha_1 d_i(x))^2} \\ q_2^* &= \alpha_1 \sum w_i \frac{d_i(x)d_i(y)}{(\alpha_2 d_i(y) + \alpha_1 d_i(x))^2}. \end{aligned} \tag{8}$$

Similarly, for quadratic costs, $c_1(q_1) = \alpha_1 q_1^2, c_2(q_2) = \alpha_2 q_2^2$ one can derive again from (6) that the optimum cost levels equal; $\alpha_1 q_1^2 = \alpha_2 q_2^2$. From this we have the explicit expressions

$$\begin{aligned} q_1^2 &= \frac{1}{2}\sqrt{\frac{\alpha_2}{\alpha_1}} \sum w_i \frac{d_i(x)d_i(y)}{(\sqrt{\alpha}_2 d_i(y) + \sqrt{\alpha}_1 d_i(x))^2} \\ q_2^2 &= \frac{1}{2}\sqrt{\frac{\alpha_1}{\alpha_2}} \sum w_i \frac{d_i(x)d_i(y)}{(\sqrt{\alpha}_2 d_i(y) + \sqrt{\alpha}_1 d_i(x))^2}. \end{aligned} \tag{9}$$

Example 1. Demand is concentrated in four points with $w_1 = 4, w_2 = 7, w_3 = 6, w_4 = 10$, demand points are $p_1 = (1,1), p_2 = (3,2), p_3 = (1,2), p_4 = (3,1)$. The firms are located at the high demand points, $x = p_4$ and $y = p_3$. The resulting optimum quality levels are $q_1 = 2.12, q_2 = 1.06$ with corresponding market capture $M = (0.5, 0.8, 0, 1)$. The resulting profit $\Pi_1 = 15.5$ and $\Pi_2 = 7.3$.

If we now change the cost functions to quadratic ones, $c_1(q_1) = q_1^2, c_2(q_2) = 2q_2^2$, the optimum quality levels are $q_1 = 1.08, q_2 = 0.76$. The resulting profit $\Pi_1 = 15.67$ and $\Pi_2 = 9.01$.

The consequence of explicit expressions is that we now also have expressions for the resulting market share. Substituting (8) for the linear model into (1) gives

$$M_i(x, y, q_1^*, q_2^*) = \frac{\alpha_2 d_i(y)}{\alpha_2 d_i(y) + \alpha_1 d_i(x)} \tag{10}$$

and for the quadratic cost substituting (9) gives

$$M_i(x, y, q_1^*, q_2^*) = \frac{\sqrt{\alpha}_2 d_i(y)}{\sqrt{\alpha}_2 d_i(y) + \sqrt{\alpha}_1 d_i(x)}. \tag{11}$$

The most important is that this leads to explicit expressions for the profit functions. Substitution of (10) and (8) or (9) in (3) gives for the linear case

$$\Pi_1(x, y, q_1^*, q_2^*) = \alpha_2^2 \sum w_i \frac{d_i(y)^2}{(\alpha_2 d_i(y) + \alpha_1 d_i(x))^2} \quad \text{if } x \neq y. \tag{12}$$

and for the quadratic cost function

$$\Pi_1(x, y, q_1^*, q_2^*) = \sum w_i \frac{\alpha_2 d_i(y)^2 + \frac{1}{2}\sqrt{\alpha_1 \alpha_2} d_i(x) d_i(y)}{(\sqrt{\alpha}_2 d_i(y) + \sqrt{\alpha}_1 d_i(x))^2} \quad \text{if } x \neq y. \tag{13}$$

Given the optimum levels of quality, the next question is what are the Nash equilibria in terms of location. We can use the profit functions, where Π_2 is symmetric to Π_1. We are dealing with a kind of bi-level program where given locations the optimum quality levels are generated and substituted.

3.2. No co-location, deterministic model

The analysis is not simple for the binary model due to discontinuities in M_i with respect to the location vectors x, y and quality level. Intuitively, one invests in quality up to the moment the market share gain is smaller than the marginal cost of investment. It is convenient now to define index sets with respect to the captured customers. Let $I_1(x, y, q_1, q_2)$ be the index set of captured demand points by 1, $I_2(x, y, q_1, q_2)$ the index set of demand points captured by 2 and $II(x, y, q_1, q_2)$ the indefinite one, where they break tie and do fifty-fifty.

$$\begin{aligned} I_1(x,y,q_1,q_2) &:= \{i \mid q_1 d_i(y) > q_2 d_i(x)\} \\ II(x,y,q_1,q_2) &:= \{i \mid q_1 d_i(y) = q_2 d_i(x)\} \end{aligned} \tag{14}$$

The concept of the index sets is illustrated in Figure 1 and Figure 2. One can observe the patronising behaviour where customers choose for one of the facilities. By increasing the quality, a company can attract customers that are relatively far away. Whether this is profitable depends on the cost functions c.

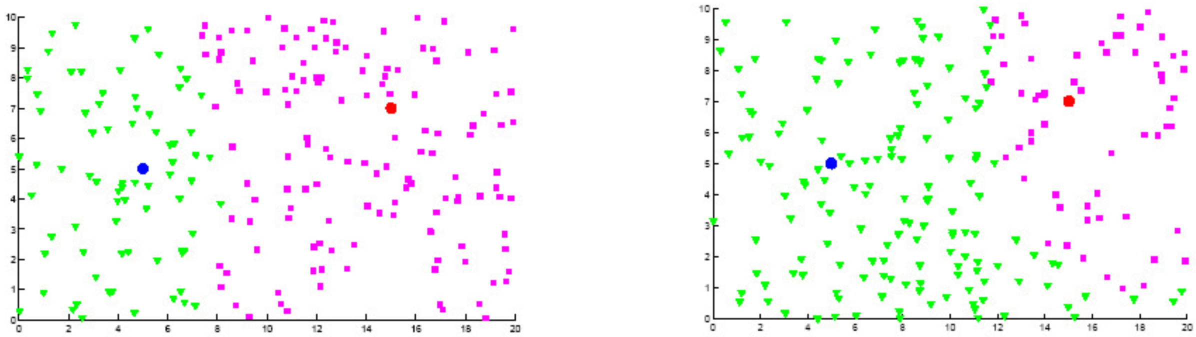

Figure 1 Patronising behaviour. Left figure: $q_1 = 1$ (facility x, blue); $q_2 = 2$ (facility y, red); Right figure: $q_1 = 2$; $q_2 = 1$

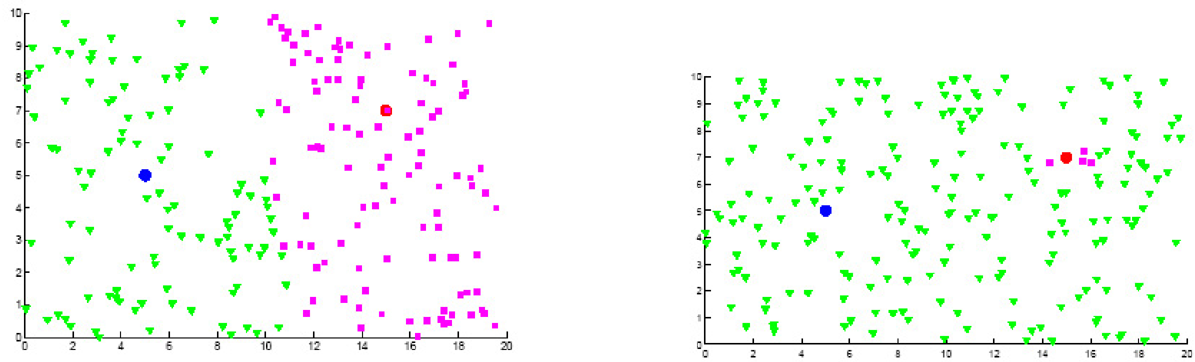

Figure 2 Patronising behaviour. Left figure: $q_1 = 1$ (facility x, blue); $q_2 = 2$ (facility y, red); Right figure: $q_1 = 2$; $q_2 = 1$

First of all, more clear than in the probabilistic model is that the maximum market capture is $\sum w_i$. Therefore, the values of the quality q_j should be situated in the range $0 \le q_j \le c_j^{-1}(\sum w_i)$.

If no co-location takes place, locating at a demand point leads to capturing its full demand. Let us focus on one (last) demand point p_i which firm 1 and 2 want to concur. The other demand points are already divided and for that firm 1 has reached q_1' and firm 2 q_2'. Firm 1 wants to set $q_1 > \frac{d_i(x)}{d_i(y)} q_2$ and is prepared to invest additionally at most w_i and firm 2 wants to set $q_2 > \frac{d_i(y)}{d_i(x)} q_1$. No Nash equilibrium is reached. Both firms will set their quality ε more up to the increase in investment costs $c_j(q_j) - c_j(q_j') = w_i$. At that moment the relatively cheaper firm (or being closer to the demand point) let say again firm 1, is going to a higher level capturing all demand w_i. This is not optimal for the more expensive firm 2 setting again $q_2 = q_2'$. This makesd firm 1 decrease the necessary quality, where firm 1 starts to invest again. This means no stable Nash equilibrium exists.

3.3. Co-location, probabilistic model

In the case of co-location, both competitors locate at the same point $x = y$. For the attraction this means that $d_i(x) = d_i(y)$. This gives in the probabilistic model $M_i = \frac{q_1}{q_1+q_2}$. An interesting question is what this means for the optimum quality level in case $c_1 \neq c_2$. Let us consider the basic linear case where Exercising with the first order conditions (6) gives optimum quality levels

$$q_1^* = \frac{\alpha_2 \sum_i w_i}{(\alpha_1 + \alpha_2)^2} \quad ; \quad q_2^* = \frac{\alpha_1 \sum_i w_i}{(\alpha_1 + \alpha_2)^2}. \tag{15}$$

For the quadratic cost function $c_j(q_j) = \alpha_j q_j^2$, first order conditions give

$$q_1^2 = \sqrt{\frac{\alpha_2}{\alpha_1}} \times \frac{\sum_i w_i}{2(\sqrt{\alpha}_1 + \sqrt{\alpha}_2)^2} \quad ; \quad q_2^2 = \sqrt{\frac{\alpha_1}{\alpha_2}} \times \frac{\sum_i w_i}{2(\sqrt{\alpha}_1 + \sqrt{\alpha}_2)^2}. \tag{16}$$

The optimum costs are the same for both firms. In the linear case $c_1(q_1^*) = c_2(q_2^*) = \frac{\alpha_1 \alpha_2}{(\alpha_1+\alpha_2)^2} \sum_i w_i$ and for the quadratic cost function $c_1(q_1^*) = c_2(q_2^*) = \frac{\sqrt{\alpha}_1 \alpha_2}{2(\alpha_1+\alpha_2)} \sum_i w_i$. the market share differs for both firms. In the linear case, $M_i = \frac{\alpha_2}{\alpha_1+\alpha_2}$. This means that if firm 1 is the cheapest, it obtains more profit. To be exact for the linear cost function

$$\Pi_1 = \frac{\alpha_2^2}{(\alpha_1 + \alpha_2)^2} \sum_i w_i \quad \text{if } x = y. \tag{17}$$

and for the quadratic cost function

$$\Pi_1 = \sum w_i \frac{\alpha_2 + \frac{1}{2}\sqrt{\alpha}_1}{(\sqrt{\alpha}_2 + \sqrt{\alpha}_1)^2} \quad \text{if } x = y. \tag{18}$$

Is it rational to co-locate? For instance if we have one demand point, or all demand is concentrated around it, this seems natural. Moreover, notice that after deciding to co-locate, the set of optimum solutions x^*, y^* is the complete plane. Let us consider the case of two demand points p_1, p_2 with corresponding demand w_1, w_2.

Example 2. Let $w_1 = 10$ and $w_2 = 17$. Consider the linear cost model with $\alpha_1 = 1$ and $\alpha_2 = 2$. If both firms co-locate they both have a quality cost of $c_1(q_1^*) = c_2(q_2^*) = \frac{2}{9} 27 = 6$ and the profit is $\Pi_1 = \frac{4}{9} 27 = 12$ and $\Pi_2 = \frac{1}{9} 27 = 3$.

For this data, the Nash equilibrium is $(x, q_1) = (p_2, 0)$ and $(y, q_2) = (p_1, 0)$ giving $\Pi_1 = 17$ and $\Pi_2 = 10$, because this is the best for both; none of the firms has any gain to deviate from that. Moreover, in this two point case, the exact locations p_1, p_2 do not matter. The decision to co-locate depends heavily on the cost and demand data of the instance; if we double demand at both points, it is better for firm 1 to co-locate; its profit becomes $\Pi_1 = 24$. Similarly, if firm 1 could decrease cost to $\alpha_1 = 0.5$, it also becomes better for it to co-locate giving $\Pi_1 = 17.24$

3.4. Co-location, deterministic model

Let us now consider the case of co-location, not at a demand point p_i. This means that market capture is either 0, 1 or $\frac{1}{2}$. In the non-symmetric case where $c_1 \neq c_2$ (let firm 1 again be the cheaper), this leads to a situation where we don't have a Nash equilibrium. Having the same level $q_1 = q_2$ is not an equilibrium, as both firms can gain half the market share by increasing the quality by a very small amount ε. Having $q_1 > q_2 > 0$ neither is, as the loosing firm 2 will prefer not to make any costs and goes for $q_2 = 0$. That leads to a nonoptimal strategy for firm 1, etc. One can observe now that a model where co-location at a demand point gives deterministic patronising according to equation (2), does lead to a model without any Nash equilibrium in terms of quality choice following the same reasoning.

The last case where a Nash equilibrium might exist is in the smoothed variant where the market share of the point $x = y = p_i$ where both facilities are located is divided proportionally. Consider again $q_1 > q_2$ with a "small" value for q_1. Now firm 2 gets a sales of $w_i \frac{q_2}{q_1+q_2}$ with a small investment q_2. The linear model gives an optimum close to the earlier result of (15)

$$q_1^* = \frac{\alpha_2 w_i}{(\alpha_1+\alpha_2)^2} \quad ; \quad q_2^* = \frac{\alpha_1 w_i}{(\alpha_1+\alpha_2)^2}. \tag{19}$$

Notice that an equilibrium does not exist here for $\alpha_1 = \alpha_2$, due to the discontinuity of reaching half of the rest of the market by increasing the level a bit. So, the deterministic model has only a quality equilibrium for the nonsymmetric case, co-locating at one of the demand points. Notice that this is most attractive for the more expensive firm at a highest demand point. Keeping the situation $\alpha_1 < \alpha_2$ gives payoff

$$\Pi_1 = \frac{\alpha_2^2}{(\alpha_1+\alpha_2)^2} w_i + \sum_{j \neq i} w_j, \Pi_2 = \frac{\alpha_1^2}{(\alpha_1+\alpha_2)^2} w_i \quad \text{if } x = y = p_i. \tag{20}$$

Concluding, the deterministic model has a different nature than the probabilistic model from the point of view of Nash equilibria. Only in the second variant when both firms locate at a point with highest demand, a Nash equilibrium exists, where the payoff is given by (20).

4. Location equilibria

In Section 3 we found expressions for the optimum level of quality investment $q_1^*(x,y), q_2^*(x,y)$. For the probabilistic model using a linear and quadratic cost function these are explicit like in (12) and (13). For the deterministic model, we found that only co-location at a demand point may give a stable result. As the more expensive firm wants to reach the highest profit, it locates at a point with highest demand. The next question is to find the Nash equilibria given the resulting profit functions for the probabilistic model.

4.1. Nash locations probabilistic model

A usual procedure is to iterate in a local search fashion; given location of y we optimise over x, then we optimise over y again etc. In our case, the profit (payoff) to be optimised is a global optimisation problem. We focus on the two cases we derived in Section 3. The specific subproblems to be solved are the following. Given total demand $\sum w_i$, cost coefficient vector a, and location of competitor v

$$\max_u \pi(u) = \begin{cases} a_2^2 \sum w_i \frac{d_i(v)^2}{(a_2 d_i(v)+a_1 d_i(u))^2} & \text{if } u \neq v \\ \frac{a_2^2}{(a_1+a_2)^2} \sum w_i & \text{if } u = v \end{cases} \tag{21}$$

for the linear cost function and

$$\max_u \pi(u) = \begin{cases} \sum w_i \frac{\alpha_2 d_i(v)^2 + \frac{1}{2}\sqrt{\alpha}_1 d_i(u) d_i(v)}{(\sqrt{\alpha}_2 d_i(v)+\sqrt{\alpha}_1 d_i(u))^2} & \text{if } u \neq v \\ \frac{a_2+\frac{1}{2}\sqrt{a}_1}{(\sqrt{a}_2+\sqrt{a}_1)^2} \sum w_i & \text{if } u = v \end{cases} \tag{22}$$

for the quadratic cost function.

Problem (21) is a Global Optimisation problem as illustrated in Figure 4.1. We developed a specific Branch-and-Bound procedure where upper bounds are based on interval considerations like in Sáiz et al. (2009). Such an algorithm guarantees that we reach the global optimum wiht a predefined accuracy. However, the results for all instances show that the optimum is attained in demand points.

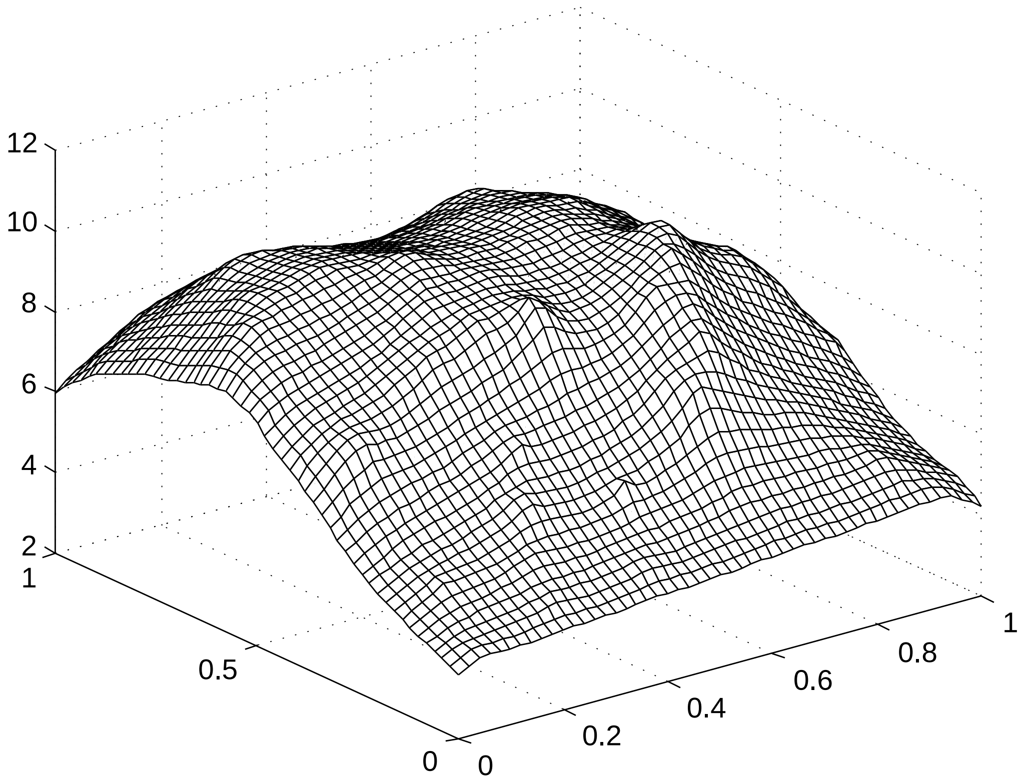

Figure 3 Level of payoff of firm 1 of an instance with 100 random demand points and firm 2 is already located

The consequence of this result is that it is feasible to enumerate all demand points as candidates. One can take the value or co-location if the oponent is met as given in (21). This operation requires $k*n$ function evaluations, where k is the number of iterations needed to converge to a Nash equilibrium.

Alternatively, one can also do a full enumeration. In a similar recent study, Sáiz and Hendrix (2008), a 2 level game required all possible configurations to be enumerated and compared to detect Nash equilibria. Alternatively, also a local search approach was proposed which always converged to a Nash equilibrium, where due to symmetry, several may exist.

In our case, the result of the calculation is a bi-matrix game, where firm 1 is maximising over the columns of a payoff matrix $V1$ and firm 2 is maximising over the rows of a payoff matrix $V2$. First of all, we define matrix $V1$ with entrances $V1_{ij} = \Pi_1(p_i, p_j, q_1^*, q_2^*)$ and its equivalent $V2$. This requires the generation of $2n \times (n-1)$ evaluations. The diagonal elements are given by the co-location result and all have the same value. An algorithm to find all Nash equilibria is given by Algorithm 1.

Example 3. Consider again the instance of Example 2. The resulting payoff matrices are

$$V1 = \begin{pmatrix} 12 & 10 \\ 17 & 12 \end{pmatrix} \text{ and } V2 = \begin{pmatrix} 3 & 17 \\ 10 & 3 \end{pmatrix}.$$

Algorithm 1 Nashloc($V1, V2, p$)

for $(i := 1 \text{ to } n)$ do
  for all $j \in \arg\max_l V2_{il}$
    if $(i \in \arg\max_k V1_{kj})$
      store $x = p_i, y = p_j$ as Nash equilibrium
endfor

The Nash equilibrium is $x = p_1$ giving the maximum for firm 1 over the first column of $V1$ and $y = p_2$ giving the maximum for firm 2 over the second row of $v2$. To illustrate a situation with co-locating equilibrium we change the demand data into $w = (1, 17)$. The resulting payoff is

$$V1 = \begin{pmatrix} 8 & 1 \\ 17 & 8 \end{pmatrix} \text{ and } V2 = \begin{pmatrix} 2 & 17 \\ 1 & 2 \end{pmatrix},$$

where the equilibrium is $x = y = p_2$; only co-locating at p_2 gives an equilibrium. The most interesting case is when there is no Nash equilibrium. This happens if the costs differ substantially. Consider $w = (10, 15)$ and $\alpha = (1, 4)$. The payoff matrices are

$$V1 = \begin{pmatrix} 16 & 15 \\ 10 & 16 \end{pmatrix} \text{ and } V2 = \begin{pmatrix} 1 & 10 \\ 15 & 1 \end{pmatrix},$$

The result is that the strong (low cost) firm wants to co-locate and the expensive firm prefers to be alone.
From the formulas, the linear cost model teaches us that the cheapest firm (say 1) is only not interested to co-locate, if it can get more than $\Pi_1 = (\frac{\alpha_2}{\alpha_1+\alpha_2})^2 \sum w_i$. This is less likely to happen if the difference between α_1 and α_2 is bigger. As the more expensive firm wants to be alone, there is less tendency of having a Nash equilibrium. If both firms have equal costs, $\alpha_1 = \alpha_2$, each instance where the demand weight can be split in (nearly) equal weighted sets $\sum_{I1} w_i \approx \sum_{I2} w_i$ seperated by a line, it is better for the firms to separate such that $d_i(x) < d_i(y), i \in I1$. This results into a higher profit than $\Pi_1 = \frac{1}{4} \sum w_i$ which can be reached by co-location, as can be directly observed from (21). This is more likely to happen, if the number of demand points is bigger, or the demand is more similar.

The same tendency can be obtained by analysing the quadratic cost case. If costs are similar, co-location is less interesting if demand can be split more or less equally, i.e. having more similar demand points. Equal costs give a co-location result of $\Pi_1 = (\frac{1}{4} + \frac{1}{8\sqrt{\alpha}}) \sum w_i$. Higher costs makes co-location less attractive and splitting demand is better. Compared to the linear case, the eagerness to want to co-locate for the cheaper firm is less with differing cost parameters α_1, α_2. This means, there is less tendency not to reach a Nash equilibrium than the linear case.

4.2. Numerical experiments

To observe the tendency of existence of Nash equilibria and appearance of co-location we did a numerical experiment generating 1000 instances with $n = 5$ and $n = 50$ demand points, where weights are uniformly randomly generated from $[0, 10]$. We took $\alpha_1 = 1$ and systematically varied $\alpha_2 = 1, 2, 4, 16$. Running Algorithm 1, we obtained the corresponding Nash equilibria which are classified in Tables 1 and 2.

The results show that co-location does not take place, unless the costs of the two firms are similar, where the quadratic cost give more tendency to lead to Nash equilibria with co-located firms. At most one Nash equilibrium exists in case of co-location. One can observe that for randomly generated cases, as soon as costs differ substantially, no Nash equilibrium exists, due to the effect we described before; firm 1 prefers to co-locate and firm 2 is running away to be alone. Notice that as soon as cost equal completely, at least two symmetric Nash equilibria exist if the firms do not co-locate.

Table 1 Tendency of Nash equilibria, linear cost model, 1000 randomly generated instances. nonash: number of instances without Nash equilibrium, col: co-location, nc: no co-location, avn: average number of nc equilibria

	$n=5$				$n=50$			
α_2	nonash	col	nc	avn	nonash	col	nc	avn
1	2	6	992	2	19	0	981	2.05
2	1	0	999	1.4	9	0	991	1.16
4	825	0	175	1	666	0	334	1.003
16	999	0	1	1	998	0	2	1

Table 2 Tendency of Nash equilibria, quadratic cost, 1000 randomly generated instances. nonash: number of instances without Nash equilibrium, col: co-location, nc: no co-location, avn: average number of nc equilibria

	$n=5$				$n=50$			
α_2	nonash	col	nc	avn	nonash	col	nc	avn
1	3	148	849	2	42	26	932	2.01
2	1	28	971	1.6	3	0	997	1.54
4	31	11	958	1.002	152	0	848	1.03
16	919	0	81	1	882	0	118	1

5. Conclusions

A location-quality model with a Huff like market capture has been described where two competing supplier firms locate in a planar market and decide on the investment of quality. The question is what characterises the Nash equilibria in such a model and how can they be determined. Another question is whether co-location is a natural Nash equilibrium in such models. Two variants are described; one where demand capture is probabilistic taking a value typically between 0 and 1, and a deterministic model, where customers mainly choose for the most attractive supplier.

We found the following results.

• first order conditions on optimum quality level are relatively easy for the probabilistic model. Given location decisions, they lead to a set of equalities with two variables.

• for the probabilistic model with a linear and quadratic cost function explicit expressions are derived for the payoff given the location decisions.

• in the deterministic model optimality conditions are not that easy due to disconituities.

• a Nash equilibrium on quality decisions in the deterministic model usually does not exist. Only if the breaking tie for co-locating at a demand point is described proportionally and their cost function is different, the competing firms decide to co-locate at one of the demand points.

• only in the latter case, both firms co-locate at a demand point with highest demand.

• for the probabilistic model with linear and quadratic cost functions the firms decide to locate at demand points.

• Nash equilibria can be found by solving a bi-matrix game for the probabilistic model.

• if cost structures differ more, there are more randomly generated instances that do not have a Nash equilibrium.

• co-location in the probabilistic model appears more in randomly generated instances if the number of demand points is low and the cost functions of the competing firms are more alike.

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