Journal of Control and Decision

ISSN: 2330-7706 (Print) 2330-7714 (Online) Journal homepage: www.tandfonline.com/journals/tjcd20

Existence and computation of linear state-transformations into the controllability/observability canonical forms of large-scale nonlinear systems

L. Gerard Van Willigenburg

JOURNAL OF CONTROL AND DECISION
https://doi.org/10.1080/23307706.2025.2546098

Existence and computation of linear state-transformations into the controllability/observability canonical forms of large-scale nonlinear systems

L. Gerard Van Willigenburg

Mathematical and Statistical Methods Group (Biometris), Wageningen University, Wageningen, The Netherlands

ABSTRACT

By circumventing symbolic Lie algebraic computations, sensitivity-based algorithms very efficiently compute controllability/observability of *large-scale nonlinear* dynamical systems. Also, they provide controllability/observability signatures, pinpointing the state variables involved in uncontrollable/unobservable modes. One contribution of this paper is to show that these algorithms can also be used to determine whether the transformation into the controllability/observability canonical form is linear or not. If linear, this paper also reveals how this transformation is obtained from the sensitivity-based algorithm. Another contribution is to show that this linear transformation can also be obtained from standard algorithms, putting linear time-invariant systems into the controllability/observability canonical form. Examples of medium- and large-scale nonlinear dynamical systems are presented to illustrate these contributions.

ARTICLE HISTORY
Received 18 March 2025
Accepted 6 August 2025

KEYWORDS
Canonical forms; observability; accessibility; sensitivity rank condition (SERC); state-transformation; large-scale nonlinear dynamical systems

1. Introduction

Canonical forms of both linear and nonlinear dynamical systems reveal in a straightforward manner whether individual state-variables are connected to the system input/output or not. The unconnected state-variables are the ones that are uncontrollable/unobservable (Isidori, 1995; Kwatny & Blankenship, 2000; Nijmeijer & van der Schaft, 1990; Van Willigenburg, 2024). The lack of a connection to the input causes a zero sensitivity of the corresponding state-variable to the input. Similarly, a lack of a connection to the output causes a zero sensitivity of the output to the corresponding state-variable. If the system is not represented in a canonical form, uncontrollable/unobservable state-variables turn into uncontrollable/unobservable modes being combinations of all state-variables as determined by the state-transformation into the canonical form, see Figure 1.

Zero sensitivities to the input of uncontrollable modes, and zero sensitivities of the output to unobservable modes, lie at the basis of sensitivity-based algorithms to determine controllability/observability of possibly large-scale nonlinear dynamical systems. Zero sensitivities cause rank-deficiency of the sensitivity matrix computed by the sensitivity-based algorithm indicating uncontrollability/unobservability. Moreover, the singular value decomposition (SVD) used to determine the (numerical) rank of the sensitivity matrix also provides right singular vectors. The nonzero components of the right singular vectors, corresponding with zero singular values indicate the state-variables involved in the uncontrollable/unobservable modes (Stigter et al., 2018; Van Willigenburg, 2024; Van Willigenburg et al., 2022). Generally, the latter is sufficient information for control system analysis and design.

To obtain further insight, in some cases, one would like to obtain the controllability/observability canonical form explicitly. To obtain these, one requires the state-transformation into the canonical form, the computation of which is avoided by sensitivity-based algorithms. This enables them to be very efficient, even for large-scale systems (Wieland et al., 2021) by avoiding symbolic Lie algebraic computations (Stigter et al., 2018; Van Willigenburg et al., 2022). One contribution of this paper is to show how sensitivity-based algorithms *can* be used to determine whether this state-transformation is linear, or not. If linear, this paper also shows how to *obtain* this transformation from the sensitivity-based algorithm. Another contribution of this paper is to show that the existence and computation of such a linear state-transformation can also be decided and performed by standard algorithms putting linear time-invariant systems into the controllability/observability canonical form, such as ctrbf/obsvf in Matlab and the ones in Van Dooren (1981). Examples of medium- and large-scale nonlinear dynamical systems are presented to illustrate these results.

Although there is a large body of literature on controllability, observability and corresponding canonical forms, the results presented in this paper seem to have

CONTACT L. Gerard Van Willigenburg gerard.vanwilligenburg@wur.nl Mathematical and Statistical Methods Group (Biometris), Wageningen University, P.O. Box 17, Wageningen 6700 AA, The Netherlands

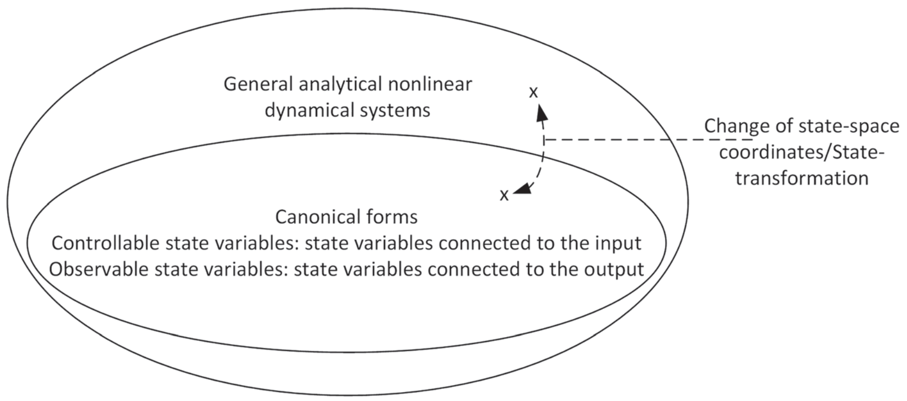

Figure 1. Canonical forms facilitating simple definitions/explanations of controllability and observability as connectivity's to the input and output. Changes of coordinates/state-transformations connect analytical nonlinear dynamical systems to their canonical forms.

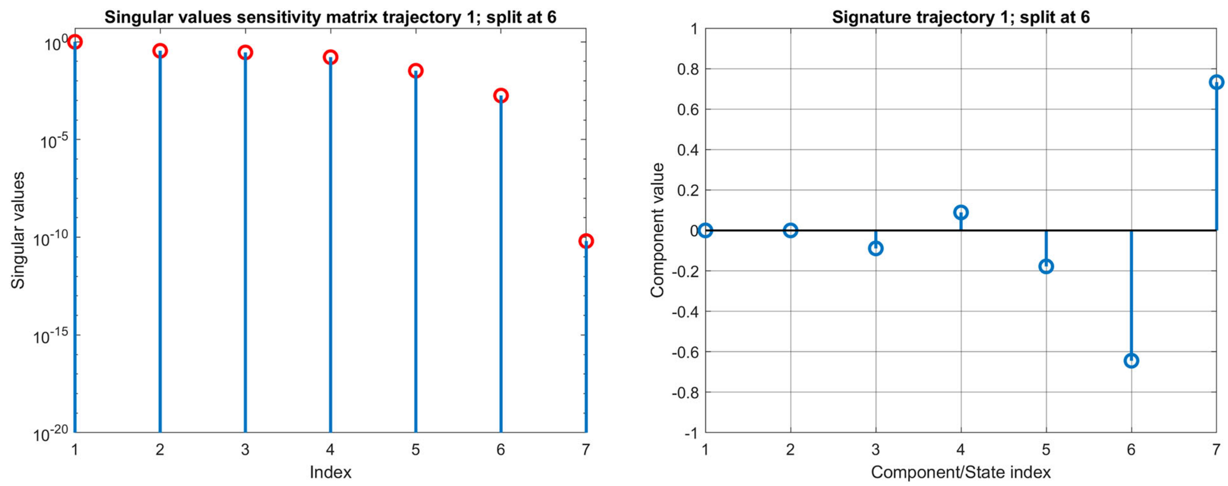

Figure 2. Singular values and controllability signature of the CSTR with three control variables as obtained from the sensitivity-based algorithm. The left panel shows one numerically zero singular value corresponding to a single uncontrollable mode. The right panel shows the corresponding controllability signature indicating that the single uncontrollable mode is a combination of state-variables 3–7.

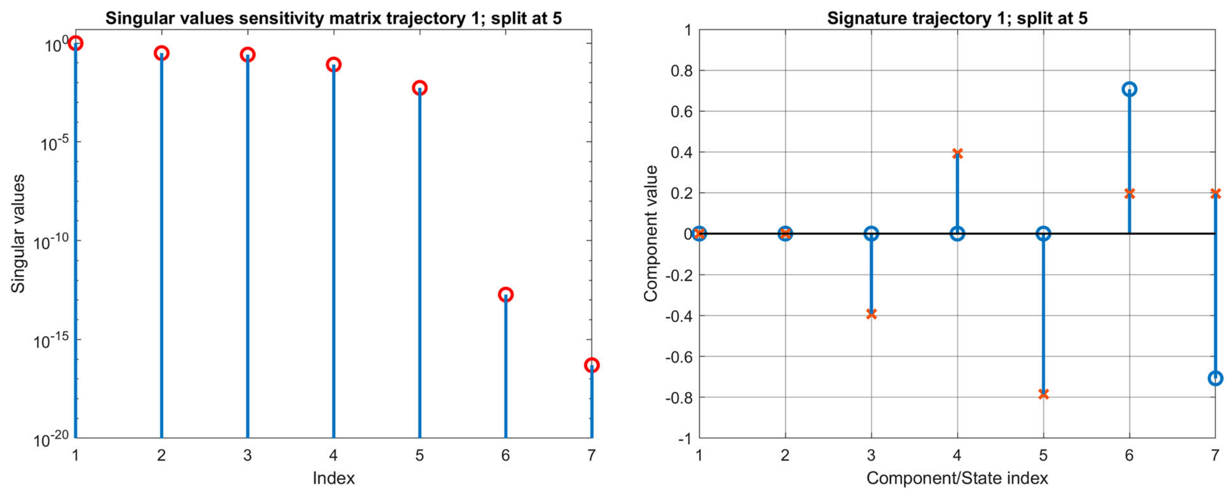

Figure 3. *Singular values and controllability signature of the CSTR with two control variables as obtained from the sensitivity-based algorithm. The left panel shows two numerically zero singular values corresponding to two uncontrollable modes. The right panel shows the corresponding controllability signature indicating that the two uncontrollable modes involve state-variables 3–7.*

gone unnoticed (Boley, 1980, 1984; Boutat & Busawon, 2011; Conte et al., 1995; Kalman, 1962; Zeitz, 1990; Zhirabok, 2008, 2010). As to the terminology used in this paper, what is called controllability and observability for linear and nonlinear systems in this paper, for nonlinear systems is most often called local strong accessibility and local observability in the literature (Nijmeijer & van der Schaft, 1990).

Section 2 presents the state-space representation of nonlinear dynamical systems together with trajectories along which sensitivity-based algorithms perform computations. It ends with an important remark concerning the local character of controllability/observability. Section 3 introduces sensitivity-based algorithms computing controllability/observability of nonlinear, possibly large-scale, dynamical systems. Section 4 provides the main results that are illustrated in Section 5 with small and large-scale examples. Finally, Section 6 presents conclusions.

2. Trajectories of nonlinear dynamical systems and the local character of controllability/observability

To facilitate their analysis, numerical solution and control, dynamical systems described by ordinary differential equations are often represented in the so-called state-space form given by

$$\dot{x}(t) = f(x(t), u(t)),\ \ x \in \mathbb{R}^{n_x},\ \ u \in \mathbb{R}^{n_u} \tag{1}$$

$$y(t) = g(x(t)),\ \ y \in \mathbb{R}^{n_y}. \tag{2}$$

Within this formulation t denotes time, $x \in \mathbb{R}^{n_x}$ is the *state-vector* or *state* collecting all *state-variables* x_i, $i = 1, 2, .., n_x$, $u \in \mathbb{R}^{n_u}$ is the *input-vector* or *input* collecting all *input-variables* $u_i \in \mathbb{R}$, $i = 1, 2, .., n_u$ and $y \in \mathbb{R}^{n_y}$ is the *output-vector* or *output* collecting all *output-variables* y_i, $i = 1, 2, .., n_y$. Equation (1) describes how the state $x(t)$ propagates and depends on the input $u(t)$ and is called the *state-equation*. Equation (2) describes how the state $x(t)$ maps on the output $y(t)$ and is called the *output-equation*. To be able to use results from nonlinear system theory, f is assumed to be affine in the input

$$f(x(t), u(t)) = f_0(x(t)) + \sum_{i=1}^{n_u} f_i(x(t)) u_i(t),$$
$$i = 1, 2, .., n_u,$$
$$f_i(x(t)) \in \mathbb{R}^{n_x}, i = 0, 1, 2, .., n_u. \tag{3}$$

Let

$$x^*(t), u^*(t), y^*(t), t_0 \leq t \leq t_N \tag{4}$$

denote a state, control and output trajectory satisfying Equations (1)–(3) with t_0, t_N representing the initial and final time, respectively.

Remark 2.1: Controllability/observability properties of dynamical systems (1)–(3) are determined by the number of what are called uncontrollable/unobservable modes in Van Willigenburg et al. (2022); Van Willigenburg (2024), see also Figure 1. Only if there are no such modes, the system is controllable/observable.

Remark 2.2: According to the Hermann-Nagano theorem (Hermann & Krener, 1977), if the system (1)–(3) is analytic, meaning f, g in (1)–(3) are analytic vector functions, as assumed throughout this paper, the number of uncontrollable/unobservable modes does not change along trajectories (4). But this number may still depend on the initial state $x(t_0)$. In the latter case, there is a part of the state-space consisting of what are called *singular states/points*, where this number is increased. This part of the state-space always is very small in the sense that when selecting the initial state $x(t_0)$ of trajectory (4) randomly, almost surely guarantees the trajectory (4) to consist of regular states $x(t)$ called *a regular trajectory*. Still, this phenomenon causes controllability/observability to be a local property in general.

Remark 2.3: : Despite a random choice of $x(t_0)$, one may still be in doubt about the regularity of trajectory (4). When selecting more trajectories, the large majority will be regular, all having the same minimal number of uncontrollable/unobservable modes.

Remark 2.4: : Sensitivity-based algorithms compute sensitivities along short trajectories (4) providing a sensitivity rank condition (SERC) that replaces the Lie algebraic rank condition (LARC). For their outcome to be correct, these trajectories must have the property that SERC and LARC provide the same outcome. Such trajectories are called *non-singular* in Van Willigenburg et al. (2022). These non-singular trajectories are *different* from the regular trajectories mentioned in Remark 2.2.

Remark 2.5: : Given $x^*(t_0)$, choosing $u^*(t)$ of trajectory (4) *randomly*, almost surely guarantees it to be non-singular, as specified in Remark 2.4.

Remark 2.6: : From Remarks 2.1–2.5, choosing both $x^*(t_0)$ and $u^*(t)$ of trajectory (4) *randomly* almost surely guarantees it to be regular, as mentioned in Remark 2.2, as well as non-singular, as mentioned in Remark 2.4. Similar to what is mentioned in Remark 2.3, when selecting several trajectories the large majority will be both regular and non-singular providing the same outcome of SERC and LARC.

3. Sensitivity-based algorithms

A nonlinear dynamical system failing controllability/observability may be interpreted as certain internal connections to the input/output failing, causing corresponding sensitivities to be zero (Van Willigenburg, 2024; Van Willigenburg et al., 2022), see Figure 1. Sensitivity-based algorithms are able to detect these zero sensitivities very efficiently, even for large-scale nonlinear dynamical systems. In this section, the important link between controllability/observability of nonlinear dynamical systems and the variational system along trajectories, that is a linear time-varying system, will be established. This link underlies sensitivity-based algorithms. Considering only observability suffices, since dual results will hold for controllability.

For the nonlinear dynamical system (1)–(3), let $x_{x_0}(t) \overset{\text{def}}{=} dx(t)/dx(t_0) \in \mathbb{R}^{n_x \times n_x}$ denote the sensitivity of state $x(t)$ for the initial state $x(t_0) \overset{\text{def}}{=} x_0$, called the state sensitivity. Similarly, let $y_{x_0}(t) \overset{\text{def}}{=} dy(t)/dx_0 \in \mathbb{R}^{n_y \times n_x}$ denote the sensitivity of the output $y(t)$ for the initial state, called the output sensitivity. State sensitivity $x_{x_0}(t)$ and output sensitivity $y_{x_0}(t)$ are both matrix functions related as follows

$$y_{x_0}(t) = \frac{\partial g}{\partial x} x_{x_0}(t). \tag{5}$$

where $x_{x_0}(t)$ is governed by the dynamics

$$\frac{d}{dt} x_{x_0}(t) = \frac{\partial f}{\partial x} x_0(t), x_{x_0}(t_0) = I_{n_x}, t \geq t_0. \tag{6}$$

The initial condition $x_{x_0}(t)$ is the sensitivity of the initial state to itself and so I_{n_x} denotes the identity matrix of dimension n_x. When evaluated along trajectories (4)

$$\frac{\partial f}{\partial x} = \left. \frac{\partial f}{\partial x} \right|_{x(t)=x^*(t),\ u(t)=u^*(t)} \overset{\text{def}}{=} A(t). \tag{7}$$

Then differential Equation (6) becomes linear, time-varying and autonomous

$$\frac{d}{dt} x_{x_0}(t) = A(t) x_{x_0}(t),\ \ x_{x_0}(t_0) = I_{n_x}. \tag{8}$$

Concerning the sensitivity-based algorithms a key observation (Van Willigenburg et al., 2022) is that

$$x_{x_0}(t) = \Phi(t, t_0),\ \ t \geq t_0 \tag{9}$$

where $\Phi(t, t_0)$ is the state-transition matrix of the linear time-varying autonomous system

$$\frac{d\delta x(t)}{dt} = A(t)\delta x(t),\ \ \delta x(t) \in \mathbb{R}^{n_x}. \tag{10}$$

where $\delta x(t)$ represent *infinitesimal* deviations of state $x(t)$ from $x^*(t)$ of trajectory (4). When evaluating $y_{x_0}(t)$ in Equation (5) along trajectories (4)

$$\frac{dg}{dx} = \left. \frac{dg}{dx} \right|_{x(t)=x^*(t),\ u^*(t)=u(t)} \overset{\text{def}}{=} C(t). \tag{11}$$

If the trajectory (4) is non-singular, observability of the nonlinear dynamical system (1)–(3) along this trajectory is equivalent with sensitivity matrix function $y_{x_0}(t)$ having n_x independent columns (Van Willigenburg et al., 2022). From (5), (8), (9), (11), this is equivalent with the linear time-varying variational system (10) with output equation

$$\delta y(t) = C(t)\delta x(t) \tag{12}$$

being observable, because for this, matrix function $C(t)\Phi(t, t_0) = y_{x_0}(t)$ should have n_x independent columns (Weiss & Kalman, 1965) (where it is called C-observable). Similar to (10), $\delta y(t)$ represents *infinitesimal* deviation of the output $y(t)$ from $y^*(t)$ of trajectory (4).

In summary, *observability along a non-singular trajectory (4) of the nonlinear dynamical system (1)–(3) is equivalent to the linear time-varying variational system (10), (12) being observable. Dual results* (Van Willigenburg et al., 2021, 2022) *hold for controllability along a non-singular trajectory (4) and controllability of the time-varying linear variational system given by*

$$\begin{aligned}
\frac{d\delta x(t)}{dt} &= A(t)\delta x(t) + B(t)\delta u(t), \\
A(t) &= \left. \frac{\partial f}{\partial x} \right|_{x(t)=x^*(t),\ u(t)=u^*(t)}, \\
B(t) &= \left. \frac{\partial f}{\partial u} \right|_{x(t)=x^*(t),\ u(t)=u^*(t)}.
\end{aligned} \tag{13}$$

Sensitivity-based algorithms numerically integrate the nonlinear system dynamics (1) along a short trajectory (4) together with differential (8). This provides $x_{x_0}(t)$, $t_0 \leq t \leq t_N$. To maximise accuracy, the numerical integration is performed demanding a high accuracy and $\partial f/\partial x$ in (7) is computed using automatic differentiation (Neidinger, 2010). To establish whether matrix function $y_{x_0}(t)$ has n_x independent columns, samples at times $t_0 < t_1 < .. < t_N$ of $y_{x_0}(t)$, given by Equation (5), are concatenated. Within Equation (5), $\partial g/\partial x$ is also computed using automatic differentiation. This provides output sensitivity matrix

$$S_o = \begin{bmatrix} y_{x_0}(t_0) \\ y_{x_0}(t_1) \\ y_{x_0}(t_2) \\ \vdots \\ y_{x_0}(t_N) \end{bmatrix}, N + 1 \geq n_x, \tag{14}$$

Dualising this result (Van Willigenburg et al., 2022) provides the input sensitivity matrix

$$S_c = \begin{bmatrix} u_{x_N}(t_N) \\ u_{x_N}(t_{N-1}) \\ u_{x_N}(t_{N-2}) \\ \vdots \\ u_{x_N}(t_0) \end{bmatrix}, x_N = x(t_N), N+1 \geq n_x. \quad (15)$$

where $u_{x_N}(t) \stackrel{\text{def}}{=} du(t)/dx(t_N)$. Starting from terminal state $x(t_N)$ of trajectory (4), values of $u_{x_N}(t)$ are obtained from backward in time integration of the nonlinear dynamics (1), (3) together with the dual version of sensitivity dynamics (8), (12) being

$$\frac{d}{dt}x_{x_N}(t) = A^T(t)x_{x_N}(t), x_{x_N}(t_N) = I_{n_x}, u_{x_N}(t) = B^T(t)x_{x_N}(t). \quad (16)$$

The nonlinear system (1)–(3) is controllable/observable along trajectory (4), if and only if rank$(S_c) = n_x$/*rank* $(S_o) = n_x$. In that case it satisfies what is called the *sensitivity rank condition* (SERC). If not, the sensitivity-based algorithm provides a *signature* indicating the cause of uncontrollability/unobservability.

Both the rank and signature are obtained from a singular value decomposition (SVD) of the sensitivity matrix S_c, respectively S_o. The number of zero singular values equals the number of uncontrollable/unobservable modes of the nonlinear system (1)–(3) along the trajectory (4). The signature consists of *the right singular vectors v_i corresponding to (numerically) zero singular values* of the sensitivity matrix. These right singular vectors are also obtained from the SVD and represent directions in which the state cannot be controlled/observed locally. Moreover, the *non-zero components* of these right singular vectors correspond to the state-variables of the nonlinear system (1)–(3) involved in the uncontrollable/unobservable modes (Van Willigenburg et al., 2022).

4. Computing linear state-transformations into the controllability and observability canonical form of nonlinear systems

Changes of coordinates representing a dynamical system will not change controllability and observability of such a system. A change of coordinates is realised by a state-transformation

$$x' = \Psi(x). \quad (17)$$

as Ψ is a diffeomorphism. By means of such state-transformations, controllability/observability canonical forms can be obtained (Isidori, 1995; Kwatny & Blankenship, 2000; Nijmeijer & van der Schaft, 1990). The controllability canonical form of system (1) is given by

$$\dot{x}' = f'(x', u), x' = \begin{bmatrix} x'^u \\ x'^{\bar{u}} \end{bmatrix} = \begin{bmatrix} \Psi^u(x) \\ \Psi^{\bar{u}}(x) \end{bmatrix},$$
$$f'(x', u) = \begin{bmatrix} f'^u(x'^u, x'^{\bar{u}}, u) \\ f'^{\bar{u}}(x'^{\bar{u}}) \end{bmatrix}. \quad (18)$$

Within the controllability canonical form (18), the transformed state x' separates into x'^u containing *controllable state-variables*, that depend on the input u and $x'^{\bar{u}}$ containing *uncontrollable state-variables*, that do not depend on the input u. A corresponding split of the state-transformation $\Psi(x)$ into $\Psi^u(x)$ and $\Psi^{\bar{u}}(x)$ is specified in (18). Each $\Psi_i(x)$, $i = 1, 2, .., n_x$ is a scalar function called a *mode* of system (1), (3). Therefore, $\Psi^u(x)$ are called the *controllable modes* and $\Psi^{\bar{u}}(x)$ the *uncontrollable modes* of system (1), (3) (Nijmeijer & van der Schaft, 1990). Their number $\dim(x'^u)$, $\dim(x'^{\bar{u}})$ can be obtained from the sensitivity-based algorithm applied to the nonlinear system (1) since $\dim(x'^{\bar{u}})$ is the number of (numerically) zero singular values $\sigma_i = 0$ obtained from the SVD of the sensitivity matrix S_c in (15) (Van Willigenburg et al., 2022) and $\dim(x'^u) = n_x - \dim(x'^{\bar{u}})$.

The observability canonical form of system (1)–(3) is given by

$$x' = \begin{bmatrix} x'^y \\ x'^{\bar{y}} \end{bmatrix} = \begin{bmatrix} \Psi^y(x) \\ \Psi^{\bar{y}}(x) \end{bmatrix}, f(x', u) = \begin{bmatrix} f^y(x'^y, u) \\ f^{\bar{y}}(x'^y, x'^{\bar{y}}, u) \end{bmatrix},$$
$$g(x') = g(x'^y) \quad (19)$$

Within the observability canonical form (19), x'^y contains the *observable state variables* on which the output y depends, whereas $x'^{\bar{y}}$ are *unobservable state variables* on which the output y does not depend. Given the corresponding split of $\Psi(x)$ in (19), $\Psi^y(x)$ contain the *observable modes* and $\Psi^{\bar{y}}(x)$ the *unobservable modes* of system Eqauitons (1)–(3). Their number $\dim(x'y)$, $\dim(x'^{\bar{y}})$ can be obtained from the sensitivity-based algorithm since $\dim(x'^{\bar{y}})$ is the number of (numerically) zero singular values $\sigma_i = 0$ obtained from the SVD of sensitivity matrix S_o in (14) and $\dim(x'y) = n_x - \dim(x'^{\bar{y}})$. Van Willigenburg (2024), states, inputs and outputs that do or do not depend on one another, are called *connected* and *disconnected* from one another.

In this paper, the special case where a *linear* state-transformation exists that realises these canonical forms is considered i.e.

$$\Psi(x) = Tx, \ T \in \mathbb{R}^{n_x \times n_x}, \ T \text{ non-singular}. \quad (20)$$

Lemma 4.1: *The linear time-invariant state-transformation* (20) *transforms the nonlinear system Equations* (1)–(3) *into,*

$$\dot{x}'(t) = f'(x'(t), u(t)) = Tf(T^{-1}x'(t), u(t)), \quad (21)$$
$$y(t) = g'(x'(t), u(t)) = g(T^{-1}x'(t), u(t)), \quad (22)$$

and the time-varying linear variational system (13), (12) *into,*

$$\delta\dot{x}'(t) = A'(t)\delta x'(t) + B'(t)\delta u(t), A'(t) = TA(t)T^{-1},$$
$$B'(t) = TB(t) \quad (23)$$
$$\delta y(t) = C'(t)\delta x'(t), C'(t) = C(t)T^{-1} \quad (24)$$

Proof: Equations (21), (22) follow from standard rules of differentiation and matrix inversion. The effect on the variational system follows from the fact that $\delta x'(t) = T\delta x(t)$.

When the linear time-invariant transformation (20) transforms system (1), (3) into the controllability canonical form (18), system matrices $A'(t)$, $B'(t)$, $C'(t)$ given by Lemma 4.1, can also be obtained directly from (18)

$$A'(t) = \left.\frac{\partial f'}{\partial x'}\right|_{*'} = \left.\begin{bmatrix} \partial f'^{u}/dx'^{u} & \partial f'^{u}/dx'^{\bar{u}} \\ 0 & \partial f'^{\bar{u}}/dx'^{\bar{u}} \end{bmatrix}\right|_{*'},$$
$$B'(t) = \left.\begin{bmatrix} \partial f'^{u}/du \\ 0 \end{bmatrix}\right|_{*'}, C'(t) = \left.\frac{\partial g'}{\partial x'}\right|_{*'}, \quad (25)$$

In case the linear time-invariant transformation (20) transforms system (1)–(3) into the observability canonical form (19)

$$A'(t) = \left.\begin{bmatrix} \partial f'^{y}/\partial x^{y} & 0 \\ \partial f'^{\bar{y}}/\partial x^{y} & \partial f'^{\bar{y}}/\partial x^{\bar{y}} \end{bmatrix}\right|_{*'}, B'(t) = \left.\frac{\partial f'}{\partial u}\right|_{*'},$$
$$C'(t) = \left.\begin{bmatrix} \partial g'/\partial x^{y} \\ 0 \end{bmatrix}\right|_{*'}, \quad (26)$$

with $*'$ the shorthand for $x'(t) = x'^{*}(t) = Tx^{*}(t)$, $u(t) = u^{*}(t)$.

To further develop the results in this paper, it is convenient to state two assumptions, as well as their dual versions, explicitly. ■

Assumption 4.1: A linear state-transformation (20) exists that transforms the nonlinear system (1), (3) into its controllability canonical form (18).

Assumption 4.2: The non-zero parts of matrices $A'(t)$, $B'(t)$ of the variational system (25) do not cause additional uncontrollable modes on top of the $\dim(x'^{\bar{u}})$ uncontrollable modes of the variational system. In other words, $\dim(x'^{\bar{u}}) = \dim(\delta x'^{\bar{u}})$ holds.

Dually there are,

Assumption 4.3: A linear state-transformation (20) exists that transforms the nonlinear system (1)–(3) into its observability canonical from (19).

Assumption 4.4: The non-zero parts of matrices $A'(t)$, $C'(t)$ of the variational system (26) do not cause additional unobservable modes on top of the $\dim(x'^{\bar{y}})$ unobservable modes of the variational system. In other words, $\dim(x'^{\bar{y}}) = \dim(\delta x'^{\bar{y}})$ holds.

Lemma 4.2: *Suppose Assumption* 4.1 *and Assumption* 4.2 *hold. Then the uncontrollable state-variables $x^{\bar{u}}$ in the controllability canonical form* (18) *correspond one-to-one with their uncontrollable state-increments $\delta x'^{\bar{u}}$ of the time-varying linear variational system* (23), (25), *irrespective of $u^{*}(t)$, $x'^{*}(t)$.*

Dually, assume Assumption 4.3 *and Assumption* 4.4 *hold. Then the unobservable state variables $x'^{\bar{y}}$ in the observability canonical form* (19), *correspond one-to-one with their unobservable state-increments $\delta x'^{\bar{y}}$ of the time-varying linear variational system* (23),(24), (26), *irrespective of $u^{*}(t)$, $x'^{*}(t)$.*

Proof: The zero parts of matrices $A'(t), B'(t), C'(t)$ in Equations (25), (26) reveal that the input $\delta u(t)$ does not affect $\delta x'^{\bar{u}}$ in (3), (25) and that the output in (23), (24), (26) is not affected by $\delta x'^{\bar{y}}$. Also, these zero parts of matrices $A'(t), B'(t), C'(t)$ in Equations (25), (26) do not depend on $u^{*}(t)$, $x'^{*}(t)$. ■

The linear variational system (11)–(13) is *time-varying* whenever $u^{*}(t)$, $x^{*}(t)$ in (4) are not constant. In that case, a linear *time-varying* state-transformation is generally required to transform the variational system into its controllability/observability canonical form (25)/(26) (Kalman, 1962). But here, as seen from Lemma 4.1, this state-transformation equals (20) that is linear *time-invariant*. Such a transformation is only obtained from a *time-invariant* variational system. Since Lemma 4.2 holds *irrespective* of $u^{*}(t)$, $x'^{*}(t) = Tx(t)$, and thus *irrespective* of $u^{*}(t)$, $x^{*}(t)$ in (4), the following idea presents itself.

Select $u^{*}(t)$, $x^{*}(t)$ in (4) *randomly and constant, no longer satisfying (1), (3) in general.* Then a 'variational system' is obtained that now is *time-invariant* and, therefore, the state transformation into the controllability/observability canonical form (25)/(26) will now also be *time-invariant*. This state-transformation can be computed by algorithms readily available in the control literature, such as ctrbf/obsv in Matlab or the ones in Van Dooren (1981). The one-to-one correspondence mentioned in Lemma 4.2 suggests that the same linear state transformation will put the nonlinear system (1) in the controllability/observability canonical form (18)/(19), if this can be done by a linear state-transformation, as was assumed. To check the latter, apply the same linear time-invariant state-transformation to the nonlinear system (1)–(3) to see if it provides the independence of $f'^{\bar{u}}$ from x'^{u} and u in (8) or dually the independence of f'^{y} and g' from $x'^{\bar{y}}$ in (19). The checks for independence require knowledge of the number of uncontrollable/unobservable state-variables $\dim(x'^{\bar{u}})/\dim(x'^{\bar{y}})$. From Lemma 4.2, these numbers can be taken to be $\dim(\delta x'^{\bar{u}})/\dim(\delta x'^{\bar{y}})$ as obtained

Algorithm 4.1. Finding linear state-transformations into the controllability canonical form.

1. *Randomly* select *constant* values for $u^*(t)$, $x^*(t)$ in (4).
2. Obtain the time-invariant 'variational system' of the nonlinear system (1), (3), i.e. obtain system matrices A, B in (13) e.g. by automatic differentiation (Neidinger, 2010).
3. Use a standard algorithm (ctrbf in Matlab or the one from Van Dooren (1981)), to find the linear state-transformation $x' = Tx$, $T \in \mathbb{R}^{n_x \times n_x}$, T non-singular, that transforms A, B in (13) into the controllability canonical form A', B' in (25). Also the number of uncontrollable modes of A, B is obtained from this algorithm being the number of uncontrollable states $\dim(\delta x'^{\bar{u}})$ as determined by A', B'. Although the random choice in step 1 is generally sufficient for Assumption 4.2 to hold, to better trust this, one can repeat steps 1–3 and take one giving the smallest outcome of $\dim(\delta x'^{\bar{u}})$ here.
4. Apply the same linear state-transformation to the nonlinear system (1), (3), in an attempt to obtain the controllability canonical form (18).
5. See if $f'^{\bar{u}}$ in (18) is indeed independent of x'^{u} and u by evaluating $f'^{\bar{u}}$ for random values x',u, and then changing only x'^{u}, u randomly to see if $f'^{\bar{u}}$ stays the same. Knowing f and T, use (21) to evaluate f'. Take for the number of uncontrollable modes $\dim(x'^{\bar{u}})$ =$\dim(\delta x'^{\bar{u}})$, found under step 3.

Algorithm 4.2. Finding linear state-transformation into the observability canonical form

1. *Randomly* select *constant* values for $u^*(t)$, $x^*(t)$ in (4).
2. Find the corresponding constant $y^*(t)$ from the output (2).
3. Obtain the time-invariant 'variational system' of the nonlinear system (1)–(3), i.e. obtain system matrices A, C from (11)–(13), e.g. by automatic differentiation (Neidinger, 2010).
4. Use a standard algorithm (obsvf from Matlab or the one from Van Dooren (1981)), to find the linear state-transformation $x' = Tx$, $T \in \mathbb{R}^{n_x \times n_x}$, T non-singular, that transforms A, C in (13), (12) into the observability canonical form A', C' in (26). Also the number of unobservable modes $\dim(\delta x'^{\bar{y}})$ is obtained from this algorithm being the number of unobservable states determined by A', C'. Although the random choice in step 1 is generally sufficient for Assumption 4.4 to hold, to better trust this, one can repeat steps 1–4 and take one giving the smallest outcome of $\dim(\delta x'^{\bar{y}})$ here.
5. Apply the same linear state-transformation to the nonlinear system (1)–(3), in an attempt to obtain the observability canonical form (19).
6. See if f'^{y} and g' in (19) are indeed independent of $x'^{\bar{y}}$ by evaluating f'^{y}, g' for randomly selected x' and then changing only $x'^{\bar{y}}$ randomly to see if f'^{y}, g' stay the same. Knowing f, g and T, use Equations (21), (22) to evaluate f',g'. Take for the number of unobservable modes $\dim(x'^{\bar{y}}) = \dim(\delta x'^{\bar{y}})$ found under step 4.

from the algorithms putting the linear time-invariant variational system into the controllability/observability canonical form. If the check comes out positive, a linear state-transformation is found transforming the nonlinear system into the corresponding canonical form. Algorithm 1, Algorithm 4.2 and Theorem 4.1 formalise this procedure and result.

Theorem 4.1: *If step 5 of Algorithm 4.1 comes out positive, step 3 finds a linear state-transformation $x' = Tx$, $T \in \mathbb{R}^{n_x \times n_x}$, T non-singular, that transforms the nonlinear system (1), (3) into the controllability canonical form (23) with $\dim(x'^{\bar{u}}) = \dim(\delta x'^{\bar{u}})$. If not, then no linear state-transformation into the controllability canonical form exists and $\dim(x'^{\bar{u}}) = \dim(\delta x'^{\bar{u}})$ need not hold.*

Dually, if step 6 of Algorithm 4.2 comes out positive, step 4 finds a linear state-transformation $T \in \mathbb{R}^{n_x \times n_x}$, T non-singular, that puts the nonlinear system (1)–(3) into the observability canonical form (19) with $\dim(x'^{\bar{y}}) = \dim(\delta x'^{\bar{y}})$. If not, then no linear state-transformation into the observability canonical form exists and $\dim(x'^{\bar{y}}) = \dim(\delta x'^{\bar{y}})$ need not hold.

Proof: Given in Appendix 1.

As the next theorem states, a candidate linear state-transformation into the controllability/observability canonical form can be obtained directly from the sensitivity-based algorithm. ■

Theorem 4.2: *Consider the linear state-transformation $x' = Vx$, with $V \in \mathbb{R}^{n_x \times n_x}$ the matrix $\begin{bmatrix} v_1 & v_2 & .. & v_{n_x} \end{bmatrix}$ found from the singular value decomposition of the sensitivity matrix S_c in (15) as obtained from the sensitivity-based algorithm checking controllability of the nonlinear system (1), (3). The numbering $i = 1, 2, .., n_x$ must be such that the singular values $\sigma_i \geq 0$ appear in decreasing order, as in most algorithms computing an SVD (such as svd in Matlab). Let steps 4–5 of Algorithm 4.1 check if this linear state-transformation transforms the nonlinear system (1), (3) in the controllability canonical form (18). In step 5, take the number of uncontrollable modes $\dim(x'^{\bar{u}})$ to be the one obtained from the sensitivity-based algorithm. If so, a linear state-transformation into the controllability canonical form (18) exists and is found. If not, no such linear state-transformation exists.*

Dually, for observability of the nonlinear system (1), (3) steps 4–6 of Algorithm 4.2 check if the linear state-transformation $x' = Vx$, found from the singular value decomposition of the sensitivity matrix S_o in (14), transforms it in the observability canonical form (19). In step 5, take the number of unobservable modes $\dim(x'^{\bar{y}})$ to be the one obtained from the sensitivity-based algorithm. If so, a linear state-transformation into the observability canonical form (19) exists and is found. If not, no such linear state-transformation exists.

Proof: Given in Appendix 1. ■

5. Examples

Two examples illustrating the two algorithms and theorems in this paper, are presented in this section. The first concerns the control and startup of a continuous stirred tank chemical reactor. In dynamically modelling chemical reaction systems redundant states often occur. These cause uncontrollability as well as unobservability. These redundant states can very often be removed after application of a linear state-transformation. These issues have been beautifully analysed and mathematically formalised in Amrhein et al. (2010); Bhatt et al. (2010) from which the next example is taken. This allows us to verify the outcome of the algorithms and theorems in this paper with theirs.

Example 5.1: CSTR control and startup, taken from Amrhein et al. (2010); Bhatt et al. (2010).

The CSTR reaction system considered is the ethanolysis of phthalyl chloride. It involves 7 chemical species denoted by *A,B,C,D,E,F,G* and 3 chemical reactions given by,

$$\begin{aligned} R_1: &\quad A+B \xrightarrow{r_1} C+D \\ R_2: &\quad C+B \xrightarrow{r_2} E+D \\ R_3: &\quad D+B \overset{r_3}{\rightleftarrows} F+G. \end{aligned} \tag{27}$$

The reaction rates satisfy

$$\begin{aligned} &r_1 = k_1 C_A C_B,\ r_2 = k_2 C_B C_C, \\ &r_3 = k_3 C_B C_D - k_4 C_F C_G, \end{aligned} \tag{28}$$

as C_A denotes the concentration $[kmol\, m^{-3}]$ of species A in the CSTR and similar for the other species. The stoichiometric matrix indicates the number of species gained in each reaction and is, therefore, given by

$$\begin{array}{cc} & \begin{array}{ccccccc} A & B & C & D & E & F & G \end{array} \quad \text{gained} \\ N = & \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 1 \end{bmatrix} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \end{array} \tag{29}$$

The reactor is fed with two input streams $u_{1,in}, u_{2,in}$ $[kg]$ of pure species A and B and has an output stream $u_{3,out}$ $[kg]$ The system dynamics are represented by the mole balance equation

$$\dot{n}(t) = N^T V(t) r(t) + W_{in} \begin{bmatrix} u_{1,in} \\ u_{2,in} \end{bmatrix} - \frac{u_{3,out}}{m(t)} n(t) \tag{30}$$

where $V(t)$ $[m^3]$ is the reactor liquid volume and $m(t)$ $[kg]$ are total mass in the reactor. Furthermore

$$n = [n_A, n_B, n_C, n_D, n_E, n_F, n_G]^T \tag{31}$$

is the state vector in which n_A $[kmol]$ is the amount of species A in the reactor and similarly for the other species. $V(t)$ and $m(t)$ depend algebraically on $n(t)$,

$$m(t) = M_{w,1} n_A + M_{w,2} n_B + \ldots + M_{w,7} n_G, \tag{32}$$

$$V(t) = \frac{M_{w,1}}{\rho_1} n_A + \frac{M_{w,2}}{\rho_2} n_B + \ldots + \frac{M_{w,7}}{\rho_7} n_G, \tag{33}$$

with molecular weights $[kg\ kmol^{-1}]$,

$$\begin{aligned} &M_{w,1} = 203, M_{w,2} = 46, M_{w,3} = 212.5, M_{w,4} = 36.5 \\ &M_{w,5} = 222, M_{w,6} = 64.5, M_{w,7} = 18 \end{aligned} \tag{34}$$

and liquid densities $[kg\ m^{-3}]$,

$$\begin{aligned} &\rho_1 = 1400, \rho_2 = 790, \rho_3 = 1118, \rho_4 = 1486, \\ &\rho_5 = 1118, \rho_6 = 921.4, \rho_7 = 998 \end{aligned} \tag{35}$$

From Equations (28), the reaction rate vector $r(t) = [r_1, r_2, r_3]^T$ $[kmol\ h^{-1}\ m^{-3}]$ is given by

$$r(t) = [k_1 n_A n_B,\ k_2 n_D n_B,\ k_3 n_D n_B - k_4 n_F n_G]^T / V^2(t) \tag{36}$$

with reaction constants $[m^3\ kmol^{-1}\ h^{-1}]$,

$$k_1 = 0.127,\ k_2 = 0.023,\ k_3 = 11.97,\ k_4 = 8.01 \tag{37}$$

Furthermore

$$W_{in} = \begin{bmatrix} 0.0049 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0217 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \tag{38}$$

and $u_{1,in}$, $u_{2,in}$ and $u_{3,out}$ are the control variables. From the analysis in Amrhein et al. (2010); Bhatt et al. (2010) it follows that with these three control variables, the system has one uncontrollable mode. This is confirmed by the singular values and controllability signature obtained from the sensitivity-based algorithm, depicted in Figure 2.

If we consider the chemical reactor without an outflow, i.e. without $u_{3,out}$, according to the analysis in Amrhein et al. (2010); Bhatt et al. (2010) two uncontrollable modes result. Again, this is confirmed by the singular values and controllability signature obtained from the sensitivity-based algorithm, depicted in Figure 3.

In case the CSTR has two control variables Amrhein et al. (2010); Bhatt et al. (2010) specifies a linear state-transformation that transforms the system into the controllability canonical form (18). According to Theorem 4.2, the matrix $V = \begin{bmatrix} v_1 & v_2 & .. & v_{n_x} \end{bmatrix}$ obtained from the SVD of the sensitivity-based algorithm, must also be a state-transformation that transforms the CSTR dynamics into the controllability canonical form (18). To verify this, steps 4–5 of Algorithm 4.1 were executed. Numerical evaluations of f', for significantly different values of x'^u and u, confirmed that $f'^{\bar{u}}$ is independent of both, since the maximum absolute difference recorded for components of $f'^{\bar{u}}$ was 3.3769e-12, which is considered zero numerically. Following Theorem 4.1, by executing Algorithm 4.1 entirely, we find this to be 6.9389e-17. This outcome is even more close to zero and therefore more accurate.

In case the CSTR has an outflow, i.e. with $u_{3,out}$ being the third control variable, according to the analysis in Amrhein et al. (2010); Bhatt et al. (2010), the state-transformation into the controllability canonical form (18) is no longer linear. With $V = \begin{bmatrix} v_1 & v_2 & .. & v_{n_x} \end{bmatrix}$ taken from the sensitivity-based algorithm, steps 4–5 of Algorithm 4.1 produce 1.6216e-04 as the maximum absolute difference of components of $f'^{\bar{u}}$. This complies with Theorem 4.2 stating that, in this case, no linear state-transformation exists into the controllability canonical form (18). Following Theorem 4.1, by executing Algorithm 4.1 entirely, gives 1.3591e-04 as the maximum absolute difference for components of $f'^{\bar{u}}$ for

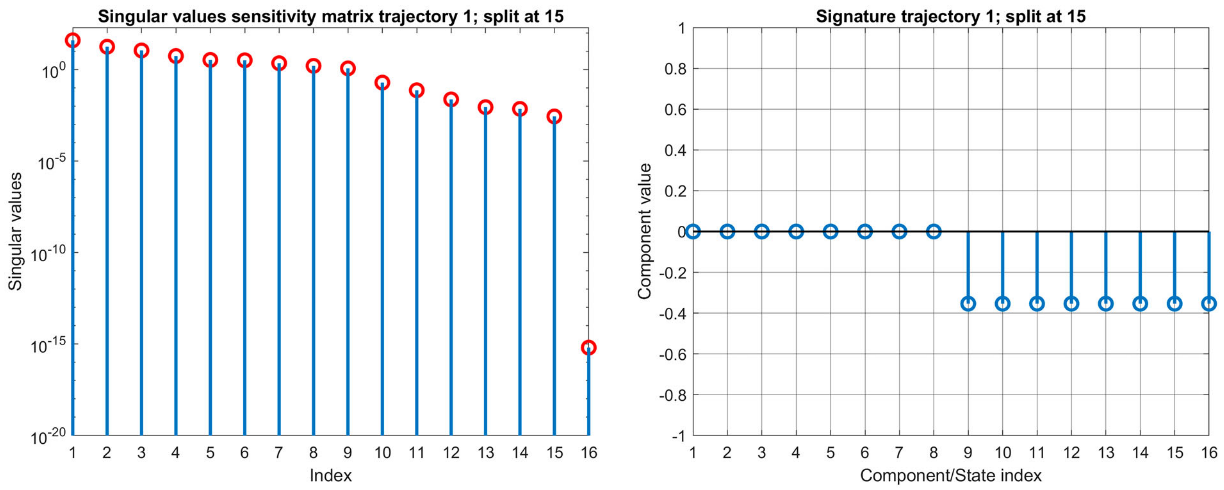

Figure 4. Singular values indicating a single numerically zero singular value and corresponding unobservable mode (left panel). From the observability signature (right panel), the single unobservable mode is a combination of state-variables 9–16 being all phases $\phi_i,\ i = 1, 2, .., 8$ *of all oscillators. The CPU-time on an ordinary Windows 10 PC was 2.62 s.*

the case with three control variables. This also confirms there is no linear state-transformation into the controllability canonical form (18) in this case.

The previous example concerned transformation into the controllability canonical form (18) for what might be called a medium-scale system having seven state-variables. The next example is chosen to both challenge the sensitivity-based algorithm with a large-scale system and to illustrate the dual case of transformation into the observability canonical form (19).

Example 5.2: Coupled nanoelectromechanical oscillators taken from (Emenheiser et al., 2016; Matheny et al., 2019; Mesbahi et al., 2021)

The example concerns an arbitrary number of reactively coupled nanoelectromechanical oscillators numbered $i = 1, 2, .., M$. Each oscillator is coupled with its neighbours $i-1$, $i+1$ while for oscillator $i = 1$, $i-1$ becomes M, and for oscillator $i = M$, $i+1$ becomes $i = 1$. The amplitude a_i and phase ϕ_i of the oscillators satisfy the ordinary differential equations

$$\frac{da_i}{dt} = -\frac{a_i - 1}{2} - \frac{\beta}{2}[a_{i+1}\sin(\phi_{i+1} - \phi_i) + a_{i-1}\sin(\phi_{i-1} - \phi_i)] \tag{39}$$

$$\frac{d\phi_i}{dt} = \alpha a_i^2 - \beta + \frac{\beta}{2\alpha_i}[a_{i+1}\cos(\phi_{i+1} - \phi_i) + a_{i-1}\sin(\phi_{i-1} - \phi_i)] \tag{40}$$

A single variable is measured given by

$$y = \sum_{i=1}^{N} \cos(\phi_i - \phi_{i+1}). \tag{41}$$

In Emenheiser et al. (2016); Matheny *et al.* (2019); Mesbahi et al. (2021), $M = 8$ coupled oscillators are considered, as is also done here. This makes up a state-space model having 16 state-variables. To challenge the sensitivity-based algorithms the number of oscillators was also raised to $M = 100$ making up a state-space model having 200 state-variables. By selecting $x_i = a_i$, $x_{i+N} = \phi_i$, $i = 1, 2, .., M$, $p_1 = \alpha$, $p_2 = \beta$, a state-space representation (1), (2) is obtained.

For $M = 8$ the results obtained with the sensitivity-based algorithm are depicted in Figure 4. The existence of one unobservable mode complies with the result in Mesbahi et al. (2021) stating that the system is unobservable. Applying Theorem 4.2 presents and interesting case. With the linear state-transformation $x' = Vx$, $V = \begin{bmatrix} v_1 & v_2 & .. & v_{n_x} \end{bmatrix}$ obtained from the sensitivity-based algorithm, running through steps 5–6 of Algorithm 4.2 results in a maximum difference in f'^y, g' of 2.6694e-14 which is considered zero numerically. According to Theorem 4.2, this implies that this linear transformation transforms the system into the observability canonical form (19). Following Theorem 4.1, by fully applying Algorithm 4.2, results in a maximum difference in f'^y, g' of 5.9715e-15. Since this number is slightly more close to zero, another, slightly more accurate, linear state-transformation into the observability canonical form (19) is found.

A comparable and similar result is obtained for $M = 100$ oscillators. Due to the high state-dimension of two hundred, to obtain this result, it is now necessary to invoke the feature of concatenating the result of several trajectories, when applying the sensitivity-based algorithm. This enlarges and moves up the gap within the singular values (Van Willigenburg et al., 2022), so that it becomes clearly visible, as can be seen from Figure 5.

Again, a single unobservable mode is present which is a combination of all state-variables representing phases, as seen from the observability signature depicted in Figure 6. With the linear state-transformation $x' = Vx$, $V = \begin{bmatrix} v_1 & v_2 & .. & v_{n_x} \end{bmatrix}$ from the sensitivity based algorithm, running through steps

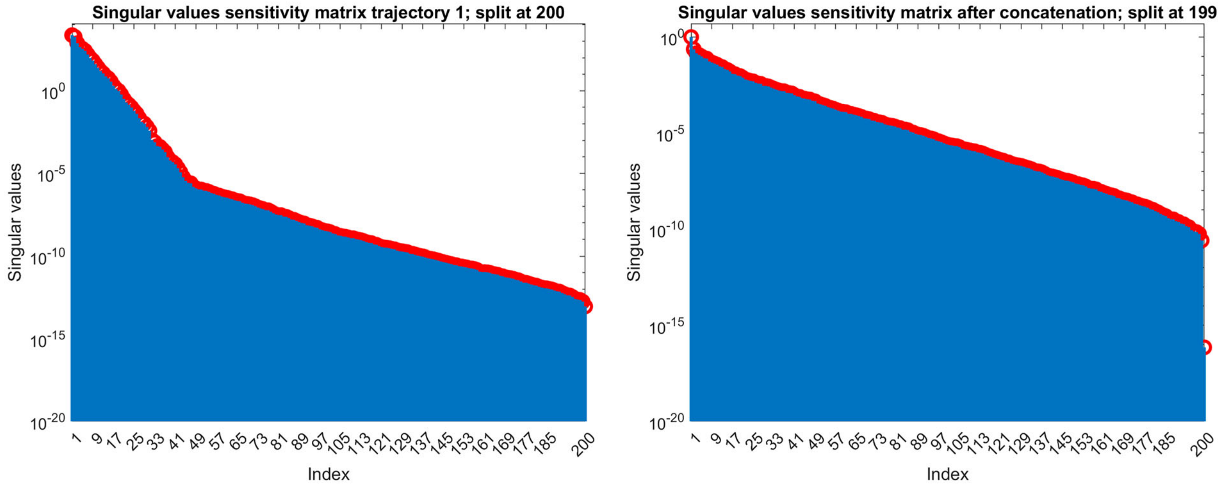

Figure 5. Two hundred singular values obtained from the sensitivity-based algorithm for one trajectory (left panel) and when concatenating the result of six trajectories (right panel). Only the latter correctly shows a single numerically zero singular value and corresponding unobservable mode. Each trajectory started from the same initial condition and had a final time 10, but used different values for the two parameters α, β. *The total CPU time on an ordinary Windows 10 PC to compute all six trajectories was 364 s.*

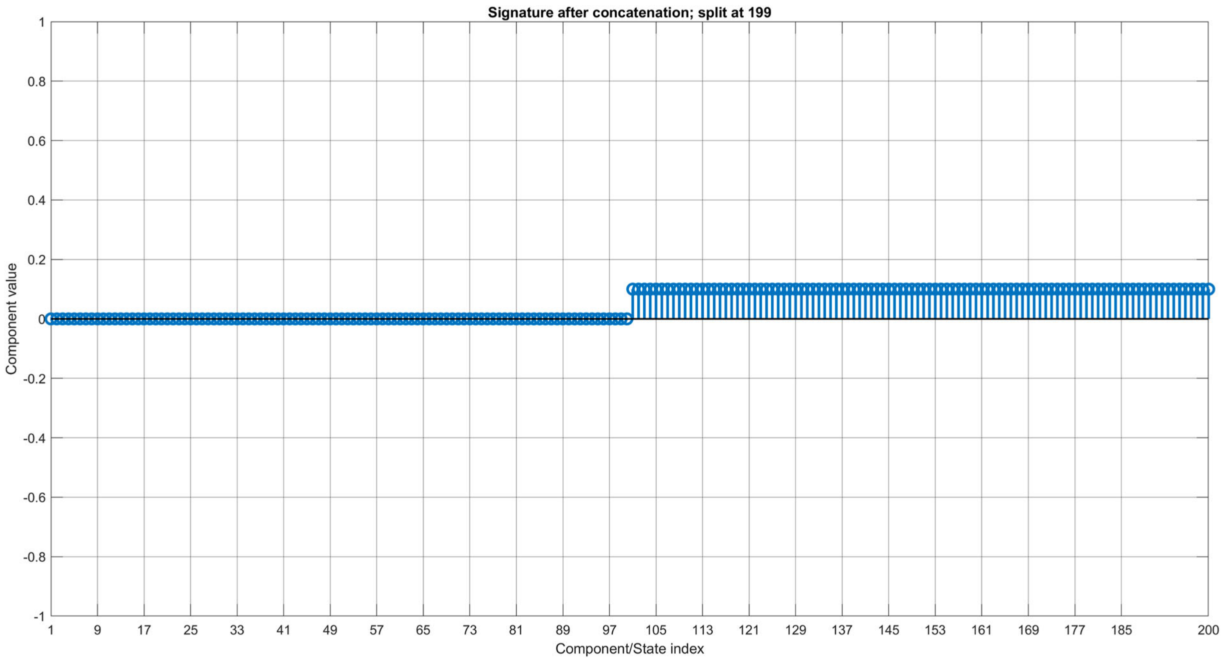

Figure 6. Observability signature for the system containing one hundred oscillators showing that the single unobservable mode is a combination of all phases $x_{i+100} = \phi_i, i = 1, 2, .., 100$ *being the last one hundred state-variables.*

5–6 of Algorithm 4.2 results in a maximum difference in f_1', g' of 9.3866e-11, which is considered zero numerically. This implies a linear state-transformation into the observability canonical form (19) is found. Following Theorem 4.1, by fully applying Algorithm 4.2, results in a maximum difference in f_1', g' of 4.4179e-14. Since this number is more close to zero, another, slightly more accurate, linear state-transformation into the observability canonical form (19) is found.

6. Conclusion

For ordinary and large-scale nonlinear dynamical systems, sensitivity-based algorithms provide most information concerning controllability/observability a control engineer wants. They not only determine whether the system is controllable/observable, but also provide the state-variables involved in uncontrollable/unobservable modes. If the system is not controllable/observable, in some cases, one would additionally want a state-transformation, i.e. a change of coordinates, representing the system into the controllability/observability canonical form. This paper showed that, if a linear state-transformation exists that does this job, it can be directly obtained from the sensitivity-based algorithm. Moreover, simple computational steps have been provided to check if this linear state-transformation does the job. If not, no linear state-transformation exists that does the job.

Instead of using sensitivity-based algorithms, for ordinary and large-scale nonlinear systems an alternative way to compute candidate linear state-transformations to do the job has been presented. It only requires running standard controllability/observability algorithms for linear time-invariant systems, such as ctrbf/obsvf in Matlab. After a candidate linear state-transformation has been obtained this way, the same simple computational steps can be used to check if it provides the appropriate canonical form of the nonlinear system. If not, no linear state-transformation exists that does the job.

Because the two types of algorithms presented in this paper are of a numerical nature, they demand acceptable numerical conditioning of the system dynamics and all related computations. Like deciding on the numerical rank of a matrix, the decision on whether a linear state transformation is found from the algorithm relies on the decision whether very small numbers may be considered zero numerically. It appears that the type of algorithms that use standard controllability/observability algorithms for linear time-invariant systems are more accurate numerically, as they compute smaller numbers for those that should be considered zero numerically. This might be because they linearise along a single state, instead of a state-trajectory, as the sensitivity-based algorithms do.

The results of this paper naturally raise the question of how often state-transformations into controllability/observability canonical forms are linear. Roughly speaking, this does not happen that often. This paper identified two types of dynamical systems having this property, namely dynamical models of chemical reactions having redundant states and coupled nanoelectromechanical oscillators. Although not shown in this paper, by *starting* from a canonical representation followed by a linear state-transformation, examples are obtained that all have this property. Moreover, the state, input and output dimension as well as the number of uncontrollable/unobservable modes can then be selected at will. This allows one to further validate the results presented in this paper.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Notes on contributor

L. Gerard van Willigenburg received his M.Sc. in electrical engineering and his Ph.D. in control engineering at Delft University of Technology, The Netherlands, in 1983 and 1991, respectively. In 1990, he became an Assistant Professor within the Systems & Control group of Wageningen University. In 2015, he moved to the Mathematical and Statistical Methods group of Wageningen University. His professional research interests include digital optimal control, reduced-order control, model predictive control, adaptive dual control, systems biology and model structure. The application areas range from indoor climate control (storage rooms, greenhouses, stables and buildings), robot control, automatic guidance (field vehicles, multicopters), to process control (fermentation, sterilisation, food) and the control of economic and biological systems (crops, plants, cells, networks and diabetes). Since 2006, the modelling of fundamental physics (thermodynamics, quantum mechanics, relativity) has become a private research interest.

ORCID

L. Gerard Van Willigenburg http://orcid.org/0000-0002-9370-4503

References

Amrhein, M., Bhatt, N., Srinivasan, B., & Bonvin, D. (2010). Extents of reaction and flow for homogeneous reaction systems with inlet and outlet streams. *AIChE Journal*, *56*(*11*), 2873–2886. http://doi.org/10.1002/aic.v56:11

Bhatt, N., Amrhein, M., & Bonvin, D. (2010, June 6). Minimal state representation of homogeneous reaction systems. *20th European Symposium on Computer Aided Process Engineering*, ESCAPE20, Naples, Italy.

Boley, D. L. (1980). On Kalman's procedure for the computation of the controllable/observable canonical form. *SIAM Journal on Control and Optimization*, *18*(6), 624–626. https://doi.org/10.1137/0318047

Boley, D. L. (1984). Computing the Kalman decomposition: An optimal method. *IEEE Transactions on Automatic Control*, *29*(1), 51–53. https://doi.org/10.1109/TAC.1984.1103365

Boutat, D., & Busawon, K. (2011). On the transformation of nonlinear dynamical systems into the extended nonlinear observable canonical form. *International Journal of Control*, *84*(1), 94–106. https://doi.org/10.1080/00207179.2010.541285

Conte, G., Perdon, A., & Moog, C. (1995). Toward a notion of canonical form for nonlinear systems. *Banach Center Publications*, *32*(1), 149–165. https://doi.org/10.4064/-32-1-149-165

Emenheiser, J., Chapman, A., Pósfai, M., Crutchfield, J. P., Mesbahi, M., & D'Souza, R. M. (2016). Patterns of patterns of synchronization: Noise induced attractor switching in rings of coupled nonlinear oscillators. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, *26*(*9*), 4935. http://doi.org/10.1063/1.4960191

Hermann, R., & Krener, A. (1977). Nonlinear controllability and observability. *IEEE Transactions on Automatic Control*, *22*(5), 728–740. https://doi.org/10.1109/TAC.1977.1101601

Isidori, A. (1995). Local decompositions of control systems. In A. Isidori (Ed.), *Nonlinear control systems* (pp. 1–76). Springer. https://doi.org/10.1007/978-1-84628-615-5_1

Kalman, R. E. (1962). Canonical structure of linear dynamical systems. *Proceedings of the National Academy of Sciences of the United States of America*, *48*(4), 596–600. https://doi.org/10.1073/pnas.48.4.596

Kwatny, H. G., & Blankenship, G. L. (2000). *Nonlinear control & analytical mechanics: A computational approach*. Springer.

Matheny, M. H., Emenheiser, J., Fon, W., Chapman, A., Salova, A., Rohden, M., Li, J., Hudoba de Badyn, M., Pósfai, M., Duenas-Osorio, L., Mesbahi, M., Crutchfield, J. P., Cross, M. C., D'Souza, R. M., & Roukes, M. L. (2019). Exotic states in a simple network of nanoelectromechanical oscillators. *Science*, *363*(*6431*), eaav7932. http://doi.org/10.1126/science.aav7932.

Mesbahi, A., Bu, J., & Mesbahi, M. (2021). Nonlinear observability via Koopman analysis: Characterizing the role of symmetry. *Automatica*, *124*, 109353. https://doi.org/10.1016/j.automatica.2020.109353

Neidinger, R. D. (2010). Introduction to automatic differentiation and MATLAB object-oriented programming. *Siam Review*, *52*(3), 545–563. https://doi.org/10.1137/080743627

Nijmeijer, H., & van der Schaft, A. (1990). *Nonlinear dynamical control systems* (H. Nijmeijer & A. van der Schaft, Eds.). Springer. https://doi.org/10.1007/978-1-4757-2101-0_3

Stigter, J. D., van Willigenburg, L. G., & Molenaar, J. (2018). An efficient method to assess local controllability and observability for non-linear systems. *IFAC-PapersOnline*, *51*(2), 535–540. https://doi.org/10.1016/j.ifacol.2018.03.090

Van Dooren, P. (1981). The generalized eigenstructure problem in linear system theory. *IEEE Transactions on Automatic Control*, *26*(1), 111–129. https://doi.org/10.1109/TAC.1981.1102559

Van Willigenburg, L. G. (2024). To be or not to be connected: Reconstructing nonlinear dynamical system structure. In P. Y. P. Chen & V. Martinez-Luaces (Eds.), *Nonlinear systems and matrix analysis – recent advances in theory and applications* [Working Title] (pp. 1–22). InTech. https://www.intechopen.com/online-first/1184633

Van Willigenburg, L. G., Stigter, J. D., & Molenaar, J. (2021). Establishing local strong accessibility of large-scale nonlinear systems by replacing the Lie algebraic rank condition. *Proceedings European Control Conference 2021*, 2645–2650. https://doi.org/10.23919/ECC54610.2021.9655043

Van Willigenburg, L. G., Stigter, J. D., & Molenaar, J. (2022). Sensitivity matrices as keys to local structural system properties of large-scale nonlinear systems. *Nonlinear Dynamics*, *107*(3), 2599–2618. https://doi.org/10.1007/s11071-021-07125-4

Weiss, L., & Kalman, R. E. (1965). Contributions to linear system theory. *International Journal of Engineering Science*, *3*(2), 141–171. https://doi.org/10.1016/0020-7225(65)90042-X

Wieland, F.-G., Hauber, A. L., Rosenblatt, M., Tönsing, C, & Timmer, J. (2021). On structural and practical identifiability. *Current Opinion in Systems Biology*, *25*, 60–69. http://doi.org/10.1016/j.coisb.2021.03.005

Zeitz, M. (1990). Canonical forms for nonlinear systems. In A. Isidori (Ed.), *Nonlinear control systems design 1989* (pp. 33–38). Pergamon. https://doi.org/10.1016/B978-0-08-037022-4.50012-0

Zhirabok, A. N. (2008). *Observability and controllability analysis of nonlinear systems by linear methods*. 10th Intl. Conf. on Control, Automation, Robotics and Vision, Hanoi, Vietnam, 17–20 December. IEEE.

Zhirabok, A. N. (2010). Analysis of observability and controllability of nonlinear dynamic systems by linear methods. *Journal of Computer and Systems Sciences International*, *49*(1), 8–15. https://doi.org/10.1134/s1064230710010028

Appendix 1. Proof

In this appendix both Theorem 4.1 and Theorem 4.2 are proved. Both proofs require the following lemma.

Lemma A.1: *Suppose Assumption 4.1 and Assumption 4.2 hold. Then all linear state-transformations satisfying Assumption 4.1 are given by,*

$$T' = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} T, \ \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \ 0 \in \mathbb{R}^{n_x^{\bar{u}} \times (n_x - n_x^{\bar{u}})},$$
$$T_{11} \in \mathbb{R}^{(n_x - n_x^{\bar{u}}) \times (n_x - n_x^{\bar{u}})},$$
$$T_{12} \in \mathbb{R}^{(n_x - n_x^{\bar{u}}) \times n_x^{\bar{u}}}, T_{22} \in \mathbb{R}^{n_x^{\bar{u}} \times n_x^{\bar{u}}} \text{ such that}$$
$$\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \text{non-singular}, \tag{42}$$

where $n_x^{\bar{u}} = \dim(x'^{\bar{u}})$ is the number of uncontrollable modes of the nonlinear system (1), (3). *Moreover, by Lemma 4.2, these linear state-transformations $\Psi(x) = T'x$ also represent all state-transformations that transform the linearisation into the controllability canonical form* (23) *with $\dim(\delta x'^{\bar{u}}) = \dim(x'^{\bar{u}})$.*

Dually, assume Assumption 4.3 and Assumption 4.4 hold. Then all linear state-transformations satisfying Assumption 4.3 are given by,

$$T' = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix} T, \ T_{11} \in \mathbb{R}^{(n_x - n_x^{\bar{y}}) \times (n_x - n_x^{\bar{y}})},$$
$$0 \in \mathbb{R}^{(n_x - n_x^{\bar{y}}) \times n_x^{\bar{y}}} T_{21} \in \mathbb{R}^{n_x^{\bar{y}} \times (n_x - n_x^{\bar{y}})},$$
$$T_{22} \in \mathbb{R}^{n_x^{\bar{y}} \times n_x^{\bar{y}}} \text{ such that } \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}$$
$$\text{non-singular}, \tag{43}$$

with $n_x^{\bar{y}} = \dim(x'^{\bar{y}})$ the number of unobservable modes of the nonlinear system (1)–(3). Moreover, by Lemma 4.2, these linear state-transformations $\Psi(x) = T'x$ also represent all state-transformations that transform the linearisation into the observability canonical form (23), (24) with $\dim(\delta x'^{\bar{y}}) = \dim(x'^{\bar{y}})$.

Proof: The controllability canonical form (18) is fully characterised by the property that the controllable state-variables x'^{u} as well as the input u, do not influence $x'^{\bar{u}}$. Equation (42) captures *all* linear state-transformations that leave this property unaffected because of the non-singularity and zero part of the matrix $\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$. As to the linearisations, the result follows from the one-to-one correspondence in Lemma 4.2 of all uncontrollable state-variables $x'^{\bar{u}}$ with their uncontrollable state-increments $\partial x'^{\bar{u}}$.

Dually, the observability canonical form (19) is fully characterised by the property that the unobservable state-variables x'_2 do not influence x'_1. Equation (43) captures *all* linear state-transformations that leave this property unaffected because of the non-singularity and zero part of the matrix $\begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}$. As to the linearisations, the result follows from the one-to-one correspondence in Lemma 4.2 of unobservable state-variables $x'^{\bar{y}}$ with their unobservable state-increments $\delta x'^{\bar{y}}$. ■

Proof of Theorem 4.1: Algorithm 4.1 computes a linear time-invariant state-transformation (20) that transforms A, B in (13) into the canonical form A', B' in (25). When Assumption 4.1 and Assumption 4.2 hold, according to Lemma 4.2, $\dim(\delta x'^{\bar{u}}) = \dim(x'^{\bar{u}})$. Moreover, because of Lemma 6.1, this linear state-transformation transforms the nonlinear system (1), (3) into the controllability canonical form (18). Suppose Assumption 4.1 holds. Then Assumption 4.2 holds due to step 3 and the random selection of $u^*(t)$, $x^*(t)$ in step 1 of Algorithm 4.1. Next, if step 5 in Algorithm 4.1 fails, then by Lemma 4.2, Assumption 4.1 must fail.

Dually, Algorithm 4.2 computes a linear time-invariant state-transformations (20) that transforms A, B, C in (13), (12) into the observability canonical form A', B', C' in (26). When Assumption 4.3 and Assumption 4.4 hold, according to Lemma 4.2, $\dim(\delta x'^{\bar{y}}) = \dim(x'^{\bar{y}})$. Moreover, because of Lemma 6.1, this linear state-transformation transforms the nonlinear system (1)–(3) into the observability canonical form (19). Suppose Assumption 4.3 holds. Then Assumption 4.4 holds due to step 4 and the random selection of $u^*(t)$, $x^*(t)$ in step 1 of Algorithm 4.2. Next, if step 6 in Algorithm 4.2 fails, then by Lemma 4.2, Assumption 4.3 must fail. ■

Proof of Theorem 4.2: If Assumption 4.1 holds, a linear state-transformation $x' = Tx$ exists that transforms the original dynamics (1), (3) into the controllability canonical form (18). If Assumption 4.2 also holds, from Lemma 4.2 and Lemma 6.1 the same state transformation transforms A, B into the canonical form A', B'. From the zero parts of $A'(t)$, $B'(t)$ in (25), the first $\dim(\delta x^u) = \dim(x^u)$ columns of the transformation matrix T are seen to be directions along which infinitesimals ∂x can be steered whereas the last $\dim(\delta x^{\bar{u}}) = \dim(x^{\bar{u}})$ columns are directions along which infinitesimals δx cannot be steered. Matrix $V = \begin{bmatrix} v_1 & v_2 & .. & v_{n_x} \end{bmatrix}$, obtained from the sensitivity-based algorithm checking controllability through the SVD applied to (15), has exactly these properties (Van Willigenburg et al., 2022). Assumption 4.2 holds because of step 3 and the random selection of $u^*(t)$, $x^*(t)$ in step 1 of Algorithm 4.1. Then, if step 5 of Algorithm 4.1 fails, from Lemma 4.2 and Lemma 6.1, Assumption 4.1 must be violated.

Replacing Assumption 4.1 and Assumption 4.2 above with the dual Assumption 4.3 and Assumption 4.4, and step 3 and step 5 of Algorithm 4.1 by step 4 and step 6 of Algorithm 4.2, dual arguments hold for the matrix $V = \begin{bmatrix} v_1 & v_2 & .. & v_{n_x} \end{bmatrix}$ obtained from the sensitivity-based algorithm checking observability through the SVD applied to (14) (Van Willigenburg et al., 2022). ■