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ON COURNOT OLIGOPOLIES WITH A BICONCAVE PRICE FUNCTION

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Abstract. The power of a recent result for Nash equilibria of sum-aggregative games is illustrated by providing a variant of Ewerhart's equilibrium uniqueness theorem for Cournot oligopolies with a biconcave essential price function. This variant, contrary to the original result, allows for industry revenue that is discontinuous at 0 and can be considered a major generalization of the classical uniqueness result of Szidarovszky and Okuguchi for rent-seeking games. It assumes that the elasticity of the price function has a limit when its argument tends to zero.

1. **Introduction.** It has turned out that when dealing with Cournot equilibria, generalized concavity is quite useful. This was again made clear by Ewerhart's equilibrium uniqueness theorem [6, Theorem 4.2] for Cournot oligopolies with a price function p and cost functions c_i. The essential price function $\hat{p}$, i.e., the restriction of p to $\{y \neq 0 \mid p(y) > 0\}$, was assumed to be biconcave, and non-convex cost functions were allowed. This theorem, again presented in Theorem 2.10, is a milestone in oligopoly theory as it resumes various results in the literature. In this result, the payoff functions are continuous, quasi-concave in their own strategies, and subject to an effective compactness condition. Therefore, equilibrium existence could be proved by referring to (a variant of) the generalized Nikaido-Isoda theorem [7]. Thus, the main issue was equilibrium semi-uniqueness, i.e., that at most one equilibrium exists. This was proved by using backward-response functions from the Selten-Szidarovszky technique, more or less in the same way as in [8]. Ewerhart's equilibrium uniqueness result excludes price functions p for which industry revenue $r_p(y) = p(y)y$ is discontinuous at 0, such as $p(y) = y^s$ $(y \neq 0)$ with $s \leq -1$ and $p(y) = e^{-y}/y$ $(y \neq 0)$.

With Theorem 4.1 below, we provide a variant of Theorem 2.10 that allows for industry revenue r_p to be discontinuous at 0 under the assumption that $\lim_{y \downarrow 0} \frac{p'(y)y}{p(y)}$ i.e., the limit of the elasticity of p as y tends to zero, exists. Allowing for this possibility, as first shown in [10], is especially important for dealing with contest-like games. The proof of Theorem 4.1 refers to a recent result in [13, 14] on sum-aggregative

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games, where each player has $\mathbb{R}_+$ as a strategy set. This result is presented in Theorem 3.5. Therefore, in Theorem 4.1, contrary to Theorem 2.10, capacity constraints are not allowed. Theorem 3.5 is stated in terms of game primitives and has been proven completely by the Selten-Szidarovszky technique. In particular, it does not refer to a deep mathematical result, such as the generalized Nikaido-Isoda theorem. The reason for this is that this technique reduces the n-dimensional fixed-point problem for the joint best-response correspondence to a 1-dimensional problem for the aggregate backward-response correspondence. The power of Theorem 3.5 has already been illustrated in [14] by its application to a Cournot oligopoly with the Bulow-Pfleiderer price function (see [3]) and to an asymmetric contest with endogenous valuations. In this sense, the present article provides another illustration of its power.

The remainder of the article is organized as follows. Ewerhart's equilibrium uniqueness theorem, together with some simple but useful results on Cournot oligopolies, is presented in Section 2. There, the notion of biconcavity is also reconsidered. Section 3 presents the above-mentioned theorem on sum-aggregative games from [14]. A variant of Ewerhart's equilibrium uniqueness theorem is presented and proved in Section 4. Concluding remarks and suggestions for further research are provided in Section 5.

2. **Ewerhart's equilibrium uniqueness theorem.**

2.1. **Cournot oligopolies.** Here, we fix the setting for the Cournot oligopolies we deal with.

Consider a game in strategic form where each player i has $X_i = \mathbb{R}_+$ or $X_i = [0, m_i]$ with $m_i > 0$ as a strategy set and a payoff function $f_i : X_1 \times \cdots \times X_n \to \mathbb{R}$ given by

$$f_i(\mathbf{x}) = p(\sum_l x_l)x_i - c_i(x_i),$$

where, with $Y := \sum_i X_i$, $p : Y \to \mathbb{R}$, and $c_i : X_i \to \mathbb{R}$. This game concerns a (homogeneous) Cournot oligopoly with price function p, the firms as players, and for firm i, c_i its cost function, and f_i its profit function. In the case where $X_i = [0, m_i]$, one says that player i has a capacity constraint. A Nash equilibrium of this game is also referred to as Cournot equilibrium.

The presentation will benefit by introducing the following terminology. First, it is very important to note that f_i does not depend on the value $p(0)$ of p at 0, so neither does the equilibrium set. With $Y^\star := Y \setminus \{0\}$, only the function

$$\tilde{p} := p \restriction_{Y^\star},$$

i.e. the restriction of p to $Y^\star$, matters. We refer to this function as the PROPER PRICE FUNCTION. Next, the ESSENTIAL DOMAIN of p is defined as the set

$$I_p := \{y \in Y^\star \mid p(y) > 0\}.$$

We call the restriction

$$\hat{p} := p \restriction_{I_p}$$

the ESSENTIAL PRICE FUNCTION. So if $\tilde{p} > 0$, then $\hat{p} = \tilde{p}$.

The industry revenue function is the function $r_p : Y \to \mathbb{R}$ given by

$$r_p(y) := p(y)y.$$

Thus, $r_p(0) = 0$. Also, let

$$r_{\hat{p}} := r_p \restriction_{I_p} .$$

If $\tilde{p}$ is decreasing, then

$$\underline{p} := \lim_{y \downarrow 0} \tilde{p}(y) \in \mathbb{R} \cup \{+\infty\}$$

is well-defined and

$$r'_p(0) = \lim_{y \downarrow 0} (r_p(y) - r_p(0)/(y - 0) = \underline{p}. \tag{1}$$

If I_p is a proper real interval and $\hat{p}$ is differentiable, then define $\epsilon_{\hat{p}} : I_p \to \mathbb{R}$ by

$$\epsilon_{\hat{p}} := p'(y) \frac{y}{p(y)};$$

$\epsilon_{\hat{p}}$ is the elasticity of $\hat{p}$. For us, the case where I_p is a proper real interval with $0 \in \overline{I_p}$, i.e. with 0 in its topological closure, is important. Sufficient for this to hold is that $I_p \neq \emptyset$ and $\tilde{p}$ is decreasing; then, $\underline{p} \in \,]0, +\infty]$. If I_p is a proper real interval with $0 \in \overline{I_p}$ and $\hat{p}$ is differentiable, then let

$$\underline{\epsilon}_{\hat{p}} := \lim_{y \downarrow 0} \epsilon_{\hat{p}}(y)$$

in case this limit is well-defined, i.e. there exists as an element of $\mathbb{R} \cup \{-\infty, +\infty\}$.

The following result seems to be overlooked in the literature; however, see [11, Proposition 11]. To keep this work self-contained, we give a proof.

Proposition 2.1. *Suppose $Y = \mathbb{R}_+$ and p is on $\mathbb{R}_{++}$ positive, differentiable, and bounded. Suppose $c_i : X_i \to \mathbb{R}$ is twice differentiable. Finally, suppose that for every $(x_i, y) \in X_i \times \mathbb{R}_{++}$ with $0 \leq x_i \leq y$, the strict inequality $p'(y) - c''_i(x_i) < 0$ holds. Then, c_i is convex.* $\diamond$

Proof. Fix $x_i \in X_i$. We show $c''_i(x_i) \geq 0$. We do this by proving that $c''_i(x_i) \geq -\epsilon$ for every $\epsilon > 0$. So, fix $\epsilon > 0$. As $\lim_{y \to +\infty} \frac{p(y)-p(x_i)}{y-x_i} = 0$, we can fix $\tilde{y} > x_i$ such that $\frac{p(\tilde{y})-p(x_i)}{\tilde{y}-x_i} \geq -\epsilon$. As p is differentiable on $[x_i, \tilde{y}]$, the first mean value theorem implies the existence of $\xi \in \,]x_i, y\,[$ with $p'(\xi) = \frac{p(\tilde{y})-p(x_i)}{\tilde{y}-x_i}$. Thus, $p'(\xi) \geq -\epsilon$. As $p'(\xi) - c''_i(x_i) < 0$, we obtain, as desired, that $c''_i(x_i) > p'(\xi) \geq -\epsilon$. $\square$

The following useful result is not new. For the readers' convenience, we give a proof.

Proposition 2.2. *Suppose each cost function has 0 as unique minimizer.*

1. *$p(\sum_l e_l) > 0$ for each non-zero Cournot equilibrium $\mathbf{e}$.*
2. *Suppose the price function $p : Y \to \mathbb{R}$ is such that $p = \max(\breve{p}, 0)$ for some function $\breve{p} : Y \to \mathbb{R}$. Then, the Cournot equilibrium set is unchanged if p is replaced by $\breve{p}$.* $\diamond$

Proof. 1. By way of contradiction, assume that $\mathbf{e} \neq \mathbf{0}$ is an equilibrium with $p(\sum_l e_l) \leq 0$. Fix a firm i. Then, $f_i(\mathbf{e}) = p(\sum_l e_l)e_i - c_i(e_i) \leq -c_i(e_i)$. It follows that (see the beginning of Subsection 3.1 for some notation) $-c_i(e_i) \geq f_i(\mathbf{e}) \geq f_i(0; \mathbf{e}_{\hat{\imath}}) = -c_i(0)$, and thus $c_i(e_i) \leq c_i(0)$. As c_i has 0 as a unique minimizer, we obtain $e_i = 0$. Thus, $\mathbf{e} = \mathbf{0}$ follows, which is a contradiction.

2. Denote in the case of $p\,(\breve{p})$ the payoff functions by $f_i\,(\breve{f}_i)$ and the equilibrium set by $E\,(\breve{E})$.

$\breve{E} \subseteq E$: Suppose $\mathbf{e} \in \breve{E}$. Fix a firm i and $x_i \in X_i$, and let $a = \sum_{j \neq i} e_j$. We distinguish between four cases. In the third and fourth case, we shall use (as part 1 guarantees): $\mathbf{e} \neq \mathbf{0} \;\Rightarrow\; 0 < \breve{p}(e_i + a) = p(e_i + a)$. We have $f_i(x_i; \mathbf{e}_{\hat{\imath}}) = p(x_i + a)x_i - c_i(x_i)$.

Case where $\mathbf{e} = 0$ and $\breve{p}(x_i) > 0$. Now, $a = 0$ and $p(x_i + a) = p(x_i) = \breve{p}(x_i)$, and so

$$f_i(x_i; \mathbf{e}_{\hat{\imath}}) = \breve{p}(x_i)x_i - c_i(x_i) = \breve{f}_i(x_i; \mathbf{e}_{\hat{\imath}}) \leq \breve{f}_i(e_i; \mathbf{e}_{\hat{\imath}}) = -c_i(0) = f_i(e_i; \mathbf{e}_{\hat{\imath}}).$$

Case where $\mathbf{e} = 0$ and $\breve{p}(x_i) \leq 0$. Now, $a = 0$, and $p(x_i + a) = p(x_i) = 0$ and so

$$f_i(x_i; \mathbf{e}_{\hat{\imath}}) = -c_i(x_i) \leq -c_i(0) = f_i(e_i; \mathbf{e}_{\hat{\imath}}).$$

Case where $\mathbf{e} \neq \mathbf{0}$ and $\breve{p}(x_i + a) > 0$. Now, $p(x_i + a) = \breve{p}(x_i + a)$, and so

$$f_i(x_i; \mathbf{e}_{\hat{\imath}}) = \breve{p}(x_i + a)x_i - c_i(x_i) = \breve{f}_i(x_i; \mathbf{e}_{\hat{\imath}}) \leq \breve{f}_i(e_i; \mathbf{e}_{\hat{\imath}})$$

$$= \breve{p}(e_i + a)e_i - c_i(e_i) = p(e_i + a)e_i - c_i(e_i) = f_i(e_i; \mathbf{e}_{\hat{\imath}}).$$

Case where $\mathbf{e} \neq \mathbf{0}$ and $\breve{p}(x_i + a) \leq 0$. Now $p(x_i + a) = 0$, and so

$$f_i(x_i; \mathbf{e}_{\hat{\imath}}) = -c_i(x_i) \leq -c_i(0) = \breve{f}_i(0; \mathbf{e}_{\hat{\imath}})$$

$$\leq \breve{f}_i(e_i; \mathbf{e}_{\hat{\imath}}) = \breve{p}(e_i + a)e_i - c_i(e_i) = p(e_i + a)e_i - c_i(e_i) = f_i(e_i; \mathbf{e}_{\hat{\imath}}).$$

$E \subseteq \breve{E}$: Suppose $\mathbf{e} \in E$. Fix $i \in N$, $x_i \in X_i$ and let $a = \sum_{j \neq i} e_j$. Part 1 guarantees: $\mathbf{e} \neq \mathbf{0} \;\Rightarrow\; 0 < p(e_i + a) = \breve{p}(e_i + a)$. We have $\breve{f}_i(x_i; \mathbf{e}_{\hat{\imath}}) = \breve{p}(x_i + a)x_i - c_i(x_i)$. We distinguish between two cases.

Case where $\mathbf{e} \neq \mathbf{0}$.

$$\breve{f}_i(x_i; \mathbf{e}_{\hat{\imath}}) = \breve{p}(x_i + a)x_i - c_i(x_i) \leq p(x_i + a)x_i - c_i(x_i) = f_i(x_i; \mathbf{e}_{\hat{\imath}}) \leq f_i(e_i; \mathbf{e}_{\hat{\imath}})$$

$$= p(e_i + a)e_i - c_i(e_i) = \breve{p}(e_i + a)e_i - c_i(e_i) = \breve{f}_i(e_i; \mathbf{e}_{\hat{\imath}}).$$

Case where $\mathbf{e} = \mathbf{0}$. Now $a = 0$, and so

$$\breve{f}_i(x_i; \mathbf{e}_{\hat{\imath}}) = \breve{p}(x_i)x_i - c_i(x_i)$$

$$\leq p(x_i)x_i - c_i(x_i) = f_i(x_i; \mathbf{e}_{\hat{\imath}}) \leq f_i(e_i; \mathbf{e}_{\hat{\imath}}) = -c_i(0) = \breve{f}_i(e_i; \mathbf{e}_{\hat{\imath}}).$$

$\square$

2.2. **Biconcavity.** Generalized concavity with respect to a pair of means is a substantial generalization of concavity and was introduced by [2] in 1933. Generalized concavity (for continuous positive functions) for a pair of Hölder means has been coined "biconcavity" in [6].

In this subsection, I is a proper real interval with $I \subseteq \mathbb{R}_{++}$ and $g : I \to \mathbb{R}$ is a positive continuous function.

Let $H_p : \mathbb{R}_{++} \times \mathbb{R}_{++} \to \mathbb{R}_{++}$ be the Hölder mean of order $p \in \mathbb{R}$, i.e.

$$H_p(x_1, x_2) := \begin{cases} \left(\frac{x_1^p + x_2^p}{2}\right)^{1/p} & \text{if } p \neq 0, \\ x_1^{1/2} x_2^{1/2} & \text{if } p = 0. \end{cases}$$

Definition 2.3. a. For $\alpha, \beta \in \mathbb{R}$, g is (α, β)-BICONCAVE if for all $y_1, y_2 \in I$, $g(H_\beta(y_1, y_2)) \geq H_\alpha(g(y_1), g(y_2))$.

b. g is BICONCAVE if there are $\alpha, \beta \in \mathbb{R}$ such that g is (α, β)-biconcave. $\diamond$

We note that the definition of biconcavity in [6] (for a continuous function) looks different, but in fact is equivalent.

In particular, biconcavity is defined for the essential price function $\hat{p}$ of a Cournot oligopoly in case its domain I_p is a proper real interval and $\hat{p}$ is continuous.

The following lemmas can be found in [6].

Lemma 2.4. *Suppose g is twice differentiable. Then, g is (α, β)-biconcave if and only if for every $y \in I$, $(1-\beta)g(y)g'(y) + y\Big(g(y)g''(y) + (\alpha-1)g'(y)g'(y)\Big) \leq 0$.* $\diamond$

In case in the above lemma the strict inequality holds, we say that g is STRONGLY (α, β)-BICONCAVE. And, g is STRONGLY BICONCAVE if there exist $\alpha, \beta \in \mathbb{R}$ such that g is strongly (α, β)-biconcave.

Lemma 2.4 gives the following calculus characterizations in the case g is twice differentiable:

$$\begin{gathered} g \text{ is } (1,1)\text{-biconcave} \Leftrightarrow g'' \leq 0, \\ g \text{ is } (0,1)\text{-biconcave} \Leftrightarrow g''g - g'g' \leq 0, \\ g \text{ is } (-1,1)\text{-biconcave} \Leftrightarrow g''g - 2g'g' \leq 0, \\ g \text{ is } (1,0)\text{-biconcave} \Leftrightarrow yg'' + g' \leq 0, \\ g \text{ is } (0,0)\text{-biconcave} \Leftrightarrow y(g''g - g'g') + g'g \leq 0, \\ g \text{ is } (1,-1)\text{-biconcave} \Leftrightarrow yg'' + 2g' \leq 0. \end{gathered}$$

The above six cases got special attention in the context of the essential price function $\hat{p}$ of a Cournot oligopoly; see also [1]. They respectively concern the concavity of $\hat{p}$, log-concavity of $\hat{p}$, -1-concavity of $\hat{p}$, "marginal revenue condition" for $\hat{p}$, decreasing elasticity $\epsilon_{\hat{p}}$, and concave industry revenue $r_{\hat{p}}$.

Lemma 2.5. *Let $\alpha', \beta' \in \mathbb{R}$ with $\alpha' \leq \alpha$ and $\beta' \leq \beta$. Suppose g is decreasing. Then, g is (α, β) biconcave $\Rightarrow$ g is (α', β') biconcave.* $\diamond$

Lemma 2.6. *Suppose g is decreasing, I is unbounded, and g is (α, β)-biconcave with $\alpha > 0$ and $\beta \geq 0$. Then, g is constant.* $\diamond$

Lemma 2.7. *If g is decreasing, $0 \in \overline{I}$, and g is (α, β)-biconcave with $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta > 0$, then $\lim_{y \downarrow 0} g(y)y = 0$.* $\diamond$

Lemma 2.8. *If g is decreasing and strongly biconcave, then $g' < 0$ on $I \setminus \inf(I)$.* $\diamond$

Suppose g is decreasing and $g' < 0$ on $I \setminus \inf(I)$. The below formula (2) together with Lemma 2.4 shows that the following statement is not true: If g is (α, β)-biconcave, then g is strongly (α, β)-biconcave.

Example 2.9. Consider $g(y) = y^s$ $(y \in I)$ with $s \in \mathbb{R}$. Then:

1. g is $(0, 0)$-biconcave.
2. If $s \leq -1$, then there are no $\alpha \geq 0$ and $\beta \in \mathbb{R}$ with $\alpha + \beta$ $(\geq) > 0$ such that g is (strongly) (α, β)-biconcave.

Indeed, a calculation gives the formula

$$(1-\beta)g(y)g'(y) + y\Big(g(y)g''(y) + (\alpha-1)g'(y)g'(y)\Big) = y^{2s-1}s(\alpha s - \beta). \quad (2)$$

So, by Lemma 2.4 and the above formula, the first statement follows. We prove the second by contradiction. So, suppose $s \leq -1$ and g is (strongly) (α, β)-biconcave with $\alpha \geq 0$, $\beta \in \mathbb{R}$, and $\alpha + \beta\,(\geq) > 0$. By Lemma 2.4 and the above formula, $\alpha s\,(\geq) > \beta$. Thus, $0\,(\leq) < \alpha + \beta\,(<) \leq \alpha + s\alpha = (1+s)\alpha$. So, $0 < (1+s)\alpha$. It follows that $s < -1$ and $\alpha < 0$, a contradiction.

2.3. **The equilibrium uniqueness theorem.** With Theorem 2.10 we present here, using the notations in Subsection 2.1, Ewerhart's equilibrium uniqueness theorem.

Theorem 2.10. *Consider a Cournot oligopoly with price function* $p : Y \to \mathbb{R}$ *and cost functions* $c_i : X_i \to \mathbb{R}$. *Suppose* $p \geq 0$, p *is continuous and decreasing,* $I_p = \{y \neq 0 \mid p(y) > 0\} \neq \emptyset$ *and the essential price function* $\hat{p}$, *i.e.* $p : I_p \to \mathbb{R}$, *is twice continuously differentiable. Suppose every* c_i *is increasing and twice continuously differentiable. In the case where* $Y = \mathbb{R}_+$ *and* $p(y) > 0$ $(y \neq 0)$, *suppose there exists* $\overline{x} > 0$ *such that for every* $i \in N$, $z \in \sum_{j \neq i} X_j$ *and* $x_i \in X_i$

$$x_i > \overline{x} \Rightarrow p(x'_i + z)x'_i - c_i(x'_i) \geq p(x_i + z)x_i - c_i(x_i) \text{ for some } x'_i \in X_i \text{ with } x'_i \leq \overline{x} \tag{3}$$

and suppose for every $(x_1, \ldots, x_n) \in \times_{l=1}^{n} X_l$, *with* $p(\sum_l x_l) = 0$ *that there exists* $i \in N$ *with* $c_i(x_i) > c_i(0)$. *In each of the following two cases I and II the game has a unique Cournot equilibrium.*

I. $\hat{p}$ *is* (α, β)*-biconcave with* $0 \leq \alpha \leq 1$, $\alpha + \beta \leq 1$ *and for every* $i \in N$ *and* $0 \leq x_i \leq y$ *with* $y \in I_p$
$$(\alpha + \beta)p'(y) - c''_i(x_i) < 0.$$

II. $\hat{p}$ *is strongly* (α, β)*-biconcave with* $0 \leq \alpha \leq 1$ *and* $\alpha + \beta < 1$ *and for every* $i \in N$ *and* $0 \leq x_i \leq y$ *with* $y \in I_p$
$$(\alpha + \beta)p'(y) - c''_i(x_i) \leq 0. \ \diamond$$

Remark 2.11. Lemma 2.8 implies that in Theorem 2.10(I, II) one has $p'(y) - c''_i(x_i) < 0$ for every $i \in N$, $y \in I_p$ and $0 \leq x_i \leq y$.

In Theorem 2.10, in case $Y = \mathbb{R}_+$, i.e. in case at least one firm does not have a capacity constraint, and $I_p = \mathbb{R}_{++}$, i.e. $p(y) > 0$ $(y > 0)$, Proposition 2.1 together with Remark 2.11 imply that every c_i is convex.

In Theorem 2.10(I, II) consider the case where $I_p = \mathbb{R}_{++}$. If there $\alpha > 0$ and $\beta \geq 0$, then, by Lemma 2.6, the proper price function $p : \mathbb{R}_{++} \to \mathbb{R}$ is constant.

3. Recent Nash equilibrium result for sum-aggregative games.

3.1. **Setting.** Here, we closely follow the presentation in [14].

Consider a game in strategic form with player set $N := \{1, \ldots, n\}$ with $n \geq 2$ where each player i has as strategy set X_i a (non-empty) subset of $\mathbb{R}$, and, with $\mathbf{X} := \times_{l=1}^{n} X_l$, a payoff function $f_i : \mathbf{X} \to \mathbb{R}$.

For $i \in N$, let $\mathbf{X}_{\hat{\imath}} := X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$, and identify $\mathbf{X}$ with $X_i \times \mathbf{X}_{\hat{\imath}}$; accordingly, write $\mathbf{x} \in \mathbf{X}$ as $\mathbf{x} = (x_i; \mathbf{x}_{\hat{\imath}})$. For a player i and a strategy profile $\mathbf{z}$ of the other players, i.e. $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$, denote by $f_i^{(\mathbf{z})} : X_i \to \mathbb{R}$ the CONDITIONAL PAYOFF FUNCTION of player i, i.e. the payoff function of this player as a function of the strategies given the strategy profile $\mathbf{z}$ of the other players. For $i \in N$, let $T_i := \sum_{j \neq i} X_j$. We denote such a game by Γ.

Definition 3.1. The game Γ is called SUM-AGGREGATIVE if, for every $i \in N$, there exist REDUCED CONDITIONAL PAYOFF FUNCTIONS, i.e. functions $\tilde{f}_i^{(z)} : X_i \to \mathbb{R}$ $(z \in T_i)$, such that, for every $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$, $f_i^{(\mathbf{z})} = \tilde{f}_i^{(\sum_j z_j)}$. (So $f_i^{(\mathbf{0})} = \tilde{f}_i^{(0)}$.) $\diamond$

Our variant of Ewerhart's equilibrium uniqueness deals with a sum-aggregative game Γ where $X_i = \mathbb{R}_+$ ($i \in N$). We denote a sum-aggregative game with such strategy sets by

$$\overline{\Gamma}.$$

For $\overline{\Gamma}$, a player i is called ACTIVE (in $\mathbf{x}$) if $x_i \neq 0$ and INACTIVE (in $\mathbf{x}$) if $x_i = 0$. And, a strategy profile $\mathbf{x}$ is called ACTIVE if at least one player is active, i.e. if $\mathbf{x} \neq \mathbf{0}$.

Definition 3.2. A sum-aggregative game $\overline{\Gamma}$ is called ALMOST SMOOTH if every $f_i^{(\mathbf{z})}$ with $\mathbf{z} \neq 0$ is differentiable, $f_i^{(\mathbf{0})}$ is differentiable at every $x_i \neq 0$, and the (right) derivative of $f_i^{(\mathbf{0})}$ at 0 exists as an element of $\mathbb{R} \cup \{+\infty\}$. $\diamond$

Let $\Delta \ := \ \{(x, y) \in \mathbb{R}^2_+ \mid 0 \leq x \leq y\}$ and $\Delta^+ \ := \ \Delta \setminus \{(0, 0)\}$.

Definition 3.3. Consider an almost smooth sum-aggregative game $\overline{\Gamma}$. Its MARGINAL REDUCTIONS are defined as the functions $t_i : \Delta \to \mathbb{R} \cup \{+\infty\}$ ($i \in N$) given by

$$t_i(x_i, y) \ := \ (\tilde{f}_i^{(y-x_i)})'(x_i). \ \diamond$$

The formula $(\tilde{f}_i^{(y-x_i)})'(x_i)$ has to be interpreted as follows: First, one considers the derivative of the function $\tilde{f}_i^{(z)} : \mathbb{R}_+ \to \mathbb{R}$ at x_i. Then, one substitutes in this derivative $y - x_i$ for z. Note that t_i may assume the value $+\infty$ only at $(0, 0)$, and if $t_i(0, 0) \neq +\infty$, then $\tilde{f}_i^{(0)}$ is differentiable and thus continuous at 0. So,

$$\tilde{f}_i^{(0)} \text{ is discontinuous at } 0 \ \Rightarrow \ t_i(0, 0) = +\infty.$$

Further, in this subsection consider an almost smooth sum-aggregative game $\overline{\Gamma}$. Define the function $\bar{t}_i : \mathbb{R}_{++} \to \mathbb{R}$ by

$$\bar{t}_i(\lambda) \ := \ t_i(\lambda, \lambda) = (f_i^{(\mathbf{0})})'(\lambda) = (\tilde{f}_i^{(0)})'(\lambda).$$

In order to deal with a discontinuity of the payoff function $\tilde{f}_i^{(0)} = f_i^{(\mathbf{0})}$ at zero, player i is classified as type + or type -.

Definition 3.4. Player $i \in N$ is called TYPE + (TYPE -) if $\bar{t}_i(\lambda) > (<) \, 0$ for $\lambda > 0$ small enough. $\diamond$

If $t_i : \Delta \to \mathbb{R}$ is continuous, then the situation is simple:

$$t_i(0, 0) \ (<) > 0 \ \Rightarrow \ i \text{ is (type -) type +}. \tag{4}$$

Finally, wo have two notations:

$N_0 \ := \ \{i \in N \mid t_i(0, 0) > 0\}$ and $\tilde{N} \ := \ \{i \in N \mid t_i(0, y) > 0 \text{ for some } y > 0\}$.

3.2. **Assumptions.** Consider an almost smooth sum aggregative game $\overline{\Gamma}$. For its marginal reductions $t_i : \Delta \to \mathbb{R} \cup \{+\infty\}$, the following five assumptions A–E will be dealt with. They are supposed to hold for every $i \in N$. In case such an assumption is supposed to hold for a particular i, we add "[i]" to it as in, for example, A[i]. (Below, D_1 and D_2 denote partial derivatives.)

A. $t_i : \Delta^+ \to \mathbb{R}$ is continuous and continuously partially differentiable.
B. For every $(x_i, y) \in \Delta^+$: $t_i(x_i, y) = 0 \ \Rightarrow \ D_1 t_i(x_i, y) < 0$.
C. For every $(x_i, y) \in \Delta^+$ with $x_i > 0$: $t_i(x_i, y) = 0 \Rightarrow (x_i D_1 + y D_2) t_i(x_i, y) < 0$.
D. There exists $\overline{x}_i > 0$ such that $t_i(x_i, y) < 0$ for every $(x_i, y) \in \Delta^+$ with $x_i > \overline{x}_i$.
E. $t_i(0, y) > 0$ for some $y > 0 \Rightarrow t_i(0, 0) > 0$, i.e. $\tilde{N} \subseteq N_0$.

Note that in assumptions A–D, the value of t_i at $(0,0)$ does not play a role.

As marginal reductions relate to marginal payoffs, the above assumptions can be expressed in terms of primitives of the game. For example, for the Cournot oligopoly, this concerns the price function and cost functions; see (8).

Assumption A provides sufficient differentiability for the Selten-Szidarovszky technique to work. Assumptions B and C are local assumptions. In the Selten-Szidarovszky technique, assumption B guarantees that backward-response correspondences are at most single-valued (and continuous). Assumption C is the most interesting. It is a generalization of the global assumption $D_1 t_i < 0 \wedge D_2 t_i < 0$ in [4]. Assumption D is used for guaranteeing effective compactness (see the next subsection). Assumption E is a quite subtle one, but is for various concrete games, like the Cournot oligopoly a weak one (see Lemma 4.8): Among other things, it is used to handle the possibility of an inactive equilibrium (see Theorem 3.5(1)).

3.3. **Uniqueness result.**

Theorem 3.5. *Consider an almost smooth sum-aggregative game $\overline{\Gamma}$. Suppose assumptions A, B, and C hold.*

1. *Suppose assumption E holds. Then, $N_0 = \emptyset \Leftrightarrow \mathbf{0}$ is an equilibrium $\Rightarrow \mathbf{0}$ is the unique equilibrium.*
2. *$\overline{\Gamma}$ has at most one active equilibrium.*
3. *If assumptions D and E hold and at least one player is type $+$, then $\overline{\Gamma}$ has a unique equilibrium and this equilibrium is active.*
4. *If $\overline{\Gamma}$ is a symmetric game, then each equilibrium $\mathbf{e}$ is symmetric, i.e. $e_1 = \cdots = e_n$.*
5. *Suppose each player is type and assumption E holds. Then:*
   - *(a). For every $i \in \tilde{N}$ and sufficiently small $y > 0$, there exists a unique $\xi_i(y) \in]0, y[$ with $t_i(\xi_i(y), y) = 0$.*
   - *(b). For every $i \in \tilde{N}$, the limit $\underline{s}_i := \lim_{y \downarrow 0} \xi_i(y)/y$ exists and $\underline{s}_i \in]0,1]$.*
   - *(c). $\overline{\Gamma}$ has an active equilibrium $\Rightarrow \sum_{i \in \tilde{N}} \underline{s}_i > 1$.*
   - *(d). Suppose assumption D holds. Then, $\sum_{i \in \tilde{N}} \underline{s}_i > 1 \Leftrightarrow \overline{\Gamma}$ has a unique active equilibrium.*
   - *(e). In an active equilibrium, at least two players are active.* $\diamond$

Proof. For part 1, see [14, Proposition 1](2), and for parts 2–5, see [14, Theorem 1]. Note: At the webpage of the author, some minor corrections that apply to that article have been stated. Of course, the theorem here takes them into account. $\square$

We note that in Theorem 3.5 it is not (explicitly) assumed that payoff functions $f_i : \mathbb{R}^n_+ \to \mathbb{R}$ are continuous on $\mathbb{R}^n_{++}$.

As mentioned in [14], assumptions A–C guarantee that each conditional payoff function is strictly quasi-concave. Each conditional payoff function $f_i^{(\mathbf{z})}$ with $\mathbf{z} \neq \mathbf{0}$ is strictly pseudo-concave, and $f_i^{(\mathbf{0})}$ is strictly pseudo-concave on $\mathbb{R}_{++}$.

3.4. **Effective compactness.**

Definition 3.6. Consider a game in strategic form Γ where each player has $\mathbb{R}_+$ as the strategy set. Then, $k_i > 0$ is an EFFECTIVE COMPACTNESS LEVEL of player i if each strategy x_i with $x_i > k_i$ is strongly dominated by some element in $[0, k_i]$, i.e., there exists $d_i \in [0, k_i]$ (depending on x_i) such that $f_i^{(\mathbf{z})}(d_i) > f_i^{(\mathbf{z})}(x_i)$ for each strategy profile $\mathbf{z}$ of the other players. $\diamond$

Proposition 3.7. *Consider an almost smooth sum-aggregative game $\overline{\Gamma}$.*

1. *Suppose assumption D[i] holds; fix an $\overline{x}_i$ in this assumption. Then player i has $\overline{x}_i$ as an effective compactness level.*
2. *Suppose $i \in N_0$ and assumptions A[i], B[i], and C[i] hold and k_i is an effective compactness level of player i. Then, assumption D[i] holds with $\overline{x}_i = k_i$.* $\diamond$

Proof. See [14, Proposition 3]. □

4. **Variant of Ewerhart's equilibrium uniqueness theorem .**

4.1. **The theorem.**

Theorem 4.1. *Consider a Cournot oligopoly without capacity constraints and at least two firms. Suppose the proper price function, i.e. $p : \mathbb{R}_{++} \to \mathbb{R}$, is twice continuously differentiable with $p' < 0$. Suppose $I_p \neq \emptyset$ and the essential price function $\hat{p}$, i.e. $p : I_p \to \mathbb{R}$, is (α, β)-biconcave with $\alpha \in [0, 1]$ and $\beta \in \mathbb{R}$. Suppose each cost function c_i is increasing and twice continuously differentiable. For every $i \in N$, suppose*

$$p'(y) - c_i''(x_i) < 0 \text{ for every } 0 \leq x_i \leq y \text{ with } y > 0 \tag{5}$$

and that at least one of the following three conditions I, II, and III hold:

- I. $(\alpha + \beta)p'(y) - c_i''(x_i) < 0$ *for every* $0 < x_i \leq y$ *with* $y \in I_p$.
- II. $\hat{p}$ *is strongly* (α, β)*-biconcave and* $(\alpha+\beta)p'(y) - c_i''(x_i) \leq 0$ *for every* $0 < x_i \leq y$ *with* $y \in I_p$.
- III. $\alpha \neq 1$, c_i' *is positive on* $\mathbb{R}_{++}$, *and* $(\alpha+\beta)p'(y) - c_i''(x_i) \leq 0$ *for every* $0 < x_i \leq y$ *with* $y \in I_p$.

Then:

1. *There is at most one active Cournot equilibrium.*
2. *If $c_1 = \cdots = c_n$, then each Cournot equilibrium is symmetric.*
3. *Suppose $\underline{\epsilon}_{\hat{p}} = \lim_{y \downarrow 0} \frac{p'(y)y}{p(y)}$ is well-defined and $\underline{\epsilon}_{\hat{p}} < -1$. Then, in an active Cournot equilibrium, at least two firms are active.*
4. *Suppose $\underline{\epsilon}_{\hat{p}} = \lim_{y \downarrow 0} \frac{p'(y)y}{p(y)}$ is well-defined, and that for every $i \in N$, it holds that*

$$p(x_i)x_i - c_i(x_i) < -c_i(0) \text{ for } x_i \text{ large enough.} \tag{6}$$

   - (a) *If $-1 < \underline{\epsilon}_{\hat{p}} \leq 0$, then there is a unique Cournot equilibrium and this equilibrium is active.*
   - (b) *If $\underline{\epsilon}_{\hat{p}} < -1$, then each Cournot equilibrium is active and the game has a unique Cournot equilibrium if and only if $n > -\underline{\epsilon}_{\hat{p}}$.*
   - (c) *If $\underline{\epsilon}_{\hat{p}} = -1$ with $\epsilon_{\hat{p}}(y) \leq -1$ for $y > 0$ small enough and every c_i' is positive on $\mathbb{R}_{++}$, then the game has a unique Cournot equilibrium and this equilibrium is active.* $\diamond$

In Theorem 4.1(3,4), it is important that the limit $\underline{\epsilon}_{\hat{p}}$ is well-defined. The next proposition gives various sufficient conditions for this to hold. Also, see the concluding remarks III and IV in Section 5.

Proposition 4.2. *Consider a Cournot oligopoly. Suppose the proper price function $\tilde{p}$ is decreasing, $I_p \neq \emptyset$, and the essential price function $\hat{p}$ is differentiable.*

1. *If $\underline{\epsilon}_{\hat{p}}$ is well-defined, then $\underline{\epsilon}_{\hat{p}} \in [-\infty, 0]$.*
2. *If $\epsilon_{\hat{p}}$ is decreasing, then $\underline{\epsilon}_{\hat{p}}$ is well-defined and $\underline{\epsilon}_{\hat{p}} \in \,]-\infty, 0]$.*

3. *If $\tilde{p}'/\tilde{p}$ is bounded at 0, then $\underline{\epsilon}_{\hat{p}}$ is well-defined and $\underline{\epsilon}_{\hat{p}} = 0$.*
4. *If p is continuously differentiable, then $\underline{\epsilon}_{\hat{p}}$ is well-defined and $\underline{\epsilon}_{\hat{p}} = 0$.*
5. *If $\hat{p}$ is (α, β)-biconcave with $\alpha, \beta \geq 0$, then $\underline{\epsilon}_{\hat{p}}$ is well-defined and $\underline{\epsilon}_{\hat{p}} \in]-\infty, 0]$.*
6. *Suppose $\hat{p}$ is continuously differentiable, p is differentiable at 0, and $\epsilon_{\hat{p}}$ is decreasing. Then, $\underline{\epsilon}_{\hat{p}}$ is well-defined and $\underline{\epsilon}_{\hat{p}} = 0$.* $\diamond$

Proof. 1. This result follows as $\epsilon_{\hat{p}} \leq 0$.

2. As $\epsilon_{\hat{p}}$ is decreasing, $\epsilon_{\hat{p}} \leq 0$.

3. Suppose $\tilde{p}'/\tilde{p}$ is bounded at 0. Let $\epsilon > 0$. That $\tilde{p}'/\tilde{p}$ is bounded at 0 means that there exists $l > 0$ and $m > 0$ such that $|\tilde{p}'/\tilde{p}| \leq m$ on $]0, l]$. We obtain for $0 < y < \min(l, \epsilon/m)$ that $| \tilde{p}'(y) \frac{y}{\tilde{p}(y)} | < m \min(l, \epsilon/m) \leq \epsilon$. Thus, $\underline{\epsilon}_{\hat{p}} = 0$.

4. Suppose p is continuously differentiable. Fix $\xi \in I_p$. As $\tilde{p}$ is decreasing, $p(y) \geq \xi$ $(0 < y \leq \xi)$ follows. Now, $p(0) = \lim_{y \downarrow 0} p(y) > 0$ and $\underline{\epsilon}_{\hat{p}} = \lim_{y \downarrow 0} \frac{y p'(y)}{p(y)} = \frac{0 p'(0)}{p(0)} = 0$.

5. Suppose $\hat{p}$ is (α, β)-biconcave with $\alpha \geq 0$ and $\beta \geq 0$. By Lemma 2.5, $\hat{p}$ is $(0, 0)$-biconcave. This implies, as noted in Subsection 2.2, that $\epsilon_{\hat{p}}$ is decreasing. Now, apply part 2.

6. First note that p is differentiable on $I_p \cup \{0\}$ and that $p(0) > 0$. Define $\epsilon : I_p \cup \{0\} \to \mathbb{R}$ by $\epsilon(y) = p'(y)y/p(y)$. So, $\epsilon(0) = 0$. We now show by contradiction that ϵ is continuous at 0, and then, as desired, $\underline{\epsilon}_{\hat{p}} = \lim_{y \downarrow 0} \epsilon_{\hat{p}}(y) = \lim_{y \downarrow 0} \epsilon(y) = \epsilon_p(0) = 0$. So, suppose ϵ is not continuous at 0. Using $\epsilon(0) = 0$ and the decreasingness of ϵ on I_p, we see that ϵ is decreasing. As ϵ is decreasing and discontinuous at 0, there exists $\delta > 0$ such that $yp'(y)/p(y) < -\delta$ $(y \in I_p)$. Fix $y_0 \in I_p$. So, $p'(y) < -\delta p(y)/y \leq -\delta p(y_0)/y$ $(0 < y < y_0)$. As $\lim_{y \downarrow 0} -\delta p(y)/y = -\infty$, this implies that for $y > 0$ small enough, $p'(y)$ is less than, say, $p'(0) - 137$. This is a contradiction with, as p is differentiable on $I_p \cup \{0\}$, $p' : I_p \cup \{0\} \to \mathbb{R}$ being a Darboux function. $\square$

Remark 4.3. In Theorem 4.1, in the case $I_p = \mathbb{R}_{++}$, i.e. $p(y) > 0$ $(y > 0)$, Proposition 2.1 implies that every c_i is convex.

Some discussion of Theorem 4.1 is in order. First, if therein $\hat{p}$ is (α, β)-biconcave with $\alpha \geq 0, \beta \geq 0$ such that $\alpha + \beta > 0$, then Lemma 2.7 implies that the industry revenue r_p is continuous. In addition, Lemma 4.10(3) below guarantees that in Theorem 4.1(4a,4b) the industry revenue function r_p is continuous and that in Theorem 4.1(4c) it is is discontinuous at 0.

Second, in Theorem 4.1(I,II), consider the case where $I_p = \mathbb{R}_{++}$. If $\alpha > 0$ and $\beta \geq 0$, then, by Lemma 2.6, $p : \mathbb{R}_{++} \to \mathbb{R}$ is constant. Therefore, as there is also $\alpha \in [0, 1]$, only two situations are interesting: $\alpha = 0 \wedge \beta \in \mathbb{R}$ and $\alpha \in]0, 1] \wedge \beta < 0$. In particular, these situations cover $(0, 0)$-biconcavity, $(0, 1)$-biconcavity, and $(1, -1)$-biconcavity.

Third, if in Theorem 4.1 the interval I_p is bounded, then for every $i \in N$ it holds that $p(x_i)x_i - c_i(x_i) < -c_i(0)$ for x_i large enough. So, the second condition in part 4 automatically holds.

The following examples highlight some results for Cournot oligopolies.

Example 4.4. Suppose $X_i = \mathbb{R}_+$ $(i \in N)$, $n \geq 2$, $p(y) = y^s$ $(y > 0)$ with $s < -1$, and every c_i is increasing, convex, and twice continuously differentiable with $c_i'(x_i) > 0$ $(x_i \neq 0)$. Then, each Cournot equilibrium is active and the game has a unique Cournot equilibrium if and only if $n > -s$.

Proof. First note that $I_p = \mathbb{R}_{++}$, that $\epsilon_{\hat{p}} = s$, and that $\hat{p}$ is $(0,0)$-biconcave. So, part III of Theorem 4.1 applies. Next, note that $\underline{\epsilon}_{\hat{p}} = s < -1$ and that (6) holds. Now apply Theorem 4.1(4b). □

Remark 4.5. The result in Example 4.4 cannot be proved by referring to parts I and II of Theorem 4.1. In order to see this, suppose $c_i(x_i) = x_i$. Thus, $c_i'' = 0$. So, in part I, it has to hold that $\hat{p}$ is (α,β)-biconcave with $\alpha+\beta > 0$, and in part II that $\hat{p}$ is strongly (α,β)-biconcave with $\alpha+\beta \geq 0$. Finally, note that $\alpha \in [0,1]$, and apply the second statement in Example 2.9.

Example 4.6. Suppose $X_i = \mathbb{R}_+$ $(i \in N)$, $n \geq 2$, $p(y) = y^{-1}$ $(y > 0)$, and every c_i is increasing, convex, and twice continuously differentiable with $c_i'(x_i) > 0$ $(x_i \neq 0)$. Then, the game has a unique Cournot equilibrium, and this equilibrium is active.

Proof. Note that $I_p = \mathbb{R}_{++}$, and that $\hat{p}$ is $(0,0)$-biconcave. Part III of Theorem 4.1 applies. Also, note that $\epsilon_{\hat{p}} = s$, and that (6) holds. Now, apply Theorem 4.1(4c). □

The next simple example shows that, despite Remark 4.3, Theorem 4.1 can handle non-convex cost functions.

Example 4.7. Suppose $X_i = \mathbb{R}_+$ $(i \in N)$, $p(y) = \max(1+ay, 0)$ with $a < 0$, and $c_i'' < 0$ $(i \in N)$. Further, suppose c_i and c_i'' are strictly increasing with $c_i''(0)/a < 1$. (For example: $a = -2$ and $c_i = \frac{2}{3}x_i/(1+x_i)$.) Then, the game has a unique Cournot equilibrium, and this equilibrium is active.

Proof. First, by Proposition 2.2(2) we may assume that $p(y) = 1+ay$. Note that $I_p =]0, -1/a[$. Fix $\alpha \in [0,1]$ and $\beta \geq 0$ with $c_i''(0)/a \leq \alpha+\beta \leq 1$. As $\hat{p}$ is concave, it is $(1,1)$-biconcave. By Lemma 2.5, $\hat{p}$ is also (α,β)-biconcave. Part I of Theorem 4.1 applies: For every $0 < x_i \leq y$ with $y < -1/a$, we have $(\alpha+\beta)p'(y) - c_i''(x_i) = (\alpha+\beta)a - c_i''(x_i) \leq c_i''(0) - c_i''(x_i) < 0$. Also, note that $\epsilon_{\hat{p}} = 0$ and that (6) holds. Now, apply Theorem 4.1(1). □

4.2. **Proof.** Consider Theorem 4.1 with its general assumptions. As we shall, assumptions A, B, C, and E in Subsection 3.2 also hold. For assumption D to hold, further assumptions have to be made.

Almost smooth sum-aggregative game

Note that reduced conditional payoff functions are well-defined:

$$\tilde{f}_i^{(z)}(x_i) \ := \ p(x_i+z)x_i - c_i(x_i).$$

In particular,

$$\tilde{f}_i^{(0)}(x_i) = \begin{cases} p(x_i)x_i - c_i(x_i) = (r_p - c_i)(x_i) \text{ if } x_i \neq 0, \\ -c_i(0) \text{ if } x_i = 0. \end{cases} \tag{7}$$

Also, noting that $\tilde{f}_i^{(0)} = r_p - c_i$ and using (1),

$$(\tilde{f}_i^{(z)})'(x_i) = \begin{cases} p'(x_i+z)x_i + p(x_i+z) - c_i'(x_i) \text{ if } z \neq 0, \\ p'(x_i)x_i + p(x_i) - c_i'(x_i) \text{ if } z = 0 \text{ and } x_i \neq 0, \\ \underline{p} - c_i'(0) \text{ if } z = 0 \text{ and } x_i = 0. \end{cases}$$

This makes that we have to do with an almost smooth sum-aggregative game with marginal reductions $t_i : \Delta \to \mathbb{R} \cup \{+\infty\}$ given by

$$t_i(x_i, y) = \begin{cases} p'(y)x_i + p(y) - c_i'(x_i) \text{ if } (x_i, y) \neq (0,0), \\ \underline{p} - c_i'(0) \text{ if } (x_i, y) = (0,0). \end{cases} \tag{8}$$

This formula implies, for $(x_i, y) \in \Delta^+$,

$$D_1 t_i(x_i, y) = p'(y) - c_i''(x_i) \text{ and } D_2 t_i(x_i, y) = x_i p''(y) + p'(y). \tag{9}$$

Assumptions A, B, and E

Assumption A holds by formula (8) as $\tilde{p}$ and every c_i are twice continuously differentiable. Assumption B holds by formulas (9) and (5). Assumption E holds as $\tilde{N} = N_0$, as the following lemma shows.

Lemma 4.8. $\tilde{N} = N_0$. $\diamond$

Proof. $\tilde{N} \subseteq N_0$: Suppose $t_i(0, y) > 0$ for some $y > 0$. So, by (8), $t_i(0, y) = p(y) - c_i'(0) > 0$. As $\tilde{p}$ is decreasing, we have $t_i(0, 0) = \underline{p} - c_i'(0) \geq p(y) - c_i'(0) = t_i(0, y) > 0$. Thus, $i \in N_0$.

$N_0 \subseteq \tilde{N}$: Suppose $t_i(0, 0) > 0$, i.e. $\underline{p} > c_i'(0)$. So, there exists $y > 0$ with $p(y) > c_i'(0)$, and therefore $t_i(0, y) = p(y) - c_i'(0) > 0$. Thus, $i \in \tilde{N}$. $\square$

Assumption C

Suppose $(x_i, y) \in \Delta^+$ with $x_i > 0$ and $t_i(x_i, y) = 0$. We have to show that $x_i D_1 + y D_2) t_i(x_i, y) < 0$ in each of the cases I, II, and III. As $c_i'(x_i) \geq 0$, $x_i > 0$, and $p'(y) < 0$, we obtain $p(y) = c_i'(x_i) - x_i p'(y) > 0$, and therefore y belongs to the domain I_p of $\hat{p}$.

Cases I, II. The proof is complete if we show that

$$(x_i D_1 + y D_2) t_i(x_i, y) \ (<) \leq x_i \Big((\alpha + \beta) p'(y) - c_i''(x_i) \Big)$$

(with "$\leq$" in case I and "$<$" in case II).

First, we proof the following inequality that also can be found in [6]:

$$x_i p''(y) + p'(y) (<) \leq \Big(\alpha - \frac{x_i}{y} (1 - \beta) \Big) p'(y) \tag{10}$$

Then, multiply the (strict) inequality for (strong) biconcavity of $\hat{p}$ in Definition 2.3 with $x_i / y > 0$ and obtain

$$\frac{x_i}{y} (1 - \beta) p(y) p'(y) + x_i \Big(p(y) p''(y) + (\alpha - 1) p'(y) p'(y) \Big) \ (<) \leq 0.$$

As

$$x_i p'(y) + p(y) = c_i'(x_i) \geq 0$$

and $\alpha \leq 1$, we have $(1 - \alpha) p'(y)(x_i p'(y) + p(y)) \leq 0$. Adding this inequality to the previous one gives the desired result.

Next, noting that $\alpha \geq 0$ and $x_i \leq y$, we obtain with (10) and (9),

$$\begin{aligned} (x_i D_1 + y D_2) t_i(x_i, y) &= x_i (p'(y) - c_i''(x_i)) + y (x_i p''(y) + p'(y)) \\ (<) \leq x_i (p'(y) - c_i''(x_i)) &+ (\alpha y - x_i (1 - \beta)) p'(y) = (\beta p'(y) - c_i''(x_i)) x_i + \alpha y p'(y) \\ &\leq (\beta p'(y) - c_i''(x_i)) x_i + \alpha x_i p'(y) = x_i \Big((\alpha + \beta) p'(y) - c_i''(x_i) \Big). \end{aligned}$$

Case III. The proof is complete if we show that $(x_i D_1 + y D_2) t_i(x_i, y) < x_i \Big((\alpha + \beta) p'(y) - c_i''(x_i) \Big)$. We proceed as before. Now, note that

$$x_i p'(y) + p(y) = c_i'(x_i) > 0$$

and that, as $\alpha < 1$ and $p'(y) < 0$, we now have $(1 - \alpha) p'(y)(x_i p'(y) + p(y)) < 0$. This means that the strict inequality in (10) holds.

Assumption D

The following lemma gives a sufficient condition for Assumption D[i] to hold.

Lemma 4.9. *If the inequality $p(x_i)x_i - c_i(x_i) < -c_i(0)$ holds for x_i large enough, then assumption D[i] holds.* $\diamond$

Proof. Suppose $\overline{x}_i$ is such that $p(x_i)x_i - c_i(x_i) < -c_i(0)$ $(x_i > \overline{x}_i)$. As $p'(y) - c_i''(x_i) < 0$ $((x_i, y) \in \Delta^+)$, the function $p - c_i' : \mathbb{R}_{++} \to \mathbb{R}$ is strictly decreasing, which implies that $\lim_{x_i \downarrow 0}(p - c_i')(x_i) = \underline{p} - c_i'(0) = t_i(0,0)$. We distinguish between two cases.

Case where $i \notin N_0$**.** Now, $t_i(0,0) \leq 0$, and so for $(x_i, y) \in \Delta^+$ with $x_i > 0$, we obtain $t_i(x_i, y) = p'(y)x_i + p(y) - c_i'(x_i) < p(y) - c_i'(x_i) \leq (p - c_i')(x_i) \leq \underline{p} - c_i'(0) = t_i(0,0) \leq 0$. Thus, for such an i, assumption D[i] holds.

Case where $i \in N_0$**.** Remember that assumptions A[i], B[i], and C[i] hold. We show that $\overline{x}_i$ is an effective compactness level (see Definition 3.6). Then, by Proposition 3.7(2), Assumption D[i] holds as well. Suppose $x_i > \overline{x}_i$. Then, as $\tilde{p}$ is decreasing, $f_i^{(\mathbf{z})}(0) = -c_i(0) > p(x_i)x_i - c_i(x_i) \geq p(x_i + \sum_j z_j)x_i - c_i(x_i) = f_i^{(\mathbf{z})}(x_i)$. So, as desired, x_i is strongly dominated by 0. $\square$

$\underline{\epsilon}_{\hat{p}}$, types + and -

We have

$$t_i(\lambda, \lambda) = p(\lambda)(\epsilon_{\hat{p}}(\lambda) + 1) - c_i'(\lambda) \;\; (\lambda \in I_p). \tag{11}$$

Remember that $\underline{p} \in]0, +\infty]$, $t_i(0,0) = \underline{p} - c_i'(0)$, and that $\underline{\epsilon}_{\hat{p}}$, if well-defined, is an element of $[-\infty, 0]$. If $\underline{\epsilon}_{\hat{p}} \neq -1$, (11) implies

$$\lim_{\lambda \downarrow 0} t_i(\lambda, \lambda) = \underline{p}(\underline{\epsilon}_{\hat{p}} + 1) - c_i'(0). \tag{12}$$

Lemma 4.10.
1. *Suppose $\underline{\epsilon}_{\hat{p}} = 0$. Then, $t_i(0,0)\,(<) > 0 \;\Rightarrow\; i$ is type $(-)+$.*
2. *Suppose $\underline{\epsilon}_{\hat{p}} < 0$. $\underline{p} = +\infty$, $t_i(0,0) = +\infty$ $(i \in N)$, and $\tilde{N} = N_0 = N$.*
3. *Suppose $-1 < \underline{\epsilon}_{\hat{p}} \leq 0$. Then, r_p is continuous.*
4. *Suppose $\underline{\epsilon}_{\hat{p}} < -1$. Then, r_p is discontinuous at* 0 *and i is type -.*
5. *Suppose $-1 < \underline{\epsilon}_{\hat{p}} < 0$. Then, every $i \in N$ is type +.*
6. *Suppose $\epsilon_{\hat{p}}(y) \leq -1$ for $y > 0$ small enough and $c_i' > 0$ on $\mathbb{R}_{++}$. Then, i is type -.* $\diamond$

Proof. 1. Note that by (12), $\lim_{\lambda \downarrow 0} t_i(\lambda, \lambda) = \underline{p} - c_i'(0) = t_i(0,0)$.

2. Fix $q \in I_p$. The function $g : I_p \to \mathbb{R}$ is well-defined by $g(y) = p(q)e^{\int_y^q -\frac{\epsilon_{\hat{p}}(\xi)}{\xi}\, d\xi} = p(q)e^{\int_y^q -\frac{p'(\xi)}{p(\xi)}\, d\xi}$. It follows that $g' = p'$ on I_p. As $g(q) = p(q)$, we obtain

$$p(y) = p(q)e^{\int_y^q -\frac{p'(\xi)}{p(\xi)}\, d\xi} \;(y \in I_p).$$

Fix $\delta > 0$ with $\underline{\epsilon}_{\hat{p}} < -\delta$. As $\epsilon_{\hat{p}}$ is continuous, there exists $\tau \in]0, q[$ such that $\epsilon_{\hat{p}}(\xi) = p'(\xi)\frac{\xi}{p(\xi)} < \underline{\epsilon}_{\hat{p}} + \delta$ $(0 < \xi < \tau)$. So, for all such ξ, we have with $b = -\underline{\epsilon}_{\hat{p}} - \delta$ that $b > 0$ and

$$-p'(\xi)/p(\xi) > b/\xi.$$

Now, for $y \in]0, \tau[$ and $s = e^{\int_\tau^q -\frac{p'(\xi)}{p(\xi)}\, d\xi} > 0$,

$$p(y) = p(q)e^{\int_y^q -\frac{p'(\xi)}{p(\xi)}\, d\xi} = p(q)e^{\int_y^\tau -\frac{p'(\xi)}{p(\xi)}\, d\xi + \int_\tau^q -\frac{p'(\xi)}{p(\xi)}\, d\xi} \geq p(q)se^{\int_y^\tau \frac{b}{\xi}\, d\xi} = p(q)s\tau^b y^{-b}.$$

This implies $\underline{p} = +\infty$, and therefore $t_i(0,0) = \underline{p} - c_i'(0) = +\infty$. So, $N_0 = N$. By Lemma 4.8, $\tilde{N} = N_0 = N$.

3. Fix $\delta > 0$ such that $\underline{\epsilon}_{\hat{p}} - \delta > -1$. As $\epsilon_{\hat{p}}$ is continuous, there exists $\tau \in \,]0, q[$ such that $\epsilon_{\hat{p}}(\xi) = p'(\xi)\frac{\xi}{p(\xi)} > \underline{\epsilon}_{\hat{p}} - \delta > -1$ for all $\xi \in \,]0, \tau[$. So, for all such ξ we have with $b = -\underline{\epsilon}_{\hat{p}} + \delta$ that $b < 1$ and

$$0 \leq -p'(\xi)/p(\xi) < b/\xi.$$

Now, for $y \in \,]0, \tau[$ and $s = e^{\int_\tau^q -\frac{p'(\xi)}{p(\xi)}\,d\xi} > 0$, we find as above

$$p(y) \leq p(q) s \tau^b y^{-b}.$$

It follows that $r_p(y) \leq p(q) s \tau^b y^{1-b}$. As $0 < 1 - b$, we obtain $\lim_{y \downarrow 0} r_p(y) = 0 = r_p(0)$.

4. So, $\underline{\epsilon}_{\hat{p}} + 1 < 0$. By (12), $\lim_{\lambda \downarrow 0} t_i(\lambda, \lambda) = \underline{p}(\underline{\epsilon}_{\hat{p}} + 1) - c_i'(0) < -c_i'(0) \leq 0$. From this, the second statement follows.

Let $a = \underline{\epsilon}_{\hat{p}}$. Suppose $a < -1$. Fix $\delta > 0$ such that $a < -1 - \delta$. As $\epsilon_{\hat{p}}$ is continuous, there exists $\tau \in \,]0, q[$ such that $\epsilon_{\hat{p}}(\xi) = p'(\xi)\frac{\xi}{p(\xi)} < -1 - \delta$ for all $\xi \in \,]0, \tau[$. So, for all such ξ, we have, letting $b = -1 - \delta$, that $b < -1$ and

$$-p'(\xi)/p(\xi) > -b/\xi.$$

Now, for $y \in \,]0, \tau[$ and $s = e^{\int_\tau^q -\frac{p'(\xi)}{p(\xi)}\,d\xi} > 0$, we find as above

$$p(y) \geq p(q) s e^{\int_y^\tau -\frac{b}{\xi}\,d\xi} = p(q) s \tau^{-b} y^b.$$

It follows that $r_p(y) \geq p(q) c \tau^{-b} y^{1+b}$. As $1 + b < 0$, we obtain $\lim_{y \downarrow 0} r_p(y) = +\infty$.

5. As $\underline{\epsilon}_{\hat{p}} + 1 > 0$ and, by part 2, $\underline{p} = +\infty$, (12) implies $\lim_{\lambda \downarrow 0} t_i(\lambda, \lambda) = \underline{p}(\underline{\epsilon}_{\hat{p}} + 1) - c_i'(0) = +\infty$. Thus, i is type +.

6. By (11), for $\lambda > 0$ small enough, $t_i(\lambda, \lambda) = p(\lambda)(\epsilon_{\hat{p}}(\lambda) + 1) - c_i'(\lambda) \leq -c_i'(\lambda) < 0$. So, i is type -. □

The limit $\underline{s}_i$

Remember the definition of $\underline{s}_i$ in Theorem 3.5(5b).

Lemma 4.11. *Suppose $\underline{\epsilon}_{\hat{p}}$ is well-defined with $\underline{\epsilon}_{\hat{p}} < 0$ and each player is type -. Then, $\underline{s}_i = -1/\underline{\epsilon}_{\hat{p}}$ $(i \in \tilde{N})$.* ⋄

Proof. By Lemma 4.10(2), $\tilde{N} = N_0 = N$. By Theorem 3.5(5a), for every $i \in \tilde{N}$ and $y > 0$ small enough there exists a unique $\xi_i(y) \in \,]0, y]$ with $t_i(\xi_i(y), y) = 0$, so $0 = p'(y)\xi_i(y) + p(y) - c_i'(\xi_i(y))$. A little calculation shows

$$\frac{\xi_i(y)}{y} = -\frac{1}{\epsilon_{\hat{p}}(y)}\left(1 - \frac{c_i'(\xi_i(y))}{p(y)}\right)$$

for y small enough. This formula implies, noting that $0 < \xi_i(y) \leq y$ and c_i' is continuous at 0, $\underline{s}_i = \lim_{y \downarrow 0} \xi_i(y)/y = -1/\underline{\epsilon}_{\hat{p}}$. □

Finishing the proof

We already know that we have to do with an almost smooth sum-aggregative game with marginal reductions given by (8), for which Assumptions A, B, C, and E hold. By Lemma 4.9, assumption D also holds in part 4 of Theorem 4.1.

1. Apply Theorem 3.5(2).

2. Suppose $c_1 = \cdots = c_n$. This implies that the game is symmetric. Apply Theorem 3.5(4).

3. By Lemma 4.10(4), each player is type -. Apply Theorem 3.5(5e).

4a. We distinguish between three cases.

Case $\underline{\epsilon}_{\hat{p}} = 0$ and $N_0 = \emptyset$. $\mathbf{0}$ is the unique equilibrium by Theorem 3.5(1).

Case $\underline{\epsilon}_{\hat{p}} = 0$ **and** $N_0 \neq \emptyset$**.** Fix $i \in N_0$. By Lemma 4.10(1), i is type $+$. Now, apply Theorem 3.5(3).
Case $-1 < \underline{\epsilon}_{\hat{p}} < 0$**.** By Lemma 4.10(5), every $i \in N$ is type $+$. Now, apply Theorem 3.5(3).

4b. Suppose $\underline{\epsilon}_{\hat{p}} < -1$. By Lemma 4.10(2,4), $\tilde{N} = N_0 = N$ and every $i \in N$ is type -. Theorem 3.5(1) guarantees that $\mathbf{0}$ is not an equilibrium. By Lemma 4.11, $\underline{s}_i = -1/\underline{\epsilon}_{\hat{p}}$ $(i \in N)$. From this, $\sum_{i \in \tilde{N}} \underline{s}_i = -n/\underline{\epsilon}_{\hat{p}}$. Finally, apply Theorem 3.5(5d).

4c. Suppose $\underline{\epsilon}_{\hat{p}} = -1$ with $\epsilon_{\hat{p}}(y) \leq -1$ for $y > 0$ small enough and every c_i' is positive on $\mathbb{R}_{++}$. By Lemma 4.10(6), every $i \in N$ is type -. By Lemma 4.10(2), $\tilde{N} = N_0 = N$. Theorem 3.5(1) guarantees that $\mathbf{0}$ is not an equilibrium. By Lemma 4.11, $\underline{s}_i = -1/\underline{\epsilon}_{\hat{p}} = 1$ $(i \in N)$. From this, $\sum_{i \in \tilde{N}} \underline{s}_i = n > 1$. Finally, apply Theorem 3.5(5d). $\square$

5. **Concluding remarks .** Below, we compare Ewerhart's equilibrium uniqueness theorem (i.e., Theorem 2.10) with our variant (i.e., Theorem 4.1). We also provide some comments that may be useful for further research.

I. In Theorem 2.10, each firm has $\mathbb{R}_+$ or $[0, m_i]$ as its strategy set. As became clear from [12], this leads to various technicalities that must be addressed. In this context, the proof given in [6] of equilibrium semi-uniqueness in Theorem 2.10 is quite interesting, as it is conceptual and concise. (However, for the author of the present article, it was (too) difficult to verify all details of this proof.)
II. Theorem 4.1 does not assume that $\alpha + \beta \leq 1$; Theorem 2.10 does. However, (5) together with Lemma 2.5 makes this weaker assumption less interesting than it may at first sight appear.
III. For parts 3 and 4 of Theorem 4.1, the limit $\underline{\epsilon}_{\hat{p}} = \lim_{y \downarrow 0} p'(y)y/p(y)$ should be well-defined. Although Proposition 4.2 provides various conditions ensuring this, it remains unclear whether this limit is always well-defined in Theorem 2.10. In this regard, observe, for example, that Theorem 2.10 does not state that p is differentiable at 0. See also the following concluding remark.
IV. Consider a positive, twice continuously differentiable function $g : I \to \mathbb{R}$, where I is a proper real interval with $I \subseteq \mathbb{R}_{++}$ and $0 \in \overline{I}$. If g is decreasing and biconcave, does the limit $\lim_{y \downarrow 0} g'(y)y/g(y)$ exist in $\mathbb{R} \cup \{-\infty, +\infty\}$?
V. Footnote 13 in [6] states that Theorem 2.10 also holds if we replace the sentence "Suppose $p \geq 0$, p is continuous and decreasing" with "Suppose $\tilde{p} \geq 0$, $\tilde{p}$ is continuous and decreasing, and r_p is continuous (at 0)".
VI. In Theorem 4.1, the proper price function is assumed to be twice continuously differentiable, whereas in Theorem 2.10, only the essential price function must satisfy this condition. This stronger assumption arises from the setting of Theorem 3.5 (on which Theorem 4.1 relies), which concerns abstract sum-aggregative games involving marginal reductions $t_i(x_i, y)$, where all positive values of y occur. However, as suggested by Proposition 2.2(2), for many Cournot oligopolies, the values of y for which $p(y) < 0$ may be irrelevant. It would be interesting to develop a variant of Theorem 3.5 in which, loosely speaking, in the $t_i(x_i, y)$, such values of y do not appear. (Such a result, although less powerful than Theorem 3.5, can be found in [15].)
VII. The equilibrium semi-uniqueness proof of Theorem 2.10 uses backward-response correspondences. For the proofs of Theorem 4.1, share correspondences were used.

VIII. An equilibrium existence result for Cournot oligopolies with a $(-1, 1)$-biconcave price function can be found in [5]. Its proof consists of verifying the conditions of the generalized Nikaido-Isoda theorem. Interestingly, [9] proves an equilibrium uniqueness result in the case of $(-1/n, 1)$-biconcave price functions. Since $-1/n < 0$, neither Theorem 2.10 nor Theorem 4.1 applies to such biconcave price functions. See also the next concluding remark.

IX. Finally, consider the following observation for a $(-1/n, 1)$-biconcave essential price function and convex cost functions. Using (10), which is indeed applicable, we obtain

$$(x_i D_1 + y D_2) t_i(x_i, y) = x_i(p'(y) - c_i''(x_i)) + y(x_i p''(y) + p'(y))$$

$$\leq x_i(p'(y) - c_i''(x_i)) + (\alpha y - x_i(1-\beta))p'(y) = (\beta p'(y) - c_i''(x_i))x_i + \alpha y p'(y)$$

$$= (p'(y) - c_i''(x_i))x_i - \frac{y}{n} p'(y) = p'(y)(x_i - y/n) - c''(x_i)x_i < p'(y)(x_i - y/n).$$

(Here, the strict inequality applies if the function is strongly $(-1/n, 1)$-biconcave if $c_i''(x_i) < 0$ $(x_i \neq 0)$.) Therefore,

$$\sum_{i \in N} (x_i D_1 + y D_2) t_i(x_i, y) < p'(y) \sum_{i \in N} (x_i - y/n) = 0.$$

Given [15, Theorem 4.1] (which generalizes [8]) and [13, Theorem 4.4], this observation may be useful for handling the $(-1/n, 1)$-biconcave case along the lines of these two articles.

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