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Output controllability of large-scale nonlinear dynamical systems: analysis, computation and examples

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Abstract: A sensitivity-based algorithm to establish state controllability is extended to establish output controllability being the ability to control the outputs of a nonlinear dynamical system instead of the full state. Due to the exceptional efficiency of the sensitivity-based algorithm, large-scale nonlinear dynamical systems can be handled, as demonstrated by several examples in this paper. As a final contribution, this paper starts with a simple analysis of state and output controllability properties of nonlinear dynamical systems in terms of connectivity's and sensitivities. The latter relate directly to the sensitivity-based algorithm.

Keywords: Output controllability, state controllability, large-scale nonlinear dynamical systems, sensitivity-based algorithms, sensitivity rank condition (SERC), local strong accessibility.

1. INTRODUCTION

State controllability and observability, being the ability to control and observe the state of a dynamical system, were initially developed for linear dynamical systems requiring results from linear algebra (Kalman, 1960, 1962, 1963; Kreindler and Sarachik, 1964; Weiss and Kalman, 1965; Ho and Kalman, 1966). The same holds for output controllability that appears to be introduced in Kreindler and Sarachik (1964). The extension of these and other properties to nonlinear dynamical systems requires linear algebra to be replaced with differential geometry. This significantly complicates the mathematics and computations such as the definition and computation of state and output controllability that now rely on Lie algebras (Hermann and Krener, 1977; Nijmeijer, 1983; Nijmeijer and van der Schaft, 1990; Kwatny and Blankenship, 2000).

On the other hand, canonical representations of both linear and nonlinear dynamical systems obtained from state controllability and observability are remarkably simple and similar. This observation was exploited in Van Willigenburg (2024) providing simple definitions and computations for state controllability and observability based on connectivity's and sensitivities. Triggered by the development of a sensitivity-based algorithm to establish identifiability of large-scale nonlinear systems (Stigter and Molenaar, 2015), sensitivities also appeared as keys to computing state controllability and observability of large-scale nonlinear dynamical systems. Remarkably, these computations essentially concern the controllability and observability of *linearizations* along trajectories of the nonlinear dynamical system, describing *exactly* the sensitivity dynamics (Stigter, van Willigenburg and Molenaar, 2018; Van Willigenburg, Stigter and Molenaar, 2022). In this paper these results are extended from state to output controllability.

The results in this paper apply to *analytical* dynamical systems because these guarantee controllability properties to be invariant over time. In practice, if dynamical systems are not analytical, they are usually piecewise analytical and the analysis can be applied to each separate interval over which the system is analytical.

State and output controllability are presented and analyzed in section 2. The sensitivity-based algorithm for state controllability and its extension to output controllability are specified in section 3. Section 4 considers numerical aspects of the algorithms and their implementation. In section 5 small as well as large-scale examples are presented showing that nonlinear dynamical systems can be state controllable but not output controllable and vice versa. Section 6 provides conclusions.

2. STATE AND OUTPUT CONTROLLABILITY

The controllability canonical form of analytical dynamical systems facilitates a simple definition and understanding of state controllability, see Fig. 1. Not applying the appropriate state-transformation in Fig. 1, controllable/uncontrollable state-variables turn into controllable/uncontrollable *modes* being combinations of all state-variables, see also (Van Willigenburg, 2024). These modes represent directions in the state-space in which one can/cannot steer.

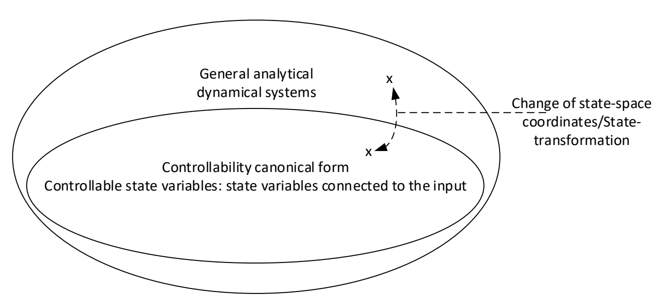

Fig. 1: *An appropriate change of state-space coordinates (state-transformation) realizes a canonical form in which individual state-variables are either connected to the input/controllable or not.*

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Obviously, state-variables or modes that are not connected to the input are *insensitive* to the input. The sensitivity-based algorithm detects the modes/state-variables that are insensitive to the input (Van Willigenburg, Stigter and Molenaar, 2022). If there are no such modes/state variables the system is state controllable.

A key insight, that seems to have been overlooked for a long time in the literature, was obtained from the development of sensitivity-based algorithms to establish state controllability of analytical *nonlinear* dynamical systems (often called local strong accessibility). Sensitivity dynamics concern the propagation of *infinitesimal* deviations from trajectories of the nonlinear system. Therefore, linearized dynamics along the trajectory are an *exact description* of sensitivities, not an approximation. As a result, controllability of nonlinear dynamical systems is equivalent with state controllability of linearizations along *non-singular* trajectories of the nonlinear system. Such a linearization is a *time-varying linear* system in general, also called the *variational system*. Moreover, trajectories of the nonlinear system are *generically* non-singular (Sontag, 1992; Van Willigenburg, Stigter and Molenaar, 2021, 2022).

State and output controllability of time-varying linear systems were discovered and developed using linear algebra (Kalman, 1960; Kreindler and Sarachik, 1964; Weiss and Kalman, 1965). Given the time-varying linear system

$$\begin{gathered}\dot{x}(t)=A(t)x(t)+B(t)u(t),\ y(t)=C(t)x(t),\\ x\in\mathbb{R}^{n_x},\ y\in\mathbb{R}^{n_y},\ u\in\mathbb{R}^{n_u},\end{gathered}\tag{2.1}$$

its state response is

$$x(t)=\Phi(t,t_0)x(t_0)+\int_{t_0}^{t}\Phi(t,\tau)B(\tau)u(\tau)d\tau,t\geq t_0 .\tag{2.2}$$

with $\Phi(t,t_0)$ being the state-transition matrix of system (2.1) satisfying the differential equation

$$\frac{d\Phi(t,t_0)}{dt}=A(t)\Phi(t,t_0),\ t\geq t_0,\ \Phi(t_0,t_0)=I_{n_x}\tag{2.3}$$

with I_{n_x} representing the square identity matrix of dimension n_x. Similarly, the output response is

$$y(t)=C(t)\left(\Phi(t,t_0)x(t_0)+\int_{t_0}^{t}\Phi(t,\tau)B(\tau)u(\tau)d\tau\right)\tag{2.4}$$

From (2.2), an analytical time-varying linear system (2.1) is state controllable if and only if the n_x rows of matrix function $\Phi(\tau,t_0)B(\tau)$ are independent over an interval $\tau\in[t_0,t]$, $t>t_0$ (Weiss and Kalman, 1965). Similarly from (2.4), an analytical time-varying linear system is output controllable if the n_y rows of matrix function $C(t)\Phi(\tau,t_0)B(\tau)$, $\tau\in[t_0,t]$, $t>t_0$ are independent (Kreindler and Sarachik, 1964).

3. SENSITIVITY-BASED ALGORITHM FOR STATE AND OUTPUT CONTROLLABILITY

The sensitivity-based algorithm for state controllability of analytical *nonlinear* dynamical systems was developed as the dual of the algorithm for state observability (Stigter, van Willigenburg and Molenaar, 2018; Van Willigenburg, Stigter and Molenaar, 2022). Therefore it integrates the state-equation of the nonlinear dynamical system

$$\dot{x}(t)=f\left(x(t),u(t)\right)\tag{3.1}$$

backward in time from a fixed terminal state $x(t_f)$ over a small time-interval $t\in[t_0,t_N]$, $t_0<t_N$ together with equation (2.3) in which $A(t)=\partial f/\partial x\big|_{x=x(t),u=u(t)}$. Next, taking $B(t)=\partial f/\partial u\big|_{x=x(t),u=u(t)}$, matrix function $\Phi(t_N,t)B(t)$ is evaluated at increasing time instants t_i, $i=0,1,2,..,N$, $N=n_x-1$. Each evaluation $\Phi(t_N,t_i)B(t_i)$ represents the state response at time t_N of the *variational system* resulting from impulses $\delta(t_i)$ to each of its inputs. Concatenating these evaluations provides sensitivity matrix

$$\begin{gathered}S_x=\left[\Phi(t_N,t_0)B(t_0),\Phi(t_N,t_1)B(t_1),..,\right.\\ \left.\Phi(t_N,t_N)B(t_N)\right]\in\mathbb{R}^{n_x\times n_u n_x}\end{gathered}\tag{3.2}$$

A singular value decomposition (SVD) applied to S_x provides n_x singular values and corresponding left singular vectors. Taking n_x evaluations of $\Phi(t_N,t)B(t)$ in (3.2) prevents S_x to be a-priori rank deficient if $B(t)$ has minimal rank 1 (making n_u effectively 1). The number of zero singular values of S_x equals the number of uncontrollable modes of the nonlinear system (3.1). The corresponding left singular vectors span the null-space representing all directions in which state $x(t_N)$ of system (3.1) cannot be controlled locally. The non-zero components of these left singular vectors indicate the state-variables together making up all uncontrollable modes of system (3.1) (Van Willigenburg, Stigter and Molenaar, 2022).

Similarly, output controllability of the nonlinear dynamical system (3.1) with output equation

$$y(t)=g\left(x(t)\right)\in\mathbb{R}^{n_y}\tag{3.3}$$

uses n_y evaluations at t_i, $i=0,1,2,..,N=n_y-1$ of matrix function $C(t_N)\Phi(t_N,t)B(t)$ where $C(t_N)=dg/dx\big|_{x=x(t_N)}$. This provides sensitivity matrix

$$\begin{gathered}S_y=\left[C(t_N)\Phi(t_N,t_0)B(t_0),C(t_N)\Phi(t_N,t_1)B(t_1),..,\right.\\ \left.C(t_N)\Phi(t_N,t_N)B(t_N)\right]=C(t_N)S_x\in\mathbb{R}^{n_y\times n_u n_y}\end{gathered}\tag{3.4}$$

An SVD is now applied to matrix S_y. Its interpretation now concerns output controllability and equals the one above for state controllability with state x replaced by output y.

4. NUMERICAL ASPECTS AND IMPLEMENTATION OF THE SENSITIVITY-BASED ALGORITHMS

A most important numerical aspect of the sensitivity-based algorithm concerns the detection of *numerically* zero singular values, since each one corresponds to an uncontrollable mode/state variable. One generally considers singular values to be *numerically zero* if these are below a *significant gap* within all singular values. A rule of thumb is that this gap should be at least as large as 4 to 5 decades. In general, as system dimensions grow, the *non-zero* singular values cover a wider range. Given the finite machine precision, to detect a possible gap, one prefers the range of non-zero singular values to be as small as possible.

One technique to decrease this range and enlarge a possible gap is to concatenate the results of several trajectories, not just one (Stigter, Joubert and Molenaar, 2017; Van Willigenburg, Stigter and Molenaar, 2022). For state controllability, when concatenating the results of $N>1$ trajectories the algorithm computes sensitivity matrix

$$\begin{aligned} S_x &= [S_x^1,\, S_x^2,..,S_x^M], \\ S_x^k &= \left[\Phi(t_N,t_0)B(t_0),\Phi(t_N,t_1)B(t_1),..,\right. \\ &\qquad \left.\Phi(t_N,t_N)B(t_N)\right]^k,\ k=1,2,..,M. \end{aligned} \tag{3.5}$$

$$\begin{aligned} S_y &= [S_y^1,\, S_y^2,..,S_y^M] = C(t_N)S_x, \\ S_y^k &= \left[C(t_N)\Phi(t_N,t_0)B(t_0),C(t_N)\Phi(t_N,t_1)B(t_1),..,\right. \\ &\qquad \left.C(t_N)\Phi(t_N,t_N)B(t_N)\right]^k = C(t_N)S_x^k,\ k=1,2,..,M. \end{aligned} \tag{3.6}$$

An alternative to computing S_x and S_y in (3.2), (3.4) and (3.5), (3.6) would be to compute the Gramian matrices

$$W_x(t_N,t)=\int_t^{t_N}\Phi(\tau,t_f)B(\tau)B^T(\tau)\Phi^T(\tau,t_f)\,d\tau \tag{3.7}$$

and

$$W_y(t_N,t)=C(t_N)W_x(t_N,t)C^T(t_N) \tag{3.8}$$

respectively (Kreindler and Sarachik, 1964). However, the *squaring* of the matrix function $\Phi(\tau,t_N)B(\tau)$ within W_x and W_y respectively, causes their non-zero singular values to be spread over a wider range. This is undesirable from the point of view of detecting a possible gap as already noted in the pioneering paper (Moore, 1981).

Numerical integration of the nonlinear dynamics (3.1), (3.3) together with (2.3) is performed over time-interval $t\in[t_0,t_N]$. This time-interval may be very short, partly causing the algorithms exceptional efficiency. But it must be long enough to capture all relevant system and sensitivity dynamics. Enlarging this time-interval is also another way of decreasing the range of non-zero singular values.

Within (2.3), $A(t)=\partial f/\partial x\big|_{x=x(t),u=u(t)}$ is computed using automatic differentiation (Neidinger, 2010; Margossian, 2019) or using complex derivatives (Martins, Sturdza and Alonso, 2022) providing exceptionally accurate derivatives. The same applies to $C(t_N)=dg/dx\big|_{x=x(t_N)}$ required to compute S_y in (3.4), (3.6). Constant system inputs $u(t)$ often suffice. Only in exceptional cases they may cause the trajectory to be singular or not sufficiently exciting. The fixed terminal state $x(t_N)$ of the nonlinear system (3.1) should preferably be obtained from integrating just (3.1) forward in time over $t\in[t_0,t_N]$ starting from an arbitrary, but realistic, state $x(t_0)$. And finally, when concatenating the results of several trajectories in (3.5), (3.6), these trajectories should be such that they do not change the null-space made up by the left singular vectors corresponding to zero singular values of the SVD (Stigter, Joubert and Molenaar, 2017).

5. EXAMPLES

Example 1: Example 6.1 from (Kwatny and Blankenship, 2000).

$$f(x,u)=\begin{bmatrix} x_1x_3+x_2e^{x_2} \\ x_3 \\ x_4-x_2x_3 \\ x_3^2+x_2x_4-x_2^2x_3 \end{bmatrix}+\begin{bmatrix} x_1 \\ 1 \\ 0 \\ x_3 \end{bmatrix}u \tag{3.9}$$

The outcome of the SVD obtained from the sensitivity-based algorithm computing state controllability is listed in Table 1. From it we conclude that system (3.9) has two uncontrollable modes involving state-variables 2, 3 and 4, since the first component of both left singular vectors is numerically zero.

Table 1: *Singular values for state controllability of system (3.9) (left column) and the two left singular vectors corresponding to the two numerically zero singular values.*

2.3212e+00	-3.9881e-13	1.4134e-15
2.8240e-02	-4.1173e-02	2.2967e-01
2.2726e-14	9.8431e-01	1.7646e-01
2.5034e-17	1.7159e-01	-9.5714e-01

Next consider output controllability of system (3.9) with

$$y=\left[x_1^2,x_2^2\right]^T \tag{3.10}$$

as well as

$$y=\left[x_1^2,x_3^2\right]^T. \tag{3.11}$$

Table 2 lists the outcome of the SVD for output-controllability in both cases. From it we may conclude that system (3.9) with outputs (3.10) is output controllable whereas with outputs (3.11) it is not. In the latter case the uncontrollable output mode consists solely of output 2 being x_3^2. This complies with the fact that for state controllability, x_1 is not involved in uncontrollable modes. Moreover, using the Lie algebraic rank condition (LARC) we verified the outcome of state and output controllability obtained from our sensitivity rank condition (SERC). For state controllability, LARC computes the rank of a matrix containing concatenated Lie brackets that we computed using Hall bases

(Duleba, 1997). For output controllability LARC computes the rank of the same matrix pre-multiplied with dy/dx.

All computations were performed on a 3,10 gigahertz Intel Core i5-8600 PC running Windows 10 and Matlab 2024a.

Table 2: *Singular values for output controllability of system (3.9) corresponding with outputs (3.10), (3.11) respectively (left two columns) and the left singular vector corresponding to the numerically zero singular value of output (3.11) (last column).*

4.1374e+00	7.3527e-01	1.3417e-21
5.3098e-02	9.5159e-22	-1.0000e+00

In order to illustrate the capability of sensitivity-based algorithms to handle large-scale systems very efficiently consider the next example.

Example 2: Ring of coupled Kuramoto oscillators (Kuramoto, 1975, 1984; Strogatz, 2000; Baggio, Bassett and Pasqualetti, 2021). The dynamics of the ring of n coupled oscillators is represented by

$$\dot{\theta}_i(t) = \omega_i + \sin\left(\theta_{i-1}(t) - \theta_i(t)\right) + \sin\left(\theta_{i+1}(t) - \theta_i(t)\right),\quad i = 1,2,..,n \tag{3.12}$$

with i periodic mod n. One easily recognises that the phases of the oscillators θ_i may be considered state variables. The oscillator frequencies ω_i are constant parameters apart from the first that is slightly modified by the single control input

$$\omega_1 \to \omega_1 + 0.01u_1(t). \tag{3.13}$$

As a first case take $n = n_x = 10$, terminal time $t = 1$ and the components of terminal state $x(t)$ random numbers in between 0 and 1. Furthermore, for ω_i, $i = 2,3,..,n$ take random numbers in between 1 and 1.01 since the model applies only to oscillators who's angular frequencies are very close together (Strogatz, 2000). And finally take $u(\tau) = u_1(\tau)$ a constant randomly selected between 0 and 1. For state controllability the singular values are plotted on a logarithmic scale in Fig. 1 revealing controllability. Taking $\text{round}(n_x/3)$ to be controlled outputs

$$y_i = e^{-x_{i-1+j}},\ i = 1,2,..,\text{round}(n_x/3),\ j = \text{round}(n_x/3) \tag{3.14}$$

the singular values obtained for output controllability are plotted in Fig. 2.

Next introduce a redundant output that is nonlinearly related to two other outputs as follows

$$y_1 = y_2^2 + y_3^2. \tag{3.15}$$

Equation (3.15) overrules the first output as specified by (3.14). As seen from Fig. 3a, the system is no longer output controllable because of the redundant output (3.15). The corresponding signature in Fig. 3b correctly indicates that the single uncontrollable output mode involves outputs 1,2 and 3.

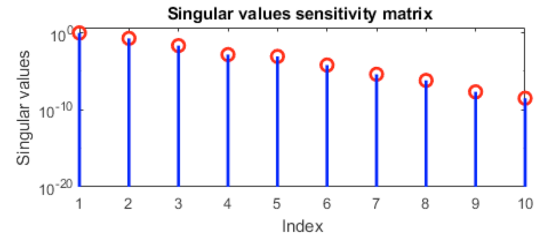

Fig. 1: *Normalized singular values determining state controllability for system (3.12), (3.13) with $n = 10$. Being well within the machine precision range of $2.2\,10^{-16}$, and not showing a gap, there are no numerically zero singular values so the system is state controllable. CPU time 0.491 s.*

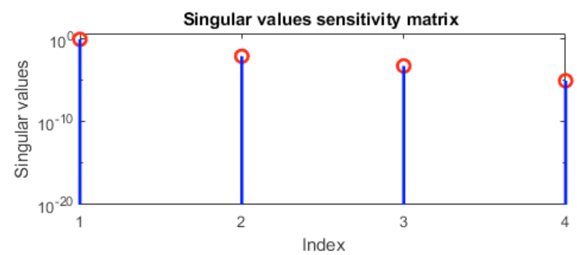

Fig. 2: *Normalized singular values determining output controllability for system (3.12), (3.13), (3.14) with $n = 10$. Being well within the machine precision range of $2.2\,10^{-16}$, and not showing a gap, there are no numerically zero singular values so the system is output controllable. CPU time 0.573 s.*

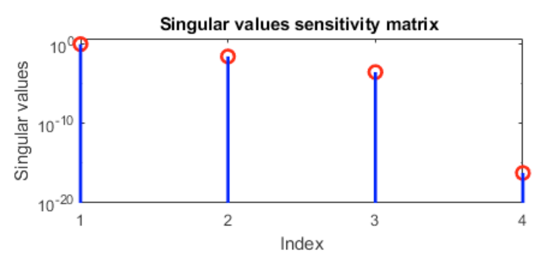

Fig. 3a: *Normalized singular values determining output controllability for system (3.12), (3.13), (3.14), (3.15) with $n = 10$. Being well within the machine precision range of $2.2\,10^{-16}$, one numerically zero singular value is found so the system is not output controllable. CPU time 0.483 s.*

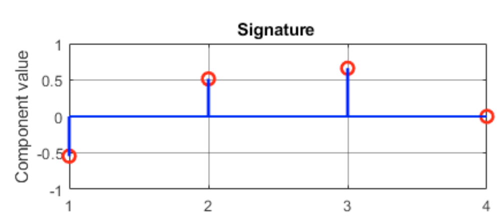

Fig. 3b: *Signature of the output uncontrollable system (3.12), (3.13), (3.14), (3.15) showing the components of the left singular vector corresponding to the single numerically zero singular value.*

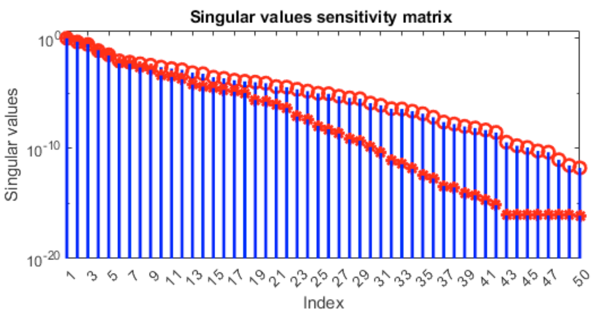

Fig. 4: *Normalized singular values determining state controllability for system (3.12), (3.13) with $n = 50$, $t = 10$ for one trajectory (lower red stars) and after concatenating the result of four trajectories (top red dots) bringing the singular values within the machine precision range. In the latter case there are no numerically zero singular values so the system again turns out state controllable. CPU time all four trajectories together 23.663 s.*

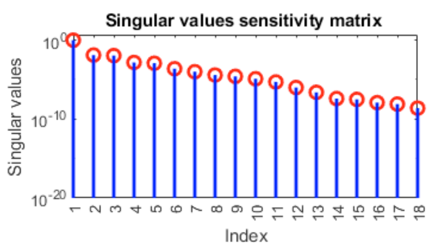

Fig. 5: *Normalized singular values determining output controllability for system (3.12), (3.13), (3.14) with $n = 50$, $t = 10$ after concatenating the result of four trajectories. There are no numerically zero singular values so the system is again output controllable. CPU time of all four trajectories 23.508 s.*

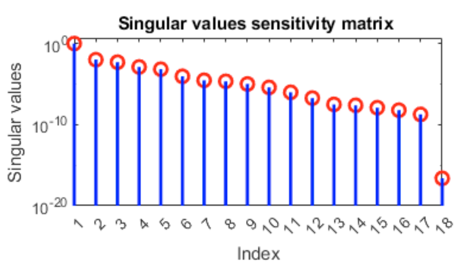

Fig. 6a: *Normalized singular values determining output controllability for system (3.12), (3.13), (3.14), (3.15) with $n = 50$, $t = 10$ computed from concatenating four trajectories. There is one numerically zero singular value so the system again turns out not output controllable with a single uncontrollable mode. CPU time all four trajectories 23.579 s.*

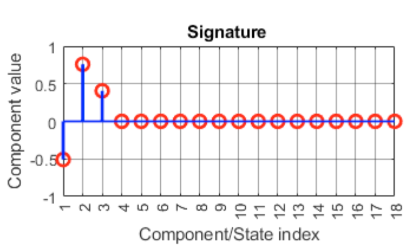

Fig. 6b: *Signature of the output uncontrollable system (3.12), (3.13), (3.14), (3.15) showing the components of the single left singular vector corresponding to the single numerically zero singular value.*

When increasing n, the singular values start to cover the full range of the machine precision causing difficulties in establishing numerically zero singular values. On the other hand, since the singular values are actually *measures of controllability*, very small singular values indicate very poor controllability in directions given by the corresponding left singular vectors.

To alleviate the problem of detecting numerically zero singular values, terminal time t_f may be increased and the results of several trajectories may be concatenated as shown in Figs. 4-6. The additional three trajectories have different parameter values ω_i, $i = 2,3,..,n$. For the second trajectory they are random numbers in between 2 and 2.01, for the third in between 3 and 3.01, and for the fourth in between 4 and 4.01. Finally, all trajectories share the same terminal state $x(t_f)$ of system (3.1) thereby fulfilling the requirement mentioned at the end of section 4.

6. CONCLUSIONS

A sensitivity-based algorithm computing state controllability of large-scale nonlinear dynamical systems has been extended to compute output controllability. As can be seen from equations (3.2), (3.4), due to the linear nature of sensitivity dynamics, the modification comes down to a pre-multiplication of state sensitivity matrix S_x, obtained for state controllability, with matrix $C(t_N)$ being the sensitivity of the output to the state at the terminal time of the trajectory used by the sensitivity-based algorithm.

Small and large-scale examples showed that nonlinear dynamical systems can be state controllable, but not output controllable and vice versa.

Sensitivity-based algorithms rely on numerical integration along trajectories and a sensitivity rank condition (SERC) requiring the detection of zero singular values of a sensitivity matrix. The singular values obtained from a singular value decomposition (SVD) are actually *measures* of controllability in directions given by the corresponding left singular vectors obtained from the SVD. These measures are highly valuable information to engineers concerning ‘practical controllability’. Theoretically, if the smallest of the singular values is very small yet non-zero, the system is controllable. Therefore the sensitivity-based algorithms may provide a *theoretically* erroneous answer if singular values fall below the range determined by the machine precision. The large-scale examples in this paper showed how prolonging trajectories and concatenating the results of several trajectories alleviate this problem.

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