

## A conceptual interpretation of the renewal theorem with applications

J.A.M. van der Weide

*Delft University of Technology, Delft, The Netherlands*

M.D. Pandey

*Department of Civil Engineering, University of Waterloo, Waterloo, Canada*

J.M. van Noortwijk

*HKV Consultants, Lelystad, The Netherlands*

*Delft University of Technology, Delft, The Netherlands*

**ABSTRACT:** The risk management of degrading engineering systems can be optimised through inspections and renewals. In the probabilistic modelling of life-cycle management, the renewal theorem plays a key role in the computation of the expected number of renewals and the cost rate associated with a management strategy. The renewal theorem is well known and its rigorous mathematical proof is presented in the literature, though probabilistic arguments associated with its derivation are not well understood by the engineering community. The central objective of this paper is to present a more lucid and intuitive interpretation of the renewal theorem and to derive asymptotic expansions for the first and second moment of the number of renewals. As far as the authors know, the latter expansion is new in the sense that it also contains a constant term.

### 1 INTRODUCTION

In industrialised nations, the infrastructure elements critical to economy, such as bridges, roads, power plants and transmission lines, are experiencing aging related degradation, which makes them more vulnerable to failure. To minimise the risk associated with failure of an infrastructure system, inspection and replacements are routinely carried out. Because of uncertainty associated with degradation mechanisms, operational loads and limited inspection data, probabilistic methods play a key role in developing cost effective management models. The theory of stochastic renewal processes and the renewal theorem have been fundamental to the development of risk-based asset management models (Rackwitz, 2001; van Noortwijk, 2003). For an example in bridge management, see van Noortwijk and Klatter (2004). Although the renewal processes have been discussed in many mathematical treatises (Feller, 1950, 1966; Karlin and Taylor, 1975; Tijms, 2003), the concepts are not amenable to the engineering community. The objective of this paper is to present a conceptually simple and intuitive interpretation of renewal processes with applications.

For sake of conceptual simplicity, we consider a discrete time scale to model the process of times at which failure occurs with an independent, identically

distributed (*iid*) sequence of 0–1 random variables, where 1 means occurrence of failure. This leads to a discrete version of the Poisson process where the times between failures are independent, geometrically distributed random variables. However, in cases where the geometric distribution of the inter-occurrence time cannot be justified, a natural way to generalise the analysis is to model it as a renewal process. The renewal theorem provides asymptotic results for the first and second moment of the number of failures. For discrete-time and continuous-time renewal processes, see Feller (1949, 1950) and Smith (1954, 1958), respectively. In this paper, we extend Feller's asymptotic expression for the second moment of the number of renewals.

We propose to model the renewal processes with a sequence of 0–1 variables which are in general neither independent nor identically distributed. The probabilities of occurrence of failure are given by the renewal sequence associated with the distribution between failures. This approach gives insight in the dependence structure of the indicator variables of the event that failure occurs. The proposed approach allows to calculate quite easily the mean and variance of the number of failures in a finite time horizon.

The proposed model is applied to develop a risk-based asset management framework for utility wood poles in an electrical transmission line network. Using the actual inspection data, we model the lifetime

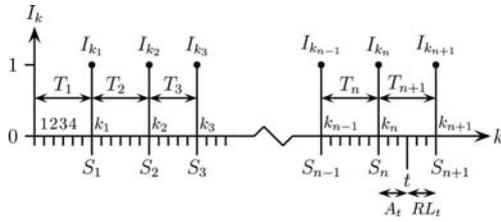


Figure 1. Terminology associated with the renewal process where  $N_t = n$ .

distribution as a discrete Weibull distribution. A second example concerns the renewal of hydraulic cylinders for which the lifetimes are distributed according to a Poisson distribution.

This paper is organised as follows. In Section 2, we present the basic properties of discrete-time renewal processes by studying the number of renewals over a finite time horizon. For an infinite time horizon, asymptotic results are obtained in Section 3. Discounted renewal cost is incorporated to the renewal model in Section 4. Two illustrations of the proposed discrete-time renewal model are presented in Section 5. Conclusions are formulated in Section 6.

## 2 THEORY OF RENEWAL PROCESSES

### 2.1 General concepts

The basic premise is that the lifetime ( $T$ ) of a component is a random variable. It is assumed that the component is replaced with a new and identical component as soon as a failure occurs. The probability distribution of  $T$  is given as

$$\mathbb{P}(T = k) = p_k \quad (k = 1, 2, \dots).$$

We will always assume that the probability distribution ( $p_k$ ) is aperiodic which is certainly the case if  $p_1 > 0$ . This process is shown in Figure 1 in which  $T, T_1, T_2, \dots$  denote an *iid* sequence of positive, integer valued random variables corresponding to inter-arrival times of failure. The probability generating function of ( $p_k$ ) is given by

$$P(z) = \sum_{k=0}^{\infty} p_k z^k,$$

where  $p_0 = 0$ . The probability distribution ( $p_k$ ) is referred to as the renewal distribution. The time of occurrence of the  $n$ th renewal (failure) is given by the partial sum

$$S_n = T_1 + \dots + T_n \quad (n \geq 1).$$

The counting process  $N = (N_t; t = 0, 1, 2, \dots)$  associated with the sequence ( $S_n$ ) is called the renewal process with renewal distribution ( $p_k$ )

$$N_t = n \Leftrightarrow S_n \leq t < S_{n+1} \quad (n \geq 0), \quad (1)$$

where  $S_0 \equiv 0$ . The simplest example of a renewal process is the discrete Poisson process with a shifted geometric renewal distribution

$$p_k = p(1-p)^{k-1} \quad (k \geq 1),$$

where  $0 < p < 1$ . However, the simplest way to define the discrete Poisson process is by introducing an *iid* sequence  $X, X_1, X_2, \dots$  of 0–1 random variables

$$\mathbb{P}(X = 1) = p = 1 - \mathbb{P}(X = 0),$$

and defining  $N_0 = 0$  and

$$N_t = X_1 + \dots + X_t \quad (t \geq 1).$$

It follows that  $N_t$  is binomially distributed and therefore the discrete Poisson process is also called the binomial process.

### 2.2 Distribution of number of renewals

The number of renewals,  $N_t$ , in the interval  $(0, t]$  is a random variable, and its distribution can be derived from probabilistic arguments. In particular  $N_t = 0$  if only if  $S_1 = T_1 > t$ , hence

$$\mathbb{P}(N_t = 0) = \mathbb{P}(T_1 > t) = 1 - \sum_{k=1}^t p_k. \quad (2)$$

For  $n \geq 1$ , it follows by conditioning on  $T_1$  and (1) that

$$\mathbb{P}(N_t = n) = \sum_{k=1}^t p_k \mathbb{P}(S_{n-1} \leq t - k < S_n).$$

It is clear that the time of occurrence of the  $(n-1)$ th failure necessarily follows an inequality,  $S_{n-1} \geq n-1$ . It means that we only have to sum over  $k$  such that  $t-k \geq n-1$  or  $k \leq t+1-n$ . In summary, it leads to a recursive formula to calculate the probability distributions of  $N_t$  as

$$\mathbb{P}(N_t = n) = \sum_{k=1}^{t+1-n} p_k \mathbb{P}(N_{t-k} = n-1), \quad (3)$$

where  $\mathbb{P}(N_t = 0)$  is given by (2).

### 2.3 Renewal function

The expected number of renewals up to time  $t$  (first moment) is referred to as the renewal function, i.e.,  $M(t) = \mathbb{E}(N_t)$ . This section presents the derivation of  $M(t)$ . An alternative way to write  $N_t$  is

$$N_t = \sum_{n=1}^{\infty} 1_{\{S_n \leq t\}}. \quad (4)$$

Note that the indicator function  $1_A = 1$  when the condition  $A$  is true, otherwise it is zero. Actually, the sum in (4) is over the finite range  $n = 1, \dots, t$ , since  $S_k \geq k$  for all  $k$ . Now taking expectations of both sides of (4) results in

$$\mathbb{E}(N_t) = \sum_{n=1}^t \mathbb{E}(1_{\{S_n \leq t\}}) = \sum_{n=1}^t F_n(t), \quad (5)$$

where  $F_n$  denotes the cumulative distribution function of  $S_n$ . A recursion equation for  $\mathbb{E}(N_t)$  can be derived using (4) and (5) as

$$\begin{aligned} \mathbb{E}(N_t; T_1 = k) \\ = \mathbb{E} \left( 1 + \sum_{n=2}^{\infty} 1_{\{T_2 + \dots + T_n \leq t-k\}}; T_1 = k \right) \end{aligned} \quad (6)$$

for  $k \in [1, t]$ . Recall from (4) that

$$\sum_{n=2}^{\infty} 1_{\{T_2 + \dots + T_n \leq t-k\}} \stackrel{d}{=} N_{t-k}.$$

This equation comes from the fact that the process starts afresh after the first failure. Since  $T_1$  is independent of other failure times, (6) is rewritten as

$$\mathbb{E}(N_t; T_1 = k) = (1 + \mathbb{E}(N_{t-k})) p_k.$$

By summation over  $k = 1, \dots, t$ , and using the law of total probability

$$\mathbb{E}(N_t) = M(t) = F(t) + \sum_{k=1}^t p_k \mathbb{E}(N_{t-k}). \quad (7)$$

Equation (7) is called the renewal equation, and the renewal function is its unique solution corresponding to the renewal distribution  $(p_k)$ .

### 2.4 Rate of renewal

The rate of occurrence of renewals can be derived through the use of a random indicator variable  $I_j$ . It is an indicator of a renewal at time  $j$ , such that  $I_j = 1$  if there is a renewal at step  $j$ , otherwise 0. Since a renewal

takes place at discrete times  $S_k$ , the indicator variable can be written as

$$I_j = \sum_{k=1}^j 1_{\{S_k=j\}} \quad (j \geq 1).$$

The mathematical properties of the sequence  $(I_j)$  are extensively studied in Kingman (1972) under the name of discrete-time regenerative phenomena. Indicators of recurrent events as defined in Feller (1949, 1950) are examples of regenerative phenomena. The renewal rate at time  $j$ , denoted as  $u_j$ , is defined as

$$u_j = \mathbb{P}(I_j = 1) \quad (j \geq 1), \quad (8)$$

and it can be expressed as

$$u_j = \sum_{k=1}^j \mathbb{P}(S_k = j) \quad (j \geq 1). \quad (9)$$

It follows from (8) that  $u_j = \mathbb{E}(I_j)$  for  $j = 1, 2, \dots$ . Let us define  $u_0 = 1$ . The renewal rate can be alternatively derived using the generating function

$$U(z) = u_0 + u_1 z + u_2 z^2 + \dots$$

which is related with the generating function of the renewal distribution as

$$U(z) = 1 + P(z)U(z). \quad (10)$$

Consider an example of renewal distribution,  $p_1 = p$  and  $p_2 = 1 - p = q$ . Its generating function is given as

$$P(z) = pz + qz^2.$$

Using Equation (10)

$$U(z) = \frac{1}{1 - pz - qz^2} = \frac{1}{(1-z)(qz+1)}$$

and partial fractions, the generating function  $U(z)$  is obtained as

$$U(z) = \sum_{n=0}^{\infty} \left( \frac{1 - (-q)^{n+1}}{1+q} \right) z^n.$$

Thus,

$$u_n = \frac{1 - (-q)^{n+1}}{1+q}.$$

The following recursion can be used to calculate the sequence as

$$u_n = \sum_{k=1}^n p_k u_{n-k} \quad (n \geq 1, u_0 = 1). \quad (11)$$

In general, the renewal indicators  $I_j$  are not independent. It turns out that

$$\mathbb{E}(I_n I_{n+m}) = u_n u_m.$$

Since

$$N_t = I_1 + \dots + I_t,$$

the mean and the factorial second moment can be expressed in terms of the sequence  $u_j$  as

$$\mathbb{E}(N_t) = \sum_{j=1}^t u_j, \quad (12)$$

and since  $I_j^2 = I_j$  for 0–1 random variables

$$\mathbb{E}(N_t(N_t - 1)) = 2 \sum_{j=1}^{t-1} u_j(u_1 + \dots + u_{t-j}). \quad (13)$$

From (12) and (13), the second moment of the number of renewals follows simply

$$\mathbb{E}(N_t^2) = \mathbb{E}(N_t(N_t - 1)) + \mathbb{E}(N_t). \quad (14)$$

So, the basic idea is that the renewal rate can be directly obtained from the generating function of the renewal distribution. Then, it can be used to compute moments of the number of renewals.

### 2.5 Distribution of remaining lifetime and age

In the context of the renewal process, it is of interest to obtain the remaining lifetime of a component at a future time  $t$ . Since it is a random variable, its probability distribution can be derived from the renewal arguments as follows.

At time  $t$ , the number of renewals is denoted as  $N_t$ . The next renewal will occur at time  $S_{N_t+1}$ . Thus the remaining lifetime (also called the excess of residual lifetime) can be written as

$$RL_t = S_{N_t+1} - t. \quad (15)$$

The time of the last renewal that occurred before time  $t$  is denoted as  $S_{N_t}$ . The age (or current lifetime) of the current component is therefore given as

$$A_t = t - S_{N_t}. \quad (16)$$

If there was a renewal precisely at time  $t$ , then  $S_{N_t} = t$  and the age of the current component is 0. The age of the current component is also referred to as the backward recurrence time at  $t$ .

Consider the event

$$\{A_t = i, N_t = k, RL_t = j\}$$

It follows from (16) that the event can only occur if  $i = t - S_k$ . So  $S_k = t - i$  and, since  $S_k \geq k$ , we get that  $i + k \leq t$ . If  $RL_t = j$  and  $N_t = k$ , then (15) implies that  $S_{k+1} = t + j$  or  $T_{k+1} = S_{k+1} - S_k = i + j$ . The joint distribution of  $(A_t, N_t, RL_t)$  is given by

$$\begin{aligned} \mathbb{P}(A_t = i, N_t = k, RL_t = j) \\ \mathbb{P}(S_k = t - i, T_{k+1} = i + j) = p_{i+j} \mathbb{P}(S_k = t - i), \end{aligned}$$

where  $i + k \leq t$  and  $j \geq 1$ . By the law of total probability,

$$\mathbb{P}(A_t = i, RL_t = j) = p_{i+j} \sum_{k=1}^{t-i} \mathbb{P}(S_k = t - i),$$

and (9) implies that the joint distribution of  $(A_t, RL_t)$  is given by

$$\mathbb{P}(A_t = i, RL_t = j) = u_{t-i} p_{i+j}. \quad (17)$$

By the law of total probability, the marginal distributions of the age and the remaining life are

$$\mathbb{P}(A_t = i) = u_{t-i} \sum_{j=1}^{\infty} p_{i+j} = u_{t-i} (1 - F(i)), \quad (18)$$

with  $0 \leq i \leq t$  and

$$\mathbb{P}(RL_t = j) = \sum_{k=0}^t u_k p_{t+j-k} \quad (j \geq 1). \quad (19)$$

## 3 ASYMPTOTIC RESULTS

This section analyses the generalised renewal equation

$$x_t = a_t + \sum_{k=1}^t p_k x_{t-k} \quad (t \geq 0), \quad (20)$$

where  $(a_t)$  is some given sequence. When  $a_t = F(t)$ , the solution is the renewal function. If  $a_0 = 1$  and  $a_t = 0$  for all  $t > 0$ , the solution is the renewal sequence  $(u_j)$ . Using generating functions, we can show that the solution of (20) is the convolution of the sequence  $(a_t)$  and the renewal sequence  $(u_j)$

$$x_t = \sum_{k=0}^t a_k u_{t-k}.$$

If the  $k$ th moment of the renewal distribution exists, it will be denoted by  $\mu_k$ . One of the most important theorems about renewal processes is the renewal theorem. If the first moment of the renewal distribution exists, then

$$\lim_{t \rightarrow \infty} u_t = \frac{1}{\mu_1}. \quad (21)$$

For the discrete renewal theorem, see Feller (1950, Chapters 12 & 13) and Karlin and Taylor (1975, Chapter 3). It follows for the renewal function  $M(t) = \mathbb{E}(N_t)$  that

$$\lim_{t \rightarrow \infty} \frac{M(t)}{t} = \frac{1}{\mu_1}. \quad (22)$$

The discrete version of the key renewal theorem follows now directly. Let  $(x_t)$  be the solution of the generalised renewal equation (20). If the first moment of the renewal distribution exists and  $\sum_{t=0}^{\infty} |a_t|$  is convergent, then

$$\lim_{t \rightarrow \infty} x_t = \lim_{t \rightarrow \infty} \sum_{k=0}^t a_k u_{t-k} = \frac{1}{\mu_1} \sum_{k=0}^{\infty} a_k. \quad (23)$$

Following Tijms (2003, Chapter 8), we use the discrete key renewal theorem to derive asymptotic expansions for the first and the second moment of the renewal process  $N$ . Suppose that the renewal distribution  $(p_k)$  has a finite second moment  $\mu_2$ . From (22), it follows that  $M(t) \approx t/\mu_1 + o(t)$  for large  $t$ , where the term  $o(t)$  is of lower order than  $t$ . Define

$$Z_0(t) = M(t) - \frac{t}{\mu_1}, \quad t = 0, 1, 2, \dots$$

$Z_0$  is the solution of the generalised renewal equation (20) with

$$a_t = F(t) - \frac{t}{\mu_1} + \frac{1}{\mu_1} \sum_{k=1}^t (t-k)p_k.$$

and

$$\sum_{t=0}^{\infty} a_t = -\mu_1 + \frac{\mu_2 + \mu_1}{2\mu_1}.$$

It follows from an application of the key renewal theorem that (Feller, 1949)

$$\lim_{t \rightarrow \infty} \left[ M(t) - \frac{t}{\mu_1} \right] = \frac{\mu_2}{2\mu_1^2} - 1 + \frac{1}{2\mu_1}. \quad (24)$$

An asymptotic expansion of the second moment of the number of renewals  $N_t$  can be found along the same

lines but is much more technical. We only give the result. Let  $M_2(t) = \mathbb{E}(N_t^2)$  and assume that the renewal distribution has a finite third moment  $\mu_3$ . Then

$$\begin{aligned} \lim_{t \rightarrow \infty} \left[ M_2(t) - \left\{ \frac{t^2}{\mu_1^2} + \left( \frac{2\mu_2}{\mu_1^3} - \frac{3}{\mu_1} + \frac{1}{\mu_1^2} \right) t \right\} \right] \\ = \frac{3\mu_2^2}{2\mu_1^4} - \frac{2\mu_3 - 3\mu_2}{3\mu_1^3} + \frac{1 - 9\mu_2}{6\mu_1^2} - \frac{3}{2\mu_1} + 1. \end{aligned} \quad (25)$$

The asymptotic expansion for the first moment can be found in Feller (1949) with a derivation using generating functions. In Feller's paper, there is also an expansion for the second moment but the constant term is not explicitly given and it is not clear how this term can be found using generating functions. Note that the coefficients in the expansion for the discrete case and the continuous case are different. The results for the continuous case can be found in Tijms (2003, Chapter 8).

#### 4 DISCOUNTED COST

In this section, we consider the case of constant renewal cost  $c > 0$  with exponential discounting

$$K_t = \sum_{j=1}^t ce^{-rj} I_j,$$

where  $r > 0$  is the discount rate and  $e^{-r}$  is the discount factor. It follows that

$$\mathbb{E}(K_t) = \sum_{j=1}^t ce^{-rj} u_j,$$

and

$$\lim_{t \rightarrow \infty} \mathbb{E}(K_t) = \sum_{j=1}^{\infty} ce^{-rj} u_j = c(U(e^{-r}) - 1),$$

where  $U$  is the generating function of the renewal sequence  $(u_j)$ . It follows from the renewal equation (10) that

$$\lim_{t \rightarrow \infty} \mathbb{E}(K_t) = \frac{cP(e^{-r})}{1 - P(e^{-r})}.$$

The variance of  $K_t$  is given by

$$\begin{aligned} \text{Var}(K_t) = c^2 \left\{ \sum_{i=1}^t e^{-2ri} u_i + \right. \\ \left. 2 \sum_{i=1}^{t-1} u_i e^{-2ri} \left( \sum_{k=1}^{t-i} e^{-rk} u_k \right) - \left( \sum_{j=1}^t e^{-rj} u_j \right)^2 \right\}, \end{aligned}$$

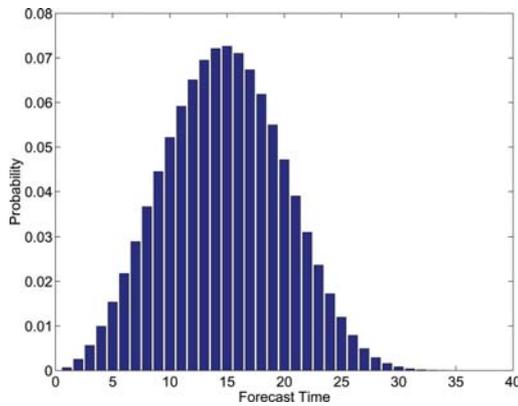


Figure 2. Lifetime distribution of the component.

and with  $\alpha = e^{-r}$  we get, after some simplification,

$$\lim_{t \rightarrow \infty} \text{Var}(K_t) = c^2 \frac{P(\alpha^2) - (P(\alpha))^2}{(1 - P(\alpha^2))(1 - P(\alpha))^2}.$$

When the renewal cost depends on the renewal time  $T$  and is denoted by  $c_T$ , then the mean and variance of the discounted cost over a finite (and infinite) time horizon can be determined using the recurrence relations given in van Noortwijk (2003).

## 5 ILLUSTRATIONS

This section presents illustrations in which the renewal rate and the first and second moment of the number of renewals over a finite time horizon as well as their asymptotic expansions are derived for renewal inter-occurrence times having a discrete Weibull distribution and a shifted Poisson distribution.

### 5.1 Discrete Weibull distribution

This section illustrates an application of the theory of renewal processes to the life-cycle management of an electrical transmission line network. The lifetime distribution of electrical components in power plants, such as switches and relays, can be modelled by a discrete Weibull distribution, as shown in Figure 2.

Suppose the lifetime follows a discrete Weibull distribution with the hazard rate function given as

$$h_k = \begin{cases} (k/m)^{\alpha-1} & (k = 1, \dots, m) \text{ if } \alpha > 1, \\ \beta k^{\alpha-1} & (k = 1, 2, \dots) \text{ if } 0 < \alpha \leq 1, \end{cases}$$

where  $0 < \beta < 1$ . In the literature, this distribution is also known as the discrete Weibull distribution of type

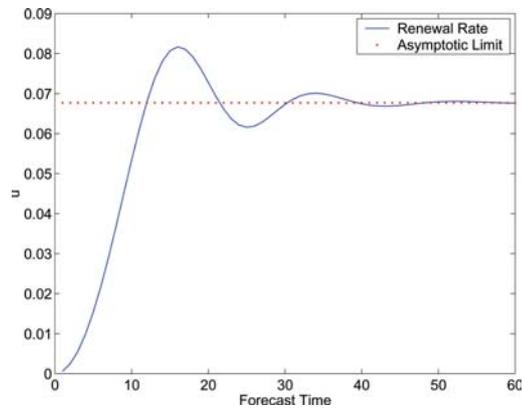


Figure 3. Renewal rate of the component during the service life.

II (Stein and Dattero 1984; Ali Khan et al., 1989). Note that when  $\alpha = 1$  the distribution becomes a geometric distribution. Useful explicit expressions for the mean and variance of the discrete Weibull distribution don't exist.

For the component at hand, let the shape parameter be  $\alpha = 3$  and the upper bound of the support of the distribution  $m = 40$ . The mean lifetime of this component is about 15 years. Because the discrete hazard rate function is defined as

$$h_k = \frac{p_k}{1 - \sum_{i=1}^{k-1} p_i} \quad (k = 1, \dots, m),$$

the probability function can be written as

$$p_k = h_k \prod_{i=1}^{k-1} [1 - h_i] \quad (k = 1, \dots, m). \quad (26)$$

The probability function of the lifetime is shown in Figure 2.

The formulas derived in the previous sections can be applied to compute the renewal rate and associated statistics. Figure 3 plots the rate of occurrence of renewal, calculated using (11), along with its asymptotic limit ( $1/\mu_1 = 0.0677$ ). The mean and standard deviation of the number of renewals is plotted in Figure 4, which were calculated using (12) and (14).

### 5.2 Shifted Poisson distribution

As a simplified example, we study the renewal of a hydraulic cylinder on a swing bridge (adapted from van Noortwijk and van der Weide, 2006). The deterioration  $X(t)$  at time  $t \geq 0$  is assumed to be distributed according to a stationary gamma process with shape function  $v(t) = (\mu/\sigma)^2 t$  and scale parameter

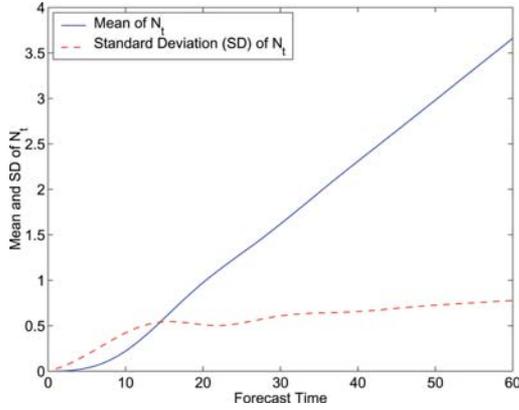


Figure 4. Mean and standard deviation of number of renewals.

$u = \mu/\sigma^2$  for  $\mu, \sigma > 0$ . Hence, the cumulative amount of deterioration  $X(t)$  has the gamma distribution

$$f_{X(t)}(x) = \frac{x^{(\mu/\sigma)^2 t - 1}}{\Gamma((\mu/\sigma)^2 t)} \left[ \frac{\mu}{\sigma^2} \right]^{(\mu/\sigma)^2 t} \exp\left\{-\frac{\mu x}{\sigma^2}\right\} \quad (27)$$

for  $\mu, \sigma > 0$  with

$$\mathbb{E}(X(t)) = \mu t, \quad \text{Var}(X(t)) = \sigma^2 t.$$

The expected condition  $R(t) = r_0 - X(t)$  is assumed to degrade linearly in time from the initial condition of  $r_0 = 100\%$  down to the failure level of  $s = 0\%$  for  $t \geq 0$ . The cumulative distribution function of the hitting time of the failure level (lifetime) can be rewritten as:

$$\mathbb{P}(T_y \leq t) = \mathbb{P}(X(t) \geq y) = \frac{\Gamma((\mu/\sigma)^2 t, y\mu/\sigma^2)}{\Gamma((\mu/\sigma)^2 t)},$$

where  $y = r_0 - s$  and  $\Gamma(a, x) = \int_{t=x}^{\infty} t^{a-1} e^{-t} dt$  is the incomplete gamma function for  $x \geq 0$  and  $a > 0$ . For the hydraulic cylinder at hand, the time at which the expected condition equals the failure level is 15 years with parameters  $\mu = 6.67$  and  $\sigma = 1.81$ .

A useful property of the gamma process with stationary increments is that the gamma density in (27) transforms into an exponential density if

$$t = (\sigma/\mu)^2 = 0.074.$$

When the unit-time length is chosen to be  $(\sigma/\mu)^2$ , the increments of deterioration are exponentially distributed with mean  $\sigma^2/\mu$  and the probability of failure in unit time  $i$  reduces to a shifted Poisson distribution (see, e.g., van Noortwijk et al., 1995):

$$p_k = \mathbb{P}(T = k) = \frac{1}{(k-1)!} \left[ \frac{y\mu}{\sigma^2} \right]^{k-1} \exp\left\{-\frac{y\mu}{\sigma^2}\right\}$$

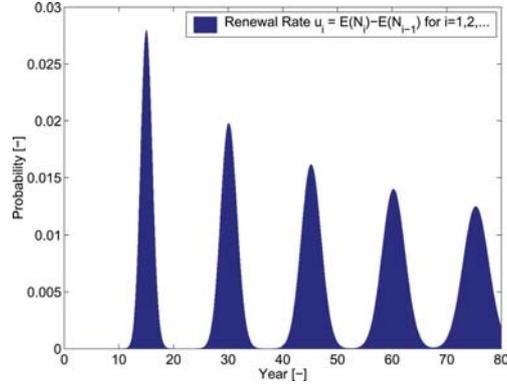


Figure 5. Rate of cylinder renewal per unit time of length  $\Delta = 0.074$  year.

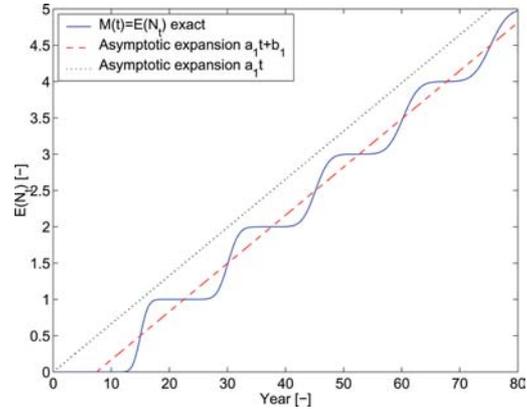


Figure 6. First moment of cumulative number of cylinder renewals.

for  $k = 1, 2, 3, \dots$ . The mean and variance of the shifted Poisson distribution are

$$\mathbb{E}(T) = 1 + \frac{y\mu}{\sigma^2}, \quad \text{Var}(T) = \frac{y\mu}{\sigma^2}.$$

The rate of cylinder renewal per unit time of length  $\Delta = 0.074$  year,

$$u_j = \mathbb{E}(I_j) = \mathbb{E}(N_j) - \mathbb{E}(N_{j-1}) \quad (j \geq 1),$$

is determined over an eighty-year design life and displayed in Figure 5. For a finite time horizon of eighty years, the first and second moment of the cumulative number of cylinder replacements are determined using (12) and (14), respectively. The first and second moment are approximated with the asymptotic expansions (24) and (25). In Figure 6, we can clearly see that the asymptotic expansion for the first moment including the constant term is a much better approximation

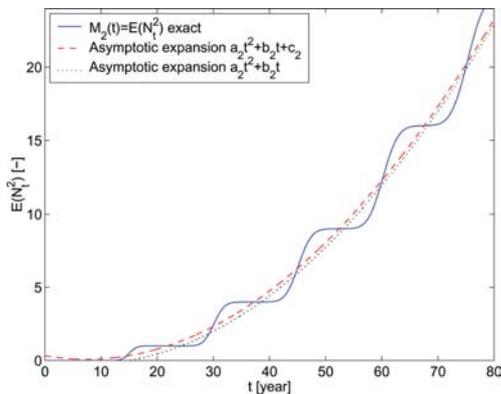


Figure 7. Second moment of cumulative number of cylinder renewals.

than the asymptotic expansion excluding the constant term. For this particular example, the constant term in the asymptotic expansion for the second moment of the number of renewals doesn't have much influence (see Figure 7).

## 6 CONCLUSIONS

In the probabilistic modelling of life-cycle management of engineering systems, the renewal theorem plays a key role in the computation of the expected number of renewals and the cost rate associated with a management strategy. The renewal theorem is well known and its rigorous mathematical proof is presented in the literature for both discrete and continuous random variables. The mathematical details and technicalities associated with its derivation are not well understood by the engineering community. This paper presents a more lucid and intuitive interpretation of the renewal theorem.

To simplify the presentation, we have utilised discrete random variables in the formulation of the renewal problem. This approach results in analytical expressions for the renewal rate and the probability distributions of the age and remaining lifetime. These formulas are very easy to compute, in contrast with the traditional continuous random variable formulation which requires solution of integral equations. In the proposed approach, the life-cycle cost in a finite time horizon can be computed in a straightforward manner using the rate of renewal. In addition to explicit formulas for a finite time horizon, the paper derives elegant asymptotic results for the renewal rate and discounted life-cycle cost over an infinite time horizon. Asymptotic expansions are derived for the first and

second moment of the number of renewals. As far as the authors know, the latter expansion is new in the sense that it also contains a constant term.

The proposed concepts are illustrated by two examples involving the discrete Weibull distribution and the shifted Poisson distribution (resulting from a discretised gamma deterioration process). More detailed practical applications of this approach to nuclear power plant systems are under investigation.

## REFERENCES

- Ali Khan, M.S., Khaliq, A., & Abouammoh, A.M. 1989. On estimating parameters in a discrete Weibull distribution. *IEEE Transactions on Reliability*, 38(3): 348–350.
- Feller, W., 1949. Fluctuation theory of recurrent events. *Transactions of the American Mathematical Society*, 67(1):98–119.
- Feller, W., 1950. *An Introduction to Probability Theory and its Applications; Volume 1*. New York: John Wiley & Sons.
- Feller, W., 1966. *An Introduction to Probability Theory and its Applications; Volume 2*. New York: John Wiley & Sons.
- Karlin, S. & Taylor, H.M., 1975. *A First Course in Stochastic Processes*, Second Edition. San Diego: Academic Press.
- Kingman, J.F.C., 1972. *Regenerative Phenomena*. New York: John Wiley & Sons.
- Rackwitz, R., 2001. Optimizing systematically renewed structures. *Reliability Engineering and System Safety*, 73(3):269–279.
- Smith, W.L., 1954. Asymptotic renewal theorems. *Proceedings of the Royal Society of Edinburgh, Section A (Mathematical and Physical Sciences)*, 64:9–48.
- Smith, W.L., 1958. Renewal theory and its ramifications. *Journal of the Royal Statistical Society, Series B (Methodological)*, 20(2):243–302.
- Stein, W.E. & Dattero, R., 1984. A new discrete Weibull distribution. *IEEE Transactions on Reliability*, 33:196–197.
- Tijms, H.C., 2003. *A First Course in Stochastic Models*. New York: John Wiley & Sons.
- van Noortwijk, J.M., 2003. Explicit formulas for the variance of discounted life-cycle cost. *Reliability Engineering and System Safety*, 80(2):185–195.
- van Noortwijk, J.M., Cooke, R.M., & Kok, M., 1995. A Bayesian failure model based on isotropic deterioration. *European Journal of Operational Research*, 82(2): 270–282.
- van Noortwijk, J.M. & Klatter, H.E., 2004. The use of lifetime distributions in bridge maintenance and replacement modelling. *Computers and Structures*, 82(13–14): 1091–1099.
- van Noortwijk, J.M. & van derWeide, J.A.M., 2006. Computational techniques for discrete-time renewal processes. In Guedes Soares, C. & Zio, E., (eds.), *Safety and Reliability for Managing Risk, Proceedings of ESREL 2006 – European Safety and Reliability Conference 2006, Estoril, Portugal, 18–22 September 2006*, pages 571–578. London: Taylor & Francis Group.