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Putting the world in mind: The case of mental representation of quantity

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ABSTRACT

A reoccurring question in cognitive science concerns the way the world is represented. Cognitive scientists quantify the contribution of a physical attribute to a sensation and try to characterize the underlying mechanism. In numerical cognition, the contribution of physical properties to quantity perception in comparison tasks was widely demonstrated albeit leaving the underlying mechanism unclear. Furthermore, it is unclear whether this contribution is related solely to comparison tasks or to a core, general ability. Here we demonstrate that the shape of the convex hull, the smallest convex polygon containing all objects in an array, plays a role in the transfer function between quantity and its mental representation. We used geometric probability to demonstrate that the shape of the convex hull is correlated with quantity in a way that resembles the behavioral enumeration curve of subitizing and estimation. Then, in two behavioral experiments we manipulated the shape of the convex hull and demonstrated its effect on enumeration. Accordingly, we suggest that humans learn the correlation between convex hull shape and numerosity and use it to enumerate.

1. Introduction

Our cognitive system is limited in its inputs, its processing resolution and its processing capacity. These limitations create an under-representation of the physical world. In order to deal with these limitations on the one hand, and world complexities on the other, the cognitive system has adopted strategies to reduce dimensionality and create cognitive shortcuts while remaining fairly accurate. This is apparent in many optical illusions and other heuristic strategies. These biases allow us to understand how the physical world is represented, revealing, at least in part, a transfer function between the physical world and its representation. This transfer function represents the cognitive mechanism transforming real world property to its representation.

Quantity perception is an interesting example. Quantity is physically represented in the world, as any numerical array is characterized by its area, density, etc. (i.e., physical properties). These properties often modulate and bias quantity perception (Clayton, Gilmore, & Inglis, 2015; Gebuis & Reynvoet, 2012b; Katzin, Salti, & Henik, 2018; Smets, Sasanguie, Szücs, & Reynvoet, 2015). However, most of the efforts were dedicated to quantifying the effects of physical properties on quantity perception, and not to understanding the underlying mechanism. Moreover, the effect of physical properties was demonstrated only in non-symbolic comparison tasks. Hence, it is unclear if the effect of physical properties is task dependent, or if they play a role in numerical perception in general. Here, we show the effect of one physical property in the emergence of quantity perception, unveiling part of the quantity transfer function.

Recently, Salti, Katzin, Katzin, Leibovich, and Henik (2017) presented a taxonomy of the physical properties commonly used in numerical cognition studies, in an effort to better understand their intercorrelations. This taxonomy divides the physical properties to intrinsic and extrinsic properties. Intrinsic properties are properties that describe the size of a single object and can be summed or averaged to describe the entire array (e.g., average diameter, total circumference and total surface area). Extrinsic properties describe an array of dots as they consider both the size and the location of the dots (e.g., density and convex hull area). Accordingly, extrinsic properties entail more information about the array as a whole. Importantly, the convex hull (CH), the smallest convex polygon containing all objects and all straight lines between any two objects in an array (see Fig. 1A), appears to have an advantage over the other properties. CH is a lower resolution version of any array of objects, as it uses the minimal number of objects that contain the array—only the items on the circumference. Thus, making it

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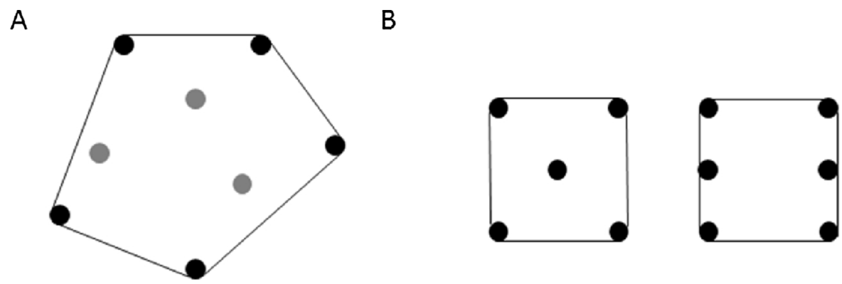

Fig. 1. ***(A)*** A set of points and its CH. CH vertices are black, interior dots are grey. **(B)** CH of five and six dots, as presented on a die, have the same CH shape.

the most economic property, that imports minimal input and exports maximum output.

Indeed, several studies found that CH area is an important property in dot-comparison tasks. Bertamini, Zito, Scott-Samuel, and Hulleman (2016), examined how different visual features of dot arrays affect perception of numerosity, density and clustering. They found that CH area affected perceived numerosity, so that arrays with larger CH were perceived as more numerous. Gebuis and Reynvoet (2012a) examined the congruency effect (in accuracy) in a non-symbolic comparison task. They found a congruency effect when only the congruency of the CH area was manipulated, however, when the congruency of other physical properties was manipulated there was a reverse congruency effect or no effect at all. In another study, when the CH area ratio was equated to the numerical ratio, it was more salient than numerosity and other physical properties (Katzin et al., 2018). Gilmore et al. (2016) compared the role of total surface area and convex hull area in the development of numerical perception. They found that the total surface area ratio predicted accuracy only in children, while the CH area ratio predicted accuracy in children and adults, and its predictive power was positively correlated with age. However, it seems unlikely that the area of CH per se underlies this behavior, as conceptually it is not different from the total surface area. Instead, manipulating the CH area might affect the saliency of another characteristic of CH, namely its shape, which in turn could drive this behavior.

Shape was shown to influence enumeration. Mandler and Shebo (1982) proposed that small quantities, even when randomly placed, are associated with specific shapes. They suggested that small quantities up to four create canonical patterns (i.e., 2 dots create a line, 3 dots create a triangle etc.). However, the association between quantity and shape was intuitive and therefore was limited to quantities up to four. Moreover, studies providing evidence in favor of the pattern recognition theory usually used symmetrical canonical arrangements (e.g., like on a die, see Fig. 1B) (Ashkenazi, Mark-Zigdon, & Henik, 2013; Mandler & Shebo, 1982a; Mandler & Shebo, 1982b; Piazza, Mechelli, Butterworth, & Price, 2002). These symmetrical arrangements encompass only a small portion of the phenomenon. The vast number of possible arrangements make it unlikely and uneconomical to learn them all. Relying on spatial arrangement would be more plausible if it could exploit a single property that can downsize the number of possible arrangements while enabling distinction between numerosities. The shape of the CH could be a candidate if it predicts numerosity.

Consequently, we aimed to examine the extent to which the shape of the CH predicts numerosity of a given set. The shape of the CH can be measured by the number of vertices on the polygon as 3 vertices create a triangle, 4 create a quadrilateral, etc. Accordingly, we ask, if an unknown number of dots (S) is randomly scattered over a region, how will knowing the shape of the CH (*CH(S)*) help in estimating the total number of dots (n) $P[n|(CH(S) = k)]$. We were interested in the way the information CH holds about numerosity changes with quantity.

In the field of geometric probability, the opposite probability question was raised: if n points are chosen independently with uniform probability on a given region, what is the probability that the CH of these points is a polygon with exactly k vertices $P[(CH(S) = k)|n]$? (Croft, Falconer, & Guy, 1991). The answer to this question depends on the shape of the given region. For the case of a rectangular region, it was solved (Buchta, 2009; Trott, 2006; Valtr, 1995). Importantly, we wanted to know how predictive the shape of the CH is of quantity, (i.e., $P[n|(CH(S) = k)]$). That is, given that exactly k points are on the CH, we ask what the probability is that the number of total points is exactly n. The answer to this question depends on how the number of points is chosen. This can be seen by the use of Bayes' theorem:

$$P[n|(CH(S) = k)] = \frac{P[(CH(S) = k)n]^{*} P(n)}{P(CH(S) = k)} \tag{1}$$

Here, the number k is fixed, so $P(CH(S) = k)$ may be viewed as a normalizing constant, that is, a scaling factor ensuring that the probabilities $P[n|(CH(S) = k)]$ are between 0 and 1. However, the choice of $P(n)$ strongly influences the values $P[n|(CH(S) = k)]$. For instance, choosing $P(n)$ as a uniform distribution (i.e., assuming all numerosities $n \geq (CH(S) = k)$ are equally probable) gives for every fixed $CH(S) = k$, $P[n|(CH(S) = k)] \propto P[(CH(S) = k)n]$(i.e., $P[n(CH(S) = k)]$is directly proportional to $P[(CH(S) = k)n]$). Unfortunately, it is impossible to assume that all numerosities greater than k are equally probable, since a discrete uniform probability on an infinite number of points does not exist. It follows that if a uniform probability is assumed, we must also assume an upper bound on the number of points. While we cannot calculate $P[n(CH(S) = k)]$ exactly, the relation $P[n|(CH(S) = k)] \propto P[(CH(S) = k)n]$allows us to know, given k, which n's are more probable, and their relative likelihood.

Table 1 depicts the calculated probabilities of CH shapes $P[(CH(S) = k)|n]$ for numerosities 3 to 20. The probability that a quantity creates its unique shape (i.e., all dots are vertices of the CH, $P(CH(S = k)) = P(n)$) drops drastically at five (a drop from 0.6944 at four to 0.3403 at five). In fact, starting at $n = 5$, the case of $k = n$ is no longer the most probable. To calculate the relative likelihood of each k, we divided the probability of the most probable numerosity by the second most probable numerosity. For $k = 3$ and $k = 4$, the most probable numerosity is $n = k$. For these cases, the ratio between the most probable and second most probable n is 3.27 and 1.25, respectively (i.e., $\frac{1}{0.3056}$ and $\frac{0.6944}{0.5556}$). For $k = 5$, not only is it less likely that $n = k$, but the ratio between the most and second most probable n is only 1.006 (i.e. , $\frac{0.4764}{0.4735}$, see Table 1). As k increases, this ratio approaches 1. Hence, for $n \leq 4$, the shape of the CH highly predicts a specific numerosity; otherwise, CH predicts an expanding range of numerosities.

The correlation between the CH shape and numerosity corresponds to seminal behavioral findings of enumeration. Up to the quantity four, where the CH shape is highly predictive, enumeration is fast and accurate (i.e., subitizing). Above four, where the CH shape predicts an expanding range of numerosities, enumeration is slower and less accurate. If indeed the shape of the CH contributes to the transfer function from the physical stimulus to its representation, then the shape of the CH should affect enumeration. Namely, when the number of vertices on the CH is the most likely amount, that is, the most probable shape, estimations should be the most accurate. A higher or lower number of vertices on the CH should yield overestimations or underestimations, respectively. For example, for numerosity 12, we would expect the most

Table 1
Probabilities of CH shapes for numerosities 3–20 (P(CHS = k)|n) calculated according to Buchta (2009).

	k									
n	3	4	5	6	7	8	9	10	11	12
3	**1**									
4	.3056	**.6944**								
5	.1042	**.5556**	.3403							
6	.0381	.3631	**.4764**	.1225						
7	.0146	.2256	**.4735**	.2528	.0336					
8	.0058	.1391	**.4129**	.3419	.0930	.0072				
9	.0024	.0868	.3388	**.3851**	.1604	.0254	.0013			
10	.0010	.0552	.2699	**.3930**	.2220	.0534	.0053	.0002		
11	.0004	.0360	.2123	**.3783**	.2709	.0878	.0133	.0009	0	
12	.0002	.0241	.1665	**.3511**	.3052	.1248	.0254	.0026	.0001	0
13	.0001	.0166	.1308	.3185	**.3258**	.1610	.0413	.0056	.0004	0
14	0	.0117	.1034	.2847	**.3348**	.1940	.0600	.0103	.0010	.0001
15	0	.0085	.0823	.2523	**.3349**	.2226	.0806	.0167	.0020	.0001
16	0	.0063	.0661	.2224	**.3284**	.2461	.1021	.0247	.0036	.0003
17	0	.0048	.0535	.1956	**.3174**	.2645	.1235	.0342	.0058	.0006
18	0	.0037	.0438	.1720	**.3035**	.2779	.1443	.0450	.0087	.0011
19	0	.0029	.0361	.1513	**.2879**	.2870	.1638	.0567	.0124	.0018
20	0	.0023	.0300	.1333	.2715	**.2923**	.1817	.0692	.0168	.0035[1]

Note. Probabilities rounded up to 4 significant digits. **Bold** probabilities mark the most probable pattern of each quantity.
[1]This value was calculated with a Monte-Carlo simulation, because computation with the formula was complicated and took too long (more than a week's work on 4 processing cores). In the simulation n points were chosen randomly with a uniform distribution on the square [0,1]X[0,1]. The number of vertices on the CH was then recorded, and the process was iterated 1000,000 times.

accurate enumerations with 6 vertices on the CH, underestimations when there are less than 6 vertices, and overestimations for more than 6 vertices. Importantly, when we refer to number of vertices, we use it as an operational definition of shape, we do not suggest that the observers count the vertices in order to represent this shape. To investigate this prediction, we conducted two behavioral experiments in which the number of vertices on the CH was manipulated.

2. Experiment 1

2.1. Method

2.1.1. Participants

Sixteen participants (4 males, average age: 23.6 years old, SD: 1.31) took part in Experiment 1. All participants had normal or corrected-to-normal vision and received course credit for participation. The experiments were approved by the university ethics committee.

2.1.2. Stimuli

The stimuli were arrays of black dots on a white background. All dots were the same size (diameter of 30 pixels). Dots were randomly placed under two limitations: the number of dots on the CH was restricted (according to the condition) and dots could not overlap. We used four numerosities (i.e., n's): 7, 8, 9, and 10; and five different k's: 3, 4, 5, 6, and 7. In total, we had 20 conditions created by the combination of all chosen n's and k's. For each condition we presented 20 stimuli, resulting in 400 trials in total.

2.1.3. Procedure

Participants were asked to estimate the number of dots presented on the screen. They were told the dots would appear for a very short duration so they could not count them, but would need to estimate. Each trial began with a black fixation point presented on a white screen for 1000 ms. This was followed by a blank white screen for another 1000 ms and then the stimulus was presented for 16 ms. After the stimulus was presented, the participants had 4800 ms to respond vocally. After the participants responded, an experimenter coded the response. The experiment began with five practice trials. After the practice block, 400 trials were split into 4 blocks of 100 trials each. Participants' estimation and response times (RT) were recorded. Duration of presentation was short for two reasons. First, we wanted to prevent a counting strategy. Second, we hypothesized that the shape of the CH is extracted relatively early in the perceptual process.

2.2. Results

2.2.1. Pre-processing

We removed trials in which participants gave estimations in the subitizing range (lower than 5). In addition, we removed trials in which participants gave estimations higher than 15 and trials in which they did not give a response, resulting in a total of 108 trials (0.02%) that were removed.

2.2.2. Analysis

We conducted a two-way repeated measures analysis of variance with total numerosity (7–10) and CH configuration (3–7) as independent within subject variables, and participant estimation as the dependent variable. Assumption of sphericity was violated, however, results remained the same after correction.

We found a significant main effect for total numerosity, F (3, 45) = 160.48, $p < .001$, $\eta_p^2 = .91$, so that estimations were higher as numerosity was higher (see Fig. 2). In addition, a main effect for number of vertices on the CH was found, F (4, 60) = 5.9, $p < .001$, $\eta_p^2 = .28$. Namely, estimations were higher as the number of vertices on the CH grew (see Fig. 3). The 2-way interaction between total numerosity and number of vertices on the CH was significant as well, F (12, 180) = 1.98, $p < .05$, $\eta_p^2 = .12$. The interaction did not reveal a clear pattern, and is not relevant to our hypothesis, and as such, will not be discussed further.

2.2.3. Post-hoc analysis

To rule out the possibility that participants counted the dots, we conducted a similar analysis with RT as the dependent variable. Counting is a serial strategy that is manifested in an increase of RT as numerosity increases (Kaufman, Lord, Reese, & Volkmann, 1949; Piazza et al., 2002). Accordingly, if participants were counting the array or the number of vertices of the CH, we would expect to see an effect in RT. None of the effects were significant: total numerosity, F (3, 45) = 1.71, $p > .1$; number of vertices on the CH, F (4, 60) = 1.07, $p > .1$; and interaction effect, F (12, 180) = 1.66 $p > .05$.

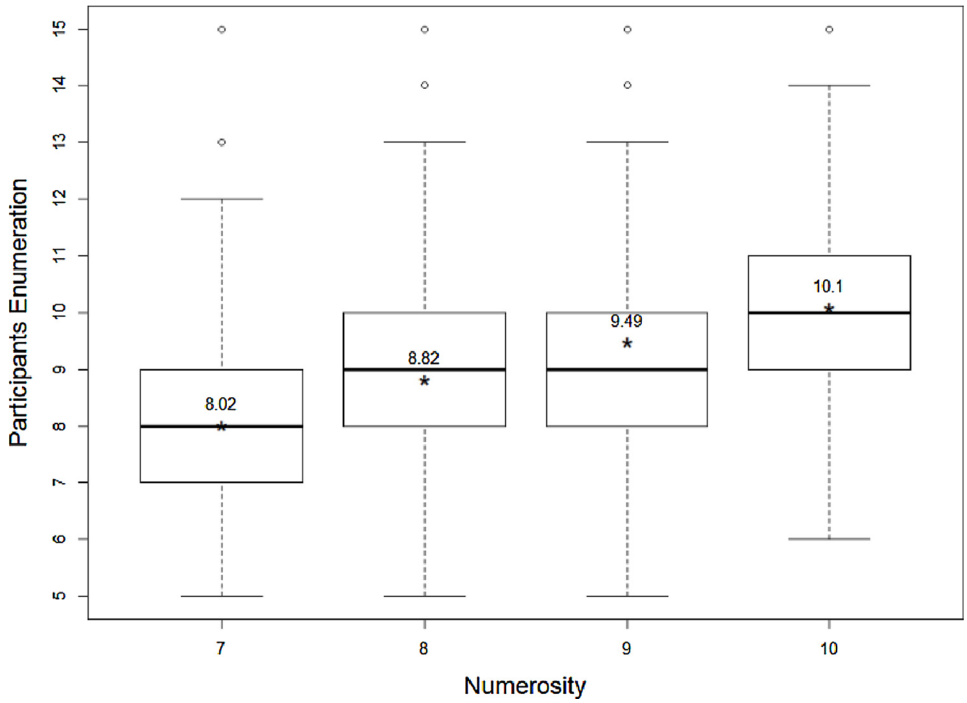

Fig. 2. Accuracy of participants' enumeration in Experiment 1. The black horizontal lines represent the group median (i.e., 50th percentile), and the bottom and top of the box represent the 25th and 75th percentile (i.e., the lower and upper quartile), respectively. The asterisk represents the mean, which is also written in the boxplot.

In order to support the null hypothesis, we computed Bayes factors for all possible models based on all combinations of factors (with the caveat that interaction terms were only included together with their main effects). Bayes factors were computed in JASP, using default JZS priors (Rouder & Morey, 2012; Rouder, Morey, Speckman, & Province, 2012). We then computed Inclusion Bayes factors for each factor by comparing the average change in posterior between models that include and models that exclude said factor (Clyde, Ghosh, & Littman, 2011). We found the following Bayes factors: for the effect of total numerosity, $BF_{Inclusion}$ = 1.01; for the effect of number of vertices on the CH, $BF_{Inclusion}$ = .03; and for the interaction effect, $BF_{Inclusion}$ = .01. These results reflect, in general, medium to strong evidence in favor of the null hypotheses.

2.3. Discussion

The results support our hypothesis; shape of CH is positively correlated with participants' enumeration. However, overall, participants tended to overestimate in this experiment and so, the most accurate results were actually when the number of vertices on the CH was low. In order to rule out the possibility that participants were counting the dots, we examined their RT. A counting strategy would have been reflected in longer RT as numerosity and/or the number of vertices on the CH increased. We did not find any effects of RT, suggesting that the shape of the CH was perceived as a whole, and the vertices were not counted.

In the second experiment we had several goals. First, we wanted to replicate the results of the first experiment and expand the results to larger numerosities. Second, we wanted to make the arrays more ecological, so we varied the size of the dots. Third, in the first experiment we created a factorial design by keeping the number of vertices on the CH similar for all numerosities. This design had two limitations. First, it constrained the number of numerosities. Second, it did not control for the probability of appearance of each condition. For example, for numerosity 7, the probability for the most probable configuration is 0.4735 (5 vertices on the CH), and the least probable is 0.0146 (3

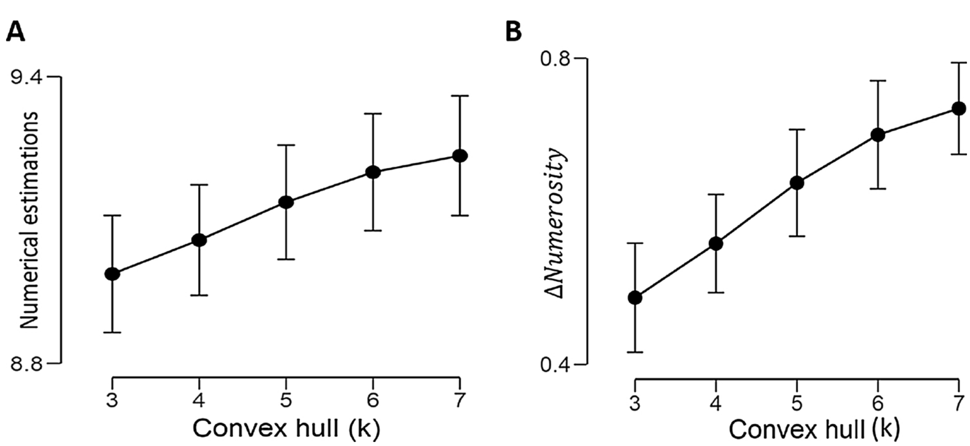

Fig. 3. Main effect of the number of vertices on the CH in Experiment 1. A) Y-axis is participants' numerical estimations. B) Y-axis is the delta between perceived numerosity and actual numerosity. Accordingly, positive values represent overestimation.

vertices on the CH). For numerosity 10, the values are 0.3930 (6 vertices on the CH) and 0.0002 (10 vertices on the CH), respectively.

Accordingly, in the second experiment we used numerosities 7–20. For each numerosity, we chose three different patterns that varied in the number of vertices on the CH. The ‘standard’ pattern was the most probable pattern. The ‘low’ and ‘high’ patterns were patterns that were at least 30% less likely to appear compared to the standard, with a lower or higher number of vertices on the CH, respectively. For example, for numerosity 14, the ‘low’ pattern has 4 vertices on the CH and its probability is 0.117, the ‘standard’ pattern has 7 vertices on the CH and its probability is 0.3348, and the ‘high’ pattern has 9 vertices on the CH and its probability is 0.06.

We did not measure RT in this experiment for two reasons. First, we did not find an effect of RT in the first experiment. Second, in this experiment the CH conditions were determined based on probability (e.g., low, standard and high), accordingly, the number of vertices on the CH varied between numerosities. To illustrate, the ‘low’ condition of quantity 15 and the ‘standard’ condition of quantity 8 both have 5 vertices on the CH. This in turn, does not allow us to replicate the RT analysis.

3. Experiment 2

3.1. Method

3.1.1. Participants

A power analysis using G*Power 3.1 (Faul, Erdfelder, Lang, & Buchner, 2007) indicated that the needed sample size for examining the effect of the shape of the CH at a power > 90% to test effect-size similar to Experiment 1 (0.28) with a Type 1 error ($\alpha < 0.05$), is 35 participants. Thirty-six (13 males, average age: 22.92 years old, SD: 1.63) participants took part in Experiment 2. All participants had normal or corrected-to-normal vision and received course credit for participation. The experiments were approved by the university ethics committee.

3.1.2. Stimuli

The stimuli were arrays of black dots on a white background. Dot diameter varied between 10–69 pixels. We used fourteen numerosities (i.e., n’s): 7–20 and three k’s for each numerosity (see Table 2), that is, the k representing the most probable number of vertices and k’s representing at least 30% more and 30% less than the most probable number of vertices on the CH (see Table 1). In total, we had 42 conditions created by the combination of all chosen n’s and k’s. For each condition we presented 10 stimuli, resulting in 420 trials in total.

3.1.3. Procedure

The procedure was similar to that of Experiment 1 except that 420 trials were split into 4 blocks of 105 trials each. Participants’ estimation and RT were recorded.

Table 2
Number of vertices on the convex hull for each numerosity and type for the stimuli of Experiment 2.

Numerosity	Low	Standard	High
7	3	5	7
8	3	5	7
9	4	6	8
10	4	6	8
11	4	6	8
12	4	6	9
13	4	6	9
14	4	7	9
15	5	7	9
16	5	7	10
17	5	7	10
18	5	7	10
19	5	7	10
20	5	8	10

3.2. Results

3.2.1. Pre-processing

We removed trials in which participants gave estimations in the subitizing range (lower than 5). We also removed trials in which participants gave estimations of 40 and higher and trials in which they did not give a response, resulting in a total of 273 trials (0.02%) that were removed.

3.2.2. Analysis

We conducted a two-way repeated measures analysis of variance with total numerosity (7–20) and CH configuration (low/standard/high) as independent within subject variables, and participant estimation as the dependent variable. Assumption of sphericity was violated, however, results remained the same after correction.

The results of Experiment 2 replicated the results of Experiment 1. We found a significant main effect for total numerosity, $F\ (13, 455) = 245.6$, $p < .001$, $\eta_p^2 = .87$, so that estimations were higher as numerosity was higher (see Fig. 4). A main effect for number of vertices on the CH was significant as well, $F\ (2, 70) = 10.84$, $p < .001$, $\eta_p^2 = .24$. Namely, estimations were higher as the number of vertices on the CH grew (see Fig. 5A). In addition, the interaction was significant, $F\ (26, 910) = 2.81$, $p < .001$, $\eta_p^2 = .07$. The interaction did not reveal a clear pattern, and is not relevant to our hypothesis, and as such, will not be discussed further (see Fig. 5B).

3.2.3. Post hoc analysis

We conducted a hierarchical stepwise regression to examine the contribution of various physical properties (average diameter, total circumference, total surface area, density, CH area, CH shape) and total numerosity, with participants’ estimation as the dependent measure. The final model included only 2 predictors: total numerosity and shape of convex hull *Adjusted* $R^2 = 0.923$, $F(1,417) = 12.78$, $p < .001$.

3.3. Discussion

The results of Experiment 2 replicate the results of Experiment 1, indicating that the shape of the CH modulates perceived numerosity. Contrary to Experiment 1, here we found that participants’ overall tendency was to underestimate the number of dots. This might have occurred due to the fact that numerosities in this experiment were higher than those in the first experiment and the range was wider. Indeed, examining the numerosities of experiment 1 (7–10), within experiment 2, reveals a replication. That is, in both experiments numerosities 7–10 are overestimated (see Fig. 4). In fact, underestimation is first apparent at numerosity 13. A similar pattern is evident in previous studies. Kaufman et al. (1949) examined enumeration in quantities between 1–210, and report underestimation beginning at 20∼. Mandler and Shebo (1982) used numerosities 1–20, and see underestimation starting at 5–9 depending on duration of presentation. Jevons (1871), presented numerosities 3–15 and found exact enumeration for quantities 3–5, overestimation for quantities 5–8, and underestimation for quantities 9 and above. Accordingly, under and overestimation of quantities in different ranges is a documented but overlooked phenomenon. Importantly, the effect of CH shape occurs both in under and overestimation ranges.

In the first experiment we ruled out the possibility that participants are counting the number of vertices on the CH. In this experiment we added a regression analysis showing that other physical properties did not predict participants’ enumeration, including another aspect of the CH, its area. This finding further supports our hypothesis that CH shape is perceived holistically and is used for enumeration.

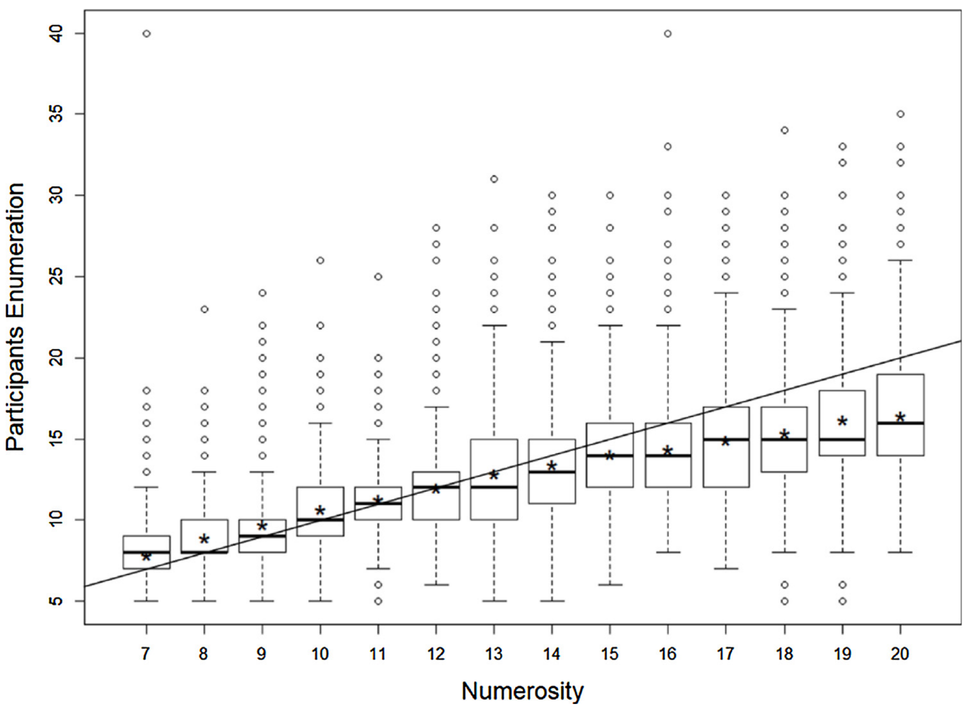

Fig. 4. Accuracy of participants' enumeration in Experiment 2. The bold black horizontal lines represent the group median (i.e., 50th percentile), and the bottom and top of the box represent the 25th and 75th percentile, respectively (i.e., the lower and upper quartile). The asterisk represents the mean. The diagonal represents the 100% correct.

4. General discussion

In the literature dealing with quantity perception, physical attributes are treated as a covariate whose influence is measured and quantified. In this paper we attempted to go beyond this quantification by characterizing a plausible mechanism through which a specific physical attribute, namely the CH, influences quantity perception. Inspired by the psychophysical tradition that aimed at quantifying the sensation of a stimulus and characterizing the change in subjective experience as a function of objective change in intensity, we examined the natural correlation between shape and quantity, and whether this correlation is mirrored in behavior.

Using Buchta's formulas (Buchta, 2009), we showed that in the physical world, CH shape predicts quantity. Interestingly, the predictive power of CH changes. Small numerosities, up to 4, usually appear in their unique shape (i.e., 3 dots create a triangle, 4 create a quadrilateral). As quantity increases, there are more possible CH shapes per quantity that are likely to appear. These results are similar to behavioral enumeration findings of subitizing and estimation and shed light on a theoretical controversy in numerical cognition: are subitizing and estimation two distinct mechanisms or a product of a single mechanism that operates with varying efficiency? (Piazza et al., 2002). Our results show that a single mechanism is plausible. As such, we propose that the CH shape is utilized as a heuristic for enumeration in all ranges. Namely, subitizing and estimation both reflect the correlation between shape and quantity.

We hypothesized that observers would rely on the natural probabilities of CH's number vertices and that deviation up or down would cause over and underestimation, respectively. However, there are other mechanisms influencing enumeration, such as the ones that underlie the tendency to over or underestimate that we did not take into consideration in our hypothesis. Importantly, we see that probability of CH shape modulates enumeration in a linear fashion, as we hypothesized. Moreover, we relate to CH shape as a "quick and dirty" mechanism for enumeration that enables crude estimations that later on are refined. We test our hypothesis, however, in a very stringent manner as we try to predict exact numerosity.

The behavioral experiments showed that the CH shape modulates enumeration. In line with our hypothesis, participants' enumerations increased as the number of vertices on the CH increased. One possibility is that the number of vertices on the CH was counted. We did not find any support for this possibility as RT did not increase with the number

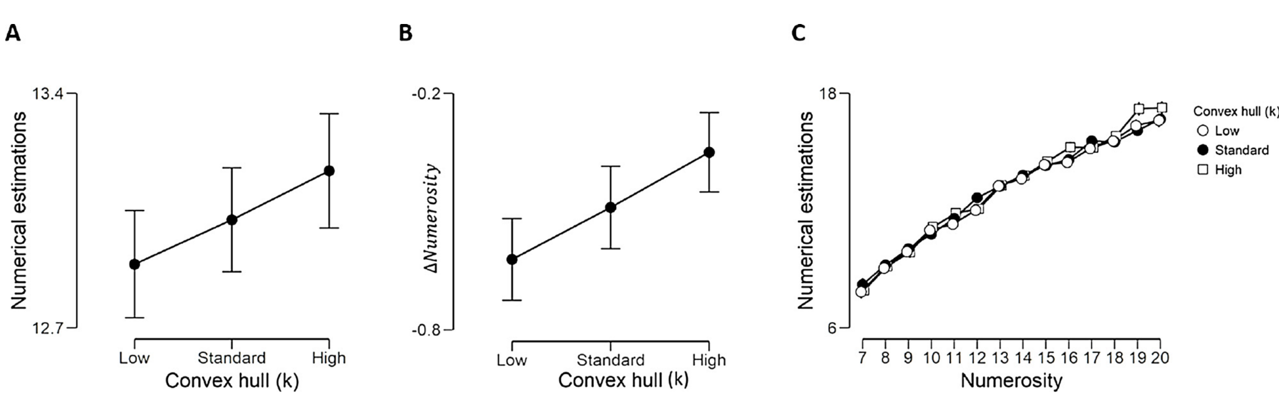

Fig. 5. A) Main effect of the number of vertices on the CH in Experiment 2. Y-axis is participants' numerical estimations. B) Main effect of the number of vertices on the CH in Experiment 2. Y-axis is the delta between perceived numerosity and actual numerosity. Accordingly, negative values represent underestimation. C) Interaction effect of total numerosity and number of vertices on the CH. Y-axis is participants' numerical estimations.

of vertices on the CH. Moreover, Bayesian analysis supported the notion that shape was perceived holistically. We suggest that the shape is perceived automatically, similar to Gestalt ideas, and is used to infer numerosity. Visual objects are recognized easily from their outlines (Wagemans et al., 2008). Moreover, convexities are important for shape recognition. In one study, participants were presented with segmented shapes of either the convexities, concavities or the lines in between of shapes of varying complexity and were asked to match the segmented shape to one of two whole-contour shapes. Results showed that shape recognition was significantly more accurate for segmented shapes of convexities than for concavities or intermediate lines, corroborating the importance of convexities in shape recognition (Schmidtmann, Jennings, & Kingdom, 2015).

The current study focused on the physical world in order to understand its psychophysical representation. The underlying rational was finding a correlation between the physical properties of numerical arrays and their representation. This notion corresponds to the idea that through the process of statistical learning, humans learn the correlation between CH shape and numerosity and use it to enumerate. However, the present study does not exclude other possibilities such as the correlation between CH shape and numerosity is hardwired. Future studies should examine these possibilities.

Theorizing CH shape as a transfer function between the physical world and its representation allows a unique glimpse to a plausible mechanism of abstraction. Numerosity is an abstract concept representing a specific quantity. There is a hierarchy of abstraction within quantities; small quantities are more concrete than large quantities. To illustrate, it is easier to imagine exactly 3 apples than imagining exactly 30 apples. As reflected in the computed probabilities, the less informative the CH becomes, the more abstract the numerosity. Perhaps, a concept is abstract because its physical properties are less informative.

To conclude, we chose an approach that focused on the transfer function between the physical world and its representation, looking for the natural distribution of CH vertices and its correlation with numerosity. This proved beneficial as it complied with behavioral hallmarks, allowing us not only to suggest a mechanism through which this physical property affects quantity perception but also to expand the effect of physical properties from comparison tasks to a general numerical domain.

Code and data availability

Data files are available here: https://osf.io/pe6qm/.

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References

Ashkenazi, S., Mark-Zigdon, N., & Henik, A. (2013). Do subitizing deficits in developmental dyscalculia involve pattern recognition weakness? *Developmental Science, 16*(1), 35–46. https://doi.org/10.1111/j.1467-7687.2012.01190.x.

Bertamini, M., Zito, M., Scott-Samuel, N. E., & Hulleman, J. (2016). *Spatial clustering and its effect on perceived clustering, numerosity, and dispersion.* https://doi.org/10.3758/s13414-016-1100-0.

Buchta, C. (2009). On the number of vertices of the convex hull of random points in a square and a triangle. *Fachbereich Mathematik, 143*, 3–10.

Clayton, S., Gilmore, C., & Inglis, M. (2015). Dot comparison stimuli are not all alike: The effect of different visual controls on ANS measurement. *Acta Psychologica, 161*, 177–184. https://doi.org/10.1016/J.ACTPSY.2015.09.007.

Clyde, M. A., Ghosh, J., & Littman, M. L. (2011). Bayesian adaptive sampling for variable selection and model averaging. *Journal of Computational and Graphical Statistics, 20*(1), 80–101. https://doi.org/10.1198/jcgs.2010.09049.

Croft, H. T., Falconer, K. J., & Guy, R. K. (1991). *Convexity. Unsolved problems in geometry*. New York, NY: Springer6–47. https://doi.org/10.1007/978-1-4612-0963-8_2.

Gebuis, T., & Reynvoet, B. (2012a). The interplay between nonsymbolic number and its continuous visual properties. *Journal of Experimental Psychology. General, 141*(4), 642–648. https://doi.org/10.1037/a0026218.

Faul, F., Erdfelder, E., Lang, A. G., & Buchner, A. (2007). G* Power 3: A flexible statistical power analysis program for the social, behavioral, and biomedical sciences. *Behavior Research Methods, 39*(2), 175–191.

Gebuis, T., & Reynvoet, B. (2012b). The role of visual information in numerosity estimation. *PloS One, 7*(5), https://doi.org/10.1371/journal.pone.0037426.

Gilmore, C., Cragg, L., Hogan, G., Inglis, M., Gilmore, C., Cragg, L., et al. (2016). *Congruency effects in dot comparison tasks: Convex hull is more important than dot area.* 5911 (September)https://doi.org/10.1080/20445911.2016.1221828.

Jevons, W. S. (1871). The power of numerical discrimination. *Nature, 3*(67), 281–282. https://doi.org/10.1038/003281a0.

Katzin, N., Salti, M., & Henik, A. (2018). Holistic processing of numerical arrays. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 45*(6), 1014–1022.

Kaufman, E. L., Lord, M. W., Reese, T. W., & Volkmann, J. (1949). The discrimination of visual number. *The American Journal of Psychology, 62*(4), 498. https://doi.org/10.2307/1418556.

Mandler, G., & Shebo, B. J. (1982). Subitizing: An analysis of its component processes. *Journal of Experimental Psychology, 111*(1), 1–21. Retrieved from http://psycnet.apa.orgjournals/xge/111/1/1.

Piazza, M., Mechelli, A., Butterworth, B., & Price, C. J. (2002). Are subitizing and counting implemented as separate or functionally overlapping processes? *NeuroImage, 15*(2), 435–446. https://doi.org/10.1006/nimg.2001.0980.

Rouder, J. N., & Morey, R. D. (2012). Default bayes factors for model selection in regression. *Multivariate Behavioral Research, 47*(6), 877–903. https://doi.org/10.1080/00273171.2012.734737.

Rouder, J. N., Morey, R. D., Speckman, P. L., & Province, J. M. (2012). Default bayes factors for ANOVA designs. *Journal of Mathematical Psychology, 56*(5), 356–374. https://doi.org/10.1016/J.JMP.2012.08.001.

Salti, M., Katzin, N., Katzin, D., Leibovich, T., & Henik, A. (2017). One tamed at a time: A new approach for controlling continuous magnitudes in numerical comparison tasks. *Behavior Research Methods, 49*, 1120–1127. https://doi.org/10.3758/s13428-016-0772-7.

Schmidtmann, G., Jennings, B. J., & Kingdom, F. A. A. (2015). *Shape recognition: Convexities, concavities and things in between.* Nature Publishing Group1–11. https://doi.org/10.1038/srep17142.

Smets, K., Sasanguie, D., Szücs, D., & Reynvoet, B. (2015). The effect of different methods to construct non-symbolic stimuli in numerosity estimation and comparison. *Journal of Cognitive Psychology, 27*(3), 310–325. https://doi.org/10.1080/20445911.2014.996568.

Trott, M. (2006). *The mathematical guide book for symbolics. The mathematical guide book for symbolics.* Springer-Verlag298–311.

Valtr, P. (1995). Probability that random points are in convex position. *Discrete & Computational Geometry, 13*(3–4), 637–643. https://doi.org/10.1007/BF02574070.

Wagemans, J., Winter, J., De Beeck, H., Op De Ploegerô, A., Beckers, T., & Vanroose, P. (2008). Identification of everyday objects on the basis of silhouette and outline versions. *Perception, 37*(1980), https://doi.org/10.1068/p5825.