Principles of the unsaturated flow and their application to the penetration of moisture into the soil

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IV. PRINCIPLES OF THE UNSATURATED FLOW AND THEIR APPLICATION TO THE PENETRATION OF MOISTURE INTO THE SOIL

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1. Introduction

It is a well-known fact that the moisture content of the soil changes throughout the year under the influence of evapotranspiration, rainfall and drainage. Circumstances with static equilibrium of soil water rarely occur. Under climatological conditions in the Netherlands they are only present during the early spring.

The soil above the water table being unsaturated, changes in moisture content throughout the profile depend on the flow of water in the unsaturated medium. Under natural conditions steep potential gradients in the unsaturated part of the profile are only build up in vertical direction. Generally two cases may be distinguished.

In the first case the soil surface is maintained at saturation, as for example in the case of heavy rainfall or irrigation. Then water will enter into the soil. The development of the moisture profile during this infiltration determines the rate of entry of water. This infiltration rate is a very important factor in hydrological studies. In the context of irrigation it firstly determines the intensity with which water can be applied at the soil surface. Secondly the total amount of irrigation water is dependent on the previous history of the water regime in the soil. In drainage problems we are concerned with the control of soil moisture throughout the profile, while it affects both biological processes and soil water suction which in turn controls the mechanical strength of the soil. The latter factor dominates the bearing strength of the soil (implements, cattle).

During heavy rainfall the infiltration rate determines the rainfall intensity at which surface run-off and formation of pools occur and the part of the rainfall which will flow over the surface.

In the second case the soil surface is free to dry out as is generally the case during dry weather. A potential gradient will then be built up in the opposite direction and water will move vertically upward. This so-called "capillary rise" of moisture is very important for the water supply of the crop. In areas with high evapotranspiration rates this phenomenon is for a great deal responsible for the salinization of the upper layers of the soil.

When the entry of water at the soil surface ceases, the development of the moisture profile still continues and a certain redistribution of water takes place. During this period again there may be a surface drying out and consequently capillary rise occurs so that the moisture profile is changing into two ways. Thus in the same profile both directions of flow can occur simultaneously. The exact flow of water in unsaturated soil under natural conditions has a very complex nature. In this paper the physical laws governing the flow will be treated. Only the infiltration phenomena will be discussed while the capillary rise will be dealt with by Wind (1960).

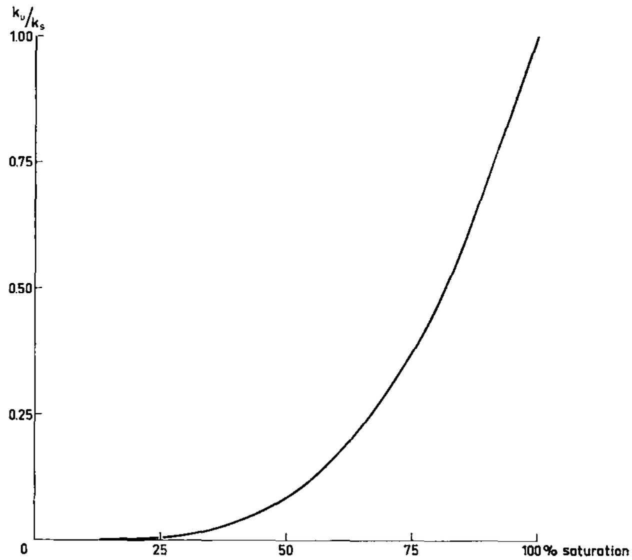


FIG. 1.
The relation between the ratio k_u/k_s and saturation for unconsolidated sands according to experiments from WYCKOFF and BOTSET (1936) and theoretical computation from IRMAY (1954)

2. THE LAWS GOVERNING THE FLOW OF WATER IN UNSATURATED SOIL

GARDNER (1920) was among the first who developed a theory about the flow of water in unsaturated soil in analogy with heat flow. In his concept the so-called "capillary transmission constant" was assumed to be independent of the moisture content of the soil. In a later article GARDNER (1936) follows the theory of BUCKINGHAM (1907) who stated that the capillary conductivity must be dependent on the moisture content.

In his pioneer work on unsaturated flow, RICHARDS (1931) developed a theory from which the validity was later on proved by CHILDS and COLLIS-GEORGE (1950). According to this theory the flow of water in unsaturated soil obeys DARCY's law

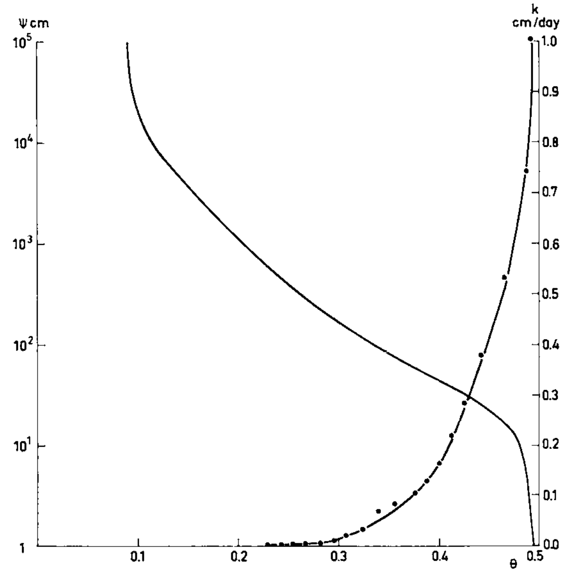
$$v = -k \frac{d\varphi}{dx} \tag{1}$$

where v is the rate at which water crosses a section of unit area perpendicular to the flow direction when $d\varphi/dx$ is the potential gradient. The negative sign indicates that the direction of flow is opposite to that in which the potential increases.

During the flow in saturated soil the whole pore space is effective in conducting water. If the soil is unsaturated the total effective cross sectional area available for the flow, however, is reduced while a certain part of the pore space is filled with air. With decreasing moisture content the effective cross sectional area will reduce.

If in the steady state flow the air-spaces could in some way be filled with solid, the conditions of flow and therefore the relation between the flow and water moving forces would remain unchanged. Compared with the saturated steady state flow in

FIG 2.
The relation between moisture content θ and moisture tension ψ (left hand scale) and hydraulic conductivity (right hand scale) for Yolo light clay, taken from MOORE (1939) as referred to by PHILIP (1957a)



the same medium, the same law holds true, except that in the case of unsaturated flow k will be smaller owing to the reduction in the effective cross sectional area. Henceforth k is no longer a constant as in the case of saturated flow but is now a function of the moisture content Θ of the soil and therefore of the suction ψ.

While the same law is true for both saturated and unsaturated flow we will follow the nomenclature of CHILDS (1958) calling k the "hydraulic conductivity" in both cases.

In fig. 1 the ratio k_u/k_s (unsaturated conductivity/saturated conductivity) is plotted against the saturation (percent pores filled with water) according to experiments from WYCKOFF and BOTSET (1936) and theoretical computations from IRMAY (1954). At a certain saturation (10 to 15 %) the ratio becomes zero while the water then only occurs in pores or wedges isolated from the general three dimensional network of water films and flow channels.

In fig. 2 the relation between moisture content Θ and moisture tension ψ is given for Yolo light clay (taken from MOORE (1939) as referred to by PHILIP (1957a). In the same figure the relation between Θ and k is given. From this figure it is evident that the conductivity shows a sharp decrease in the first stages of reduction of moisture content. The reason for this phenomenon is that the largest pores are emptied first when Θ decreases. Since the contribution to conductivity per unit area varies with the square of the pore radius, k may be expected to decrease much more rapid then Θ (PHILIP, 1957a). Further if Θ decreases the pores which have been emptied have to be avoided by the remaining paths of flow which therefore become more tortuous as water removal proceeds. If Θ proceeds to decrease further once continuity breaks down, there can be no more flow in the liquid phase (see fig. 1).

The flow of a fluid in an unsaturated porous system must obey the law of conservation of matter which is expressed in the equation of continuity which in turn expresses the fact that the differences between the rates of flow into and out of an element of the soil equals the rate of storage, thus

$$\frac{\partial v}{\partial x} = - \frac{\partial \Theta}{\partial t} \tag{2}$$

where Θ is the volume of water in unit volume of soil.

Substituting eq 1 into eq 2 we arrive at

$$\frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial x} (k \frac{\partial \varphi}{\partial x}) \tag{3}$$

where the potential φ is given by the equation

$$\varphi = p + h \tag{4}$$

in which p is the pressure expressed in terms of head of water and h is the height of the point considered relative to an arbitrary zero level. Eq 4 simply expresses the fact that the total potential is the sum of pressure potential and the gravitational potential.

In unsaturated soil p is negative. For vertical flow further $\frac{\partial h}{\partial z} = 1$. Designing the negative pressure by ψ eq 3 changes into

$$\frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial z} (k \frac{\partial \psi}{\partial z}) + \frac{\partial k}{\partial z} \tag{5}$$

where z is the vertical ordinate positive upward.

For horizontal flow $\partial h / \partial x = 0$ and eq 3 changes into

$$\frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial x} (k \frac{\partial \psi}{\partial x}) \tag{5a}$$

As is pointed out above, k may be considered to be a unique function of the moisture content. This is not really true when k is expressed as a function of the section since hysteresis complicates matters. In the latter case one ought to distinguish between sorption and desorption.

Introducing now the concept of the diffusion coefficient of water in soil the problem becomes analogous to the thermal diffusion of heat. This can be done by splitting the gradient of the pressure potential into two parts, hence $\frac{\partial \psi}{\partial z} = \frac{\partial \psi}{\partial \Theta} \cdot \frac{\partial \Theta}{\partial z}$ subject to the condition that the pressure potential is a unique function of the moisture content Θ. Then eq 5 may be written

$$\frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial z} (k \frac{\partial \psi}{\partial \Theta} \cdot \frac{\partial \Theta}{\partial z}) + \frac{\partial k}{\partial z} \tag{6}$$

and for horizontal flow

$$\frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial \psi}{\partial \Theta} \cdot \frac{\partial \Theta}{\partial z}\right) \tag{6a}$$

Both k and $\frac{\partial \psi}{\partial \Theta}$ (the tangent on the moisture characteristic of the soil) are now dependent on the moisture content and therefore the product is also a moisture-dependent property of the soil. Indicating this product by D (the diffusivity) eq 6 may be written

$$\frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial z}\left(D \cdot \frac{\partial \Theta}{\partial z}\right) + \frac{\partial k}{\partial z} \tag{7}$$

and for horizontal flow

$$\frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial x}\left(D \cdot \frac{\partial \Theta}{\partial x}\right) \tag{7a}$$

The transformation described above and used by various investigators (a.o. CHILDS and COLLIS-GEORGE (1950), PHILIP (1955a) KLUTE (1952)) provides an equation from which solutions in the form of profiles of Θ at given terms t may be obtained. For these solutions one first has to know how k varies with the moisture content. Secondly with this information the differential equation has to be solved subject to the initial and boundary conditions which are defined in the particular problem under discussion. The value of $\frac{\partial \psi}{\partial \Theta}$ may be determined from the moisture characteristic, k is either determined experimentally or calculated and the product D can easily be obtained.

Besides the movement of water in the liquid phase there can be a movement in the vapour phase and in the adsorbed phase. PHILIP (1955b) presents a curve showing the dependence upon moisture content of the total diffusivity and of the relative significance of the liquid and vapour components. For small and moderate values of the tension the vapour phase movement is of little importance (RICHARDS 1931). In so far the above mentioned equations are used to calculate the conductivity or the diffusivity, no error caused by the above phenomena will be made when the calculation is based on the total moisture movement, while the movement in the vapour and in the adsorption phase obey concentration-dependent diffusion equations similar to the first part of the right hand member of eq 7 (PHILIP, 1955a).

3. THE DETERMINATION OF THE HYDRAULIC CONDUCTIVITY OF UNSATURATED SOIL

As is pointed out above, first of all the relation between the hydraulic conductivity and the moisture content must be known for solving flow problems in unsaturated soil. In order to calculate the diffusivity the moisture characteristic must be available.

Otherwise the conductivity may be derived from D-values and the moisture characteristic of the soil.

There are various methods applied in order to determine the hydraulic conductivity of unsaturated soils. The principles of the methods described in literature are given below.

RICHARDS (1931) clamped a block of soil between two hollow fired clay cells in which different suctions were maintained which were measured by means of tensiometers. Using the rate of flow and the pressure gradient throughout the soil sample the conductivity can be calculated with the aid of eq 1. The same principle was used in a somewhat modified form by CHRISTENSEN (1944). Generally the suction will vary along the axis of the column and so also will the moisture content and subsequently the conductivity. The steady state velocity being the same everywhere throughout the column it follows from eq 1 that the potential gradient varies along the axis of the column, being high where the conductivity is low and vice versa (CHILDS, 1958).

The exact conductivity at a given moisture content must be obtained by measuring the potential gradient at the point where that moisture content prevails. The method of RICHARDS, therefore, only can be used for very small differences in suction between the ends of the column, such that the conductivity may be considered to be constant. In this case the rate of flow will be very small and the measurement of it and of the suction must be very accurate.

Taking the case, that water at a constant rate is flowing down a sufficient long column to a water table maintained at a constant level below the surface, that a sufficient time has elapsed for the rate of flow to be the same everywhere in the profile so that no further changes of Θ take place, a steady state of flow is obtained. According to eq 3 and eq 4, the equation of flow is given by

$$\frac{\partial}{\partial z}\left\{ k\left(\frac{\partial \psi}{\partial z} + 1\right)\right\} = 0 \qquad (8)$$

Then the first stage of integration gives

$$k\left(\frac{\partial \psi}{\partial z} + 1\right) = k_o \qquad (9)$$

where k_0 is the constant of integration. Eq 9 simply gives an expression of DARCY'S law where k_0 is the rate of flow down the column. Taking ψ a positive value, eq 9 changes into

$$\frac{\partial \psi}{\partial z} = 1 - \frac{k_o}{k} \qquad (10)$$

Integrating between the limits of the water table (where both z and ψ are zero) and z (where the suction is ψ) gives

$$z = \int_o^{\psi} \frac{d\psi}{(1 - k_o/k)} \qquad (11)$$

CHILDS and COLLIS-GEORGE (1950) now utilized the fact that in the case described above, the moisture content and suction are uniform over an appreciable length. Zones of variable moisture content (suction) are located at the lower end of the column to the neighbourhood of the water table in a way which depends upon the pore size distribution and at the upper end to a zone in which is localized any intermittency of water supply. In the zone of uniform moisture content there is only a pressure gradient due to gravity. The moisture content adjusts itself to provide the necessary permeability to conduct the imposed flow with the gravitational potential gradient. For the computation of the conductivity one needs only to measure the rate of flow and the cross sectional area of the column. For the calculation itself eq 10 is used with $\frac{\partial\psi}{\partial z} = 0$. Gradients different from unity may be obtained by sloping the column. The uniform moisture content lends itself to estimation by indirect electrical methods (capacity or heat conduction).

MOORE (1939) used a steady surface evaporation rate E_0 to compute values for k. He made use of cylindrical tubes with a length of 90 to 120 cm and 20 cm diameter. At the lower end a constant water table was maintained and the suction along the column was measured by means of tensiometers. The conductivity then can be calculated by eq 11, if $- E_0$ is substituted for k_0. The same principle was used by WIND (1955) in his field investigations.

On the basis of the assumption that k (ψ) and $\frac{d\psi}{d\Theta}$ are constant over a small range of potential, one can derive the equation for horizontal flow:

$$\frac{\partial\psi}{\partial t} = \overline{D}\,\frac{\partial^2\psi}{\partial x^2} \tag{12}$$

when $\overline{D}$ is a mean value of D. This equation has the form of an ordinary diffusion equation and solutions are readily available for a wide variety of boundary and initial conditions (cf CARLSLAW and JAEGER, 1947). Eq 11 was used by GARDNER (1956) for the measurement of the permeability as a function of ψ but also the diffusion function may be obtained from it. The system used by GARDNER consisted of a soil sample in a pressure membrane apparatus. The outflow of water as a function of time is measured and from this it is possible to evaluate the permeability. By repeating the measurements over a succession of small increments of potentials ($\psi_2 - \psi_1$) a series of permeability-potential values is obtained.

BRUCE and KLUTE (1956) used a horizontal semi-infinite column with initial constant moisture content. Water was applied at the end $x = 0$ by maintaining this end at saturation. Water now will flow into the column and assume a distribution conditioned by the nature of the diffusivity-moisture content function. The initial and boundary conditions are

$$\begin{array}{lll} \Theta = \Theta_i & 0 < x < \infty & t = 0 \\ \Theta = \Theta_s & x = 0 & t > 0 \end{array} \tag{13}$$

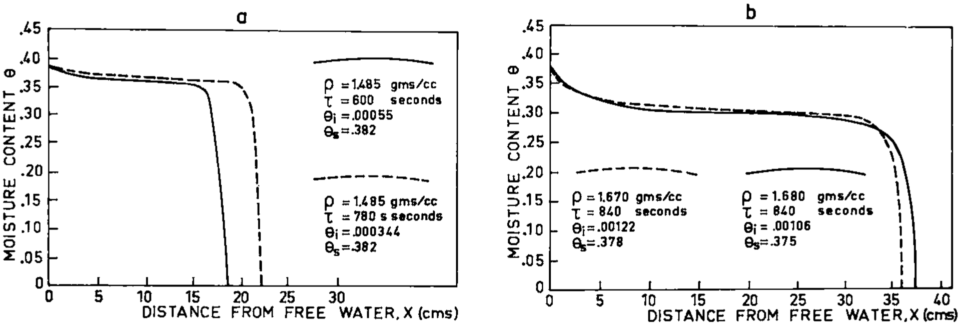


FIG. 3.
Moisture content-distance curves during infiltration into horizontal columns of 75 μ glass beads (a) and 50 to 250 μ Bloomfield sand (b) used by BRUCE and KLUTE (1956) for the computation of the diffusivity

where Θ_i is the initial moisture content and Θ_s the moisture content at saturation. The fundamental differential equation for this case is given by eq 7a. Assuming that the solution of the last mentioned equation is a function of the variable $\varphi = xt^{-\frac{1}{2}}$ (see section 5) one can derive the solution for eq 7a subject to the boundary conditions (11) (BRUCE and KLUTE, 1956)

$$D(\Theta_x) = -\frac{1}{2t}\left[\frac{\partial x}{\partial \Theta}\right]_{\Theta_x} \int_{\Theta_i}^{\Theta_x} x \, . \, \partial\Theta \tag{14}$$

The integral in this equation is evaluated from a moisture content-distance curve for the flow system described above (fig. 3). The derivate $\left[\frac{\partial x}{\partial \Theta}\right]_{\Theta_x}$ is the slope of this curve evaluated at Θ_x and t is the time at which the distribution is measured.

Besides the experimental determinations of the hydraulic conductivity of unsaturated media, calculations have been carried out based on the KOZENY-type equations for saturated flow (cf HOOGHOUDT, 1934). The simplest type of equation is that derived by IRMAY (1954)

$$\frac{k_u}{k_s} = \frac{[S - S_o]^3}{[1 - S_o]^3} \tag{15}$$

where k_u and k_s are the unsaturated and saturated conductivity respectively and S is the fraction of pores filled with water. Further S_o is the saturation at which k_u becomes zero (see also fig. 1). The equation agrees very well with experiments from WYCKOFF and BOTSET (1936).

CHILDS and COLLIS-GEORGE (1950) developed a method to calculate the conductivity based on the moisture characteristic of the soil. Here both the pore size distribution and the total pore space are taken into account. In order to get the real values of k a so-called "matching factor" must be known from experiments. The

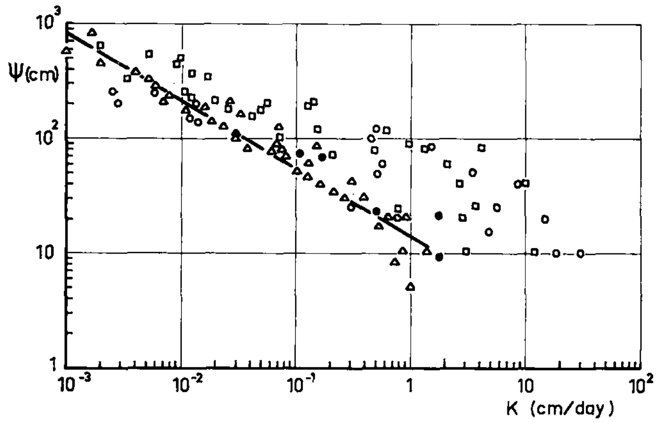


FIG. 4.
Relation between the hydraulic conductivity k and moisture tension ψ for sandy soils (o), loam (□), clay loam and clay (Δ), and peat (•)

results obtained by this method for glass beads, agree very well with those obtained by experiments carried out by BRUCE and KLUTE (1956).

In fig. 4 taken from WESSELING (1957) the results of various experiments concerning the relation between hydraulic conductivity and suction are brought together. The data are divided into four groups ranging from heavy to light soils. The figure suggests that for $\psi > 20$ cm the data obey the equation:

$$k = a\,\psi^{-n} \tag{16}$$

where a and n are constants and n ranges from 1 to 3 dependent on the heaviness of the soil. Recently GARDNER (1958) suggested the equation

$$k = a\,[\psi^{n} + b]^{-1} \tag{17}$$

where a, n and b are constants. GARDNER reports some steady state solutions of capillary rise from a water table with $n = 1, {}^3/_2, 2, 3$ and 4 which are similar to that suggested by VISSER (1957).

Although the relation between k and ψ can be given in a rather simply algebraic equation this is not the case for D and Θ. This is due to the fact that there is no simple relation between ψ and Θ. As will be shown below, this is a serious difficulty in solving flow equations in unsaturated media.

4. INFILTRATION EXPERIMENTS

When water is applied in excess at the top of a soil column which has a uniform moisture content, the water will enter into the soil with a velocity dependent on the physical properties of the soil. The entry of water into a soil column is demonstrated by fig. 5 taken from MILLER and RICHARDS (1952). The infiltration at zero time is infinite, and decreases to a constant value when time proceeds.

When the moisture profile is determined at varous time intervals a picture similar to that given in fig. 6 (MILLER and RICHARDS, 1952) will be obtained. The suction-

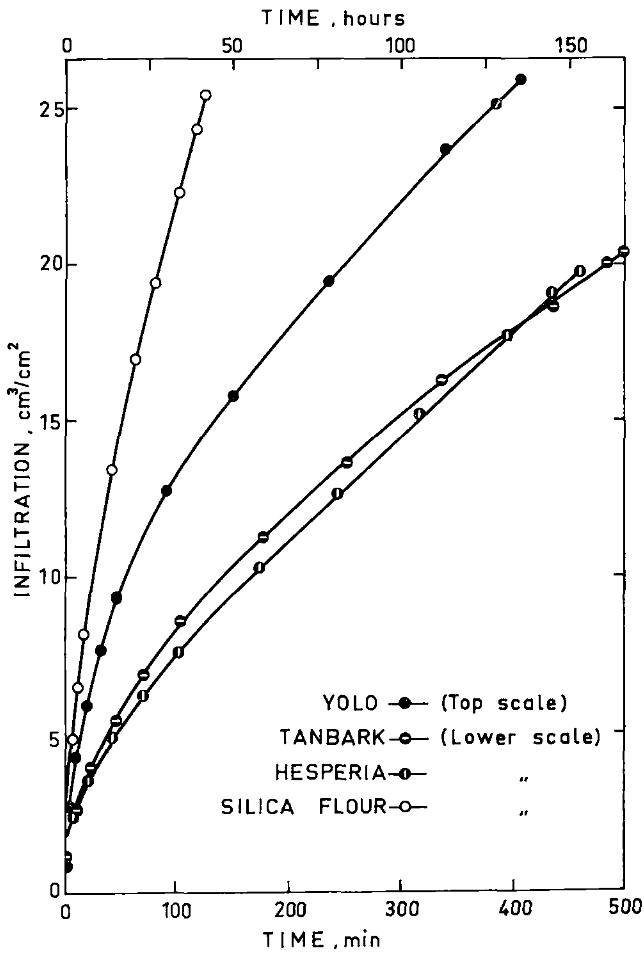


FIG. 5.
Infiltration into four soil types according to experiments of MILLER and RICHARDS (1952)

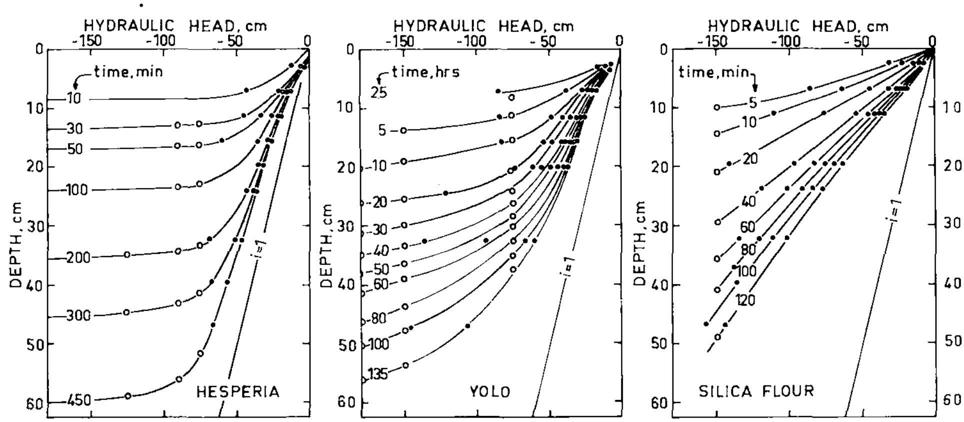
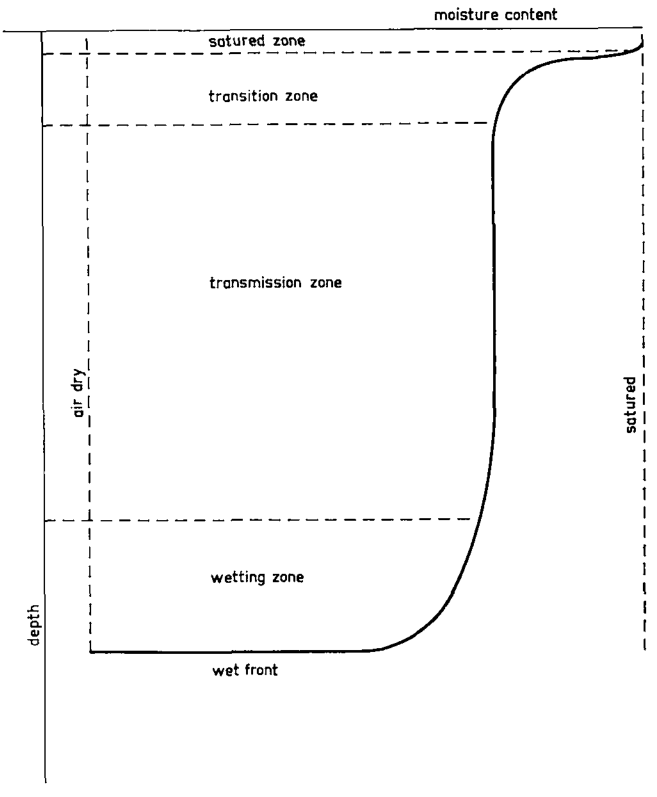


FIG. 6.
Moisture tension profiles during infiltration into Hesperia sandy loam, Yolo loam and silica flour after experiments of MILLER and RICHARDS (1952)

FIG. 7.
Infiltration zones of BODMAN and COLMAN (1943)



moisture profile gradually changes but later on this profile becomes nearly constant and moves downward with a constant velocity, indicating a constant infiltration rate as is shown by fig. 5.

In their classical work on infiltration BODMAN and COLMAN (1943) and COLMAN and BODMAN (1944) distinguished a number of zones in the soil column during infiltration. These zones are schematically given in fig. 7. From above to below this figure shows:

a) *the saturated zone* reaching a depth of about 1.5 cm,

b) *the transition zone* extending to a depth of about 5 cm in which a rapid decrease in moisture occurs,

c) *the transmission zone* increasing in length as infiltration proceeds in which Θ is nearly constant,

d) *the wetting zone*, a region of fairly rapid change of Θ,

e) *the wet front* or *wetting front*, a region of very steep moisture gradients which represents the visible limit of moisture penetration into the soil.

Evidently water is applied to the wetting front from the saturated surface across the transmission zone in which the moisture content and hence the conductivity can be considered to be constant. Making use of this assumption VAN DUIN (1955, 1956) derived an equation for the advance of the wetting front. Ignoring the water depth over the surface and assuming that the flow is caused only by capillary forces and by gravity, the advance of the wetting front is given by

$$\frac{dz}{dt} = \frac{k_t}{\mu} \cdot \frac{z + \psi}{z} \tag{18}$$

where k_t is the hydraulic conductivity, μ is the storage capacity of the transmission zone and z is the depth of the wetting front.

Integrating eq 16 with the initial conditions $z = 0$, $t = 0$ and assuming the suction at the wetting front to be constant gives

$$t = \frac{\mu}{k_t} \left[z - \psi \ln \frac{z + \psi}{\psi} \right] \tag{19}$$

Without introducing large errors, the moisture content throughout the profile may be considered to be constant. Consequently μ is nearly constant and the cumulative infiltration into the soil is given by $i = \mu . z$. Solving now μ from eq 19 and substituting this value in the last mentioned equation gives

$$i = k_t . t \left[1 - \frac{\psi}{z} \ln \frac{z + \psi}{\psi} \right]^{-1} \tag{20}$$

With increasing moisture content of the soil the suction ψ is decreasing and in wet soils $\psi \rightarrow 0$. In the case of an initially wet soil eq 20 reduces to DARCY's law for unit potential gradient

$$i = k_t . t \tag{21}$$

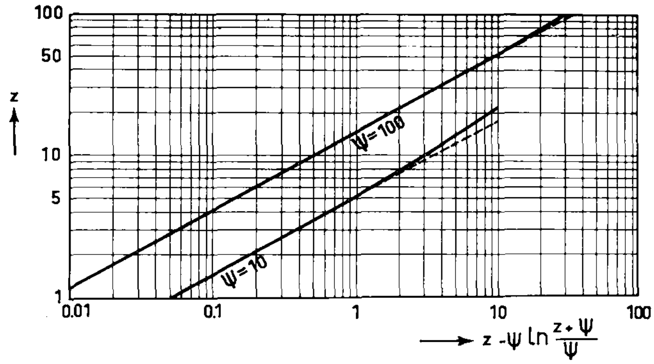
Eq 19 also holds true when $z \gg \psi$. So after a certain period of infiltration which will be shorter the higher the initial moisture content of the soil, the infiltration rate will become constant as is shown in fig. 5. This phenomenon is used to determine the hydraulic conductivity of the soil by means of an infiltrometer. While the suction ψ in the wetting zone is very small, the conductivity nearly equals the saturated value.

Considering eqs 18 and 20 it is clear that both the advance of the wetting front and the infiltration capacity are dependent on the initial moisture content of the soil. In infiltration experiments it is found that in wet soils the velocity of the wetting front is greater but the infiltration capacity is much smaller than in dry soils.

With respect to drainage problems VAN DUIN (1956) divides soils into two groups:

1. Soils with good permeability in which the storage capacity is large even at field capacity. In these soils the constant infiltration rate will be reached after a relatively long time and therefore the infiltration capacity is large. This group contains the coarser textured soils in which drainage gives no difficulties.

FIG. 8.
The relation between z and p according to eq 22 after VAN DUIN (1955)



2. Soils with low hydraulic conductivity and a small amount of non-capillary pores in which the infiltration velocity will reach its constant value after a very short time. In this group of heavy and dense soils difficulties may be expected while the infiltration capacity is small.

Various investigators tried to describe infiltration of water into the soil with rather simple algebraic equations. A review of these investigations is given by PHILIP (1957a).

When eq 19 is changed into

$$z - \psi \ln \frac{z + \psi}{\psi} = \frac{k_t \, . \, t}{\mu} = p \tag{22}$$

and p is plotted against z (fig. 8), the relation

$$\log z = \log a + 0.55 \log p$$

or

$$z = ap^{0.55} \tag{23}$$

can be derived (VAN DUIN, 1955) holding for $0 \leqq z \leqq 0.5\ \psi$. Now a is a constant dependent on the physical properties of the soil.

Substituting eq 23 in $i = \mu z$ gives

$$i = \mu a \left[\frac{k_t \, . \, t}{\mu} \right]^{0.55} \tag{24}$$

which agrees with the infiltration equation proposed by KOSTIAKOV (1932)

$$i = bt^{\alpha} \tag{25}$$

where b is assumed to depend on the hydraulic conductivity of the saturated soil and α is about 0.5.

As is pointed out above, the equation of KOSTIAKOV is only valid for the first stage of infiltration where $z \leqq 0.5\ \psi$. Further for $t \to \infty$ the value $\alpha = 1$ must occur so that the parameters in eq 25 are not constant and the extent of the equation is rather limited.

HORTON (cf PHILIP, 1957b) proposed the equation

$$V = V_{\infty} + [V_o - V_{\infty}]\, e^{-\beta t} \qquad (26a)$$

or

$$i = V_{\infty} t + \frac{1}{\beta} [V_o - V_{\infty}]\, [1 - e^{-\beta t}] \qquad (26b)$$

where V_o is the initial value, V_{∞} is the final value of V and β is a constant. While $V_{\infty} = k_t$, the equation is incapable to describe the rapid decrease of V at small values of t. It will describe the infiltration better the longer the time ranges. Nevertheless three parameters are necessary.

PHILIP (1954) proposed an equation similar to that of VAN DUIN (eq 19)

$$t = \frac{1}{k_t} \left\{ i - \beta \ln \left[1 + \frac{i}{\beta} \right] \right\} \qquad (27)$$

where β is a function of k_t and the initial- and saturated moisture content of the soil.

Recently PHILIP (1957b) derived the equation

$$i = S t^{\frac{1}{2}} + A t \qquad (28)$$

where S is a function of the initial and saturated moisture content and A is a parameter which is related to the analysis developed in PHILIP's infiltration theory (cf section 5).

PHILIP (1957b) compared the results of the above mentioned infiltration equations with those obtained from a detailed analysis of infiltration into Yolo light clay. The results are given in table 1. The KOSTIAKOV equation fits moderately well but the equation of HORTON yields bad results despite its three parameters. Further the parameters are highly dependent on the time for which the infiltration is computed except for the case of eq 28.

TABLE 1. Comparison of infiltration equations according to PHILIP (1957b)

Method of computation	for t = 10^5 sec		for t = 10^6 sec	
	i cm	% error	i cm	% error
Detailed analysis	4.447	0	18.670	0
HORTON (eq 26)	8.147	+82	29.412	+58
KOSTIAKOV (eq 25)	4.225	— 5.6	15.395	—18
PHILIP (eq 27)	4.448	— 0.56	18.206	— 2.6
PHILIP (eq 28)	4.449	— 0.63	17.753	— 4.9

5. Theoretical solutions of infiltration into a soil column with constant initial moisture content

In this section a brief review will be given of the recent development of the theory of infiltration and the possibilities of computing the rate of infiltration from known physical properties of the soil.

As is pointed out in section 2 the flow of water in unsaturated soils can be described by diffusion equations with concentration-dependent diffusivity. With respect to the solution of these equations, difficulties arise while the function D (Θ) cannot be expressed in a simple analitical form although both D and k may be considered to be unique functions of the moisture content. Therefore numerical methods should be applied.

The simplest case is represented by the movement of water in a horizontal column for which eq 7a holds

$$\frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial x}\left[D.\frac{\partial \Theta}{\partial x}\right] \tag{7a}$$

When the column is semi-infinite, initially at moisture content Θ_n and subsequently plane $x = 0$ is maintained at moisture content Θ_o, the initial and boundary conditions are

$$\begin{array}{lll} \Theta = \Theta_n & t = 0 & x > 0 \\ \Theta = \Theta_o & x = 0 & t \geqq 0 \end{array} \tag{29}$$

For $\Theta_o > \Theta_n$, (7a) and (29) describe the wetting up of a column. The removal of water by the application of a constant suction at plane $x = 0$ is indicated by $\Theta_o < \Theta_n$.

KLUTE (1952) used a numerical method due to CRANK and HENRY (1949) to solve eq 7a subject to (29). PHILIP (1955b) developed a new iterative procedure for which convergence is more rapid and each iterative step is simple. Both authors make use of BOLZMANN'S (1894) transformation

$$\varphi = xt^{-\frac{1}{2}} \tag{30}$$

in order to reduce eq 7a to an ordinary differential equation. Substituting eq 30 into 7a gives

$$-\frac{\varphi}{2}\frac{d\Theta}{d\varphi} = \frac{d}{d\varphi}\left[D\,.\,\frac{d\Theta}{d\varphi}\right] \tag{31}$$

subject to the conditions

$$\begin{array}{ll} \Theta = \Theta_o & \varphi = 0 \\ \Theta \rightarrow \Theta_n & \varphi \rightarrow \infty \end{array} \tag{32}$$

which implies $\quad \Theta \rightarrow \Theta_n\,,\ \dfrac{d\Theta}{d\varphi} \rightarrow 0$

In brief both methods give a numerical solution of the relation between Θ and φ.

In the case of PHILIP this is obtained by making the independent variable by multiplying both sides of eq 31 with $\frac{d\varphi}{d\Theta}$

$$-\frac{\varphi}{2} = \frac{d}{d\Theta}\left[D \cdot \frac{d\Theta}{d\varphi}\right] \tag{33}$$

Then integration leads to

$$\int_{\Theta_n}^{\Theta} \varphi \, d\Theta = -2\,D\,\frac{d\Theta}{d\varphi} \tag{34}$$

subject to the conditions

$$\Theta = \Theta_o\,, \quad \varphi = 0 \tag{35}$$

By PHILIP (1955b) a method is developed to obtain the solution of (34) subject to (35) by means of an iterative procedure.

Water movement in a vertical column is described by eq 7

$$\frac{\partial\Theta}{\partial t} = \frac{\partial}{\partial z}\left[D\frac{\partial\Theta}{\partial z}\right] + \frac{\partial k}{\partial z} \tag{36}$$

When the column is semi-infinite with $z \geqq 0$ and initially at moisture content Θ and subsequently plane $z = 0$ is maintained at moisture content Θ_o, the conditions governing eq 36 are the same as for horizontal flow given by (29). For the case $\Theta_o > \Theta_n$ eq 36 and eq 29 describe capillary rise in the column. For $\Theta_o < \Theta_n$ the equations describe drainage of the column by the application of a constant suction at its base.

In the special case of infiltration water is applied at the top of the column. Taking now $x = -z$ the infiltration is governed by

$$\frac{\partial\Theta}{\partial t} = \frac{\partial}{\partial x}\left[D\frac{\partial\Theta}{\partial x}\right] - \frac{\partial k}{\partial x} \tag{37}$$

where x is now the vertical ordinate positive downward.

Using an iterative procedure similar to that described above PHILIP (1957b) derived the solution for eq (37) subject to (29)

$$x = \varphi t^{1/2} + \chi t + \psi t^{3/2} + \omega t^{2} + \;\ldots. \tag{38}$$

where the coefficients φ, χ, ψ, ω .... are all function of Θ. They are solutions of a series of ordinary differential equations similar to eq 34. YOUNGS (1957) compared the results of this equation with those from infiltration experiments and found a very good agreement.

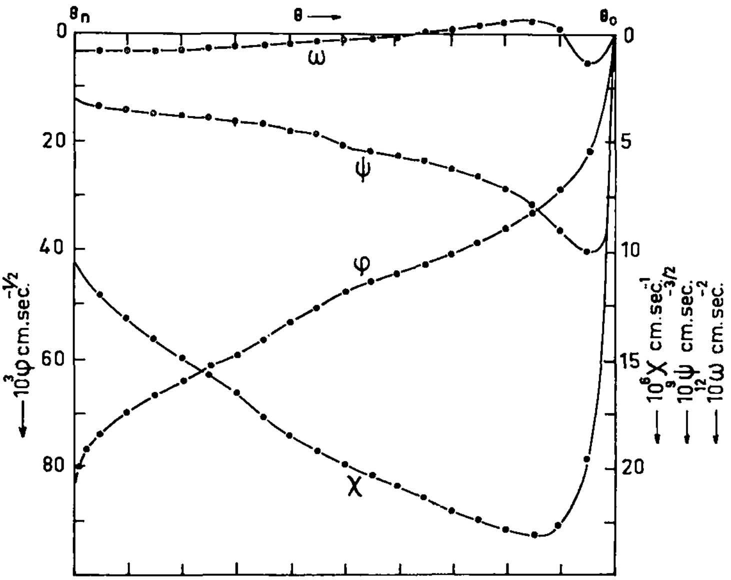


FIG. 9.
Functions φ, χ, ψ and ω for infiltration into Yolo light clay according to PHILIP (1957c)

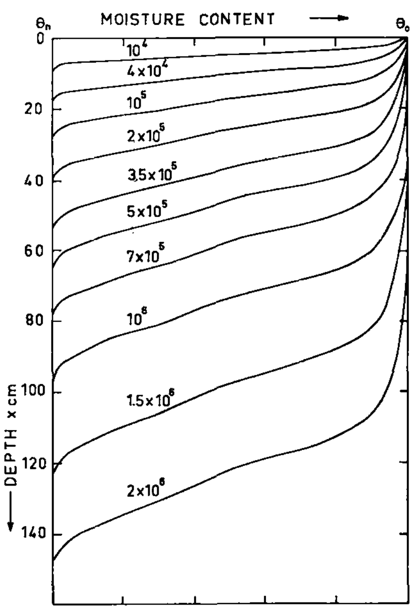


FIG. 10.
Computed moisture profiles during infiltration into Yolo light clay. The number on each profile represents the time in seconds at which the profile is realized (PHILIP, 1957a).

The series of eq 38 converges rapidly even for large t and only three or four terms have to be taken. The solution of φ, χ, ψ and ω for the infiltration into Yolo light clay is given in fig. 9. With the aid of these values the moisture profiles at various

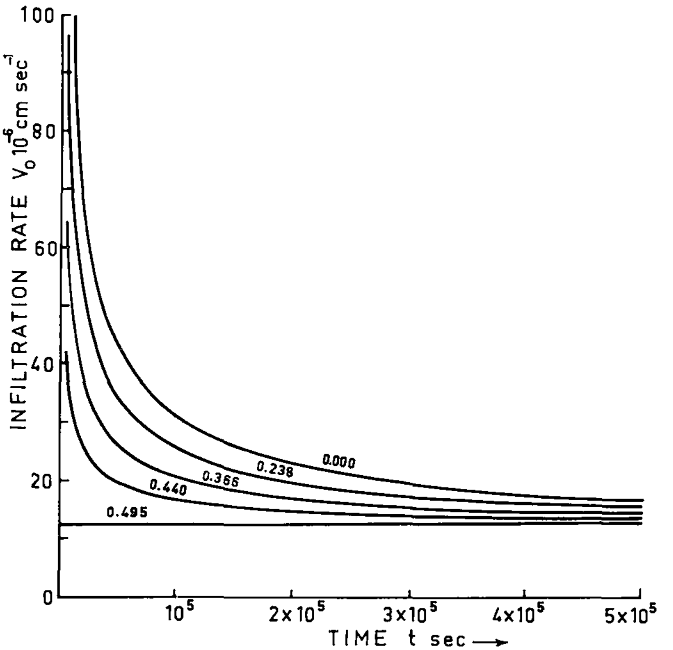


FIG. 11.
The influence of the initial moisture content of the soil on the infiltration rate after PHILIP (1957a). The numbers on each curve represent the corresponding initial moisture content.

times may be computed as is shown by fig. 10 in which the results up to $t = 2.10^6$ sec are taken from fig. 9. For larger times an other solution must be used while eq 38 does not converge for $t > 2.10^6$ even if one takes more coefficients (PHILIP, 1957c) but this is not of interest in our problems.

From fig. 10 it is evident that for small times the moisture profiles maintain a constant shape with x increasing as $t^{\frac{1}{2}}$. This indicates that the infiltration during the first stage is indistinguishable from the adsorption into a horizontal column. This is what one may expect while the potential gradients due to gravity are negligible compared to those due to capillarity. Further fig. 10 shows the same phenomena as those described in section 4.

From the theory developed by PHILIPS it is possible to determine the influence of the initial moisture content of the soil on infiltration rate. The result of such a computation taken from PHILIP (1957d) is given in fig. 11. From this figure it is clear that the infiltration rate decreases rapidly with increasing initial moisture content as is described at the end of section 4.

In sections 4 and 5 only the case of a continuous infiltration into the soil has been considered. In drainage problems this problem will only occur when rainfall intensity is larger than the infiltration rate and pools will be formed on the soil surface. After some time these pools will disappear and a certain redistribution of moisture in the soil profile will occur. A second possibility is that rainfall intensity is smaller than the infiltration capacity of the soil.

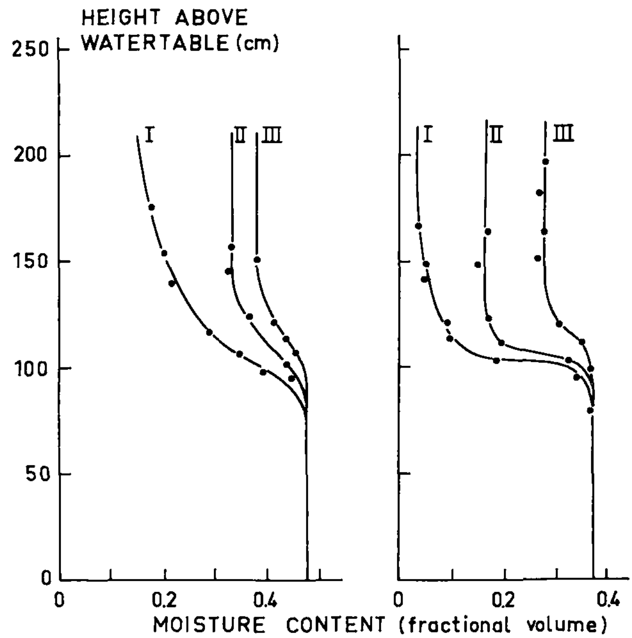


FIG. 12.
Equilibrium moisture profiles for infiltration at a constant rate into columns which drain to a water table.
Left: slate dust (I) Q = o, (II) Q = $\rho g k_o/10$, (III) Q = $\rho g k_o/5$;
Right: "Ballotini" grade 15 (I) Q = o, (II) Q = $\rho g k_o/25$, (III) Q = $\rho g k_o/5$.
• computed; — experimental. (After YOUNGS, 1957).

The case of water infiltration at a constant rate into a column which drains to a water table is dealt with by YOUNGS (1957). The infiltration into an artificial soil consisting of slate dust a material containing particles ranging in size from 0.04 mm to 0.125 mm and in "Ballotini", a soil consisting of glass beads less than 0.1 mm in diameter is given in fig. 12. The figures I, II and III near the curves indicate infiltration rates increasing from zero, so the curves I indicate the moisture characteristics.

6. REDISTRIBUTION OF MOISTURE

As is pointed out in section 2 the diffusion coefficient exhibits hysteresis. For problems involving only wetting, such as continuous infiltration, the diffusion coefficient obtained from the sorption moisture characteristic may be used, while in the case of drainage the computation has to be based on the desorption curve. During redistribution of soil moisture there will be a draining part and a sorption part in the soil as is shown by fig. 13 taken from YOUNGS (1958a). Here again the points in the figure are based on experiments while the full drawn curves indicate computation.

For redistribution of moisture after infiltration into vertical columns, YOUNGS (1958b) gives some experimental results which are given in figures 14 and 15 for slate dust and "Ballotini" respectively. In both figures profiles are given for two initial times of infiltration. The general shape of fig. 13 agrees with the curves of fig. 14. In both curves a very sleep moisture gradient occurs at the wetting front. In the left hand part of fig. 14, with small depth of infiltration, only a small amount of water is

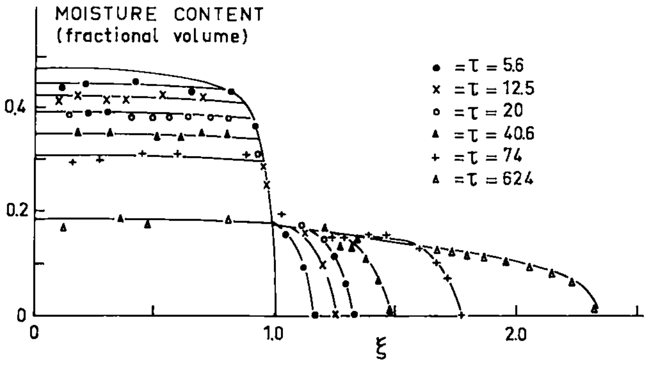


FIG. 13.

Moisture profiles during the redistribution of moisture after horizontal infiltration into slate dust. In this figure, $\tau = t/T$ were t is the time after the start of infiltration and T is the time of infiltration, $\xi = x/X$ where x is the distance measured from the origin and X is the value of x at the wetting front (after YOUNGS, 1958a).

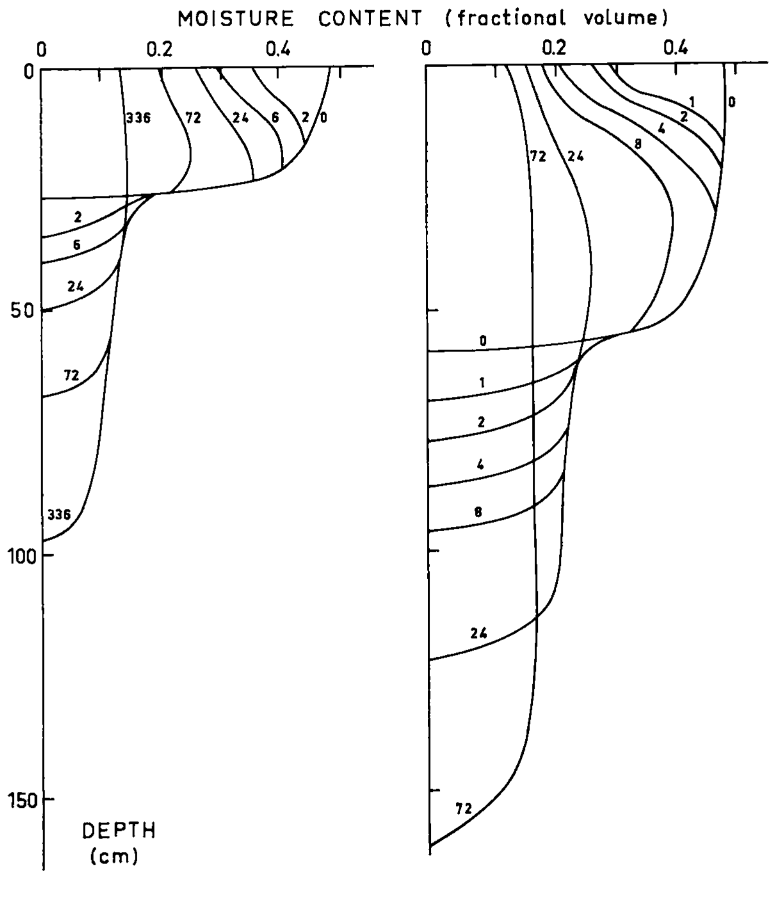


FIG. 14.

Moisture profiles during the redistribution of moisture after infiltration in the vertical direction into slate dust. The figures on the curves indicate the time in hours after the cessation of infiltration. For the left hand curve the time of infiltration is smaller than for the right hand curve (after YOUNGS, 1958b).

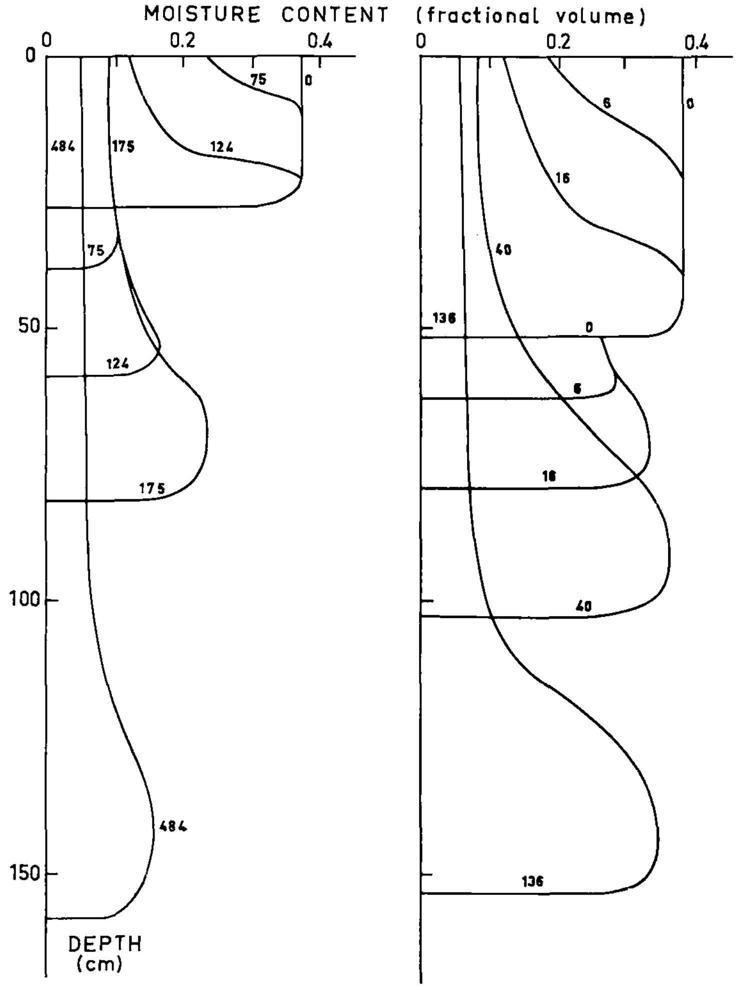


FIG. 15.
The same as fig. 14 for "Ballotini" grade 15

draining to deeper parts of the column. This amount is increasing with increasing depth of infiltration. On the other hand the time at which a constant moisture content (field capacity) throughout the profile is established again, is decreasing with increasing depth of infiltration.

During redistribution of moisture in "Ballotini", which may be compared with a sandy soil, the moisture profiles have a shape far different from those in slate dust which is comparable with a clay soil. Here again the influence of the depth of infiltration on redistribution is obvious.

The shape of the moisture profiles during redistribution will depend on what happens at the transition from the sorption curve into the desorption curve, thus on the measure in which D exhibits hysteresis. For full details of these phenomena the author refers to YOUNG's article.

Concerning the redistribution of moisture after infiltration into soils which drain to a water table, no investigations are available yet. The same holds for infiltration into soils consisting of two or more layers. Both subjects are however, very interesting features in drainage problems.

Summary

The flow of water in unsaturated soil is an important factor in plant growth, in moisture changes in the soil and in infiltration- and drainage problems. The laws governing this flow have been discussed. Introducing the concept of diffusion of soil moisture, the governing differential equations are diffusion equations with concentration-dependent diffusivity. Some methods to determine the diffusivity or the hydraulic conductivity of unsaturated soil have been discussed.

A review has been given of various infiltration experiments concerning both steady and unsteady state flow and of the algebraic equations derived from them. Further, the theoretical solutions of the infiltration into soil columns with constant initial moisture content derived by KLUTE and PHILIP have been dealt with.

In drainage problems, we are concerned with the infiltration of water towards a water table, with a changing initial moisture content and with the redistribution of soil moisture after infiltration. Attention has therefore been given to the phenomena occurring during the process of infiltration draining to a water table in soil columns and to those occurring during the redistribution of soil moisture.

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