Floquet theory and economic dynamics (extended version)

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Abstract

Floquet theory is an appropriate tool for studying ordinary linear recurrence and differential equations with periodic coefficients, and is a generalization of the theory for constant coefficients. Floquet theory has still not found its way into economics, although it seems to be relevant for economic dynamics. As well as a discussion of this relevance and an illustration of it in the context of the Samuelson-Hicks multiplier-accelerator model, this article contains an appendix that provides a quite complete exposition of Floquet theory for recurrence equations.

Key words and phrases: Samuelson-Hicks model, shock-dependency, linear recurrence equations, Floquet theory. JEL Codes: B4, C6, E3.

1 Introduction

Many models on economic dynamics are stated in terms of recurrence or differential equations. In such a model, it is usual to be interested in qualitative properties of solutions such as asymptotic, oscillatory and steady state behaviour, in their quantitative properties such as explicit expressions for solutions and in which way solutions can be controlled. Various qualitative and quantitative properties can especially be mastered in case of ordinary linear equations with constant coefficients. One knows (see, for example, [2, 12, 16]) that in this context characteristic roots play a major role. But, much less known is the fact that, in case of periodic coefficients, quite analogous results hold for such equations and that this much more general case is only slightly more difficult to handle if one uses an appropriate setting. The essential idea for such a generalization was given by Floquet [5] more than hundred years ago, and this has grown into what today is called 'Floquet theory' (see, for example, [2, 6, 11, 12, 17–19, 21]). The resemblance between constant and periodic coefficients theory is because, loosely speaking, an equation with periodic coefficients can be transformed by a substitution of variables into an equation with constant coefficients.

Floquet theory has found several applications in physics (see, for instance, [18]). However, as far as we know this theory has still not found its way into economics (nor, for example, into biology), although we can imagine it may be of interest to it, particulary for trade cycle and growth theory.[1] Indeed, as far as we know, the

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[1] However, for quite modest attempts see [21] and references therein.

standard growth and trade cycle models have not been analysed in the context of periodic coefficients.

Periodic coefficients form for (trade cycle and growth models) an interesting generalization of constant ones because they allow the inclusion of explicitly periodicity aspects. The possible consideration of periodic coefficients can already be found in the article [20] of Samuelson, in the sense of periodic government expenditures in the context of his multiplier-accelerator model.[2] Once there is a model with constant coefficients, it may be desirable to investigate whether a model with periodic coefficients is preferable. In this context we speak of 'Floquetization of a model'. Floquet theory provides some analytical tools for the analysis of such models. In this article we floquetize the Samuelson-Hicks model. In fact, the Floquet theory is appropriate to dealing with equations where all coefficients are Floquetian of the same type: we call c_t 'Floquetian of type (q, z)' if

$$c_{t+q} = zc_t$$

for all t. So, Floquetian of type $(q, 1)$ is nothing other than q−periodicity. Floquetian coefficients are thus a generalization of periodic ones. Floquetian coefficients also allow the inclusion of 'trend' aspects.

The organisation of this article is as follows. Section 2 contains a discussion of the value of Floquet theory for economic dynamics and section 3 contains an illustration of it in the context of the Samuelson-Hicks multiplier-accelerator model.[3] After the formulation of this model we consider it for, among other things, asymptotic behaviour of motions, periodic motions and shock-independency.[4] Because (the scarce) references for Floquet theory found in the mathematical and physical literature are not so appropriate for economics, we have added an appendix that provides a self-contained modern and almost complete exposition of Floquet theory for vector and scalar linear recurrence equations in a setting that may be appropriate for economics.[5] We hope that this appendix also may of interest for itself.[6]

We hope that this article may serve as a starting point for further research.

2 Floquet theory and economic dynamics

We now give some fundamental reasons why periodic coefficients may be useful.

[1. Status of constant coefficients.] In economic dynamic models, coefficients are often taken to be time-independent (i.e. constant). This may be more unrealistic the longer the time horizon (i.e. duration). Observations such as preferences may be time dependent are an argument for allowing non-constant coefficients.[7] After constant coefficients, periodic coefficients may be considered as the simplest ones. It is not for nothing that econometric articles, like [7] on periodic autoregressive time series models, allow for periodic coefficients. However, there seems to be a gap between econometric and non-econometric studies dealing with periodic coefficients equations.[8] One question to be asked is why constant coefficients (even in linear

[2] But this only leads to periodicity of the coefficients in the right-hand side of the equation.

[3] The results in subsection 3.3 are mathematically rigourous.

[4] The work we present here is a new version of [21]. As well as an amelioration of the overall presentation, it also contains new results.

[5] In fact the appendix covers more material than is necessary for sections 2 and 3. In [21] only the second order scalar case was considered.

[6] We fully realize that the appendix is straightforward from a mathematical point of view. However, we do not know a modern coherent presentation of Floquet theory like ours.

[7] Think for example on influences due to winter and summer. However, dealing with aggregate quantities, as in macro-economics, may weaken such influences.

[8] In the rest of this article we will not deal with econometric aspects.

models) remain nevertheless so popular. We collected three types of answers: 'It is tradition'; 'I don't know how to handle non-constant coefficients'; 'The real-world data are so bad such that such a refinement is not appropriate'. However, only the last answer makes sense but does not eliminate the necessity for a theory that can handle more general coefficients. In case of an infinite time horizon, a constant coefficients economic model has a good chance of having even an 'absurd' real-world interpretation. Indeed, in this case, such coefficients have in some way the appearance of constants of nature, as they appear in physics, which is very unrealistic for (contemporary) economics.[9] But even in physics there are only a few such constants. One of the reasons for dealing with infinite time horizons is the desire to discuss stability questions, notions that are primarily developed in the context of such time horizons. So, we think that it is especially for (mathematical) convenience that infinite time horizons are often dealt with.[10]

[2. Problem of too regular motions.] In reality, a wide variety of fluctuating motions[11] can be found that cannot be explained by economic growth and trade cycle models, without policy instruments or stochastic exogenous influences[12] described by low order linear equations with constant coefficients. The problem of too regular motions may not only be 'solved' by introducing stochastic exogenous influences, but also by using non-linearities in the model. In the latter case, even chaotic dynamical behaviour may occur (see [4] for a recent overview). If linear equations are allowed to be of arbitrary order (with constant coefficients) or to have non-constant like periodic coefficients, then quite 'irregular' motions can occur even with such equations.[13] In (the appendix) of this article we even combine the arbitrary order feature with the periodic feature.[14]

[3. Shortcoming of shock-dependent trade cycle models.] In [8, page 41] a trade cycle model is called 'shock-dependent' if (for reasonable values of the parameters) the generation of cycles relies on an impetus which is not explained in itself by the model. We have eliminated some vagueness in this definition of shock-dependency by restating it as follows. We call a trade cycle model[15] (with infinite time-horizon) 'shock-dependent' if a necessary condition for the existence of a cycle (for the main endogenous variable) is that policy instruments[16] that the model allows for are applied, or that stochastic exogenous influences are present. In this article we call a motion a 'cycle' if it is bounded and does not tend to a constant.[17] Moreover we

[9]Even, we consider, contrary to the usual opinion, an infinite time horizon in an economic model as very unrealistic, independent of the fact whether the model deals with constant or with non-constant coefficients, and independent of what type of real-world issue it discusses: usual economic agents still have finite lives, society changes, and recent evidence on the decay of protons suggests that the universe itself may be of finite duration ... .

[10]Also notice that (asymptotic) stability in an infinite horizon context can never be verified in practice. We are not charmed by the (vague) reinterpretation of undetermined duration of an infinite time horizon. What is needed in fact is some kind of short-time stability theory that takes into account the order of magnitude of the appearing coefficients as they may be realistic for economics. Such (mathematical) theories exist (see, for instance, [10]). However, we were not motivated enough to spend time at this stage in finding out in what sense they are appropriate for economics. Therefore we are here pragmatic and only deal with an infinite time horizon case setting.

[11]Or 'time-paths' if you prefer this terminology.

[12]Of course, the precise meaning of 'stochastic exogenous influences' has to be specified in a given concrete model.

[13]But, of course, much less irregular than can be obtained with non-linear models.

[14]And even with the vector-valued feature.

[15]Of course, a notion of shock-dependency also makes sense in other type of models, like in growth models.

[16]May be government expenditures.

[17]It may sound strange, but we could not find a (mathematically) precise definition for the economic meaning of 'cycle', neither in the literature or in discussions. The same holds for 'oscillatory', 'cyclical growth' and 'trend'. Because for the purposes of this article it is not by all means necessary to have such definitions, we do not propose to give one by adapting the mathematical

call a trade cycle model 'shock-independent' if it is not shock-dependent. Thus, a trade cycle model is shock-independent, if and only if, a cycle exists in the absence of policy instruments and stochastic exogenous influences. Traditional trade cycle models are especially shock-dependent, often because the lack of policy instruments and stochastic exogenous influences causes each motion to damp out or to explode. Shock-dependency, like instability, is sometimes considered a shortcoming of the model. Shock-dependency in linear models can generally be repaired by introducing non-linearities into the model equations.[18] We show that shock-dependency may also generally be repaired by replacing the constant coefficients by periodic ones. This is illustrated in the context of the Samuelson-Hicks model.

3 Periodic Samuelson-Hicks model

3.1 Constant coefficients

First, we consider the version of the Samuelson-Hicks model (see, for instance, [1,8,13]) that deals with the following system of equations:

(1) $$C(t) = \gamma Y(t-1) + \underline{C};$$

(2) $$I(t) = \alpha(Y(t-1) - Y(t-2)) + \underline{I};$$

(3) $$Y(t) = C(t) + I(t) + G_t.$$

More or less realistic assumptions seem to be $0 < \gamma < 1, \alpha > 1$ and non-negativeness of the $\underline{C}, \underline{I}, G_t$. $Y(t)$ (national income (in period t)) is the main endogenous variable, the others being $C(t)$ (consumption) and $I(t)$ (investment). G_t (government expenditure) is a policy instrument. γ (marginal propensity to consume), α (accelerator), $\underline{C}$ (autonomous[19] consumption) and $\underline{I}$ (autonomous investment) are parameters.[20] As usual we allow that the $C(t), I(t)$ and $Y(t)$ may be negative.

The system of equations (1) - (3) is equivalent (after a right specification of the values of t in the equations (1) - (3)) to the first order vector equation

$$\begin{pmatrix} Y(t+1) \\ C(t+1) \end{pmatrix} = \begin{pmatrix} \gamma+\alpha & -\frac{\alpha}{\gamma} \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} Y(t) \\ C(t) \end{pmatrix} +$$

$$\begin{pmatrix} 1 & \frac{\alpha}{\gamma}+1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} G_{t+1} \\ \underline{C} \\ \underline{I} \end{pmatrix} \quad (t \geq 1) \qquad [SH]$$

and also to the second order scalar equation

$$Y(t+2) - (\gamma+\alpha)Y(t+1) + \alpha Y(t) = \underline{C} + \underline{I} + G_{t+2} \; (t \geq 0) \qquad (SH).^{21}$$

The original formulation of this model was by Samuelson [20]. A difference was that (following Hanssen) he took $I(t) = \alpha(C(t) - C(t-1))$ instead of (2). The other

notion of oscillatory as found in, for instance, [16], to economic purposes. (The reader may verify for himself by drawing some figures that a cycle automatically shows some 'oscillatory' behaviour.)

[18] *Such non-linearities may not only be caused, for example, by quadratic expressions of the* endogenous variables but also by imposing restrictions on the range of these variables.

[19] 'Autonomous' here should not be confused with the mathematical one of, for instance, 'autonomous recurrence equation'.

[20] We use here the following economic (system theoretical) terminology. Endogenous variables (state variables) are determined by the model, policy instruments (control variables) have to be chosen by man and parameters (parameters) are determined by the nature of the model. In some models some other types of objects also occur.

[21] A single scalar equation obtained from a system of scalar equations, like (SH), is sometimes called 'reduced form equation'. In economics one in general prefers to work with reduced form equations.

difference was taking $\underline{C} = (\underline{I} =)0$. This model is called the 'Samuelson model'.[22] The importance of Samuelson was that he used his multiplier-accelerator model to show how cycles can occur in economics. In his fine book [13] Hicks developed the Samuelson model further.[23] Like various other macro-economic models, both models contain, the so-called multiplier-accelerator mechanism (see, for example, [1]). Such models are not only based on implicit unrealistic assumptions (like the one that there is no influence of other countries) and obscure the accumulation of information and physical flows of the underlying processes, but also allow (in principle) the endogenous variables to take unrealistic values such as negatives values for the national income.[24] This means that such models mainly have pedagogical, illustrative and historical value.

3.2 Periodic coefficients

We now introduce the periodic Samuelson-Hicks model by allowing periodic parameters in equations (1) and (2). We replace in (1), γ by γ_{t-1} and $\underline{C}$ by $\underline{C}_t$ and in (2), α by α_{t-1} and $\underline{I}$ by $\underline{I}_t$ and assume the parameters γ_t, $\alpha_t, \underline{C}_t$ and $\underline{I}_t$ to be $q-$periodic. The system of equations such obtained is equivalent to the first order vector equation

$$\begin{pmatrix} Y(t+1) \\ C(t+1) \end{pmatrix} = \begin{pmatrix} \gamma_t + \alpha_t & -\frac{\alpha_t}{\gamma_{t-1}} \\ \gamma_t & 0 \end{pmatrix} \begin{pmatrix} Y(t) \\ C(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} G_{t+1} +$$

$$\begin{pmatrix} \frac{\alpha_t}{\gamma_{t-1}} & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \underline{C}_t \\ \underline{C}_{t+1} \\ \underline{I}_{t+1} \end{pmatrix} \quad (t \geq 1) \quad [PSH]$$

and also to the second order scalar equation

$$Y(t+2) - (\gamma_{t+1} + \alpha_{t+1})Y(t+1) + \alpha_{t+1}Y(t) = \underline{C}_{t+2} + \underline{I}_{t+2} + G_{t+2} \ (t \geq 0) \quad (PSH).$$

3.3 Results

Now, we present some results that can be obtained with Floquet theory for the periodic Samuelson-Hicks model. These results are presented for the reduced form equation (PSH),[25] where, γ_t, $\alpha_t, \underline{C}_t$, $\underline{I}_t$ are $q-$periodic, $\gamma_t \in (0,1)$, $\alpha_t > 0$, and $\underline{C}_t$, $\underline{I}_t$, $G(t)$ arbitrary.[26] They are discussed also in relation to well-known ones for (SH).[27]

[22] In this case, instead of (SH) we have

$$\text{and } Y(t+2) - \gamma(1+\alpha)Y(t+1) + \gamma\alpha Y(t) = G_{t+2} (t \geq 0) \quad (S).$$

Notice that one obtains (S) from (SH) by replacing α by $\gamma\alpha$, $\underline{I}$ by 0 and $\underline{C}$ by 0, which may be useful for translating results among both models. (The same holds for the vector case.)

[23] Hicks' original non-linear trade-cycle model allowed growing autonomous investments, which makes it, in fact, a growth model. The model also dealt with an income ceiling and an investment floor, which introduce non-linearities into the model. (1) - (3) deal with a detrended version of this model in the sense that the growing autonomous investments have been removed. Furthermore, these equations deal with the so-called 'elementary case' that refers to the specific number of lags used. For recent mathematically rigourous results on the Hicks' non-linear trade-cycle model see [14].

[24] See [1] for more on the real-world interpretation of such models. (3) for example means that product market equilibrium holds in each period (national product equals national income).

[25] The reader may wish to try to imitate the coming results of the scalar recurrence equation (PSH) for the vector recurrence equation $[PSH]$. Notice that $[PSH]$ is not the with PSH associated vector recurrence equation (in the sense of proposition 29).

[26] Of course the ranges of α_t, $\underline{C}_t$, $\underline{I}_t$ and, $G(t)$ are unrealistic.

[27] Proofs are given in footnotes. These proofs may refer to results in the appendix where the vector recurrence equations ☺, ●, D, ◗, ◗̲ and scalar recurrence equations 𝅗𝅥, ♩, ♩̲ are analysed. $[PSH]$ is a special case of ◗, (PSH) is a special case of ♩, $[SH]$ is a special case of ◗̲ and (SH) is a special case of ♩̲.

A prominent role in the analysis of (PSH) is played by the so-called 'Floquet multipliers' of the homogenous equation

$$Y(t+2)-(\gamma_{t+1}+\alpha_{t+1})Y(t+1)+\alpha_{t+1}Y(t)=0\ (t\geq 0)\quad (PSH)^{(0)}.$$

Such a Floquet multiplier is[28] nothing other than an eigenvalue of the matrix

$$\prod_{m=0}^{q-1}\begin{pmatrix} 0 & 1 \\ -\alpha_{q-m} & \gamma_{q-m}+\alpha_{q-m}\end{pmatrix}. \tag{4}$$ [29]

This matrix is referred to as 'monodromy matrix of $(PSH)^{(0)}$'. Being a 2×2-matrix with determinant $\alpha_1\cdots\alpha_q$, the Floquet multipliers are the two roots of the quadratic equation

$$z^2-\left(\mathrm{Tr}\prod_{m=0}^{q-1}\begin{pmatrix} 0 & 1 \\ -\alpha_{q-m} & \gamma_{q-m}+\alpha_{q-m}\end{pmatrix}\right)z+\alpha_1\alpha_2\cdots\alpha_q=0. \tag{5}$$

For the constant coefficients equation $(SH)^{(0)}$ this equation becomes

$$z^2-(\gamma+\alpha)z+\alpha=0.$$

So, we see that the Floquet multipliers come down to the (well-known) characteristic roots of $(SH)^{(0)}$.[30]

As is also well-known[31] the position of the two Floquet multipliers of $(SH)^{(0)}$ with respect to the complex unit circle

$$\mathbb{T}=\{z\in\mathbb{C}\mid |z|=1\}$$

is completely determined by the value of the accelerator:

(6) $\alpha<1 \Leftrightarrow$ both characteristic roots are inside $\mathbb{T}$;

(7) $\alpha=1 \Leftrightarrow$ both characteristic roots are on $\mathbb{T}$;

(8) $\alpha>1 \Leftrightarrow$ both characteristic roots are outside $\mathbb{T}$.

It is thus impossible for one characteristic root to be outside $\mathbb{T}$, while the other is inside it.[32] Moreover for $(SH)^{(0)}$ one easily sees that:

(9) No characteristic root equals 1 or -1;

(10) A characteristic root is simple if and only if $\gamma\neq 2\sqrt{\alpha}-\alpha$;

(11) for $\alpha=1$ the characteristic roots are conjugate complex.

From (6) - (8) and (10) it follows that

(12) for $(SH)^{(0)}$, each characteristic root on $\mathbb{T}$ is simple.

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[28] See, for the justification, proposition 77 where $v_t^{(2)}=1$, $v_t^{(1)}=-\gamma_{t+1}-\alpha_{t+1}, v_t^{(0)}=\alpha_{t+1}$.

[29] So, we can speak, for example, about a 'semi-simple Floquet multiplier'. Remember: an eigenvalue is called 'simple' if its algebraic multiplicity equals 1, and it is called 'semi-simple' if its algebraic multiplicity equals its geometric multiplicity. Each simple eigenvalue is of course also semi-simple.

[30] In first instance one may think that there is an obvious relation between the Floquet-multipliers and the coefficients. This is thus not true: the relation is through the complicated monodromy matrix.

[31] This result again follows from proposition 92.

[32] To avoid any confusion, by 'inside (outside, on) $\mathbb{T}$' we mean $|z|<1$ ($|z|>1$, $|z|=1$).

(5) - (11) are quite specific results for $(SH)^{(0)}$. The following fundamental result is specific for scalar equations with constant coefficients.[33]

(13) A characteristic root of $(SH)^{(0)}$ is semi-simple if and only if it is simple.

New features arise for $(PSH)^{(0)}$. For example, there may be one Floquet multiplier inside and one outside $\mathbb{T}$.[34] And as a straightforward calculation (for $q = 2$) shows, a non-simple and even a non-semi-simple Floquet-multiplier on $\mathbb{T}$ is possible.[35] Also, in principle, because $(PSH)^{(0)}$, is not (in general) an equation with constant coefficients, it may very well have a semi-simple Floquet multiplier that is not simple.[36] (We did not try to find out whether this indeed may happen for our particular equation $(PSH)^{(0)}$.[37]) Floquet multipliers for $(PSH)^{(0)}$ are more difficult to determine and to locate than those for $(SH)^{(0)}$. With

$$D := \alpha_1\alpha_2\cdots\alpha_q, \;\; \Delta_q := \mathrm{Tr} \prod_{m=0}^{q-1} \begin{pmatrix} 0 & 1 \\ -\alpha_{q-m} & \gamma_{q-m} + \alpha_{q-m} \end{pmatrix},$$

(6)-(8) now become the less transparent:[38]

$$\begin{aligned} &1 + D < |\Delta_q| \Leftrightarrow \text{one Floquet multiplier is inside and the other is outside } \mathbb{T}; \\ (14) \qquad &|\Delta_q| - 1 < D < 1 \Leftrightarrow \text{both Floquet multipliers are inside } \mathbb{T}; \\ &\max(|\Delta_q| - 1,\ 1) < D \Leftrightarrow \text{both Floquet multipliers are outside } \mathbb{T}. \end{aligned}$$

And for property (6), one has:[39]

> For small enough accelerators both Floquet multipliers of $(PSH)^{(0)}$ are inside $\mathbb{T}$.

Now, let us consider the asymptotic behaviour of motions[40] for national income of (PSH). Concerning this we consider their boundedness and their vanishing[41] for $t \to \infty$. (These types of behaviour are directly related to stability notions.[42]) Two fundamental results are:[43]

I The dimension of the linear space of bounded motions of $(PSH)^{(0)}$ equals the sum of the algebraic multiplicities of the Floquet multipliers inside $\mathbb{T}$ plus the sum of the geometric multiplicities of the Floquet multipliers on $\mathbb{T}$;

II The dimension of the linear space of motions of $(PSH)^{(0)}$ that tend to zero equals the sum of the algebraic multiplicities of the Floquet multipliers inside $\mathbb{T}$.

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[33] Proof.- By proposition 80.

[34] Proof.- In case $q = 2$ with $\alpha_1 = 1/20, \alpha_2 = 40, \gamma_1 = 1/4, \gamma_2 = 4/5$, (5) becomes $z^2 + 27.81z + 2 = 0$ which, by proposition 92, has one root inside and one root outside $\mathbb{T}$.

[35] For example: In case $q = 2$ with $\alpha_1 = \frac{5}{2} + \frac{3}{4}\sqrt{17}$, $\alpha_2 = \frac{12}{53}\sqrt{17} - \frac{40}{53}, \gamma_1 = \gamma_2 = 1/2$, there is a non-semi-simple root on $\mathbb{T}$.

[36] See subsubsection B.4.6.

[37] For $q = 2$ it does not happen. Indeed: A 2×2-matrix has a semi-simple root that is not simple if and only if it has the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. For $q = 2$ the monodromy matrix of $(PSH)^{(0)}$ has the non-zero number $\gamma_2 + \alpha_2$ on the 12-place.

[38] Proof.- By proposition 92.

[39] Proof.- From $\mathrm{Tr}\prod_{t=0}^{q-1} \begin{pmatrix} 0 & 1 \\ 0 & \gamma_t \end{pmatrix} = \gamma_1 \cdots \gamma_q$, (14) and continuity.

[40] I.e. solutions.

[41] I.e. tending to 0.

[42] Also see footnote 79.

[43] Proof.- By theorem 2, proposition 14 and (18).

Easy implications of I and II are:

III Each motion of $(PSH)^{(0)}$ is bounded if and only if all Floquet multipliers of $(PSH)^{(0)}$ lie on or inside $\mathbb{T}$ and each Floquet multiplier on $\mathbb{T}$ is semisimple;

IV Each non-zero motion of $(PSH)^{(0)}$ is unbounded if and only if all Floquet multipliers of $(PSH)^{(0)}$ lie outside $\mathbb{T}$;

V Each motion of $(PSH)^{(0)}$ vanishes if and only if all Floquet multipliers of $(PSH)^{(0)}$ lie inside $\mathbb{T}$.

For (PSH) one has the following easy [44] results (H is an arbitrary given function, for example $H = 0$):

VI Each motion of (PSH) is bounded (tends to H) if and only if (PSH) has a bounded motion (motion that tends to H) and each motion of $(PSH)^{(0)}$ is bounded (vanishes);

VII Suppose (PSH) has a bounded motion B. Then: B is the unique bounded motion of (PSH) if and only if 0 is the unique bounded motion of $(PSH)^{(0)}$.

From III - VII , (6)-(8) and (12) one obtains:

Each motion of $(SH)^{(0)}$ is bounded $\Leftrightarrow$ $\alpha \leq 1$;

Each non-zero motion of $(SH)^{(0)}$ is unbounded $\Leftrightarrow$ $\alpha > 1$;

Each motion of (SH) is bounded $\Leftrightarrow$ (SH) has a bounded motion and $\alpha \leq 1$;

VIII Suppose (SH) has a bounded motion B. Then: each motion of (SH) not equal to B is unbounded $\Leftrightarrow$ $\alpha > 1$;

IX Each motion of $(SH)^{(0)}$ vanishes if and only if $\alpha < 1$;

X Each motion of (SH) tends to H $\Leftrightarrow$ (SH) has a motion that tends to H and all Floquet multipliers of $(SH)^{(0)}$ lie inside $\mathbb{T}$.

Whether (PSH) has a bounded or motion that tends to H depends on the autonomous consumption, autonomous investment and government expenditures. It is easy to see that a necessary condition for the existence of a bounded motion is bounded government expenditures. Not so clear is the following result in this context.[45]

Consider (PSH) with $q-$periodic government expenditures. (PSH) has a bounded motion if and only if it has $q-$periodic one.

Now, let us consider the existence and unicity of constant and periodic motions of (PSH) more closely. Constant motions are easy:

XI Consider (PSH). $B_t := (\underline{C}_{t+2} + \underline{I}_{t+2} + G_{t+2})/(1-\gamma_{t+1})$ $(t \geq 0)$. (PSH) has a constant motion if and only if B_t is constant. In this case B_t is the unique constant motion and government expenditures are $q-$periodic.

[44] Or see proposition 10 if wished.

[45] Proof.- By (a real solution version of) theorem 6.

So, by choosing appropriate $q-$periodic government expenditures, each constant motion can be obtained. For periodic motions, unicity is easy:[46]

XII A sufficient and necessary condition for a given $r-$periodic motion of (PSH) to be unique is that $(PSH)^{(0)}$ has 0 as unique $r-$periodic motion.

The first (easy) result we notice for existence is:[47]

XIII Necessary for the existence of a $r-$periodic motion of (PSH) is that government expenditures are $\mathrm{lcm}(r,q)-$periodic.

In particular, a necessary condition for the existence of a $q-$periodic motion is that government expenditures are $q-$periodic. A more complete result, which is evident for (SH), is:[48]

XIV (PSH) has a unique $q-$periodic motion $\Leftrightarrow$ government expenditures are $q-$periodic and $(PSH)^{(0)}$ has 0 as unique $q-$periodic motion.

One can prove that:[49]

XV If, for $(PSH)^{(0)}$, 0 is not the unique $q-$periodic motion for national income, then there exists $q-$periodic government expenditures so that (PSH) does not have a $q-$periodic motion.

In order to find out what now actually happens, we notice the following fundamental result:[50]

XVI $(PSH)^{(0)}$ has a non-zero Floquetian complex[51] motion of type (q,z) if and only if z is a Floquet multiplier of $(PSH)^{(0)}$.

So, in particular: $(PSH)^{(0)}$ has 0 as unique $q-$periodic motion $\Leftrightarrow$ 1 is not a Floquet multiplier $(PSH)^{(0)}$. We conjecture the following.[52]

Conjecture 1 *If q is the minimal period of $(PSH)^{(0)}$,[53] then 1 is not a Floquet multiplier of $(PSH)^0$.*

Now, we can deduce the following.[54]

[46] Proof.- By (a real solution version of) proposition 10(2) with W the linear space of $r-$periodic functions and B the given $r-$periodic motion.

[47] Here 'lcm' denotes the least common multiple. If all autonomous consumption and investment is zero, it is easy to see that even: a necessary condition for the existence of a motion that is Floquetian of type (q,z) is that government expenditures are Floquetian of type $(\mathrm{lcm}(r,q), z^{\frac{\mathrm{lcm}(r,q)}{q}})$.

[48] Proof.- XIII and XII imply '$\Rightarrow$'. '$\Leftarrow$': by (a real solution version of) proposition 60.

[49] Proof.- By (a real solution version of) corollary 14.

[50] Proof.- By proposition 54.

[51] We add here the word 'complex' because one may allow a motion to assume complex values. Of course, in the real world, motions are real(-valued). However, by taking real or imaginary parts of a complex motion a real one is obtained. Theoretical considerations often are simplified by allowing complex motions. The theory in the appendix is especially developed for complex solutions. This explains the above words 'a real solution version'.

[52] For $q = 1$ and $q = 2$ the conjecture is true. For $q = 1$ this follows from (9) and for $q = 2$, (5) becomes $z^2 - ((\gamma_1 + \alpha_1)(\gamma_2 + \alpha_2) - \alpha_1 - \alpha_2)z + \alpha_1\alpha_2 = 0$, which has not 1 as root, because otherwise then one would have $1 + \alpha_1 + \alpha_2 = \gamma_1\gamma_2 + \alpha_1\gamma_2 + \gamma_1\alpha_2$, which is impossible because $\gamma_i \in (0,1)$ and $\alpha_i > 0$.

[53] This means that there is no positive integer r with $r < q$ so that α_t and γ_t are $r-$periodic.

[54] Proof.- Because of (14), both Floquet multipliers are inside $\mathbf{T}$. Proposition 62 implies that both Floquet multipliers with respect to the period nq are also inside $\mathbf{T}$. XVI implies that $(PSH)^{(0)}$ has 0 as unique $nq-$periodic motion. Because of XIV, (PSH) has a unique $nq-$periodic motion. V and VI imply that each motion of (PSH) tends to this periodic motion.

> Consider (PSH) where $|\Delta_q| - 1 < D < 1$ and government expenditures that are periodic with a period that is a multiple of q, say nq. Then, there exists a unique nq-periodic motion of (PSH) so that each motion of (PSH) tends towards it.[55]

For (SH) this becomes:

> Consider (SH) with $\alpha < 1$ and n-periodic government expenditures. Then, there exists a unique n-periodic motion of (PSH) so that each motion of (SH) tends towards it.[56]

Now let us consider shock-dependency. Remember (see section 2) that (PSH) is 'shock-independent' if and only if in case of zero government expenditures (and no stochastic exogenous influences), there exists a cyclical[57] motion for national income. One has:[58]

XVII (SH) is shock-independent $\Leftrightarrow \alpha = 1$;

XVIII If conjecture 1 holds and $\frac{C_{t+2}+L_{t+2}}{1-\gamma_{t+1}}$ $(t \geq 0)$ is not constant,[59] (PSH) is shock-independent.

Finally let us look to explicit expressions for motions. First we consider the homogenous case. Consider $(SH)^{(0)}$. Suppose that there are k (different) characteristic roots. Let λ_i $(1 \leq i \leq k)$ be these characteristic roots. As is well-known,[60] in case there are two characteristic roots, the general complex motion of $(SH)^{(0)}$ is (with C_1 and C_2 complex constants) given by

$$C_1\lambda_1^t + C_2\lambda_2^t$$

and in case there is only one characteristic root the general complex motion of $(SH)^{(0)}$ is given by

$$C_1\lambda_1^t + C_2 t\lambda_1^t.$$

It is possible to adapt these results to $(PSH)^{(0)}$. However this is complicated and will therefore not be done here.[61]

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[55] I.e. the difference of the motions tends to 0.

[56] Samuelson mentioned such a result in [20, page 77] for the equation (S): 'In case the product of the marginal propensity to consume and the accelerator is less than 1, perfect periodic government expenditure will result eventually in perfect periodic fluctuations in national income.'

[57] I.e. a bounded and not towards a constant tending.

[58] Proof of XVII .- Consider (SH) with zero government expenditures. By XI , (SH) has a unique constant real motion B. B is bounded. If $\alpha > 1$, then by VIII , each non-constant motion of (SH) is unbounded and if $\alpha < 1$, then, by IX and X , each motion of (SH) tends to a constant. So, in both cases, (SH) does not have a cyclical motion. If $\alpha = 1$, then by (7) and (12) both characteristic roots lie on $\mathbf{T}$ and are simple. Let λ_1, λ_2 be these roots. By (11), $\lambda_2 = \overline{\lambda_1}$ (where the overline denotes complex conjugation). λ_1^t and $\overline{\lambda_1}^t$ are non-zero Floquetian motions of $(SH)^{(0)}$ of type $(1, \lambda_1)$. They are bounded. $g := \lambda_1^t + \overline{\lambda_1}^t$ is a real bounded motion of (SH). Because of (9), $\lambda_1 \neq 1$. This implies that g does not tend to a constant. Thus g is a cyclical

Proof of XVIII .- Consider (PSH) with zero government expenditures. If conjecture 1 holds, then by XIV , (PSH) has a unique q-periodic motion. Because of XI , this motion is not constant and thus is a cycle.

[59] Notice that this condition on $\frac{C_{t+2}+L_{t+2}}{1-\gamma_{t+1}}$ is a very weak one.

[60] This result again can be found in theorem 7.

[61] *If wished, see subsubsection B.4.6 in the appendix for the case where each Floquet multiplier is semi-simple.*

A Linear recurrence equations

A.1 Settings

We present Floquet theory for recurrence equations. Here is an informal discussion of the possible settings for this case.[62] We always assume we are dealing with a single recurrence equation, i.e. not with a systems of equations,[63] and with ordinary equations, i.e. not with partial equations.[64] The most important choices to be made then are between: scalar/vector variable, linear/non-linear equation, semi-infinite/infinite/finite domain, first/second/arbitrary order equation and elliptic/non-elliptic equations. Here are some comments on these choices. In the scalar variable case, the dependent variable takes values in $\mathbb{C}$ and in the vector variable case it takes values in $\mathbb{C}^N$ (or even in a module). By 'semi-infinite' we mean that the domain of the dependent variable is of the form $\{x|x \geq a\}$ or $\{x|x \leq a\}$.[65] For the notion of ellipticity see subsection A.2.1. Especially the treatment of non-elliptic equations may give some technical difficulties. Our proper references for recurrence equations are [6, 16, 22]

The theory of arbitrary order (linear) vector recurrence equations is contained in the theory of first order (linear) vector recurrence equations. No wonder that various results for first order linear vector recurrence equations and arbitrary order linear scalar equations are similar; Having once the results for the vector case, those for the scalar case may be obtained with less effort.[66] First order linear equations have the advantage that they are more easy to handle and that they provide a theoretical more transparent setting for a dynamical system approach and as a consequence also for stability notions.[67] However, sometimes the equation is a scalar one, like (S), and it may be desirable to have direct results in terms of (S) rather than the with (S) associated vector equation.

It is possible to present both linear scalar as the linear vector case in what we call 'operator language' and 'equation language'. In operator language the main object is an operator and in equation language it is an equation; An operator can be associated with an equation and vice versa. In operator language, it is usually not only the main operators that are introduced, but also others. Operator language has the advantage that the linear algebraic structure[68] becomes more transparent.[69] Translations between these languages is straightforward.

Finally, we want to look at other issues which may play a role in fixing the setting. I. Instead of using complex-valued coefficients, it is possible to use real-valued coefficients in which special attention can be given to real-valued solutions.[70]

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[62] Instead of 'recurrence equation' one also uses the (less correct) term 'difference equation' which is in fact an equation like $g(x+2) + \cos(\frac{2}{39}\pi)g(x) + g(x-1) = 0$ $(x \in \mathbb{R})$. Recurrence equations are simpler mathematical objects than differential equations, in the sense that in dealing with recurrence equations, some technical details that have to be handled for differential equations do not appear. Various of our considerations hold also (in some way) for differential equations.

[63] To avoid any confusion: we refer to, for example, (1)-(3) as a system of (scalar) equations and to $[SH]$ as a single (vector) equation. Written out in coordinates, an equation in an N-dimensional vector-variable corresponds to a system of N equations in N scalar variables and a system of scalar equations may, by introduction of vectors, written as a single vector equation.

[64] Partial equations are equations like $g(m, n+1) - 4\cos(m)g(m,n) + g(m, n-1) = 0$ $(m, n \in \mathbb{Z})$.

[65] Of course, to study this case it is possible to use only the form $\{x|x \geq 0\}$.

[66] Proposition 29 plays an important role here.

[67] For our purposes a dynamical system approach is too abstract.

[68] For abstract (linear) algebra we recommend [9].

[69] Operator language becomes almost obligatory if interest lies with functional analytic questions where eigenvalues of (certain restrictions of) the operator are dealt with. (We do not have such an interest in this article.) This can also be done in equation language by introducing an eigenvalue parameter. This means that on the right-hand side of the equation a term $\epsilon g(t)$ explicitly appears. (ϵ is the eigenvalue parameter.) This term can also be taken into account on the left-hand side by including it in a coefficient.

[70] For complex coefficients, real-valued solutions have less chance to exist.

For real coefficients, the presentation of the theory can benefit from also allowing complex-valued solutions. II. Certain types of second order scalar equations can be rewritten to a so-called 'self-adjoint form' (see [16, page 251]). III. It is possible to consider equations with stochastic coefficients (see for example [18]).

It is clear that a (possible) combination of all the above mentioned choices leads to many different settings. In general, a presentation for a specific setting needs at some points some minor and at at other points some major modifications for it to become a presentation for another setting. As far as we know, the whole picture of settings has not ever been discussed systematically in the literature. It would be interesting to have a (clear but not forced) presentation that enables all of these choices to be take into account (almost).

We now look at what choices may be especially appropriate for economic (trade cycle and growth) theory. In this context, it may be interesting to compare these with similar settings for (solid-state) physics. In physics, because of the Newton and Schrödinger equations, it is especially second order equations that are used. In economics, the domain is normally semi-infinite (to the right) which is related to the economists interests in the future and to infinite horizons. In physics, the infinity of the domain also occurs, because many models deal with position and self-adjoint equations are important because of quantum mechanics. In economics, only real coefficients are important. In contrast to physics, there are several applications in economics where, in first instance only real-valued solutions are dealt with. The non-elliptic case does not seem to play an important role in economics.

We present Floquet theory for settings that deal at least with the appropriate cases for economics. The formulation of results is in equation or operator language, while proofs usually use operator language.[71] The organization of the rest of this appendix is as follows. After a presentation for our purposes of useful results for linear recurrence equations with arbitrary coefficients (see [16] for additional results and for examples), we consider Floquet theory for linear recurrence equations with periodic coefficients. In each of these two settings, the first order vector and arbitrary order scalar case are dealt with.

A.2 Arbitrary order vector case

A.2.1 The object

A (N−dimensional elliptic M-th order linear) vector recurrence equation is an equation of the form

$$\sum_{s=0}^{M} \mathcal{B}_t^{(s)}(G(t+s)) \;=\; U_t \;(t \in \mathbb{N}) \qquad ☺,$$

where $M \geq 1$, each $\mathcal{B}_t^{(s)} \in \mathrm{End}(\mathbb{C}^N)$, even $\mathcal{B}_t^{(0)}, \mathcal{B}_t^{(M)} \in \mathrm{GL}(\mathbb{C}^N)$,[72] each $U_t \in \mathbb{C}^N$ and $G : \mathbb{N} \to \mathbb{C}^N$. In case $N = 1$, one speaks of a 'scalar recurrence equation'. 'Elliptic' refers to the invertibility of $\mathcal{B}_t^{(0)}$ and $\mathcal{B}_t^{(M)}$. The $\mathcal{B}_t^{(s)}$ and U_t are called 'coefficients'. In case no coefficient depends on t one speaks of an 'autonomous equation'. A mapping $G : \mathbb{N} \to \mathbb{C}^N$ that satisfies ☺ is called a 'solution of ☺'. In case we want to make explicit the dependence on U of ☺ we also use the notation ☹$^{(J)}$. In particular, ☹$^{(0)}$, is obtained by taking $U = 0$. ☹$^{(0)}$, is called the '(associated) homogeneous equation'. It is possible and often even usual to take for $\mathcal{B}_t^{(s)}$ an element of $M_N(\mathbb{C})$, i.e. a $N \times N$ matrix with complex coefficients and to interpret it in a natural way as a transformation of $\mathbb{C}^N$; In such a context $\mathbb{C}^N$ will refer to

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[71] We hope that our modern presentation of periodic coefficients theory may for various readers give a new view on and better understanding of constant coefficients theory.

[72] $\mathbb{N}$ denotes the set of non-negative integers, $\mathrm{End}(\mathbb{C}^N)$ denotes the collection of all linear transformations of $\mathbb{C}^N$ and $\mathrm{GL}(\mathbb{C}^N)$ denotes the group of bijective linear transformations of $\mathbb{C}^N$.

column vectors. However, in the following we prefer a coordinate free presentation (i.e. we try to avoid matrices).

Given $☺^{(0)}$ (or ☺ if you want), one defines the linear transformation $\mathcal{H}$ of $(\mathbb{C}^N)^{\mathbb{N}}$ by

$$(\mathcal{H}G)(t) := \sum_{s=0}^{M} \mathcal{B}_t^{(s)}(G(t+s)).$$

We call $\mathcal{H}$ 'the (with $☺^{(0)}$) associated (vector) recurrence operator'.[73]

Given $m \in \mathbb{N}$, we define for $r \in \mathbb{N}$ the linear transformation $\mathcal{T}_r$ of $(\mathbb{C}^m)^{\mathbb{N}}$ by

$$(\mathcal{T}_r h)(t) := h(t+r).$$

One has for all $r, t \geq 0$

$$\mathcal{T}_r \mathcal{T}_t = \mathcal{T}_{r+t}.$$

And we define for a given sequence $\mathcal{B} := (\mathcal{B}_t)_{t \geq 0}$ of $\mathrm{End}(\mathbb{C}^m)$ the linear transformation $\mathcal{M}_{\mathcal{B}}$ of $(\mathbb{C}^m)^{\mathbb{N}}$ by[74]

$$(\mathcal{M}_{\mathcal{B}} G)(t) := \mathcal{B}_t(G(t)).$$

Given $☺^{(0)}$, one has with these notations[75]

$$\mathcal{H} = \sum_{s=0}^{M} \mathcal{M}_{\mathcal{B}^{(s)}} \mathcal{T}_1^s.$$

We denote the set of solutions of ☺ ($☺^{(0)}$) by

$$\mathrm{SOL} \ (\mathrm{SOL}^{(0)}).$$

It is clear that

$$\mathrm{SOL}^{(0)} = \ker(\mathcal{H}) \text{ and } \mathrm{SOL} = \{G \in (\mathbb{C}^N)^{\mathbb{N}} \mid \mathcal{H}G = U\}.$$

So, $\mathrm{SOL}^{(0)}$ forms[76] a linear subspace of $(\mathbb{C}^N)^{\mathbb{N}}$ and SOL is affine. Thus for each $G \in \mathrm{SOL}$ one has

(15) $$\mathrm{SOL} = G + \mathrm{SOL}^{(0)}.$$

In this context one calls G a 'particular solution of ☺'. Here is the so-called 'superposition principle'.

Proposition 1 *If G_1 is a solution of $☺^{(U)}$ and G_2 is a solution of $☺^{(V)}$, then for all $c, d \in \mathbb{C}$, $cG_1 + dG_2$ is a solution of $☺^{(cU+dV)}$.* ⋄

Proof.— $\mathcal{H}G_1 = U, \mathcal{H}G_2 = V$, so $\mathcal{H}(cG_1 + dG_2) = c\mathcal{H}G_1 + d\mathcal{H}G_2 = cU + dV$. □

[73] Also forms for ☺ like $\sum_{s=L}^{M} \mathcal{B}_t^{(s)}(G(t+s)) = U_t$, where $L \leq M$ would be possible. However, our form has the advantage that $\mathcal{H}$ is a linear transformation of $(\mathbb{C}^N)^{\mathbb{N}}$.

[74] One may call $\mathcal{M}_{\mathcal{B}}$ a 'multiplication operator'.

[75] And $m := N$.

[76] Under the usual addition and scalar multiplication.

A.2.2 Initial value problems and dimension of $\mathrm{SOL}^{(0)}$

Proposition 2 is quite fundamental. It guarantees the existence and uniqueness of several (what is called) 'initial value problems for ☺' and in particular proves that $\mathrm{SOL} \neq \emptyset$. The ellipticity of ☺ is responsible for this result.

Proposition 2 *Consider ☺. Given $t_0 \geq 0$ and $V_0, \ldots, V_{M-1} \in \mathbb{C}^N$, ☺ has a unique solution G with $G(t_0) = V_0, \ldots, G(t_0 + M - 1) = V_{M-1}$.* ◇

Proof.— Define $G(t_0 + s) := V_s$ $(0 \leq s \leq M - 1)$. Now successively for $s = M, M+1, \ldots$, uniquely define $G(t_0 + s)$ from $G(t_0 + s - M), \ldots, G(t_0 + s - 1)$ so that ☺ is satisfied for $t = t_0 + s - M$. And successively for $s = 1, 2, \ldots, t_0$, uniquely define $G(t_0 - s)$ from $G(t_0 - s + 1), \ldots, G(t_0 - s + M)$ so that ☺ is satisfied for $t = t_0 - s$. So, G is as desired. □

Proposition 3 *Consider ☺$^{(0)}$. Given $t_0 \geq 0$ and $M_0, \ldots, M_{M-1} \in \mathrm{End}(\mathbb{C}^N)$. There exists a unique $F : \mathbb{N} \to \mathrm{End}(\mathbb{C}^N)$ with $F(t_0) = M_0, \ldots, F(t_0 + M - 1) = M_{M-1}$ and $\sum_{s=0}^{M} \mathcal{B}_t^{(s)} F(t+s) = 0$ $(t \geq 0)$.* ◇

Proof.— In the same way as proposition 2. □

An $F : \mathbb{N} \to \mathrm{End}(\mathbb{C}^N)$ with $\sum_{s=0}^{M} \mathcal{B}_t^{(s)} F(t+s) = 0$ $(t \geq 0)$ is called 'operator solution' of ☺$^{(0)}$. If F is such, then for each $Y \in \mathbb{C}^N$, $G(t) := F(t)Y$ is a solution of ☺$^{(0)}$.

Corollary 1 *Two solutions (or operator solutions) of ☺$^{(0)}$ that are equal in M consecutive points, are equal. In particular: Any solution (or operator solution) of ☺$^{(0)}$ that equals 0 in M consecutive points, is equal to 0.* ◇

A.2.3 Various fundamental mappings

Because of proposition 2 there exists for each $t_0 \geq 0$ and $V := (V_0, \ldots, V_{M-1}) \in (\mathbb{C}^N)^M$ a unique solution G of ☺ with $G(t_0 + s) = V_s$ $(0 \leq s \leq M - 1)$. We denote this solution by

$$\mathcal{G}_{t_0;V}.$$

For the homogenous equation ☺$^{(0)}$ we denote $\mathcal{G}_{t_0;V}$ also by $\mathcal{G}^{(0)}_{t_0;V}$. One thus has $(\mathcal{G}_{t_0;V}(t_0), \ldots, \mathcal{G}_{t_0;V}(t_0 + M - 1)) = V$.

Proposition 4 *If G is a solution of ☺, then $G = \mathcal{G}_{t_0;(G(t_0),\ldots,G(t_0+M-1))}$ $(t_0 \geq 0)$.* ◇

Proof.— G and $\mathcal{G}_{t_0;(G(t_0),\ldots,G(t_0+M-1))}$ are solutions of ☺ that equal $G(t_0 + s)$ in $t = t_0 + s$ $(0 \leq s \leq M - 1)$. Because of proposition 2 they are equal. □

Proposition 5 *For all $V, W \in (\mathbb{C}^N)^M$, $t_0 \geq 0$ and $c \in \mathbb{C}$ one has*

1. *$\mathcal{G}_{t_0;V} - \mathcal{G}_{t_0;W} = \mathcal{G}^{(0)}_{t_0;V-W}$, so the difference of any two solutions of ☺ is a solution of ☺$^{(0)}$.*

2. *$\mathcal{G}^{(0)}_{t_0;V+W} = \mathcal{G}^{(0)}_{t_0;V} + \mathcal{G}^{(0)}_{t_0;W}$, $\mathcal{G}^{(0)}_{t_0;cV} = c\, \mathcal{G}^{(0)}_{t_0;V}$.* ◇

Proof.— 1. The superposition principle (with $U^1 = U^2 = U$, $c_1 = 1$ and $c_2 = -1$) gives $\mathcal{G}_{t_0;V} - \mathcal{G}_{t_0;W} \in \mathrm{SOL}^{(0)}$. Because at $t = t_0 + s$ $(0 \leq s \leq M - 1)$ this solution equals $V_s - W_s$, proposition 2 implies $\mathcal{G}_{t_0;V} - \mathcal{G}_{t_0;W} = \mathcal{G}^{(0)}_{t_0;V-W}$. 2. As 1. □

Consider ☺. Define for an integer $t_0 \geq 0$,

$$\mathcal{U}_{t_0} : (\mathbb{C}^N)^M \to \mathrm{SOL}$$

by

$$\mathcal{U}_{t_0}(V) := \mathcal{G}_{t_0;V}.$$

$\mathcal{U}_0$ will also be denoted by $\mathcal{U}$ and for ☺$^{(0)}$ also the notations $\mathcal{U}_{t_0}^{(0)}$ and $\mathcal{U}^{(0)}$ will be used. So, $\mathcal{U}_{t_0}^{(0)}, \mathcal{U}^{(0)} : (\mathbb{C}^N)^M \to \mathrm{SOL}^{(0)}$. For $t, t' \geq 0$, define the mapping

$$\Phi_{t,t'} : (\mathbb{C}^N)^M \to (\mathbb{C}^N)^M$$

by

(16) $$\Phi_{t,t'}(V) := (\mathcal{G}_{t';V}(t), \ldots, \mathcal{G}_{t';V}(t+M-1)).$$

For ☺$^{(0)}$ also the notation $\Phi_{t,t'}^{(0)}$ will be used. One thus has

$$\Phi_{t,t'}(V) = ((\mathcal{U}_{t'}V)(t), \ldots, (\mathcal{U}_{t'}V)(t+M-1)).$$

We call each of the mappings $\Phi_{t,t'}$ a 'semi-flow'.

Proposition 6 *For each solution G of* ☺, *one has* $\Phi_{t,t'}(G(t'), \ldots, G(t'+M-1)) = (G(t), \ldots, G(t+M-1))$ $(t, t' \geq 0)$. ⋄

Proof.— By proposition 4. □

Proposition 7 *Consider* ☺. *One has:*

1. $\Phi_{t,t} = id$ $(t \geq 0)$;
2. $\Phi_{t,t'}\Phi_{t',t''} = \Phi_{t,t''}$ $(t, t' \geq 0)$;
3. $\Phi_{t,t'}$ *is invertible and one has* $(\Phi_{t,t'})^{-1} = \Phi_{t',t}$ $(t, t' \geq 0)$. ⋄

Proof.— 1. Because $(\mathcal{G}_{t;V}(t), \ldots, \mathcal{G}_{t;V}(t+M-1)) = V$.

2. One has

$$\Phi_{t,t''}(V) = (\mathcal{G}_{t'';V}(t), \ldots, \mathcal{G}_{t'';V}(t+M-1))$$

and one easily calculates that

$$\Phi_{t,t'}\Phi_{t',t''}V =$$

$$(\mathcal{G}_{t';(\mathcal{G}_{t'';V}(t'),\ldots,\mathcal{G}_{t'';V}(t'+M-1))}(t), \ldots\ldots (\mathcal{G}_{t';(\mathcal{G}_{t'';V}(t'),\ldots,\mathcal{G}_{t'';V}(t'+M-1))}(t+M-1)).$$

To see that these two expressions are equal we notice that
$\mathcal{G}_{t'';V} = \mathcal{G}_{t';(\mathcal{G}_{t'';V}(t'),\ldots,\mathcal{G}_{t'';V}(t'+M-1))}$ (indeed, at $t'+s$ both solutions equal $\mathcal{G}_{t'';V}(t'+s)$).

3. Because of 1 and 2 one has $\Phi_{t,t'}\Phi_{t',t} = id$ and $\Phi_{t',t}\Phi_{t,t'} = id$. □

Proposition 8 *1.* $\mathcal{U}_{t_0} - \mathcal{U}_{t_0}(0) = \mathcal{U}_{t_0}^{(0)}$.[77]

2. $\Phi_{t,t'} - \Phi_{t,t'}(0) = \Phi_{t,t'}^{(0)}$. ⋄.

Proof.— 1. If one applies the left-hand side to $V \in (\mathbb{C}^N)^M$ one obtains $\mathcal{G}_{t_0;V} - \mathcal{G}_{t_0;(0,\ldots,0)}$. By proposition 5 this equals $\mathcal{G}_{t_0;V}^{(0)}$, i.e. to the right-hand side applied to V. 2. As 1. □

[77] To avoid any confusion: $(\mathcal{U}_{t_0} - \mathcal{U}_{t_0}(0))V = \mathcal{U}_{t_0}(V) - \mathcal{U}_{t_0}$.

Proposition 9 *1. (a) Each* $\mathcal{U}_{t_0}^{(0)} : (\mathbb{C}^N)^M \to \mathrm{SOL}^{(0)}$ *is a linear isomorphism, its inverse being* $(\mathcal{U}_{t_0}^{(0)})^{-1}(G) = (G(t_0), \ldots, G(t_0 + M - 1))$.

(b) Each $\mathcal{U}_{t_0}$ *is a bijective affine mapping.*

2. (a) Each $\Phi_{t,t'}^{(0)}$ *is a linear automorphism.*

(b) Each $\Phi_{t,t'}$ *is a bijective affine mapping.*

(c) Each $\Phi_{t,t'}$ *is a homeomorphism.* ◇

Proof.— 1a. $\mathcal{U}_{t_0}^{(0)}$ is linear: by proposition 5(2), $\mathcal{U}_{t_0}^{(0)}(c_1 V_1 + c_2 V_2) = \mathcal{G}_{t_0;c_1V_1+c_2V_2}^{(0)} = c_1\mathcal{G}_{t_0;V_1}^{(0)} + c_2\mathcal{G}_{t_0;V_2}^{(0)} = c_1\mathcal{U}_{t_0}^{(0)}V_1 + c_2\mathcal{U}_{t_0}^{(0)}V_2$.

$\mathcal{U}_{t_0}^{(0)}$ is injective: if $\mathcal{U}_{t_0}^{(0)}V = 0$, then $\mathcal{G}_{t_0,V}^{(0)} = 0$, in particular $V = (\mathcal{G}_{t_0;V}^{(0)}(t_0), \ldots, \mathcal{G}_{t_0;V}^{(0)}(t_0 + M - 1)) = 0$. $\mathcal{U}_{t_0}^{(0)}$ is surjective: If $G \in \mathrm{SOL}^{(0)}$, then, by proposition 4, $G = \mathcal{G}_{t_0;(G(t_0),\ldots,G(t_0+M-1))}^{(0)} = \mathcal{U}_{t_0}^{(0)}(G(t_0), \ldots, G(t_0 + M - 1))$.

Furthermore, for $G \in \mathrm{SOL}^{(0)}$, by proposition 4, $((\mathcal{U}_{t_0}^{(0)}(\mathcal{U}_{t_0}^{(0)})^{-1})G)(t) = (\mathcal{U}_{t_0}^{(0)}(G(t_0), \ldots, G(t_0 + M - 1)))(t) = \mathcal{G}_{t_0;(G(t_0),\ldots,G(t_0+M-1))}^{(0)}(t) = G(t)$. And for $V \in \mathbb{C}^N$, $((\mathcal{U}_{t_0}^{(0)})^{-1}\mathcal{U}_{t_0}^{(0)})V = (\mathcal{U}_{t_0}^{(0)})^{-1}(\mathcal{G}_{t_0;V}^{(0)}) = V$.

1b. Because of 1a and proposition 8.

2a. Linearity follows from proposition 5(2). Bijectivity holds because of proposition 7(3).

2b. Because of 2a and proposition 8.

2c. By 2b and the fact that $(\mathbb{C}^N)^M$ has finite dimension. □

So, $\mathrm{SOL}^{(0)}$ is isomorphic to $(\mathbb{C}^N)^M$. Now from (15):

Corollary 2 SOL *and* $\mathrm{SOL}^{(0)}$ *have (affine) dimension* MN.[78] ◇

Denote by

$$e_1, \ldots, e_N$$

the canonical base of $\mathbb{C}^N$, i.e. all N coefficients of e_j are zero with the exception of the j-th one that equals 1. And, for $0 \leq i \leq M - 1$, $1 \leq j \leq N$, denoting by

$$e_j^{(i)}$$

the element of $(\mathbb{C}^N)^M$ that has e_j at place i and 0 at the other places. One easily verifies that $e_j^{(i)}$ $(0 \leq i \leq M-1,\ 1 \leq j \leq N)$ is a base of $(\mathbb{C}^N)^M$. The distinguished solutions

$$G_j^{(i)}\ (0 \leq i \leq M - 1,\ 1 \leq j \leq N)$$

of ☺$^{(0)}$ are defined by

$$G_j^{(i)} := \mathcal{G}_{0;e_j^{(i)}}^{(0)}.$$

$G_j^{(i)} = \mathcal{U}^{(0)}(e_j^{(i)})$, so by proposition 9(1a) one has that $G_j^{(i)}$ $(0 \leq i \leq M - 1,\ 1 \leq j \leq N)$ is a base of $\mathrm{SOL}^{(0)}$.

[78] The determination of these dimensions in case of ☺ where one does not suppose ellipticity is not so easy. However, one can show that corollary 2 remains valid in case all $B_t^{(M)} \in \mathrm{GL}(\mathbb{C}^N)$.

A.2.4 First order versus arbitrary order vector recurrence equations

Here we shall see that first order vector recurrence equations are in fact quite general by showing that given ☺, one can obtain a with ☺ equivalent first order vector recurrence equation. Indeed, if $G : \mathbb{N} \to \mathbb{C}^N$ is a solution of ☺, define

$$G_i(t) := G(t+i-1) \ (1 \le i \le M).$$

Then

$$G_i(t+1) = G_{i+1}(t) \ (1 \le i \le M-1),$$

$$\mathcal{B}_t^{(M)} G_M(t+1) = -\sum_{s=0}^{M-1} \mathcal{B}_t^{(s)} G_{s+1}(t) + \mathcal{U}_t^{(0)}.$$

With

$$\mathbf{G}(t) := \begin{pmatrix} G_1(t) \\ \vdots \\ G_M(t) \end{pmatrix} \in (\mathbb{C}^N)^M = \mathbb{C}^{NM},$$

this can be written as

$$\mathrm{diag}(id, id, \ldots, id, \mathcal{B}_t^{(M)})\mathbf{G}(t+1) =$$

$$\begin{pmatrix} 0 & id & 0 & \cdots & 0 \\ 0 & 0 & id & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & id \\ -\mathcal{B}_t^{(0)} & -\mathcal{B}_t^{(1)} & -\mathcal{B}_t^{(2)} & \cdots & -\mathcal{B}_t^{(M-1)} \end{pmatrix} \mathbf{G}(t) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ U_t \end{pmatrix}.$$

With the aid of (block) companion matrices (see subsection C.1) and

$$\mathbf{U}_t := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ U_t \end{pmatrix},$$

this becomes

$$\mathbf{G}(t+1) = \mathrm{diag}(id, id, \ldots, id, (\mathcal{B}_t^{(M)})^{-1})\Big(\mathrm{Comp}(\mathcal{B}_t^{(M-1)}, \ldots, \mathcal{B}_t^{(0)})(\mathbf{G}(t)) + \mathbf{U}_t\Big).$$

This is a (NM-dimensional first order linear) vector recurrence equation. The great advantage of first order vector recurrence equations is that they are more easily to solve. Indeed, for example in case $U = 0$ one has, if we abbreviate the above equation as $\mathbf{G}(t+1) = \mathcal{N}_t \mathbf{G}(t)$, $\mathbf{G}(t) = (\mathcal{N}_{t-1} \cdots \mathcal{N}_0)\mathbf{G}(0)$. However, notice that the $\mathcal{B}_t^{(s)}$ are elements of $\mathrm{End}(\mathbb{C}^N)$, which complicates things. One has to do with numbers in case $N = 1$, a case that will be made more operational in subsubsection A.4.2 in case $N = 1$.

A.2.5 Typical solutions

Proposition 10 *Consider ☺. Let $\mathcal{W} = J + \mathcal{W}_0$ be a non-empty affine subset of* $(\mathbb{C}^N)^{\mathbb{N}}$.

1. *The following statements are equivalent.*

   (a) Each solution of ☺ belongs to $\mathcal{W}$.

(b) *Each solution of* $☺^{(0)}$ *belongs to* $\mathcal{W}_0$ *and* ☺ *has a solution that belongs to* $\mathcal{W}$.

2. *Suppose* ☺ *has a solution* B *that belongs to* $\mathcal{W}$. *The following statements are equivalent.*

   (a) B *is the unique solution of* ☺ *that belongs to* $\mathcal{W}$.

   (b) 0 *is the unique solution of* $☺^{(0)}$ *that belongs to* $\mathcal{W}_0$. $\diamond$

Proof.— 1. 'a $\Rightarrow$ b': $\mathrm{SOL} \neq \emptyset$. Take $G \in \mathrm{SOL}$. Then $G \in \mathrm{SOL} \cap \mathcal{W}$. If $H \in \mathrm{SOL}^{(0)}$, then by proposition 5(1), $G - H \in \mathrm{SOL}$. Therefore $G - H \in \mathcal{W}$. Thus $H = G - (G - H) \in \mathcal{W} - \mathcal{W} = \mathcal{W}_0$.

'b $\Leftarrow$ a': Take $G \in \mathrm{SOL} \cap \mathcal{W}$. Let $H \in \mathrm{SOL}$. Then $G - H \in \mathrm{SOL}^{(0)}$, so $G - H \in \mathcal{W}_0$. Thus $H = G - (G - H) \in \mathcal{W}$.

2. 'a $\Rightarrow$ b': Take $G \in \mathrm{SOL}^{(0)}$ with $G \neq 0$. Then $B + G \in \mathrm{SOL}$ and $B + G \neq B$, so $B + G \notin \mathcal{W}$. If one would have $G \in \mathcal{W}_0$, then $B + G \in \mathcal{W}$, a contradiction. Thus $G \notin \mathcal{W}_0$.

'b $\Leftarrow$ a': Let $G \in \mathrm{SOL}$ with $G \neq B$. One has $G - B \in \mathrm{SOL}^{(0)}$ and $G - B \neq 0$. Therefore $G - B \notin \mathcal{W}_0$. If one would have $G \in \mathcal{W}$, then on would have $G - B \in \mathcal{W}_0$, a contradiction. Thus $G \notin \mathcal{W}$. $\square$

Proposition 11 *Consider* ☺. *Let* $\mathcal{W}$ *be an* $\mathcal{H}$*−invariant finite dimensional linear subspace of* $(\mathbb{C}^N)^{\mathbb{N}}$. *Then, the following statements are equivalent.*

1. *For each* $J \in \mathcal{W}$, $☺^{(J)}$ *has precisely one solution that belongs to* $\mathcal{W}$.

2. *For each* $J \in \mathcal{W}$, $☺^{(J)}$ *has a solution that belongs to* $\mathcal{W}$.

3. ☺ *has precisely one solution that belongs to* $\mathcal{W}$.

4. 0 *is the unique solution of* $☺^{(0)}$ *that belongs to* $\mathcal{W}$. $\diamond$

Proof.— All four statements are equivalent with bijectivity of $H \restriction W : W \to W$. $\square$

Important examples of $\mathcal{W}$'s are (taking $m = N$):

$$l^{\infty} := \{G : \mathbb{N} \to \mathbb{C}^m | G \text{ is bounded}\};$$

$$c_0 := \{G : \mathbb{N} \to \mathbb{C}^m | \lim_{t \to +\infty} G(t) = 0\};$$

$$\mathrm{FLOQ}_{q,z} := \{G : \mathbb{N} \to \mathbb{C}^m \mid G(t+q) = zG(t)\ (t \geq 0)\}\ (q \geq 1, z \in \mathbb{C}^{\star});$$

$$\mathrm{PER}_q := \mathrm{FLOQ}_{q,1}.$$

In particular $\mathrm{FLOQ}_{1,1}$ is the linear space of the constant functions $\mathbb{N} \to \mathbb{C}^m$. Later, in case of equations with periodic coefficients we will see how Floquet multipliers give information on the belonging to l^{∞} and c_0 of solutions. Notice that for $\mathrm{FLOQ}_{q,z}$ one has

$$\mathrm{FLOQ}_{q,z} \;=\; \{G \in (\mathbb{C}^N)^{\mathbb{N}} \mid \mathcal{T}_q G = zG\}.$$

With $\| Y \|$ we denote the Euclidean norm of $Y \in \mathbb{C}^N$.

Proposition 12 *In case there exists* $C > 0$ *so that* $\| B_t^{(s)} Y \| \leq C \| Y \|$ $(Y \in \mathbb{C}^N)$ *for all* s *and* t, *one has:*

1. $U \notin l^{\infty} \Rightarrow \mathrm{SOL} \cap l^{\infty} = \emptyset$;

2. $U \notin c_0 \Rightarrow \mathrm{SOL} \cap c_0 = \emptyset$. $\diamond$

Proof.— 1. Suppose $G \in \mathrm{SOL} \cap l^\infty$. Then also $t \mapsto (\mathcal{H}G)(t) = \sum_{s=0}^{M} B_t^{(s)}(G(t+s)) \in l^\infty$. Because $U = \mathcal{H}G$, $U \in l^\infty$, a contradiction. 2. In the same way. □

The question about the (algebraic) dimension of $\mathrm{SOL}^{(0)} \cap l^\infty$ and of $\mathrm{SOL}^{(0)} \cap c_0$ may be in general a difficult one, but in case of $☻^{(0)}$ has a simple answer as we show in theorem 2. Indeed, for linear equations with periodic coefficients a lot can be said about the asymptotic behaviour of solutions.[79]

$☺^{(0)}$ has 0 as constant solution. There is another constant solution if and only if $\sum_{s=0}^{M} B_t^{(s)}$ is singular. In general, ☺ does not have a constant solution. A necessary and sufficient condition for this is that there exists $Y \in \mathrm{Im} \sum_{s=0}^{M} B_t^{(s)}$ so that $U_t = (\sum_{s=0}^{M} B_t^{(s)})Y$; Y is then is a constant solution. In particular, in case $\sum_{s=0}^{M} B_t^{(s)} = 0$ $(t \geq 0)$, each constant $Y \in \mathbb{C}^N$ is a solution of $☺^{(0)}$.

A.2.6 Real solutions

In this subsubsection we always consider ☺ for real coefficients; i.e. one even has that each $B_t^{(s)} \in \mathrm{End}(\mathbb{R}^N)$ for each $Y \in \mathbb{R}^N$ and each $U_t \in \mathbb{R}^N$. In this case the interest may be in real(-valued) solutions of ☺, i.e. in solutions $G : \mathbb{N} \to \mathbb{R}^N$. The coming definitions and results may be useful for this.[80]

If B is an affine subset of $(\mathbb{C}^N)^{\mathbb{N}}$, define B' as the set of real-valued elements of B. B' is a $\mathbb{R}$−linear subspace of B (in particular SOL' and $\mathrm{SOL}^{(0)\prime}$ are defined).[81] Notice that in proposition 2, $G \in \mathrm{SOL}'$ if $V_0, \ldots, V_{M-1} \in \mathbb{R}^N$. Again for each $G \in \mathrm{SOL}^{(0)\prime}$:

$$\mathrm{SOL}' = G + \mathrm{SOL}^{(0)\prime}.$$

Fix a positive integer $m \geq 1$. One defines for each function $G : \mathbb{N} \to \mathbb{C}^m$ the functions

$$\Re G, \Im G : \mathbb{N} \to \mathbb{R}^m$$

by

$$(\Re G)(t) := \frac{G(t) + \overline{G(t)}}{2}, \quad (\Im g)(t) := \frac{G(t) - \overline{G(t)}}{2i}.$$

(Here, for $(x_1, \ldots, x_m) \in \mathbb{C}^m$, $\overline{(x_1, \ldots, x_m)} := (\overline{x_1}, \ldots, \overline{x_m})$.) Then

$$G = \Re G + i \Im G.$$

What is useful for operator language presentations, if one wants to deal with real solutions, is the mapping $\mathcal{C} : (\mathbb{C}^m)^{\mathbb{N}} \to (\mathbb{C}^m)^{\mathbb{N}}$ defined by

$$(\mathcal{C}G)(t) := \overline{G(t)}.$$

Notice that $\mathcal{C}^2 = id$ and that $\mathcal{C}$ is semi-linear, i.e. that $\mathcal{C}(G+H) = \mathcal{C}(G) + \mathcal{C}(H)$, $\mathcal{C}(cG) = \bar{c}\mathcal{C}(G)$. In terms of $\mathcal{C}$ one has:

$$B' = \{G \in B \mid \mathcal{C}G = G\};$$

$$\Re G = \frac{G + \mathcal{C}G}{2}, \quad \Im G = \frac{G - \mathcal{C}G}{2i}. \tag{17}$$

[79] This behaviour is (because we are dealing with linear equations) directly related to stability notions (in the sense of Lyapunov). This will not be discussed further in this article. The interested reader may find this a shortcoming of this article. In fact, the concept of stability is quite subtle and there are many different refinements of this notion. In case of differential equations there are a lot of (good) textbooks on stability, but, as far as we know, this is not the case for recurrence equations. (The stability theory for recurrence equations is not completely similar to that for differential equations.)

[80] As mentioned in subsection A.1 one can develop the theory more generally for example for $\mathbb{K}^{\mathbb{N}}$ (where $\mathbb{K}$ is a commutative field). Subsubsection A.2.6 repairs in some sense this omission.

[81] For example, denoting by 'Vect' the linear span, if $B = \mathrm{Vect}(e^{it})$ in $\mathbb{C}^{\mathbb{N}}$, then $B' = \{0\}$.

Proposition 13 *Fix $G : \mathbb{N} \to \mathbb{C}^m$. One has: For no $c \in \mathbb{T}$ one has $\mathcal{C}G \neq cG$ $\Leftrightarrow$ $\Re g$ and $\Im g$ are linearly independent $\Leftrightarrow$ G and $\mathcal{C}G$ are linear independent.* $\diamond$

Proof.— $\Re G, \Im G$ are linearly independent if and only if G and $\mathcal{C}G$ are linearly independent. This implies '$\Leftarrow$'. '$\Rightarrow$': Then $G \neq 0$. Suppose $G, \mathcal{C}G$ would be linearly dependent. Then there is $c \in \mathbb{C}$ so that $\mathcal{C}G = cG$, and we find $G = \mathcal{C}(\mathcal{C}G) = \bar{c}(\mathcal{C}G) = |c|^2 G$. Thus, $|c| = 1$, a contradiction. □

Proposition 14 $\dim(B) = \dim(B')$ *for each $\mathcal{C}-$invariant linear subspace B of $(\mathbb{C}^N)^{\mathbb{N}}$.* $\diamond$

Proof.— Let $(h_i)_{i \in I}$ be a base of B'. We shall prove that it is also a base of B. Completeness: Take $H \in B$. Now notice that one has $H = \Re H + \imath \Im H$, $\Re H \in B'$, $\Im H \in B'$. Linear independence: Suppose $\sum_{j \in K} \lambda_j h_j = 0$ with the $\lambda_j \in \mathbb{C}$ and K a finite subset of I. Then $0 = \mathcal{C}(\sum_{j \in K} \lambda_j h_j) = \sum_{j \in K} \overline{\lambda_j} \mathcal{C}(h_j) = \sum_{j \in K} \overline{\lambda_j} h_j = \sum_{j \in K} (\Re \lambda_j - \imath \Im \lambda_j) h_j = \sum_{j \in K} (\Re \lambda_j) h_j - \imath \sum_{j \in K} (\Im \lambda_j) h_j$. So $\sum_{j \in K} (\Re \lambda_j) h_j = \sum_{j \in K} (\Im \lambda_j) h_j = 0$. This implies $\Re \lambda_j = \Im \lambda_j = 0$ $(j \in K)$, thus $\lambda_j = 0$ $(j \in K)$. □

Now, consider the with $☺^{(0)}$ associated linear recurrence operator $\mathcal{H}$. $\mathcal{C}$ and $\mathcal{H}$ commute, that is one has:

Proposition 15 *For $☺^{(0)}$ with real coefficients one has* $[\mathcal{C}, \mathcal{H}] = 0$. $\diamond$

Proof.— $(\mathcal{C}\mathcal{H}G)(t) = \overline{(\mathcal{H}G)(t)} = \overline{\sum_{s=0}^{M} B_t^{(s)}(G(t+s))} = \sum_{s=0}^{M} \overline{B_t^{(s)}(G(t+s))} = \sum_{s=0}^{M} B_t^{(s)} (\overline{G(t+s)}) = (\mathcal{H}\mathcal{C}G)(t)$. □

Proposition 15 implies:

$$G \in \mathrm{SOL}^{(0)} \Rightarrow \mathcal{C}G \in \mathrm{SOL}^{(0)}, \qquad G \in \mathrm{SOL} \Rightarrow \mathcal{C}G \in \mathrm{SOL}. \tag{18}$$

Also it implies $\mathcal{C}(\ker(\mathcal{H} - \epsilon\, id)) \subseteq \ker(\mathcal{H} - \bar{\epsilon}\, id)$ and because $\mathcal{C}^2 = id$ even $\mathcal{C}(\ker(\mathcal{H} - \epsilon\, id)) = \ker(\mathcal{H} - \bar{\epsilon}\, id)$.

Proposition 16 *For $☺^{(0)}$ with real coefficients one has:*

1. *For each solution G of $☺^{(0)}$, $\Re G$ and $\Im G$ are real solutions of $☺^{(0)}$.*
2. *For each solution G of $☺$, $\Re G$ is a real solution of $☺$ and $\Im G$ is a real solution of $☺^{(0)}$.* $\diamond$

Proof.— 1. this holds because $\mathrm{SOL}^{(0)}$ is a linear space, (17) and (18). 2. Let $G \in \mathrm{SOL}$. Then $\mathcal{C}G \in \mathrm{SOL}$ and because SOL is affine one has $\Re G = \frac{1}{2}G + \frac{1}{2}\mathcal{C}G \in \frac{1}{2}\mathrm{SOL} + (1 - \frac{1}{2})\mathrm{SOL} = \mathrm{SOL}$. Moreover, $\Im G \in \mathrm{SOL}^{(0)}$ because $\mathcal{H}(\Im G) = \mathcal{H}\frac{G - \mathcal{C}G}{2i} = \frac{1}{2i}(\mathcal{H}G - (\mathcal{H}\mathcal{C})G) = \frac{1}{2i}(U - (\mathcal{C}\mathcal{H})G) = \frac{1}{2i}(U - \mathcal{C}U) = \frac{1}{2i}(U - U) = 0$.[82] □

A.2.7 Adjoint equation

Put on $\mathbb{C}^m$ the inner-product $< \mid >$ given by

$$< Y | W > := \sum_{j=1}^{m} Y_j \overline{W_j}. \tag{19}$$

[82] By taking real and imaginary parts of $☺^{(0)}$ one obtains an equation language proof of this proposition.

The adjoint equation ☺$^{(0)}_{\mathrm{ad}}$ of ☺$^{(0)}$ is by definition the equation

$$(20)\qquad \sum_{s=0}^{M} (\mathcal{B}_{t+s}^{(M-s)})^{\star}(G(t+s)) = 0 \ (t \geq 0) \qquad ☺^{(0)}_{\mathrm{ad}}$$

where $G : \mathbb{N} \to \mathbb{C}^N$ and $\mathcal{A}^\star$ is the (Hilbert-)adjoint of $\mathcal{A} \in \mathrm{End}(\mathbb{C}^N)$ with respect to $< \mid >$ (with $m := N$). We denote the set of solutions of ☺$^{(0)}_{\mathrm{ad}}$ by

$$\mathrm{SOL}^{(0)}_{\mathrm{ad}}.$$

The adjoint equation again is a vector equation of the form ☺. ☺$^{(0)}_{\mathrm{ad}}$ is equivalent[83] with the equation

$$(21)\qquad \sum_{s=0}^{M} (\mathcal{B}_{t-s}^{(s)})^{\star}(G(t-s)) = 0 \quad (t \geq M),^{84}$$

where $G : \mathbb{N} \to \mathbb{C}^N$. Notice that this equation is not (completely) of the form ☺$^{(0)}$.

A.2.8 Substitution of variables

Proposition 17 *Fix $L_t \in \mathrm{GL}(\mathbb{C}^N)$ $(t \geq 0)$. Then for each $G : \mathbb{N} \to \mathbb{C}^N$ one has: G is a solution of ☺$^{(0)}$ if and only if $J : \mathbb{N} \to \mathbb{C}^N$ defined by $J(t) := L_t(G(t))$ is a solution of the vector recurrence equation $J(t+1) = \sum_{s=0}^{M}(\mathcal{B}_t^{(s)} L_{t+s}^{-1})J(t+s) = 0$ $(t \geq 0)$.* ◇

Proof.— Evident. □

One refers to $J(t) = L_t(G(t))$ as a 'linear substitution of variables'. Notice that the vector recurrence equation $J(t+1) = \sum_{s=0}^{M}(\mathcal{B}_t^{(s)} L_{t+s}^{-1})J(t+s) = 0$ $(t \geq 0)$ is again of type ☺$^{(0)}$.

A.3 First order vector case

A.3.1 The object

We are going to study vector recurrence equations of the form

$$G(t+1) \;=\; \mathcal{A}_t(G(t)) + U_t \ (t \geq 0) \qquad \mathbb{D},$$

where each $\mathcal{A}_t \in \mathrm{GL}(\mathbb{C}^N)$, each $U_t \in \mathbb{C}^N$ and $G : \mathbb{N} \to \mathbb{C}^N$. $\mathbb{D}$ is a special case of ☺.[85] For the associated recurrence operator of $\mathbb{D}^{(0)}$ one has

$$(\mathcal{H}G)(t) = G(t+1) - \mathcal{A}_t(G(t)).$$

Proposition 2 reduces to: Given $t_0 \geq 0$ and $V_0 \in \mathbb{C}^N$, $\mathbb{D}$ has a unique solution G with $G(t_0) = V_0$. And corollary 2 reduces to: $\mathrm{SOL}^{(0)}$ and SOL have (affine) dimension N.

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[83] In the sense that if G is a solution of the one if it is a solution of the other.

[84] In case the $\mathcal{B}_t{}^{(s)}$ are given by matrices, $\mathcal{B}_t^{(s)\,\star}$ equals the transposed of the imaginary conjugated matrix of $\mathcal{B}_t^{(s)}$ and the adjoint equation is even equivalent with $\sum_{s=0}^{M} K(t-s)\mathcal{B}_{t-s}^{(s)} = 0$ $(t \geq M)$, where $K : \mathbb{N} \to \mathbb{C}^N$ (with row vectors as elements of $\mathbb{C}^N$).

[85] In terms of ☺, $\mathcal{B}_t^{(1)} = id$ and $\mathcal{B}_t^{(0)} = -\mathcal{A}_t$.

A.3.2 Semi-flows

Consider $\mathbb{D}$. For each $t, t' \geq 0$ the mapping

$$\Phi_{t,t'} : \mathbb{C}^N \to \mathbb{C}^N$$

is defined by (16), i.e. by

$$\Phi_{t,t'}(Y) := \mathcal{G}_{t';Y}(t).$$

Now one has that the mapping $t \mapsto \Phi_{t,t'}(Y)$ equals $\mathcal{G}_{t';Y}$, i.e. is a solution of ☺. Notice that if G is a solution of $\mathbb{D}$, then by proposition 6

$$\Phi_{t,t'}(G(t')) = G(t) \; (t, t' \geq 0).$$

In particular

$$\Phi_{t,t'}(\mathcal{G}_{t';Y}(t')) = \mathcal{G}_{t';Y}(t) \; (t, t' \geq 0).$$

Given $\mathbb{D}$, it us useful to define the affine mappings $\mathcal{F}_t : \mathbb{C}^N \to \mathbb{C}^N$ by

$$\mathcal{F}_t(Y) := \mathcal{A}_t(Y) + U_t.$$

One has $\mathcal{F}_t^{-1}(Y) = \mathcal{A}_t^{-1}(Y) - \mathcal{A}_t^{-1}(U_t)$.

Proposition 18 *Consider $\mathbb{D}$. One has:*

1. $\Phi_{t+1,t} = \mathcal{F}_t \; (0 \leq t' \leq t)$;
2. $\Phi_{t,t'} = \begin{cases} \mathcal{F}_{t-1} \cdots \mathcal{F}_{t'} \; (0 \leq t' \leq t) \\ \mathcal{F}_t^{-1} \cdots \mathcal{F}_{t'-1}^{-1} \; (0 \leq t < t') \end{cases}$. ◇

Proof.— 1. $\Phi_{t+1,t}(Y) = \mathcal{G}_{t;y}(t+1) = \mathcal{F}_t(\mathcal{G}_{t;Y}(t)) = \mathcal{F}_t(Y)$. 2. Because of 1 and proposition 7(2,3). □

Corollary 3 *1. For each solution G of $\mathbb{D}$ one has $G(t) = \mathcal{F}_{t-1} \cdots \mathcal{F}_{t'} G(t')$ $(0 \leq t' \leq t)$.*

2. For each solution G of $\mathbb{D}^{(0)}$ one has $G(t) = \mathcal{A}_{t-1} \cdots \mathcal{A}_{t'} G(t')$ $(0 \leq t' \leq t)$. ◇

Proposition 19 *If G, H are different solutions of $\mathbb{D}$, then $G(t) \neq H(t)$ $(t \geq 0)$.* ◇

Proof.— By corollary 1. □

Finally, consider the adjoint equation of $\mathbb{D}^{(0)}$:

$$G(t) - \mathcal{A}_{t+1}^{\star}(G(t+1)) \;=\; 0 \; (t \geq 0). \qquad \mathbb{D}_{\text{ad}}^{(0)}$$

Proposition 20 *If G is a solution of $\mathbb{D}_{\text{ad}}^{(0)}$, then $G(t) = \left(\Phi_{t'+1,t+1}^{(0)}\right)^{\star}(G(t'))$ $(t, t' \geq 0)$.* ◇

Proof.— If $t \leq t'$: $G(t) = \mathcal{A}_{t+1}^{\star}(G(t+1)) = \mathcal{A}_{t+1}^{\star}(\mathcal{A}_{t+2}^{\star}(G(t+2))) = \cdots = (\mathcal{A}_{t+1}^{\star} \cdots \mathcal{A}_{t'}^{\star})(G(t')) = (\mathcal{A}_{t'} \cdots \mathcal{A}_{t+1})^{\star}(G(t')) = \left(\Phi_{t'+1,t+1}^{(0)}\right)^{\star}(G(t'))$.

And if $t > t'$ then $t' \leq t$ and by the above $G(t') = \left(\Phi_{t+1,t'+1}^{(0)}\right)^{\star}(G(t))$. So $G(t) = \left(\Phi_{t+1,t'+1}^{(0)\,\star}\right)^{-1}(G(t')) = \left(\Phi_{t+1,t'+1}^{(0)\,-1}\right)^{\star}(G(t')) = \left(\Phi_{t'+1,t+1}^{(0)}\right)^{\star}(G(t'))$. □

A.3.3 Operator solutions and semi-flows

Consider a homogenous vector recurrence equation $\mathbb{D}^{(0)}$. We called a mapping $F : \mathbb{N} \to \mathrm{End}(\mathbb{C}^N)$ so that

$$F(t+1) = \mathcal{A}_t F(t)\ (t \geq 0),$$

an 'operator solution of $\mathbb{D}^{(0)}$'.[86] Proposition 3 becomes: For each $t_0 \geq 0$ and $M_0 \in \mathrm{End}(\mathbb{C}^N)$ there exists a unique operator solution F of $\mathbb{D}^{(0)}$ with $F(t_0) = M_0$.

Proposition 21 *Let F be an operator solution of $\mathbb{D}^{(0)}$. Then $F(t)$ is for all t or for no t invertible.* □.

Proof.— Suppose $F(t_0)$ is not invertible. Then there exists $Y_0 \in \mathbb{C}^N$ with $Y_0 \neq 0$ and $F(t_0)Y_0 = 0$. $G(t) := (F(t))Y_0$ is a solution of $\mathbb{D}^{(0)}$ that equals 0 at $t = t_0$. Also 0 is such. Unicity implies $G = 0$, i.e. $F(t)Y_0 = 0$ $(t \geq 0)$, so $F(t)$ is not invertible. □

An operator solution F of $\mathbb{D}^{(0)}$ is called 'fundamental' if each $F(t)$ is invertible. Often considered is the fundamental operator solution F:

Definition 1 The unique operator solution F of $\mathbb{D}^{(0)}$ with $F(0) = id$ is called 'the principal operator solution (of $\mathbb{D}^{(0)}$)'. ◇

It is clear that $F : \mathbb{N} \to \mathrm{End}(\mathbb{C}^N)$ is an operator solution of $\mathbb{D}^{(0)}$ if and only if $F(t) = \mathcal{A}_{t-1} \cdots \mathcal{A}_s F(s)$ $(0 \leq s \leq t)$, i.e. if and only if $F(t) = \Phi^{(0)}_{t,s} F(s)$ $(0 \leq s \leq t)$. From this:

Proposition 22 *$F : \mathbb{N} \to \mathrm{End}(\mathbb{C}^N)$ is a fundamental operator solution of $\mathbb{D}^{(0)}$ if and only if $\Phi^{(0)}_{t,s} = F(t)(F(s))^{-1}$ $(t, s \geq 0)$.* ◇

Corollary 4

1. *$\Phi^{(0)}_{t,0} = \mathcal{A}_{t-1} \cdots \mathcal{A}_0$ is the principal operator solution of $\mathbb{D}^{(0)}$.*
2. *$F(t) := \Phi^{(0)}_{t,t_0} M_0$ $(t \geq 0)$ is the unique operator solution of $\mathbb{D}^{(0)}$ with $F(t_0) = M_0$.* ◇

Proposition 23 *If F and G are fundamental operator solutions of $\mathbb{D}^{(0)}$, then there exists a unique $C \in \mathrm{GL}(\mathbb{C}^N)$ so that $G(t) = F(t)C$ $(t \geq 0)$.* ◇

Proof.— Unicity: If such a C exists, then $G(0) = F(0)C$, so $C = F(0)^{-1}G(0)$. Existence: Now, with $C := F(0)^{-1}G(0)$, $J(t) := F(t)C$ is an operator solution and is fundamental. Because $J(0) = G(0)$, it follows that $G(t) = J(t) = F(t)C$ $(t \geq 0)$. □

Of course, if $G^{(1)}, \ldots, G^{(m)}$ are solutions of $\mathbb{D}^{(0)}$, so that, for some $t_0 \geq 0$, $G^{(1)}(t_0), \ldots, G^{(m)}(t_0)$ are linearly independent, then $G^{(1)}, \ldots, G^{(N)}$ are linearly independent. A reverse result is also true:

Proposition 24 *If $G^{(1)}, \ldots, G^{(m)}$ are linearly independent solutions of $\mathbb{D}^{(0)}$, then for each $t \geq 0$ the vectors $G^{(1)}(t), \ldots, G^{(m)}(t)$ are linearly independent in $\mathbb{C}^N$.* ◇

Proof.— Suppose there would exist $t_0 \geq 0$ and $\lambda_i \in \mathbb{C}$ so that $\sum_{i=1}^m \lambda_i G^{(i)}(t_0) = 0$. Define $J := \sum_{i=1}^m \lambda_i G^{(i)}$. J is a solution of $\mathbb{D}^{(0)}$ that equals 0 at $t = t_0$. Also 0 is such. Because of unicity, $J = 0$, a contradiction. □

Proposition 25 *Consider $\mathbb{D}^{(0)}$. Let $G^{(1)}, \ldots, G^{(N)}$ be solutions of $\mathbb{D}^{(0)}$. Then*

[86] When working with matrices $(\mathcal{A}_t, F(t) \in M_N(\mathbb{C}))$ the term 'matrix solution' can be used.

1. $F : \mathbb{N} \to \mathrm{End}(\mathbb{C}^N)$ *defined by*

$$F(t)Y := \sum_{j=1}^{N} Y_j G^{(j)}(t)$$

is an operator solution of $\mathrm{D}^{(0)}$.

In case the $\mathcal{A}_t$ *are matrices* $F(t) := (G^{(1)}(t), \ldots, G^{(N)}(t)) \in M_N(\mathbb{C})$ *(i.e. the columns of* $F(t)$ *are the vectors* $G^{(i)}(t)$*), is a matrix solution of* $\mathrm{D}^{(0)}$.

2. *F as defined in 1 is fundamental if and only if* $G^{(1)}, \ldots, G^{(N)}$ *are linearly independent.* ◇

Proof.— 1. F is an operator solution because for all $Y \in \mathbb{C}^N$ and $t \geq 0$ one has $F(t+1)Y = \sum_{j=1}^N Y_j G^{(j)}(t+1) = \sum_{j=1}^N Y_j \mathcal{A}_t(G^{(j)}(t)) = \mathcal{A}_t(\sum_{j=1}^N Y_j G^{(j)}(t)) = \mathcal{A}_t(F(t)Y)$. The second statement is proven in the same way.

2. If $G^{(1)}, \ldots, G^{(N)}$ are linearly independent, then for each t, by proposition 24, $G^{(1)}(t), \ldots, G^{(N)}(t)$ are linearly independent, so each $F(t)$ is injective and thus also bijective. If F is fundamental, then $F(0)$ is invertible, so $G^{(1)}(0), \ldots, G^{(N)}(0)$ are linearly independent, thus $G^{(1)}, \ldots, G^{(N)}$ are so too. □

Proposition 26 *F is a fundamental operator solution of* $\mathrm{D}^{(0)} \Rightarrow ((F(t+1))^\star)^{-1}$ *is a fundamental operator solution of* $\mathrm{D}^{(0)}_{\mathrm{ad}}$. ◇

Proof.— One has $F(t+1) = \mathcal{A}_t F(t)$ $(t \geq 0)$. By taking $\star$ and then $^{-1}$ one obtains $(F(t+1)^\star)^{-1} = (\mathcal{A}_t^\star)^{-1}(F(t)^\star)^{-1}$ $(t \geq 0)$. With $J(t) := (F(t+1)^\star)^{-1} \in \mathrm{GL}(\mathbb{C}^N)$, this becomes $J(t) = (\mathcal{A}_t^\star)^{-1} J(t-1)$ $(t \geq 1)$, i.e. $J(t-1) = \mathcal{A}_t^\star J(t)$ $(t \geq 1)$, i.e. $J(t) = \mathcal{A}_{t+1}^\star J(t+1)$ $(t \geq 0)$. □

A.3.4 Particular solution of D

Here is an explicit relation between the $\Phi_{t,t'}$ and the $\Phi^{(0)}_{t,t'}$:

Proposition 27 $\Phi_{t,t_0} = \begin{cases} \Phi^{(0)}_{t,t_0} + \sum_{s=t_0}^{t-1} \Phi^{(0)}_{t,s+1}(U_s) & (0 \leq t_0 \leq t), \\ \Phi^{(0)}_{t,t_0} - \sum_{s=t}^{t_0-1} \Phi^{(0)}_{t,s+1}(U_s) & (0 \leq t \leq t_0 - 1). \end{cases}$ ◇

Proof.— By induction. For $t = t_0$, $\Phi_{t,t_0} = id = \Phi^{(0)}_{t,t_0}$. Suppose it is true for some $t \geq t_0$. Then $\Phi_{t+1,t_0} = \mathcal{F}_t \Phi_{t,t_0} = \mathcal{A}_t \Phi_{t,t_0} + U_t = \mathcal{A}_t(\Phi^{(0)}_{t,t_0} + \sum_{s=t_0}^{t-1} \Phi^{(0)}_{t,s+1}(U_s)) + U_t = \Phi^{(0)}_{t+1,t_0} + \sum_{s=t_0}^{t-1} \Phi^{(0)}_{t+1,s+1}(U_s) + \Phi_{t+1,t+1}(U_t) = \Phi^{(0)}_{t+1,t_0} + \sum_{s=t_0}^{t} \Phi^{(0)}_{t+1,s+1}(U_s)$. The proof for $t \leq t_0$ is similar. □

Because of proposition 7(2,3), one also has

$$\Phi_{t,t_0} = \Phi^{(0)}_{t,t_0}(id + \sum_{s=t_0}^{t-1} \Phi^{(0)}_{t_0,s+1}(U_s)) \quad (t \geq t_0 \geq 0). \tag{22}$$

This formula is one of the forms of the 'variation of constants formula for D'. The term is derived from the fact that Φ_{t,t_0} has the form $\Phi^{(0)}_{t,t_0} C(t)$. Another form is:

Proposition 28 *Let* $G^{(1)}, \ldots, G^{(N)}$ *be linear independent solutions of* $\mathrm{D}^{(0)}$. *Then there exist functions* $C_i : \mathbb{N} \to \mathbb{C}$ $(1 \leq i \leq N)$ *so that* $\sum_{i=1}^N C_i G^{(i)}$ *is a solution of* D. ◇

Proof.— Define $F : \mathbb{N} \to \mathrm{End}(\mathbb{C}^N)$ by $F(t)Y := \sum_{j=1}^{N} Y_j G^{(j)}(t)$. Because of proposition 25, F is a fundamental operator solution of $\mathsf{D}^{(0)}$. By proposition 22, $\Phi_{t,s}^{(0)} = F(t)(F(s))^{-1}$. By proposition 27,

$\Phi_{t,0} = \Phi_{t,0}^{(0)} + \sum_{s=0}^{t-1} \Phi_{t,s+1}^{(0)}(U_s) = F(t)F(0)^{-1} + \sum_{s=0}^{t-1}(F(t)(F(s+1))^{-1})(U_s)$. A solution of D is $\Phi_{t,0}(0)$, i.e. $\sum_{s=0}^{t-1}(F(t)(F(s+1))^{-1})(U_s)$. It equals

$\sum_{s=0}^{t-1} F(t)((F(s+1))^{-1}U_s) = \sum_{s=0}^{t-1}\sum_{j=1}^{N}((F(s+1))^{-1}U_s)_j G^{(j)}(t) =$
$\sum_{j=1}^{N}(\sum_{s=0}^{t-1}((F(s+1))^{-1}U_s)_j)G^{(j)}(t)$. So, take $C_j(t) := \sum_{s=0}^{t-1}((F(s+1))^{-1}U_s)_j$.
□

A.4 Scalar case

A.4.1 The object

We are going to study equations of the form

$$\sum_{s=0}^{M} v_t^{(s)} g(t+s) \;=\; w_t \ (t \geq 0) \qquad ♩,$$

where $w_t, v_t^{(0)}, \ldots, v_t^{(M)} \in \mathbb{C}$ $(t \geq 0)$, no $v_t^{(M)}$ and no $v_t^{(0)}$ is zero and $g : \mathbb{N} \to \mathbb{C}$. ♩ is a special case of a (M−th order elliptic linear) scalar recurrence equation. Notice that an equation of the form ♩ is nothing other than an equation of the form ☺ where $N = 1$.

For the with ♩$^{(0)}$ associated linear (scalar) recurrence operator $\mathcal{H}$ one has

$$(\mathcal{H}g)(t) = \sum_{s=0}^{M} v_t^{(s)} g(t+s).$$

Corollary 2 becomes: For ♩, SOL and SOL$^{(0)}$ have dimension M.

The distinguished solutions $G_1^{(0)}, \ldots, G_1^{(M-1)}$ of ♩$^{(0)}$ are denoted by

$$g^{(0)}, g^{(1)}, \ldots, g^{(M-1)}$$

respectively. So, $g^{(l)}$ is the unique solution of ♩$^{(0)}$ that satisfies

$$g^{(l)}(t) = \delta_{l,t} \quad (0 \leq t \leq M-1).$$

We already know that $g^{(0)}, \ldots, g^{(M-1)}$ is a base of SOL$^{(0)}$.

A.4.2 From arbitrary order scalar to first order vector equations

Given an integer $M \geq 1$, define the linear mappings $\mathcal{P} : \mathbb{C}^{\mathbb{N}} \to (\mathbb{C}^M)^{\mathbb{N}}$ and $Q : (\mathbb{C}^M)^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ by

$$(\mathcal{P}g)(t) := \begin{pmatrix} g(t) \\ \vdots \\ g(t+M-1) \end{pmatrix} \ (t \geq 0),$$

$$(QG)(t) := (G(t))_1.^{87}$$

Lemma 1 *1. For $G \in (\mathbb{C}^M)^{\mathbb{N}}$ one has: $G \in \mathrm{Im}(\mathcal{P}) \Leftrightarrow (G(t+s))_1 = (G(t))_{s+1}$ $(t \geq 0,\ 0 \leq s \leq M-1)$.*

[87] I.e. the first of the M coefficients coefficient of $G(t)$.

2. $\mathcal{P}$ is injective. One has $Q\mathcal{P} = id$ and $\mathcal{P}Q = id$ on $\mathrm{Im}(\mathcal{P})$. $\diamond$

Proof.— 1. If $G \in \mathrm{Im}(\mathcal{P})$, say $G = \mathcal{P}h$, then for all $t \geq 0$ and $0 \leq s \leq M-1$ one has $(G(t+s))_1 = ((\mathcal{P}h)(t+s))_1 = h(t+s) = ((\mathcal{P}h)(t))_{s+1} = (G(t))_{s+1}$. And, if $(G(t+s))_1 = (G(t))_{s+1}$ $(t \geq 0,\ 0 \leq s \leq M-1)$, then define $g \in \mathbb{C}^{\mathbb{N}}$ by $g(t) := (G(t))_1$. Now, one has

$$(\mathcal{P}g)(t) = \begin{pmatrix} g(t) \\ \vdots \\ g(t+M-1) \end{pmatrix} = \begin{pmatrix} (G(t))_1 \\ \vdots \\ (G(t+M-1))_1 \end{pmatrix} = \begin{pmatrix} (G(t))_1 \\ \vdots \\ (G(t))_M \end{pmatrix} =$$

$G(t)$. Thus $G = \mathcal{P}g$.

2. $\mathcal{P}$ is injective because $\ker(\mathcal{P}) = \{0\}$. $Q\mathcal{P} = id$: Take $g \in \mathbb{C}^M$, then $((Q\mathcal{P})g)(t) = (Q(\mathcal{P}g))(t) = (t)$. $\mathcal{P}Q = id$ on $\mathrm{Im}(\mathcal{P})$: Take $G \in \mathrm{Im}(\mathcal{P})$, then

$$((\mathcal{P}Q)G)(t) = (\mathcal{P}(QG))(t) = \begin{pmatrix} (QG)(t) \\ \vdots \\ (QG)(t+M-1) \end{pmatrix} = \begin{pmatrix} (G(t))_1 \\ \vdots \\ (G(t+M-1))_1 \end{pmatrix} =$$

$G(t)$. $\square$

Proposition 29 shows that the study of the vector recurrence equation $\mathbb{D}$ contains the study of the scalar recurrence equation $\mathbb{J}$. Given a scalar recurrence equation $\mathbb{J}$, we define[88]

$$(23)\quad \mathcal{M}_t := \mathrm{Comp}\,\Big(\frac{v_t^{(M-1)}}{v_t^{(M)}}, \ldots, \frac{v_t^{(0)}}{v_t^{(M)}}\Big) \in M_M(\mathbb{C}) \text{ and } W_t := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ w_t/v_t^{(M)} \end{pmatrix}.$$

Proposition 29 *Given a scalar recurrence equation $\mathbb{J}$, we associate to it the vector recurrence equation (of type $\mathbb{D}$)*

$$G(t+1) = \mathcal{M}_t G(t) + W_t \ (t \geq 0) \qquad <\mathbb{J}>.$$

One has:

1. *If G is a solution of $<\mathbb{J}>$, then $G \in \mathrm{Im}(\mathcal{P})$.*
2. *For each $g : \mathbb{N} \to \mathbb{C}$ one has: g is a solution of $\mathbb{J}$ $\Leftrightarrow$ $\mathcal{P}g$ is a solution of $<\mathbb{J}>$.*
3. *For each $G \in \mathrm{Im}(\mathcal{P})$ one has: G is a solution of $<\mathbb{J}>$ $\Leftrightarrow$ QG is a solution of $\mathbb{J}$.* $\diamond$

Proof.— 1. One has $G(t+1) = \mathcal{M}_t(G(t)) + W_t$ $(t \geq 0)$. So, $(G(t+s))_s = (\mathcal{M}_t(G(t)))_s + (W_t)_s$ $(t \geq 0,\ 1 \leq s \leq M-1)$, i.e. $(G(t+1))_s = (G(t))_{s+1}$ $(t \geq 0, 1 \leq s \leq M-1)$. This implies $(G(t))_{s+1} = (G(t+s))_1$ $(t \geq 0, 0 \leq s \leq M-1)$. By lemma 1, $G \in \mathrm{Im}(\mathcal{P})$.

2. Let $g : \mathbb{N} \to \mathbb{C}$. Then: $\mathcal{P}g$ is a solution of $<\mathbb{J}>$ $\Leftrightarrow$ $(\mathcal{P}g)(t+1) = \mathcal{M}_t(\mathcal{P}g)(t) + W_t$ $(t \geq 0)$ $\Leftrightarrow$ $((\mathcal{P}g)(t+1))_M = \mathcal{M}_t((\mathcal{P}g)(t)) + (W_t)_M$ $(t \geq 0)$ $\Leftrightarrow$ $g(t+M) = \sum_{j=0}^{M-1} \mathrm{Comp}_{M,j+1} g(t+j) + \frac{w_t}{v_t^{(M)}}$ $(t \geq 0)$ $\Leftrightarrow g(t+M) = \sum_{j=0}^{M-1} -\frac{v_t^{(j)}}{v_t^{(M)}} g(t+j) + \frac{w_t}{v_t^{(M)}}$ $(t \geq 0)$ $\Leftrightarrow$ g is a solution of $\mathbb{J}$.

3. $G = P(QG)$ by lemma 1(2) and $QG : \mathbb{N} \to \mathbb{C}$. Because of 2 one therefore has: QG is a solution of $\mathbb{J} \Leftrightarrow G$ is a solution of $<\mathbb{J}>$. $\square$

Notice that $<\mathbb{J}>^{(0)}$ and $<\mathbb{J}^{(0)}>$ are the same objects. We shall denote $\mathrm{SOL}^{(0)}$ for $<\mathbb{J}^{(0)}>$ by

$$<\mathrm{SOL}^{(0)}>.$$

[88] See subsection C.1 for definition and properties of Comp .

Corollary 5 $\mathcal{P} : \mathrm{SOL}^{(0)} \to < \mathrm{SOL}^{(0)} >$ *is a linear isomorphism.* $\diamond$

We call the $\mathcal{M}_t$ 'the transfer matrices associated with $\mathcal{J}^{(0)}$'. The reason for that is that for each solution g of $\mathcal{J}^{(0)}$ one has $(\mathcal{P}g)(t+1) = \mathcal{M}_t((\mathcal{P}g)(t))$ $(t \geq 0)$. Notice that each transfer matrix is (indeed) invertible, because of (40) one has for the determinant of $\mathcal{M}_t$

$$|\mathcal{M}_t| = (-1)^M \frac{v_t^{(0)}}{v_t^{(M)}} \neq 0. \tag{24}$$

Proposition 30 *1.* $\dim(< \mathrm{SOL}^{(0)} > \cap \mathrm{FLOQ}_{q,z}) = \dim(\mathrm{SOL}^{(0)} \cap \mathrm{FLOQ}_{q,z})$.

2. $\dim(< \mathrm{SOL}^{(0)} > \cap l^\infty) = \dim(\mathrm{SOL}^{(0)} \cap l^\infty)$.

3. $\dim(< \mathrm{SOL}^{(0)} > \cap c_0) = \dim(\mathrm{SOL}^{(0)} \cap c_0)$. $\diamond$

Proof.— First notice that, because $\mathcal{P} : \mathrm{SOL}^{(0)} \to < \mathrm{SOL}^{(0)} >$ is a linear isomorphism, for each linear subspace $\mathcal{W}$ of $\mathrm{SOL}^{(0)}$ one has $\mathcal{P}(\mathrm{SOL}^{(0)} \cap \mathcal{W}) =$ $< \mathrm{SOL}^{(0)} > \cap\ \mathcal{P}(\mathcal{W})$. For $\mathcal{W} = \mathrm{FLOQ}_{q,z}$, $\mathcal{W} = l^\infty$ and $\mathcal{W} = c_0$ one even has $< \mathrm{SOL}^{(0)} > \cap\ \mathcal{P}(\mathcal{W}) = < \mathrm{SOL}^{(0)} > \cap\ \mathcal{W}$. Indeed: $\subseteq$ is evident (and relies on the nature of l^∞, c_0 and $\mathrm{FLOQ}_{q,z}$) and if $G \in < \mathrm{SOL}^{(0)} > \cap\ \mathcal{W}$, then let $h \in \mathrm{SOL}^{(0)}$ so that $\mathcal{P}h = G$. Then $\mathcal{QP}h = \mathcal{Q}G$, i.e. $h = \mathcal{Q}G \in \mathcal{W}$. Thus $G \in \mathcal{P}(\mathcal{W})$. $\square$

Proposition 31 *For each non-zero solution* G *of* $< \mathcal{J}^{(0)} >$ *and* $j \in \{1, \ldots, M\}$ *there exists* $t \geq 0$ *with* $(G(t))_j \neq 0$. $\diamond$

Proof.— Suppose one would have $(G(t))_j = 0$ $(t \geq 0)$. Then by lemma 1, $(G(t+j-1))_1 = 0$ $(t \geq 0)$, i.e. $(\mathcal{Q}G)(t+j-1) = 0$ $(t \geq 0)$. Now $\mathcal{Q}G$ is a solution of $\mathcal{J}^{(0)}$ that equals 0 in M consecutive points. Therefore $\mathcal{Q}G = 0$. But then $G = \mathcal{PQ}G = \mathcal{P}0 = 0$. $\square$

A.4.3 Matrix of Casorati

Given a positive integer L, we define for each $t \geq 0$

$$\mathrm{Cas}_t : \mathbb{C}^{\mathbb{N}} \times \cdots \times \mathbb{C}^{\mathbb{N}} \text{ (L times) } \to M_L(\mathbb{C})$$

by

$$\mathrm{Cas}_t(h_1, \ldots, h_L) := \begin{pmatrix} h_1(t) & \cdots & h_L(t) \\ h_1(t+1) & \cdots & h_L(t+1) \\ \vdots & \ddots & \vdots \\ h_1(t+L-1) & \cdots & h_L(t+L-1) \end{pmatrix}. \tag{25}$$

$\mathrm{Cas}_t(h_1, \ldots, h_L)$ is called 'the matrix of Casorati (of $h_1, \ldots, h_L$ at t)'. Notice that[89]

$$\mathrm{Cas}_t(h_1, \ldots, h_L) = ((\mathcal{P}h_1)(t), \ldots, (\mathcal{P}h_L)(t)$$

(i.e. the i–th column of $\mathrm{Cas}_t(h_1, \ldots, h_L)$ equals $(\mathcal{P}h_i)(t)$). Fix a function $d : \mathbb{N} \to \mathbb{C}^\star$.[90] Denoting determinants by $|\ |$, the mapping, for fixed $h_1, \ldots, h_L$,

$$\omega(t) : \mathbb{N} \to \mathbb{C},$$

defined by

$$\omega(t) := d_t\ |\mathrm{Cas}_t(h_1, \ldots, h_L)|$$

[89] With $M = L$.

[90] $\mathbb{C}^\star = \{z \in \mathbb{C} \mid z \neq 0\}$.

is called 'Casoratian'. From the other side, for fixed $t \geq 0$, the mapping $d_t\ |\mathrm{Cas}_t(\cdot)| : (\mathbb{C}^{\mathbb{N}})^L \to \mathbb{C}$ is L-linear. Using a well-known property of determinants one obtains:

$$(26) \qquad h_1, \ldots, h_L \in \mathbb{C}^{\mathbb{N}} \text{ are linearly dependent } \Rightarrow \omega(t) = 0 \ \ (t \geq 0).$$

If $\omega(t) = 0$ for all $t \geq 0$, then $h_1, \ldots, h_L$ are not necessarily linear dependent.[91]

Now, consider $\mathcal{J}^{(0)}$. One has (with $L := M$)

$$(27) \qquad \mathrm{Cas}_0(g^{(0)}, \ldots, g^{(M-1)}) = id.$$

Proposition 32 *Fix $t \geq 0$. For all solutions $h^{(1)}, \ldots, h^{(M)}$ of $\mathcal{J}^{(0)}$ one has: $h^{(1)}, \ldots, h^{(M)}$ are linearly dependent $\Leftrightarrow$ $\omega(t) = 0$.* ◇

Proof.— $\Rightarrow$: Because of (26). $\Leftarrow$: Suppose that $\omega(t) = 0$. So, there exist $c_1, \ldots, c_M \in \mathbb{C}$, not all 0, so that $c_1 h^{(1)}(t+l) + \cdots + c_M h^{(M)}(t+l) = 0$, $(0 \leq l \leq M-1)$. Thus $h := c_1 h^{(1)} + \cdots + c_M h^{(M)}$ is a solution of $\mathcal{J}^{(0)}$ that equals zero in M consecutive points. Proposition 2 now implies $h = 0$. Thus $h^{(1)}, \ldots, h^{(M)}$ are linearly dependent. □

Proposition 33 *$\mathcal{J}^{(0)}$ has a solution which for no value of t vanishes.* ◇

Proof.— Let $h^{(1)}, \ldots, h^{(N)}$ be a base of $\mathrm{SOL}^{(0)}$. Take $c_2 \in \mathbb{C}$ with $c_2 \notin \{-h^{(1)}(t)/h^{(2)}(t) \mid t \geq 0,\ h^2(t) \neq 0\}$. Then $y^{(1)} := h^{(1)} + c_2 h^{(2)}$ vanishes at t if and only if $h^{(1)}(t) = h^{(2)}(t) = 0$. Next take $c_3 \in \mathbb{C}$ with $c_3 \notin \{-(h^{(1)}(t) - c_2 h^{(2)}(t))/h^{(3)}(t) \mid t \geq 0,\ h^{(3)}(t) \neq 0\}$. Then $y^{(2)} := h^{(1)} + c_2 h^{(2)} + c_3 h^{(3)}$ vanishes at t if and only if $h^{(1)}(t) = h^{(2)}(t) = h^{(3)}(t) = 0$. Continuing this construction one arrives at $y^{(M)} := h^{(1)} + c_2 h^{(2)} + \cdots + c_M h^{(M)}$ which vanishes at t if and only if $h^{(1)}(t) = h^{(2)}(t) = \cdots = h^{(M)}(t) = 0$. But there is by proposition 32 no value of t for which this happens. Thus $y^{(M)}$ is as desired. □

Proposition 34 *If $h^{(1)}, \ldots, h^{(M)}$ are solutions of $\mathcal{J}^{(0)}$, then $\mathrm{Cas}_t(h^{(1)}, \ldots, h^{(M)})$ is a matrix solution of $< \mathcal{J}^{(0)} >$. It is fundamental if and only if $h^{(1)}, \ldots, h^{(M)}$ are linearly independent.* ◇

Proof.— By proposition 29(2), $\mathcal{P}h^{(1)}, \ldots, \mathcal{P}h^{(M)}$ are solutions of $< \mathcal{J} >^{(0)}$. One has $\mathrm{Cas}_t = ((\mathcal{P}h^{(1)})(t), \ldots, (\mathcal{P}h^{(M)})(t))$. By proposition 25, Cas_t is a matrix solution and it is fundamental if and only if $\mathcal{P}h^{(1)}, \ldots, \mathcal{P}h^{(M)}$ are linearly independent. □

Together with (24), proposition 34 implies:

Corollary 6 *Consider $\mathcal{J}^{(0)}$. For $h_1, \ldots, h_M \in \mathrm{SOL}^{(0)}$ one has:*

1. $\mathrm{Cas}_t = \mathcal{M}_{t-1}\mathcal{M}_{t-2}\cdots\mathcal{M}_s \mathrm{Cas}_s$ $(0 \leq s \leq t)$;
2. $\omega(t) = (-1)^{M(t-s)}\frac{d_t}{d_s}\left(\prod_{l=s}^{t-1} \frac{v_l^{(0)}}{v_l^{(M)}}\right)\omega(s) \quad (0 \leq s \leq t)$. *In particular* $\omega(t+1) = (-1)^M \frac{v_t^{(0)}}{v_t^{(M)}} \frac{d_{t+1}}{d_t}\omega(t)$ $(t \geq 0)$. *And ,if* $d_t = 1$, *then* $\omega(t) = (-1)^{M(t-s)} \frac{v_s^{(0)} v_{s+1}^{(0)} \cdots v_{t-1}^{(0)}}{v_s^{(M)} v_{s+1}^{(M)} \cdots v_{t-1}^{(M)}}\omega(s)$ $(0 \leq s \leq t)$. ◇

[91] For example: $L = 2$, $d_t = 1$, $h_1 = \delta_{0,t}$, $h_2 = \delta_{9,t}$. But, this statement is true for $L = 2$ in case where h_1, h_2 are nowhere zero. Indeed, in this case $\omega(t) = 0$ implies that $h_1(t)/h_2(t)$ does not depend on t.

A.4.4 Explicit expressions for semi-flows

Proposition 35 *The semi-flow $\Phi_{t,t'}$ for $\jmath$ equals the semi-flow $\Phi_{t,t'}$ for $<\jmath>$.* ◇

Proof.— We may suppose that $t \geq t'$. For $\jmath$ one has $\Phi_{t,t'}(V) = (\mathcal{PG}_{t';V})(t)$. Because of proposition 29(2), $\mathcal{PG}_{t';V}$ is a solution of $<\jmath>$. Defining $\mathcal{I}_t(Y) := \mathcal{M}_t(Y) + W_t$ one has by corollary 3(1) that $(\mathcal{PG}_{t';V})(t) = \mathcal{I}_{t-1}\cdots\mathcal{I}_{t'}((\mathcal{PG}_{t';V})(t')) = \mathcal{I}_{t-1}\cdots\mathcal{I}_{t'}V$ which, by proposition 18(2), equals $\Phi_{t,t'}$ applied to V for $<\jmath>$. □

Corollary 7 *1. Consider $\jmath^{(0)}$. $\Phi^{(0)}_{t,t'} = \begin{cases} \mathcal{M}_{t-1}\cdots\mathcal{M}_{t'} \; (0 \leq t' \leq t) \\ \mathcal{M}_t^{-1}\cdots\mathcal{M}_{t'-1}^{-1} \; (0 \leq t < t') \end{cases}$.*

2. Consider $\jmath$. $\Phi_{t,t'} = \begin{cases} \Phi^{(0)}_{t,t'} + \sum_{s=t'}^{t-1}\Phi^{(0)}_{t,s+1}(W_s) \quad (0 \leq t' \leq t), \\ \Phi^{(0}_{t,t'} - \sum_{s=t}^{t'-1}\Phi^{(0)}_{t,s+1}(W_s)(0 \leq t \leq t'-1). \end{cases}$ ◇

Proof.— By propositions 18(2) and 27. □

A.4.5 Particular solution of $\jmath$

The variation of constants formula is used in the proof of the following result.[92]

Proposition 36 *Let $h^{(1)},\ldots,h^{(M)}$ be linearly independent solutions of $\jmath^{(0)}$. Then there exist functions $B_i : \mathbb{N} \to \mathbb{C}$ $(1 \leq i \leq M)$ so that $\sum_{i=1}^M B_i h^{(i)}$ is a solution of $\jmath$. Even, with $Y(t) := (\mathrm{Cas}_{t+1}(h^{(1)},\ldots,h^{(M)}))^{-1}W_t$ $(t \geq 0)$ and $B(t) := \sum_{s=0}^{t-1}Y(s) \in \mathbb{C}^M$, one has that $\sum_{i=1}^M B_i h^{(i)}$ is a solution of $\jmath$.* ◇

Proof.— By proposition 34, Cas_t is a fundamental matrix solution of $<\jmath^{(0)}>$. Proposition 22 gives $\Phi^{(0)}_{t,s} = \mathrm{Cas}_t\mathrm{Cas}_s^{-1}$. By proposition 27, $\Phi_{t,0} = \mathrm{Cas}_t\mathrm{Cas}_0^{-1} + \sum_{s=0}^{t-1}(\mathrm{Cas}_t\mathrm{Cas}_{s+1}^{-1})W_s = \mathrm{Cas}_t(\mathrm{Cas}_0^{-1} + \sum_{s=0}^{t-1}Y(s)) = \mathrm{Cas}_t(\mathrm{Cas}_0^{-1} + B(t))$. A solution of $<\jmath>$ is $\Phi_{t,0}(0)$, i.e. $\mathrm{Cas}_t(B(t))$. By proposition 29(3), $\mathcal{Q}(\mathrm{Cas}_t(B(t))) = (\mathcal{Q}(\mathrm{Cas}_t(B(t))))_1$ is a solution of $\jmath$. This solution equals $\sum_{i=1}^M B_i h^{(i)}$. □

B Linear recurrence equations with periodic coefficients

B.1 Settings

In dealing with recurrence or differential equations, interest is usually in the qualitative and quantitative properties of their solutions. As we have already shown, such problems can be mastered particulary in the case of linear autonomous equations. However, the same is also true for linear equations with periodic, or even with Floquetian coefficients. The fundamental reason for this is that such an equation can be transformed by a linear substitution of variables into an equation with constant coefficients. To go beyond Floquetian coefficients requires quite different and much more difficult methods.[93] Some attempts have been made to develop a Floquet theory for the (larger) class of almost periodic coefficients (see, for example, [18]), but with less satisfying results as far as we know.

Also a presentation of Floquet theory needs a fixed setting. Floquet theory seems to be only appropriate for ordinary linear differential and recurrence equations with

[92] This result also may be proven through proposition 28.

[93] However, also linear recurrence equations with coefficients that have a limit for $t \to \pm\infty$ admit relatively easy methods of analysis. And in this context also linear recurrence equations with polynomial coefficients should be mentioned (see, for example, [16, 22]).

Floquetian coefficients.[94] As usual a lot of interest is in the analysis of homogeneous equations. Although it would be possible to deal with (homogeneous) equations where all coefficients are Floquetian of some same type, the investigation of this case is closely related to the periodic case. To see this, first notice the simple fact[95] that if $f(t+q) = zf(t)$ (i.e. f is Floquetian of type (q,z)),[96] then there exists a unique q−periodic function h such that $f(t) = z^{t/q}h(t)$.[97] Therefore, from now on we only consider equations with periodic coefficients in the associated homogeneous equation.

Again, we restrict ourselves from now on to the recurrence equation case. As already mentioned in subsection A.1, a choice has to be made from semi-infinite/infinite/finite domain and from first/second/arbitrary order. Our article will deal with semi-infinite domain and with arbitrary order vector equations.

It can be said that there are two approaches to periodic coefficients: Floquet theory and (in physics very popular) Bloch theory. The most fundamental object in Floquet theory is the Floquet (and monodromy) operator while in Bloch theory it is the Bloch operators. We deal mainly with Floquet theory as the Bloch theory seems less useful in economics. Our proper references for Floquet theory for recurrence equations are [6, 11, 17].[98]

Three types of recurrence equations with periodic coefficients are considered: ☻, ◗ and ♩. ☻ is nothing other than the vector recurrence equation ☺ in case the $\mathcal{B}_t^{(s)}$ are q−periodic, ◗ is nothing other than the vector recurrence equation D in case $\mathcal{A}_t$ is q−periodic and ♩ is nothing other than the scalar recurrence equation in case the $v^{(s)}$ are q−periodic.

B.2 Arbitrary order vector case

B.2.1 The object

We are going to study the (N−dimensional elliptic M-th order linear) vector recurrence equation (with periodic coefficients)

$$\sum_{s=0}^{M} \mathcal{B}_t^{(s)}(G(t+s)) \;=\; U_t \;\; (t \geq 0) \qquad ☻,$$

where $M \geq 1$, each $\mathcal{B}_t^{(s)} \in \mathrm{End}(\mathbb{C}^N)$, even $\mathcal{B}_t^{(0)}, \mathcal{B}_t^{(M)} \in \mathrm{GL}(\mathbb{C}^N)$, each $U_t \in \mathbb{C}^N$, each $\mathcal{B}_t^{(s)}$ is q−periodic (i.e. $\mathcal{B}_{t+q}^{(s)} = \mathcal{B}_t^{(s)}$ $(t \geq 0)$) and $G : \mathbb{N} \to \mathbb{C}^N$.

Before continuing with ☻ we consider in the next subsubsection Floquetian mappings.

B.2.2 Floquetian mappings

Given q, we shall denote for an integer n by $\overline{n}$ the unique number in $\{1,\ldots,q\}$ so that $n - \overline{n}$ is divisible by q.

Proposition 37 *Fix integers $t_0 \geq 0$, $q, m \geq 1$, $z \in \mathbb{C}^*$ and $C_0,\ldots,C_{q-1} \in \mathbb{C}^m$. There exists a unique $G \in \mathrm{FLOQ}_{q,z}$ so that $G(t_0+s) = C_s$ $(0 \leq s \leq q-1)$.* ⋄

[94] However, some applications to nonlinear recurrence equations (as a perturbation of linear ones) can also be found, as in for example, [11].

[95] Also see proposition 41(2).

[96] $z^{t/q} = e^{\frac{t}{q}\log(z)}$, where log denotes the principal value of the logarithm.

[97] By writing all coefficients in this form, the $z^{t/q}$ can be divided out, leaving an equation with q−periodic coefficients.

[98] References [2, 5, 12, 18, 19, 21] for Floquet theory only deal with differential equations.

Proof.— Define $G \in (\mathbb{C}^m)^{\mathbb{N}}$ by $G(t) = z^{\frac{t-t_0+1-\overline{t-t_0+1}}{q}} C_{\overline{t-t_0+1}-1}$ $(t \geq 0)$. Then $(\mathcal{T}_q G)(t) = G(t+q) = z^{\frac{t+q-t_0+1-\overline{t+q-t_0+1}}{q}} C_{\overline{t+q-t_0+1}-1} = z z^{\frac{t-t_0+1-\overline{t-t_0+1}}{q}} C_{\overline{t-t_0+1}-1} = zG(t)$. Thus $\mathcal{T}_q G = zG$. Moreover for $s = 0, \ldots, q-1$, $G(t_0+s) = C_s$. □

Corollary 8 *The set of eigenvalues of $\mathcal{T}_q : (\mathbb{C}^m)^{\mathbb{N}} \to (\mathbb{C}^m)^{\mathbb{N}}$ equals the whole $\mathbb{C}$.*[99] ◇

Proof.— That $z \in \mathbb{C}^\star$ is an eigenvalue follows from proposition 37. Also 0 is an eigenvalue: $G(0) = \cdots = G(q-1) = e^{(1)}$, $G(t) = 0$ $(t \geq q)$ is an eigenvector belonging to this eigenvalue. □

Denote, for $z \in \mathbb{C}$, by $\mathrm{FLOQ}_{q,z}^{(n)}$ $(n \geq 1)$ the generalized eigenspaces of $\mathcal{T}_q$ corresponding to the eigenvalue z of $\mathcal{T}_q$, i.e.

$$\mathrm{FLOQ}_{q,z}^{(n)} := \ker(\mathcal{T}_q - z\, id)^n.$$[100]

Definition 2 We call any $\mathrm{FLOQ}_{q,z}$, where $q \geq 1$ and $z \in \mathbb{C}$, a 'Floquetian space' and any of its elements a 'Floquetian mapping of type (q,z)'. We call any element of $\cup_{z \in \mathbb{C}} \mathrm{FLOQ}_{q,z}$ a '$q-$Floquetian mapping'. We call any $\mathrm{FLOQ}_{q,z}^{(n)}$, where $q \geq 1, z \in \mathbb{C}$ and $n \geq 1$ a 'generalized Floquetian space' and any of its elements a 'generalized Floquetian mapping'. ◇

Proposition 38 *1. If $g : \mathbb{N} \to \mathbb{C}$ is Floquetian of type (q,z), then there exists a unique $q-$periodic function h so that $g = z^{t/q}h$, i.e. is a product of an exponential and a periodic function.*

2. If h is a $q-$periodic function and $z \in \mathbb{C}^\star$, then $z^{t/q}h$ is a Floquetian solution of type (q,z). ◇

Proof.— This is evident (and also follows from proposition 41(2) with $n = 1$). □

Proposition 39 gives a generalization of proposition 38(1).

Proposition 39 *Consider a mapping $F : \mathbb{N} \to \mathrm{End}(\mathbb{C}^N)$ for which there exists a $C \in \mathrm{GL}(\mathbb{C}^N)$ with $F(t+q) = F(t)C$ $(t \geq 0)$. Then there exists $B \in \mathrm{GL}(\mathbb{C}^N)$ and a mapping $P : \mathbb{N} \to \mathrm{End}(\mathbb{C}^N)$ with $P(t+q) = P(t)$ $(t \geq 0)$ so that $F(t) = P(t)B^t$ $(t \geq 0)$.* ◇

Proof.— Because C is invertible, one knows that there exists a $B \in \mathrm{GL}(\mathbb{C}^N)$ with $C = B^q$. Define $P(t) = F(t)B^{-t}$. Then $F(t) = P(t)B^t$ and $P(t+q) = F(t+q)B^{-(t+q)} = F(t)CB^{-(t+q)} = F(t)B^qB^{-(t+q)} = F(t)B^{-t} = P(t)$. □

Denote $\mathrm{FLOQ}_{q,z}^{(n)}$ in the case $m = 1$, by $\mathcal{FLOQ}_{q,z}^{(n)}$, i.e.

$$\mathcal{FLOQ}_{q,z}^{(n)} = \{g : \mathbb{N} \to \mathbb{C} \mid (\mathcal{T}_q - z\, id)^n g = 0\}.$$

One has

$$\mathrm{FLOQ}_{q,z}^{(n)} = \sum_{j=1,\ldots,m}^{\oplus} \mathcal{FLOQ}_{q,z}^{(n)} \cdot e^{(j)}. \tag{28}$$

[99] The given definition for $\mathrm{FLOQ}_{q,z}$ also makes sense for $z = 0$. If we do this, then for each $z \in \mathbb{C}$, $\mathrm{FLOQ}_{q,z}$ is the eigenspace associated with the eigenvalue z of $\mathcal{T}_q$. However, we do not do this in this article: we only need $z \neq 0$ because no Floquet multiplier is zero and because some of the following results have to be adapted for $z = 0$.

[100] One thus has $\mathrm{FLOQ}_{q,z}^{(1)} = \mathrm{FLOQ}_{q,z}$.

Given q, define for $z_1, z_2 \in \mathbb{C}^*$, the linear transformation $\mathcal{K}_{z_1,z_2}$ of $(\mathbb{C}^M)^{\mathbb{N}}$ by

$$(\mathcal{K}_{z_1,z_2}G)(t) := (\frac{z_2}{z_1})^{t/q} G(t).$$

Of course, $\mathcal{K}_{z_1,z_2}$ is a linear automorphism and $\mathcal{K}_{z_1,z_2}^{-1} = \mathcal{K}_{z_2,z_1}$. And one has

$$\mathcal{T}_q \mathcal{K}_{z_1,z_2} = \frac{z_2}{z_1}(\mathcal{K}_{z_1,z_2}\mathcal{T}_q). \tag{29}$$

Proposition 40 *$\mathcal{K}_{z_1,z_2}$: $\mathrm{FLOQ}^{(n)}_{q,z_1} \to \mathrm{FLOQ}^{(n)}_{q,z_2}$ is a linear isomorphism.* ◇

Proof.— Denote the mapping in question by $\mathcal{K}$. First of all, if $G \in \mathrm{FLOQ}^{(n)}_{q,z_1}$, then $\mathcal{K}G \in \mathrm{FLOQ}^{(n)}_{q,z_2}$: By (29), for each integer $k \geq 0$, $\mathcal{T}_q^k \mathcal{K} = (\frac{z_2}{z_1})^k (\mathcal{K}\mathcal{T}_q^k)$. Therefore $(\mathcal{T}_q - z_2\, id)^n (\mathcal{K}G) = (\sum_{k=0}^n \begin{pmatrix} n \\ k \end{pmatrix} \mathcal{T}_q^k (-z_2\, id)^{n-k})(\mathcal{K}G) =$
$z_2^n \mathcal{K} \sum_{k=0}^n (\begin{pmatrix} n \\ k \end{pmatrix} (-1)^{n-k} z_1^{-k} \mathcal{T}_q^k G) = (\frac{z_2}{z_1})^n \mathcal{K}(\sum_{k=0}^n \begin{pmatrix} n \\ k \end{pmatrix} (-1)^{n-k} z_1^{n-k} \mathcal{T}_q^k G)$
$= (\frac{z_2}{z_1})^n \mathcal{K}(\mathcal{T}_q - z_1\, id)^n G = (\frac{z_2}{z_1})^n \mathcal{K}(0) = 0.$

Of course, $\mathcal{K}$ is injective. Surjectivity: Take $G \in \mathrm{FLOQ}^{(n)}_{q,z_2}$, then by the above $\mathcal{K}_{z_2,z_1} G \in \mathrm{FLOQ}^{(n)}_{q,z_1}$. So, $\mathcal{K}_{z_1,z_2}(\mathcal{K}_{z_2,z_1}G) = G$. □

Proposition 41 *1. If $z_1, \ldots, z_l \in \mathbb{C}^*$ are different numbers and $n_1, \ldots, n_l$ positive integers, then $\sum_{j,1\leq j\leq l} \mathrm{FLOQ}^{(n_j)}_{q,z_j} = \sum^{\oplus}_{j,1\leq j\leq l} \mathrm{FLOQ}^{(n_j)}_{q,z_j}$.*

2. $\mathrm{FLOQ}^{(n)}_{q,z_2} = (z_2/z_1)^{t/q} \mathrm{FLOQ}^{(n)}_{q,z_1}$, in particular $\mathrm{FLOQ}^{(n)}_{q,z} = z^{t/q} \mathrm{FLOQ}^{(n)}_{q,1}$.

3. $\mathrm{FLOQ}^{(n)}_{q,z} \subseteq \mathrm{FLOQ}^{(n)}_{mq,z^m}$ $(m \geq 1)$.

4. $\mathrm{FLOQ}^{(n)}_{q,z}$ has dimension nNq.

5. For each positif integer L one has: $\mathrm{FLOQ}^{(n)}_{Lq,z} = \sum^{\oplus}_{\omega \in \mathbb{C}, \omega^L = z} \mathrm{FLOQ}^{(n)}_{q,\omega}$, i.e. $\mathrm{FLOQ}^{(n)}_{Lq,z} = \sum^{\oplus}_{j,0\leq j\leq L-1} \mathrm{FLOQ}^{(n)}_{q,z^{1/L}e^{2\pi i \frac{j}{L}}}$, in particular $\mathrm{FLOQ}^{(n)}_{q,z} = \sum^{\oplus}_{j,0\leq j\leq q-1} \mathrm{FLOQ}^{(n)}_{1,z^{1/q}e^{2\pi i \frac{j}{q}}}$. ◇

Proof.— 1. This is a general result for generalized eigenspaces.

2. Because of proposition 40.

3. Suppose $(\mathcal{T}_q - z\, id)^n G = 0$. Because $\mathcal{T}_{mq} - z^m\, id = \mathcal{T}_q^m - (z \cdot id)^m = z^{m-1}(\mathcal{T}_q - z\, id) \sum_{j=0}^{m-1} (\mathcal{T}_q/z)^j$, one obtains $(\mathcal{T}_{mq} - z^m\, id)^n G =$
$z^{(m-1)n} (\sum_{j=0}^{m-1} (\frac{\mathcal{T}_q}{z})^j)^n (\mathcal{T}_q - z\, id)^n G = 0.$

4. $(\mathcal{T}_q - z\, id)^n = \sum_{k=0}^n \begin{pmatrix} n \\ k \end{pmatrix} \mathcal{T}_q^k (-z\, id)^{n-k} = \sum_{k=0}^n \begin{pmatrix} n \\ k \end{pmatrix} (-z)^{n-k} \mathcal{T}_{kq}.$

This implies $J \in \mathrm{FLOQ}^{(n)}_{q,z} \Leftrightarrow \sum_{k=0}^n \begin{pmatrix} n \\ k \end{pmatrix} (-z)^{n-k} J(t+kq) = 0$ $(t \geq 0)$. For $z = 0$ this becomes the equation $J(t) = 0$ $(t \geq nq)$. And for $z \neq 0$, one has to do with a linear vector recurrence equation of order nq. In both cases the set of its solutions form, because of corollary 2, a nqN dimensional linear space.

5. Because of 1, the sum is a direct sum. Because of 2, '$\supseteq$' holds. Because of 4, $\mathrm{FLOQ}^{(n)}_{q,z^{1/L}e^{2\pi i \frac{j}{L}}}$ has dimension nNq. Because the sum is direct, the right-hand side has dimension $L \cdot nNq$, i.e. has the dimension of $\mathrm{FLOQ}^{(n)}_{q,z}$. This even proves that '$=$' holds. □

Define $\Delta : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ by

$$(\Delta G)(t) := G(t+1) - G(t).$$

One has

$$\Delta\lambda^t = (\lambda - 1)\lambda^t \ (z \in \mathbb{C}^*),$$

a result that also follows from:

Proposition 42 *Suppose $m = 1$. For each polynomial p with complex coefficients one has $p(\mathcal{T}_1)(\lambda^t) = p(\lambda)\lambda^t$.* $\diamond$

Proof.— Write $p = a_0 + a_1 z + \cdots + a_L z^L$. Then $p(\mathcal{T}_1)(\lambda^t) = (a_0\mathcal{T}_1^{(0)} + \cdots + a_L\mathcal{T}_1^L)(\lambda^t) = a_0\lambda^t + a_1\lambda^{t+1} + \cdots + a_L\lambda^{t+L} = (a_0 + a_1\lambda + \cdots + a_L\lambda^L)\lambda^t = p(\lambda)\lambda^t$.
□

$\Delta t^n = (t+1)^n - t^n$, so Δ of a power is complicated. In this context the next definition is useful. Define for each integer $r \geq 0$, $t^{(r)}$ as the (polynomial) function

$$t^{(r)} := t(t-1)(t-2)\cdots(t-r+1).$$

It can be easily verified that

$$\Delta t^{(r)} = r t^{(r-1)} \ (r \geq 1),\ \Delta t^{(0)} = 0. \tag{30}$$

From $\deg t^{(r)} = r$, it follows that $t^{(0)}, t^{(1)}, \ldots, t^{(r)}$ are linearly independent and that $\mathrm{Vect}(t^{(0)}, t^{(1)}, \ldots, t^{(r)}) = \mathrm{Vect}(t^0, t^1, \ldots, t^r)$.

Proposition 43 *A base of*

1. $\mathrm{FLOQ}_{1,1}^{(n)}$ *is* $t^i e^{(j)}$ $(0 \leq i \leq n-1,\ 1 \leq j \leq N)$*;*
2. $\mathrm{FLOQ}_{1,z}^{(n)}$ *is* $t^i z^t e^{(j)}$ $(0 \leq i \leq n-1,\ 1 \leq j \leq N)$*;*
3. $\mathrm{FLOQ}_{q,z}^{(n)}$ *is* $t^i z^{\frac{t}{q}} e^{2\pi i \frac{l}{q} t} e^{(j)}$ $(0 \leq i \leq n-1,\ 0 \leq l \leq q-1,\ 1 \leq j \leq N)$. $\diamond$

Proof.— 1. First we consider the case where $m = 1$. We shall prove that t^i $(0 \leq i \leq n-1)$ is a base of $\mathrm{FLOQ}_{1,1}^{(n)}$. One has $\mathrm{FLOQ}_{1,1}^{(n)} = \ker(\Delta^n)$. To see that $t^i \in \mathrm{FLOQ}_{1,1}^{(n)}$ $(0 \leq i \leq n-1)$, we write $t^i = \sum_{k=0}^{i} a_{ki} t^{(k)}$ and notice that, by (30), $\Delta^n(t^{(i)}) = \sum_{k=0}^{i} a_{ki}\Delta^{(n)}(t^{(k)}) = \sum_{k=0}^{i} a_{ki} \cdot 0 = 0$. Because Δ^n is a 1−dimensional vector recurrence operator of order n, $\ker(\Delta^n)$ has, because of corollary 2, dimension n. Because $t^{(i)}$ $(0 \leq i \leq n-1)$ are n linearly independent vectors, they are a base.

Now consider the case of general m. As above one proves that the given vectors belong to $\mathrm{FLOQ}_{1,1}^{(n)}$ and that they are linearly independent. Because we therefore we have Nn linearly independent vectors, the proof is complete by noticing that $\dim(\mathrm{FLOQ}_{1,1}^{(n)}) = Nn$.

2. Because of 1 and proposition 41(2). 3. Because of proposition 41(5) and 2.
□

Here is another type of base of $\mathrm{FLOQ}_{q,z}$:

Proposition 44 *A base of* $\mathrm{FLOQ}_{q,z}$ *is* $\delta^{(l,j)}$ $(1 \leq l \leq q,\ 1 \leq j \leq N)$*, where* $\delta^{(l,j)}$ *is the unique element of* $\mathrm{FLOQ}_{q,z}$ *so that* $\delta^{(l,j)}(t) = \delta_{lt} e^{(j)}$ $(1 \leq t \leq q)$. $\diamond$

Proof.— First of all notice that the $\delta^{(l,j)}$ are well defined because of proposition 37. It is easy to verify that they are linearly independent. Because we have $Nq = \dim(\mathrm{FLOQ}_{q,z})$ of them, they form a base. □

Lemma 2 $\mathcal{T}_s\delta^{(l,j)} = z^{-\frac{l-s-\overline{l-s}}{q}}\delta^{(\overline{l-s},j)}$ $(1 \le l \le q,\ 1 \le j \le N, s \ge 0)$. $\diamond$

Proof.— $\mathcal{T}_s\delta^{(l,j)}, z^{-\frac{l-s-\overline{l-s}}{q}}\delta^{(\overline{l-s},j)} \in \mathrm{FLOQ}_{q,z}$. And they are equal at $t = \overline{l-s} + r$ $(0 \le r \le q-1)$: $\mathcal{T}_s\delta^{(l,j)}(\overline{l-s}+r) = \delta^{(l,j)}(\overline{l-s}+r+s) = \delta^{(l,j)}(\overline{\overline{l-s}+r+s+q\frac{\overline{l-s}+r+s-\overline{\overline{l-s}+r+s}}{q}}) = z^{\frac{\overline{l-s}+r+s-\overline{r+l}}{q}}\delta^{(l,j)}(l+r) = z^{\frac{\overline{l-s}+r+s-\overline{r+l}}{q}}\delta_{r0}e^{(j)} = z^{\frac{\overline{l-s}+s-l}{q}}$ $\delta_{r0}e^{(j)} = z^{-\frac{l-s-\overline{l-s}}{q}}\delta^{(\overline{l-s},j)}(\overline{l-s}+r)$. Thus, by proposition 37, they are equal for all t. □

Generalized Floquetian mappings have a clear asymptotic behaviour:

Proposition 45 *1.* $\mathrm{FLOQ}^{(1)}_{q,z} \subseteq l^\infty$ $(|z| \le 1)$.

2. $\mathrm{FLOQ}^{(n)}_{q,z} \subseteq c_0$ $(|z| < 1)$.

3. $\mathrm{FLOQ}^{(n)}_{q,z} \cap l^\infty = \mathrm{FLOQ}^{(1)}_{q,z}$ $(|z| = 1)$.

4. $\mathrm{FLOQ}^{(n)}_{q,z} \cap l^\infty = \{0\}$ $(|z| > 1)$.

5. $\mathrm{FLOQ}^{(n)}_{q,z} \cap c_0 = \{0\}$ $(|z| \ge 1)$. $\diamond$

Proof.— 1 and 2 are evident from the explicit expressions in proposition 43.

3. '$\supseteq$': Evident. '$\subseteq$': First consider the case with $m = 1$. Take $g \in \mathrm{FLOQ}^{(n)}_{q,z} \cap l^\infty$. We may assume that $g \ne 0$. With $z_l := z^{1/q}e^{2\pi i\frac{l}{q}}$ $(0 \le l \le q-1)$, there exist because of proposition 43(3), $C^i_l \in \mathbb{C}$ $(0 \le i \le n-1,\ 0 \le l \le q-1)$ so that $g = \sum_{il} C^i_l t^i z^t_l$. Let $(0 \le)p(\le n-1)$ be the maximum value of i for which there exists a non-zero C^i_l. Let $\mathcal{J}$ be the set of l's for which $C^p_l \ne 0$. Take $j \in \mathcal{J}$. Now $g = \sum_{l \in \mathcal{J}} C^p_l t^p z^t_l + \sum_{i=1}^{p-1}\sum_{l=0}^{q-1} C^i_l t^i z^t_l$. Thus

$$g = z_j^t t^p \left(C^p_j + \sum_{l \in \mathcal{J}, l \ne j} C^p_l (\frac{z_l}{z_j})^t + \sum_{i=1}^{p-1}\sum_{l=0}^{q-1} C^i_l t^{i-p} (\frac{z_l}{z_j})^t \right). \tag{31}$$

The three terms in parentheses are bounded. Because $g \in l^\infty$, $g \ne 0$ and the term $z^t_j t^p$ is bounded if and only if $p = 0$, one has $p = 0$, i.e. $g \in \mathrm{FLOQ}^{(1)}_{q,z}$.

Now, consider the case of general m. By (28), $\mathrm{FLOQ}^{(n)}_{q,z} \cap l^\infty = (\sum^\oplus_j \mathcal{FLOQ}^{(n)}_{q,z} \cdot e^{(j)}) \cap l^\infty = \sum^\oplus_j((\mathcal{FLOQ}^{(n)}_{q,z} \cdot e^{(j)}) \cap l^\infty) = \sum^\oplus_j(\mathcal{FLOQ}^{(n)}_{q,z} \cap l^\infty) \cdot e^{(j)}$. By the result for $m = 1$ this equals $\sum^\oplus_j \mathcal{FLOQ}^{(1)}_{q,z} \cdot e^{(j)} = \mathrm{FLOQ}^{(1)}_{q,z}$.

4. As in 3. (Again we obtain (31) and the three terms in parentheses are bounded. The term $z^t_j t^p$ is unbounded.)

5. Because of 4, we may suppose that $|z| = 1$. The rest of the proof is as in 3.
□

B.2.3 Semi-flows

Proposition 46 *For* ● *with q–periodic U one has $\mathcal{G}_{t_0+q;V}(t+q) = \mathcal{G}_{t_0;V}(t)$.* $\diamond$

Proof.— One easily verifies that $t \mapsto \mathcal{G}_{t_0+q;V}(t+q)$ is a solution of ●. Also $\mathcal{G}_{t_0;V}$ is a such a solution. At $t = t_0, \ldots, t_0 + M - 1$ they are equal. Therefore they are equal everywhere. □

Proposition 47 *For* ● *with q–periodic U one has*

1. $\Phi_{t+q,t'+q} = \Phi_{t,t'}$ $(t, t' \ge 0)$;

2. $\Phi_{jq,0} = \Phi^j_{q,0}$ $(j \ge 0)$;

3. $\Phi_{t+jq,0} = \Phi_{t,0}\Phi^{j}_{q,0}$ $(t \geq 0)$;

4. $\Phi_{t,0} = \Phi_{\bar{t},0}\Phi_{q,0}^{\frac{t-\bar{t}}{q}}$ $(t \geq 0)$. ◇

Proof.— 1. $\Phi_{t+q;t'+q}(V) = (\mathcal{G}_{t'+q;V}(t+q), \ldots, \mathcal{G}_{t'+q;V}(t+q+M-1))$. By proposition 46 this equals $(\mathcal{G}_{t';V}(t), \ldots, \mathcal{G}_{t';V}(t+M-1)) = \Phi_{t,t'}(V)$.

2. By proposition 7(2), $\Phi_{jq,0} = \Phi_{jq,(j-1)q}\Phi_{(j-1)q,0}$. This equals $\Phi_{q,0}\Phi_{(j-1)q,0}$ by 1. Continuing in this way gives the desired result.

3. $\Phi_{t+jq,0} = \Phi_{t+jq,jq}\Phi_{jq,0} = \Phi_{t,0}\Phi_{jq,0} = \Phi_{t,0}\Phi^{j}_{q,0}$.

4. By 2, $\Phi_{t,0} = \Phi_{\bar{t}+\frac{t-\bar{t}}{q}q,0} = \Phi_{\bar{t},0}\Phi_{q,0}^{\frac{t-\bar{t}}{q}}$. □

If X is a set, then a family of mappings $(g_t)_{t\in\mathbb{N}}$ of X into itself is called a "one-parameter semi-group of mappings of X"[101] if

$$g_t \circ g_s = g_{t+s} \ (t, s \geq 0),$$

$$g_0 = id.$$

Corollary 9 *Consider ● with $q-$periodic U. $(\Phi_{jq})_{j\in\mathbb{N}}$ is a one-parameter semi-group of mappings of $(\mathbb{C}^N)^M$.* ◇

B.2.4 Floquet, Bloch and monodromy operators

Consider $●^{(0)}$. For the with $●^{(0)}$ associated vector recurrence operator $\mathcal{H}$ one has:

Proposition 48 $[\mathcal{H}, \mathcal{T}_q] = 0$. ◇

Proof.— $((\mathcal{H}\mathcal{T}_q)G)(t) = (\mathcal{H}(\mathcal{T}_qG))(t) = \sum_{s=0}^{M} \mathcal{B}_t^{(s)}((\mathcal{T}_qG)(t+s)) = \sum_{s=0}^{M} \mathcal{B}_t^{(s)}(G(t+s+q)) = \sum_{s=0}^{M} \mathcal{B}_{t+q}^{(s)}(G(t+q+s)) = (\mathcal{H}G)(t+q) = (\mathcal{T}_q(\mathcal{H}G))(t) = ((\mathcal{T}_q\mathcal{H})G)(t)$. □

Because of proposition 48, $\mathcal{T}_q$ lets the generalized eigenspaces of $\mathcal{H}$ invariant, in particular $\mathcal{T}_q$ lets $\mathrm{SOL}^{(0)}$ invariant. So,

$$\mathcal{T}_q \restriction \mathrm{SOL}^{(0)} : \ \mathrm{SOL}^{(0)} \to \mathrm{SOL}^{(0)}$$

is well defined and if U is $q-$periodic even

$$\mathcal{T}_q \restriction \mathrm{SOL} : \ \mathrm{SOL} \to \mathrm{SOL}$$

is well defined. Because of the same reason

$$\mathcal{H} \restriction \mathrm{FLOQ}_{q,z} : \ \mathrm{FLOQ}_{q,z} \to \mathrm{FLOQ}_{q,z}$$

is well defined. We call $\mathcal{T}_q \restriction \mathrm{SOL}^{(0)}$ 'Floquet operator' and each $\mathcal{H} \restriction \mathrm{FLOQ}_{q,z}$ 'Bloch operator'. Four our purposes the Floquet operator is the most important. However, the presentation will benefit by not forgetting the Bloch-operators.[102] By 'Floquet theory' we mean the analysis of the Floquet operator and by 'Bloch theory' we mean the analysis of the Bloch operators. There are interesting connections between objects associated with Floquet and Bloch theories. In this context we here only mention the following evident but useful result.

Proposition 49 *For each integer $n \geq 1$:*
$\mathrm{SOL}^{(0)} \cap \mathrm{FLOQ}^{(n)}_{q,z} = \ker((\mathcal{T}_q - z\, id)^n \restriction \mathrm{SOL}^{(0)}) = \ker(\mathcal{H} \restriction \mathrm{FLOQ}^{(n)}_{q,z})$. ◇

[101] This is indeed a semi-group.

[102] However, the mentioned 'eigenvalue parameter' in footnote 69 is necessary to really deal with Bloch theory.

Corollary 10 *The dimension of each of the linear subspaces* $\mathrm{SOL}^{(0)} \cap \mathrm{FLOQ}_{q,z}^{(n)}$, $\ker((\mathcal{T}_q - z\, id)^n \restriction \mathrm{SOL}^{(0)})$ *and* $\ker(\mathcal{H} \restriction \mathrm{FLOQ}_{q,z}^{(n)})$ *is at most* $\min(MN, nNq)$. ◇

$\mathcal{T}_q : (\mathbb{C}^N)^{\mathbb{N}} \to (\mathbb{C}^N)^{\mathbb{N}}$ is not injective, but one has:

Proposition 50 *The Floquet operator* $\mathcal{T}_q \restriction \mathrm{SOL}^{(0)}$ *is a linear automorphism.* ◇

Proof.— It is sufficient to prove that $\ker(\mathcal{T}_q \restriction \mathrm{SOL}^{(0)}) = \{0\}$. Take $G \in \mathrm{SOL}^{(0)}$ with $\mathcal{T}_q G = 0$. Then $(\mathcal{T}_q G)(t) = 0$ $(t \geq 0)$, i.e. $G(t+q) = 0$. In particular, $G(q) = \cdots = G(q + M - 1) = 0$, which, with corollary 1, implies that $G = 0$. □

Consider $\bullet^{(0)}$. By proposition 40, generalized Floquetian spaces $\mathrm{FLOQ}_{q,z_1}^{(n)}$ and $\mathrm{FLOQ}_{q,z_2}^{(n)}$ are isomorphic. This of course does not imply that the restrictions of $\mathcal{H} \restriction \mathrm{FLOQ}_{q,z_1}^{(n)}$ and $\mathcal{H} \restriction \mathrm{FLOQ}_{q,z_2}^{(n)}$ are similar.[103] If one prefers to work with generalized Floquetian spaces for $z = 1$, then this is possible by defining for $z \in \mathbb{C}^*$, $\mathcal{H}_z : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ by

$$\mathcal{H}_z := \mathcal{K}_{z,1} \mathcal{H} \mathcal{K}_{z,1}^{-1}.$$

One has

$$(\mathcal{H}_z G)(t) := \sum_{s=0}^{M-1} (z^{s/q} \mathcal{B}_t^{(s)}) G(t+s).$$

(Notice that for $z = 1$, $\mathcal{H}_z$ equals the with $\bullet^{(0)}$ associated linear recurrence operator $\mathcal{H}$.) For all $G \in (\mathbb{C}^N)^{\mathbb{N}}$ one then has

$$\mathcal{H}_z(z^{-t/q} G) = z^{-t/q}(\mathcal{H} G),$$

which implies in turn:

Proposition 51 $\ker(\mathcal{H} \restriction \mathrm{FLOQ}_{q,z}^{(n)}) = z^{t/q} \ker(\mathcal{H}_z \restriction \mathrm{FLOQ}_{q,1}^{(n)})$ $(z \in \mathbb{C}^*)$. ◇

Definition 3 Consider $\bullet^{(0)}$. By a 'monodromy operator (of $\bullet^{(0)}$)' one understands any $(\mathcal{U}_{t_0}^{(0)})^{-1} \mathcal{T}_q\, \mathcal{U}_{t_0}^{(0)}$. And $(\mathcal{U}^{(0)})^{-1} \mathcal{T}_q\, \mathcal{U}^{(0)}$ is called 'the monodromy operator (of $\bullet^{(0)}$)'; We denote it by $\mathcal{N}$. ◇

Corollary 11 *Any two monodromy operators of* $\bullet^{(0)}$ *are similar (with the Floquet operator)* . ◇

B.2.5 Relation between monodromy operators and semi-flows

Denote by E_n the $n \times n$–identity matrix.

Proposition 52 *The matrix of the monodromy operator of* $\bullet^{(0)}$ *with respect to the base* $e_1^{(0)}, \ldots, e_N^{(0)}, \ldots\ldots, e_1^{(M-1)}, \ldots, e_N^{(M-1)}$ *of* $(\mathbb{C}^N)^M$, *equals the matrix of the Floquet operator with respect to the base* $G_1^{(0)}, \ldots, G_N^{(0)}, \ldots\ldots, G_1^{(M-1)}, \ldots, G_N^{(M-1)}$ *of* $\mathrm{SOL}^{(0)}$. ◇

Proof.— Denoting a matrix of a linear transformation B with respect to the base x and y by $[B]_y^x$, the first matrix equals $[(\mathcal{U}^{(0)})^{-1} \mathcal{T}_q \mathcal{U}^{(0)}]_e^e = [(\mathcal{U}^{(0)})^{-1}]_e^G [\mathcal{T}_q]_G^G [\mathcal{U}^{(0)}]_G^e = E_{MN} [\mathcal{T}_q]_G^G E_{MN} = [\mathcal{T}_q]_G^G$, i.e. equals the second one. □

Theorem 1 $\mathcal{N} = \Phi_{q,0}^{(0)}$. ◇

[103] To avoid any confusion: two $H_1 \in \mathrm{End}(V_1)$ and $H_2 \in \mathrm{End}(V_2)$ are similar if there exists a linear isomorphism $U : V_1 \to V_2$ such that $UH_1 = H_2U$.

Proof.— We have to prove that $(\mathcal{U}^{(0)})^{-1}\mathcal{T}_q\mathcal{U}^{(0)} = \Phi^{(0)}_{q,0}$, i.e. that $\mathcal{T}_q \restriction \mathrm{SOL}^{(0)} = \mathcal{U}^{(0)}\Phi^{(0)}_{q,0}(\mathcal{U}^{(0)})^{-1}$. So, take $G \in \mathrm{SOL}^{(0)}$. Then $(\mathcal{U}^{(0)}\Phi^{(0)}_{q,0}(\mathcal{U}^{(0)})^{-1})G =$ $\mathcal{U}^{(0)}\Phi^{(0)}_{q,0}(G(0),\ldots,G(q+M-1)) = \mathcal{U}^{(0)}(G(q),\ldots,G(q+M-1)) =$ $\mathcal{G}^{(0)}_{0;(G(q),\ldots,G(q+M-1))}$. From the other hand $\mathcal{G}_{0;(G(q),\ldots,G(q+M-1))}$ is the unique solution of $\bullet^{(0)}$ that equals $G(t)$ at $t = q+s$ $(0 \le s \le M-1)$. Also $\mathcal{T}_qG$ is such. Thus $\mathcal{T}_qG = \mathcal{U}^{(0)}\Phi^{(0)}_{q,0}(\mathcal{U}^{(0)})^{-1}G$. □

B.2.6 Floquet multipliers and existence of Floquetian solutions of $\bullet^{(0)}$

Consider $\bullet^{(0)}$. An eigenvalue of the Floquet operator $\mathcal{T}_q \restriction \mathrm{SOL}^{(0)}$ (or of a monodromy operator) will also be called 'Floquet multiplier (of $\bullet^{(0)}$)'. Its characteristic equation (characteristic polynomial) is called 'characteristic equation (characteristic polynomial) of $\bullet^{(0)}$'. We shall denote the (different) Floquet multipliers by

$$\lambda_1,\ldots,\lambda_k,$$

their respective (algebraic) multiplicities by

$$\alpha_1,\ldots,\alpha_k,$$

and their respective geometric multiplicities by

$$\beta_1,\ldots,\beta_k.$$

Because the Floquet operator is a linear automorphism, one has:

Proposition 53 *No Floquet multiplier is zero.* ◇

Because $\mathrm{SOL}^{(0)}$ has dimension MN, one has

$$\alpha_1 + \cdots + \alpha_k = MN.$$

Concerning the existence of Floquetian solutions of $\bullet^{(0)}$, proposition 49 implies:

Proposition 54 *[Floquet. (1883)] $\bullet^{(0)}$ has a non-zero Floquetian solution of type (q,z) $\Leftrightarrow$ z is a Floquet multiplier of $\bullet^{(0)}$.* ◇

Notice that if $e^{i\pi\frac{r}{s}}$ with $\frac{r}{s} \in \mathbb{Q}$ is a Floquet multiplier of $\bullet^{(0)}$, then $\bullet^{(0)}$ has a $2sq$−periodic non-zero solution. And notice that $\dim(\mathrm{SOL}^{(0)} \cap \mathrm{FLOQ}_{Q,\lambda_i}) = \beta_i$ and $\dim(\mathrm{SOL}^{(0)} \cap \mathrm{FLOQ}_{q,z}) = 0$ if z is not a Floquet multiplier of $\bullet^{(0)}$.

Proposition 55 *For each $V \in (\mathbb{C}^N)^M$ one has: $(\mathcal{N} - z\, id)^nV = 0 \Leftrightarrow \mathcal{G}^{(0)}_{0;V} \in \mathrm{FLOQ}^{(n)}_{q,z}$.* ◇

Proof.— $(\mathcal{T}_q - z\, id \restriction \mathrm{SOL}^{(0)})^n = \mathcal{U}^{(0)}(\mathcal{N} - z\, id)^n(\mathcal{U}^{(0)})^{-1}$ and $\mathcal{G}^{(0)}_{0;V} = \mathcal{U}^{(0)}V$.

'⇒': $(\mathcal{T}_q - z\, id)^n\mathcal{G}^{(0)}_{0,V} = \mathcal{U}^{(0)}(\mathcal{N} - z\, id)^nV = \mathcal{U}^{(0)}0 = 0$.

'⇐': $(\mathcal{N} - z\, id)^nV = (\mathcal{U}^{(0)})^{-1}((\mathcal{T}_q - z\, id)^n\mathcal{G}^{(0)}_{0;V}) = (\mathcal{U}^{(0)})^{-1}0 = 0$. □

B.2.7 Asymptotic behaviour of solutions of $\bullet^{(0)}$

Consider $\bullet^{(0)}$. Denote the generalized eigenspaces of the Floquet operator belonging to the Floquet multiplier λ_j by $\mathcal{D}_j^{(n)}$, i.e.

$$\mathcal{D}_j^{(n)} = \ker(\mathcal{T}_q - \lambda_j \, id \restriction \mathrm{SOL}^{(0)})^n \quad (n \geq 1).$$

(We also write $\mathcal{D}_j$ instead of $\mathcal{D}_j^{(1)}$.) For each Floquet multiplier λ_j let

$$r_j$$

be its index.[104] It is well known that

$$\mathrm{SOL}^{(0)} = \mathcal{D}_1^{(r_1)} \oplus \cdots \oplus \mathcal{D}_k^{(r_k)}. \tag{32}$$

Proposition 56 *For $\bullet^{(0)}$ one has*
$\mathrm{SOL}^{(0)} = \sum_{j=1,\ldots,k}^{\oplus} \ker(\mathcal{T}_q - \lambda_j \, id \restriction \mathrm{SOL}^{(0)})^{r_j} = \sum_{j=1,\ldots,k}^{\oplus} \ker(\mathcal{H} \restriction \mathrm{FLOQ}_{q,\lambda_j}^{(r_j)})$. $\diamond$

Proof.— By (32) and proposition 49. □

We now want to study the dimension of of $SOL^{(0)} \cap \mathcal{W}$ for various linear subspaces $\mathcal{W}$ of $\mathbb{C}^{\mathbb{N}}$. Things become easier if $\mathcal{W}$ is $\mathcal{H}$- and $\mathcal{T}_q$-invariant. So, suppose this. Now,

$$\mathcal{T}_q \restriction \mathrm{SOL}^{(0)} \cap \mathcal{W}$$

is defined. Because $\mathrm{SOL}^{(0)} \cap \mathcal{W}$ is a linear subspace of $\mathrm{SOL}^{(0)}$, each eigenvalue of $\mathcal{T}_q \restriction \mathrm{SOL}^{(0)} \cap \mathcal{W}$ is also an eigenvalue of $\mathcal{T}_q \restriction \mathrm{SOL}^{(0)}$, i.e. is a Floquet multiplier of $\bullet^{(0)}$. Therefore, we can number the Floquet multipliers in such a way that

$$\lambda_1, \ldots, \lambda_l,$$

are the (different) eigenvalues of $\mathcal{T}_q \restriction \mathrm{SOL}^{(0)} \cap \mathcal{W}$. (Of course, $l \leq k$.) Denote their respective (algebraic) multiplicities by

$$\tilde{\alpha}_1, \ldots, \tilde{\alpha}_l$$

and their respective geometric multiplicities by

$$\tilde{\beta}_1, \ldots, \tilde{\beta}_l.$$

Furthermore, denote the generalized eigenspaces of $\mathcal{T}_q \restriction \mathrm{SOL}^{(0)} \cap \mathcal{W}$ belonging to the eigenvalue λ_j by $\tilde{\mathcal{D}}_j^{(n)}$, i.e.

$$\tilde{\mathcal{D}}_j^{(n)} = \ker(\mathcal{T}_q - \lambda_j \, id \restriction \mathrm{SOL}^{(0)} \cap \mathcal{W})^n \quad (n \geq 1).$$

And for each eigenvalue λ_j let

$$\tilde{r}_j$$

be its index. Again

$$\mathrm{SOL}^{(0)} \cap \mathcal{W} = \tilde{\mathcal{D}}_1^{(\tilde{r}_1)} \oplus \cdots \oplus \tilde{\mathcal{D}}_l^{(\tilde{r}_l)}. \tag{33}$$

With these notations one has:

[104] I.e. the least integer $n \geq 1$ so that $\mathcal{D}_j^{(n+1)} = \mathcal{D}_j^{(n)}$. Moreover, there are the general valid properties: $1 \leq \dim(\mathcal{D}_j) = \beta_j \leq \alpha_j$, $\dim(\mathcal{D}_j^{(r_j)}) = \alpha_j$, $\dim(\mathcal{D}_j^{(n)}) \geq \beta_j + n - 1$ $(1 \leq n \leq r_j)$. This implies $r_j \leq \alpha_j$ and $\alpha_j = \beta_j \Leftrightarrow r_j = 1$.

Lemma 3 *1. In case $\mathcal{W} = l^\infty$:*

(a) $|\lambda_j| \leq 1$ $(1 \leq j \leq l)$ and each Floquet multiplier not outside $\mathbb{T}$ is an eigenvalue of $\mathcal{T}_q \restriction \mathrm{SOL}^{(0)} \cap l^\infty$;

(b) $\tilde{r}_j = r_j$ if $|\lambda_j| < 1$, $\tilde{r}_j = 1$ if $|\lambda_j| = 1$;

(c) $\tilde{\alpha}_j = \alpha_j$ if $|\lambda_j| < 1$, $\tilde{\alpha}_j = \beta_j$ if $|\lambda_j| = 1$.

2. In case $\mathcal{W} = c_0$:

(a) $|\lambda_j| < 1$ $(1 \leq j \leq l)$ and each Floquet multiplier inside $\mathbb{T}$ is an eigenvalue of $\mathcal{T}_q \restriction \mathrm{SOL}^{(0)} \cap c_0$;

(b) $\tilde{r}_j = r_j$;

(c) $\tilde{\alpha}_j = \alpha_j$. ◇

Proof.— 1a. $\{0\} \neq \ker(\mathcal{T}_q - \lambda_j id \restriction \mathrm{SOL}^{(0)} \cap \mathcal{W}) = \mathrm{FLOQ}_{q,z} \cap l^\infty \cap \mathrm{SOL}^{(0)}$. Because of proposition 45(4), $|\lambda_j| \leq 1$. And, if λ is a Floquet multiplier not outside $\mathbb{T}$, then $\ker(\mathcal{T}_q - \lambda id \restriction \mathrm{SOL}^{(0)} \cap l^\infty) = \mathrm{FLOQ}_{q,\lambda} \cap l^\infty \cap \mathrm{SOL}^{(0)} = \mathrm{FLOQ}_{q,\lambda} \cap \mathrm{SOL}^{(0)} = \ker(\mathcal{T}_q - \lambda id \restriction \mathrm{SOL}^{(0)}) \neq \{0\}$ and thus λ is an eigenvalue of $\mathcal{T}_q \restriction \mathrm{SOL}^{(0)} \cap l^\infty$.

1b. One has for each $n \geq 1$: $\mathrm{SOL}^{(0)} \cap \mathrm{FLOQ}^{(n+1)}_{q,\lambda_j} = \mathrm{SOL}^{(0)} \cap \mathrm{FLOQ}^{(n)}_{q,\lambda_j} \Rightarrow \mathrm{SOL}^{(0)} \cap l^\infty \cap \mathrm{FLOQ}^{(n+1)}_{q,\lambda_j} = \mathrm{SOL}^{(0)} \cap l^\infty \cap \mathrm{FLOQ}^{(n)}_{q,\lambda_j}$. This implies $\tilde{r}_j \leq r_j$. If $|\lambda_j| < 1$, then because of proposition 45(2), '$\Leftarrow$' also holds and it follows that $r_j = \tilde{r}_j$. If $\lambda_j = 1$, then $\mathrm{SOL}^{(0)} \cap l^\infty \cap \mathrm{FLOQ}^{(2)}_{q,\lambda_j} = \mathrm{SOL}^{(0)} \cap l^\infty \cap \mathrm{FLOQ}^{(1)}_{q,\lambda_j}$, so $\tilde{r}_j = 1$.

1c. $\tilde{\alpha}_j = \dim(\mathrm{SOL}^{(0)} \cap l^\infty \cap \mathrm{FLOQ}^{(\tilde{r}_j)}_{q,\lambda_j})$. If $|\lambda_j| < 1$, this becomes by 1b, $\dim(\mathrm{SOL}^{(0)} \cap l^\infty \cap \mathrm{FLOQ}^{(r_j)}_{q,\lambda_j}) = \dim(\mathrm{SOL}^{(0)} \cap \mathrm{FLOQ}^{(r_j)}_{q,\lambda_j}) = \alpha_j$. And if $|\lambda_j| = 1$, it becomes $\dim(\mathrm{SOL}^{(0)} \cap l^\infty \cap \mathrm{FLOQ}^{(1)}_{q,\lambda_j}) = \dim(\mathrm{SOL}^{(0)} \cap \mathrm{FLOQ}^{(1)}_{q,\lambda_j}) = \beta_j$.

2. As 1. □

Theorem 2 *Consider* $\bullet^{(0)}$. *One has:*

1. $\dim(\mathrm{SOL}^{(0)} \cap l^\infty)$ *equals the sum of the algebraic multiplicities of the Floquet multipliers inside $\mathbb{T}$ plus the sum of the geometric multiplicities of the Floquet multipliers on $\mathbb{T}$;*

2. $\dim(\mathrm{SOL}^{(0)} \cap c_0)$ *equals the sum of the algebraic multiplicities of the Floquet multipliers inside $\mathbb{T}$. ◇*

Proof.— Using (33) and lemma 3, $\dim(\mathrm{SOL}^{(0)} \cap l^\infty) = \sum_{j=1}^{l} \dim(\tilde{\mathcal{D}}_j^{(\tilde{r}_j)})$ $= (\sum_{j=1,|\lambda_j|<1}^{l} + \sum_{j=1,|\lambda_j|=1}^{l})\tilde{\alpha}_j = \sum_{j=1,|\lambda_j|<1}^{l} \alpha_j + \sum_{j=1,|\lambda_j|=1}^{l} \beta_j$, which is the desired result. 2. As 1. □

Corollary 12 *Consider* $\bullet^{(0)}$. *One has:*

1. $\dim(\mathrm{SOL}^{(0)} \cap l^\infty) = 0 \Leftrightarrow$ *each Floquet multiplier lies outside $\mathbb{T}$;*

2. $\dim(\mathrm{SOL}^{(0)} \cap l^\infty) \geq 1 \Leftrightarrow$ *at least one Floquet multiplier lies inside or on $\mathbb{T}$;*

3. $\dim(\mathrm{SOL}^{(0)} \cap l^\infty) = MN \Leftrightarrow$ *all Floquet multipliers lie on or inside $\mathbb{T}$ and each Floquet multiplier on $\mathbb{T}$ is semi-simple;*

4. $\dim(\mathrm{SOL}^{(0)} \cap c_0) = 0 \Leftrightarrow$ *each Floquet multiplier lies on or outside $\mathbb{T}$;*

5. $\dim(\mathrm{SOL}^{(0)} \cap c_0) \geq 1 \Leftrightarrow$ *at least one Floquet multiplier lies inside* $\mathbb{T}$;

6. $\dim(\mathrm{SOL}^{(0)} \cap c_0) = MN \Leftrightarrow$ *each Floquet multiplier lies inside* $\mathbb{T}$. $\diamond$

Notice that we have obtained corollary 12 without using an explicit expression for the general solution of $\ominus^{(0)}$.

B.2.8 Adjoint on PER_q

On PER_q an inner-product $< \mid >_q$ is defined by

$$< G|J >_q := \sum_{t=0}^{q-1} < G(t)|J(t) >,$$

where $< | >$ is given by (19).

Consider for an integer $s \geq 0$ the linear transformation $\mathcal{T}_s \restriction \mathrm{PER}_q$. Define the linear transformation $\overline{\mathcal{T}_s}$ of PER_q by: given $G \in \mathrm{PER}_q$, $\overline{\mathcal{T}_s}G$ is the unique element of PER_q for which

$$(\overline{\mathcal{T}_s}G)(t) = G(t-s) \ (t \geq s).$$

Consider for a $q-$periodic sequence $B := (\mathcal{B}_t)_{t \geq 0}$ of $\mathrm{End}(\mathbb{C}^m)$ (the multiplication operator) $\mathcal{M}_B \restriction \mathrm{PER}_q$. Define the linear transformation $\overline{\mathcal{M}_B}$ of PER_q by

$$\overline{\mathcal{M}_B} := \mathcal{M}_{B^\star} \restriction \mathrm{PER}_q$$

where $B^\star := (\mathcal{B}_t^\star)_{t \geq 0}$. With these notations one has:

Proposition 57 *1. The adjoint* $(\mathcal{T}_s \restriction \mathrm{PER}_q)^\star$ *is (nothing else than)* $\overline{\mathcal{T}_s}$.

2. The adjoint $(\mathcal{M}_B \restriction \mathrm{PER}_q)^\star$ *is (nothing else than)* $\overline{\mathcal{M}_B}$. $\diamond$

Proof.— 1. Take $G, L \in \mathrm{PER}_q$. Then $< \mathcal{T}_sG|L >_q = \sum_{t=0}^{q-1} < G(t+s)|L(t) >$ $\sum_{t=s}^{s+q-1} < G(t)|L(t-s) > = \sum_{t=s}^{s+q-1} < G(t)|(\overline{\mathcal{T}_s}L)(t) >$ Because of periodicity of G and $\mathcal{T}_sL$ this equals $\sum_{t=0}^{q-1} < G(t)|(\overline{\mathcal{T}_s}L)(t) > = < G|\overline{\mathcal{T}_s}L >_q$.

2. Take $G, L \in \mathrm{PER}_q$. Then $< \mathcal{M}_BG|L >_q = \sum_{t=0}^{q-1} < \mathcal{B}_t(G(t))|L(t) > =$ $\sum_{t=0}^{q-1} < G(t)|\mathcal{B}_t^\star(L(t)) > = \sum_{t=0}^{q-1} < G(t)|\mathcal{M}_{B^\star}(L(t)) > = < G|\mathcal{M}_{B^\star}L >_q$. $\Box$

Now, consider $\bullet^{(0)}$.

Corollary 13 *One has* $(\mathcal{H} \restriction \mathrm{PER}_q)^\star = \overline{\mathcal{H}}$, *where the linear transformation* $\overline{\mathcal{H}}$ *of* PER_q *is defined by: For* $G \in \mathrm{PER}_q$, $\overline{\mathcal{H}}G$ *is the unique element of* PER_q *with* $(\overline{\mathcal{H}}G)(t) := \sum_{s=0}^{M} (\mathcal{B}_{t-s}^{(s)})^\star (G(t-s))$ $(t \geq M)$. $\diamond$

Proof.— $(\mathcal{H} \restriction \mathrm{PER}_q)^\star = (\sum_{s=0}^{M} \mathcal{M}_{B^{(s)}} \mathcal{T}_s \restriction \mathrm{PER}_q)^\star = \sum_{s=0}^{M} \overline{\mathcal{T}_s}\, \overline{\mathcal{M}_{B^{(s)}}}$. $\Box$

(21) implies

$$\mathrm{SOL}_{\mathrm{ad}}^{(0)} \cap \mathrm{PER}_q = \ker(\overline{\mathcal{H}}). \tag{34}$$

Proposition 58 *The dimension of the linear space of $q-$periodic solutions of* $\bullet^{(0)}$ *is the same as that of its adjoint equation* $\bullet_{\mathrm{ad}}^{(0)}$. $\diamond$

Proof.— By (34) and corollary 13, one obtains $\mathrm{SOL}_{\mathrm{ad}}^{(0)} \cap \mathrm{PER}_q = \ker(\overline{\mathcal{H}}) =$ $\ker((\mathcal{H} \restriction \mathrm{PER}_q)^\star)$. The dimension of this last space equals $\dim(\ker(\mathcal{H} \restriction \mathrm{PER}_q))$, which in turn, by proposition 49, equals $\dim(\mathrm{SOL}^{(0)} \cap \mathrm{PER}_q)$. $\Box$

B.2.9 Floquetian solutions of ●

One may look for Floquetian solutions of ●. We consider this question for $q-$Floquetian solutions in case U is $q-$periodic.

Notice that the set of Floquetian solutions of type (r, z) of $●^{(0)}$ equals $\mathrm{SOL}^{(0)} \cap \mathrm{FLOQ}_{r,z}$ which is a linear space and that the set of Floquetian solutions of type (r, z) of ● equals $\mathrm{SOL} \cap\ \mathrm{FLOQ}_{r,z}$ which, like SOL, is affine.

Proposition 59 *Consider ● with non-zero q-periodic U. Then each $q-$Floquetian solution of ● is $q-$periodic.* ◇

Proof.— Suppose $G \in \mathrm{FLOQ}_{q,z} \cap\ \mathrm{SOL}$. One has $U \in \mathrm{FLOQ}_{q,1}$, $\mathcal{H}G \in \mathrm{FLOQ}_{q,z}$ and $\mathcal{H}G = U$. Thus $\mathrm{FLOQ}_{q,1} \cap\ \mathrm{FLOQ}_{q,z} \neq \{0\}$. Proposition 41(1) implies $z = 1$, i.e. G is $q-$periodic. □

Proposition 60 *Consider ● with $q-$periodic U. Then: 1 is not a Floquet multiplier of $●^{(0)}$ $\Leftrightarrow$ 0 is the unique $q-$periodic solution of $●^{(0)}$ $\Leftrightarrow$ ● has a unique $q-$periodic solution.* ◇

Proof.— This is a direct consequence of proposition 54 and proposition 11 with $\mathcal{W} = \mathrm{PER}_q$. □

Corollary 14 shows that in case 0 is not the unique $q-$periodic solution of $●^{(0)}$, ● (with $q-$periodic U) does not have necessarily a $q-$periodic solution.

Proposition 61 *Consider ● with $q-$periodic U. One has: ● has a $q-$periodic solution $\Leftrightarrow$ $< G|U >_q = 0$ for each $q-$periodic solution G of* $\mathrm{SOL}^{(0)}_{\mathrm{ad}}$. ◇

Proof.— ● has a $q-$periodic solution $\Leftrightarrow$ $U \in \mathrm{Im}(\mathcal{H} \restriction \mathrm{PER}_q)$ $\Leftrightarrow$ $U \in (\ker(\mathcal{H} \restriction \mathrm{PER}_q)^\star)^\perp$ $\Leftrightarrow$ $U \in (\ker(\overline{\mathcal{H}}))^\perp$ $\Leftrightarrow$ $U \in (\mathrm{SOL}^{(0)}_{\mathrm{ad}} \cap\ \mathrm{PER}_q)^\perp$ $\Leftrightarrow$ $< V^{(j)}|U >_q = 0\ (1 \leq j \leq m)$. □

Corollary 14 *If* $\dim(\mathrm{SOL}^{(0)} \cap\ \mathrm{PER}_q) \neq 0$, *there exists $U \in \mathrm{PER}_q$ so that $●^{(U)}$ does not have a $q-$periodic solution.* ◇

Proof.— Let $V^{(1)}, \ldots, V^{(m)}$ be a base of $\mathrm{SOL}^{(0)}_{\mathrm{ad}} \cap\ \mathrm{PER}_q$. Take for U one of the $V^{(j)}$. □

Much more can be said on typical solutions of ●,[105] but this will not be done here.

B.2.10 Periods

Given $●^{(0)}$, the set consisting of 0 and the positive values of r such that each $B^{(s)}_t, \ldots, B^{(M)}_t$ is $r-$periodic forms (under the addition) a semi-group Λ. Denote $\Lambda \setminus \{0\}$ by $\Lambda^\star$. An element of $\Lambda^\star$ is called 'a period of $●^{(0)}$'. Let $q_{\min}$ be the minimal period of $●^{(0)}$. Then

$$\Lambda = q_{\min} \cdot \mathbb{N}.$$

In the above we have always automatically worked with the period q of $●^{(0)}$. However, q is not necessarily the minimal period. Sometimes it might be desirable to work with $q_{\min}$.[106]

[105] For example, one may consider the following questions. If all Floquet multipliers of $●^{(0)}$ are inside $\mathbf{T}$ and $U \in l^\infty$ (c_0), does then each solution of ● belong to l^∞ (c_0)? If each solution of $●^{(0)}$ is bounded, is then also each solution of $●^{(17)}$ bounded? (See for for example [12, chapter IV.1] in a differential equation context.)

[106] For instance as in conjecture 1.

Consider $\bullet^{(0)}$. Suppose $r \in \Lambda^\star$ and l is a positive integer. Then the Floquet operators $\mathcal{T}_Q \restriction \mathrm{SOL}^{(0)}$ and $\mathcal{T}_{lQ} \restriction \mathrm{SOL}^{(0)}$ are well defined and one has:

$$\mathcal{T}_{lQ} \restriction \mathrm{SOL}^{(0)} = \mathcal{T}_Q^l \restriction \mathrm{SOL}^{(0)}.$$

This implies:

Proposition 62 *If $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of $\mathcal{T}_Q \restriction \mathrm{SOL}^{(0)}$, then $\lambda_1^l, \ldots, \lambda_k^l$ are those for $\mathcal{T}_{lQ} \restriction \mathrm{SOL}^{(0)}$.* ◇

B.2.11 Real solutions

In addition to the results in subsubsection A.2.6, we want to make some further remarks on real solutions for $\bullet$. We suppose in this subsubsection real coefficients.

One has

$$[\mathcal{C}, \mathcal{T}_q] = 0. \tag{35}$$

This implies

$$\mathcal{C}(\ker{(\mathcal{T}_q - z\ id)^n}) = \ker{(\mathcal{T}_q - \bar{z}\ id)^n},$$

i.e. $\mathcal{C}(\mathrm{FLOQ}_{q,z}^{(n)}) = \mathrm{FLOQ}_{q,\bar{z}}^{(n)}$. In particular $G \in \mathrm{FLOQ}_{q,z} \Rightarrow \mathcal{C}G \in \mathrm{FLOQ}_{q,\bar{z}}$.

Proposition 63 *For $\bullet^{(0)}$ with real coefficients one has: λ is a Floquet multiplier $\Rightarrow \overline{\lambda}$ is a Floquet multiplier. And if G is a non-zero Floquetian solution of $\bullet^{(0)}$ of type (q, λ), then $\mathcal{C}G$ is a non-zero Floquetian solution of $\bullet^{(0)}$ of type $(q, \overline{\lambda})$.* ◇

Proof.— Let $G \in \mathrm{SOL}^{(0)}$ with $G \neq 0$ such that $\mathcal{T}_q G = \lambda G$. Then $(\mathcal{C}\mathcal{T}_q)G = \mathcal{C}(\lambda G)$. Because of (35), and the semi-linearity of $\mathcal{C}$ this becomes $\mathcal{T}_q(\mathcal{C}G) = \overline{\lambda}(\mathcal{C}G)$. So, $\mathcal{C}G$ is Floquetian of type $(q, \overline{\lambda})$. Because of (18) one has that $\mathcal{C}G \in \mathrm{SOL}^{(0)}$. Because $\mathcal{C}G \neq 0$, $\overline{\lambda}$ is a Floquet multiplier. □

Because of proposition 63 the Floquet multipliers occur in complex conjugate pairs.

Proposition 64 *For each real Floquet-multiplier λ of $\bullet^{(0)}$ with real coefficients, there exists a real-valued non-zero Floquetian solution of type (q, λ).* ◇

Proof.— Let G be a non-zero Floquetian solution of type $\bullet^{(0)}$ of type (q, λ). By proposition 16, $\Re G$ and $\Im G$ are solutions of $\bullet^{(0)}$. Using proposition 63 one finds that they are also Floquetian of type (q, λ). The proof is complete by noting that one has $\Re G \neq 0$ or $\Im G \neq 0$.

B.3 First order vector case

B.3.1 The object

Denote $\bullet$ for $M = 1$ by ◗:

$$G(t+1) = \mathcal{A}_t(G(t)) + U_t \ (t \geq 0) \quad ◗.$$

So, each $\mathcal{A}_t \in \mathrm{GL}(\mathbb{C}^N), \mathcal{A}_{t+q} = \mathcal{A}_t \ (t \geq 0)$, each $U_t \in \mathbb{C}^N$ and $G : \mathbb{N} \to \mathbb{C}^N$.

B.3.2 Monodromy operators and semi-flows

Consider ▶. Because of propositions 18(2), 27 and theorem 1 one has the following explicit expressions for the monodromy operator $\mathcal{N}$ and the semi-flow $\Phi_{q,0}$:

Proposition 65 *1.* $\mathcal{N} = \mathcal{A}_{q-1} \cdots \mathcal{A}_1 \mathcal{A}_0$.

2. $\Phi_{q,0} = \mathcal{F}_{q-1} \cdots \mathcal{F}_1 \mathcal{F}_0 = \mathcal{N} + \sum_{s=0}^{q-1} \Phi^{(0)}_{q,s+1}(U_s)$. ◇

Proposition 66 *If F is a fundamental operator solution of ▶$^{(0)}$, then there exists a unique $C \in \mathrm{GL}(\mathbb{C}^N)$ so that $F(t+q) = F(t)C$ $(t \geq 0)$. In fact $C = F(0)^{-1}\mathcal{N}F(0)$.* ◇

Proof.— $F(t+q)$ is also a fundamental operator solution of ▶$^{(0)}$ because $F(t+1+q) = \mathcal{A}_{t+1+q}F(t+q) = \mathcal{A}_{t+1}F(t+q)$ $(t \geq 0)$. By proposition 22 there exists a unique $C \in \mathrm{GL}(\mathbb{C}^N)$ so that $F(t+q) = F(t)C$. In particular $F(q) = F(0)C$. So, by proposition 22, $F(0)CF(0)^{-1} = F(q)F(0)^{-1} = \Phi^{(0)}_{q,0} = \mathcal{N}$. □

Theorem 3 *Every fundamental operator solution F of ▶$^{(0)}$ has the form*

$$F(t) \;=\; P(t)B^t \;(t \geq 0)$$

where $B \in \mathrm{GL}(\mathbb{C}^N)$, $P(t) \in \mathrm{End}(\mathbb{C}^N)$ $(t \geq 0)$ and $P(t+q) = P(t)$ $(t \geq 0)$. ◇

Proof.— By proposition 66, there exists a unique $C \in \mathrm{GL}(\mathbb{C}^N)$ with $F(t+q) = F(t)C$ $(t \geq 0)$. Now apply proposition 39. □

Denote the generalized eigenspaces of the monodromy operator $\mathcal{N}$ belonging to the Floquet multiplier λ_j by $\mathcal{E}_j^{(n)}$, i.e.

$$\mathcal{E}_j^{(n)} \;=\; \ker\,(\mathcal{N} - \lambda_j\; id)^n \;(n \geq 1)$$

(we also write $\mathcal{E}_j$ instead of $\mathcal{E}_j^{(1)}$). For each Floquet multiplier λ_j let

$$r_j$$

be the index of $\mathcal{N}$.[107] One has

$$\mathbb{C}^N = \mathcal{E}_1^{(r_1)} \oplus \cdots \oplus \mathcal{E}_k^{(r_k)}. \tag{36}$$

Because of $\mathcal{N} = (\mathcal{U}^{(0)})^{-1}\mathcal{T}_q\,\mathcal{U}^{(0)}$ one has $\mathcal{D}_j^{(n)} \;=\; \mathcal{U}^{(0)}(\mathcal{E}_j^{(n)})$.

B.3.3 Explicit expression for the general solution of ▶$^{(0)}$

Proposition 67 *Let $V_0 \in \mathcal{E}_j$. Then:* $\mathcal{G}^{(0)}_{0;V_0}(t) = \lambda^{\frac{t-\bar{t}}{q}}(\mathcal{A}_{\bar{t}-1} \cdots \mathcal{A}_0)V_0$. ◇

Proof.— By proposition 47(4), $\mathcal{G}^{(0)}_{0;V_0}(t) = \Phi^{(0)}_{t,0}V_0 = (\Phi^{(0)}_{\bar{t},0}\mathcal{N}^{\frac{t-\bar{t}}{q}})V_0 = \lambda_j^{\frac{t-\bar{t}}{q}}\Phi^{(0)}_{\bar{t},0}V_0 = \lambda_j^{\frac{t-\bar{t}}{q}}(\mathcal{A}_{\bar{t}-1} \cdots \mathcal{A}_0)V_0$. □

In case each Floquet multiplier is semi-simple,[108] proposition 67 leads to an explicit expression for the general solution of ▶$^{(0)}$ (i.e. to a base of $\mathrm{SOL}^{(0)}$):

[107] Of course, because of corollary 11, this r_j is the same as the index of the Floquet operator, for which we also use the symbol r_j.

[108] I.e. its algebraic equals its geometric multiplicity.

Proposition 68 *Suppose each Floquet multiplier λ_j of $\mathbf{\Phi}^{(0)}$ is semi-simple. For each $j \in \{1, \ldots, k\}$, let F_i^j $(0 \le i \le \alpha_j - 1)$ be a base of $\mathcal{E}_j$. Then, a base for* $\mathrm{SOL}^{(0)}$ *equals* $\lambda_j^{\frac{t-\bar{t}}{q}}(\mathcal{A}_{\bar{t}-1} \cdots \mathcal{A}_0)F_i^j$ $(1 \le j \le k,\ 0 \le i \le \alpha_j - 1)$. $\diamond$

Proof.— $H_i^j(t) := \lambda^{(t-\bar{t})/q}(\mathcal{A}_{\bar{t}-1} \cdots \mathcal{A}_0)F_i^j$. By proposition 67, each $H_i^j \in \mathrm{SOL}^{(0)}$. Because $\sum_{j=1}^k \alpha_j = N = \dim(\mathrm{SOL}^{(0)})$, the proof is complete if we show that the H_i^j are linear independent. For this it is sufficient to show the linear independency of their values (vectors from $\mathbb{C}^N$) at $t = 0$. These values equal F_i^j $(1 \le j \le k, 1 \le i \le \alpha_i)$, which are linearly independent. $\square$

Things are more complicated when not every Floquet multiplier is semi-simple. To handle this general case, the appropriate generalization of proposition 67 is:

Lemma 4 *Let $V_0 \in \mathcal{E}_j^{(n)}$. Then*
$\mathcal{G}_{0;V_0}^{(0)}(t) = (\mathcal{A}_{\bar{t}-1} \cdots \mathcal{A}_0) \sum_{l=0}^{\min(\frac{t-\bar{t}}{q}, n-1)} \lambda_j^{\frac{t-\bar{t}}{q}-l} \begin{pmatrix} \frac{t-\bar{t}}{q} \\ l \end{pmatrix} (\mathcal{N} - \lambda_j\, id)^l V_0.$[109] $\diamond$

Proof.— $\mathcal{G}_{0;V_0}^{(0)}(t) = \Phi_{t,0}^{(0)} V_0 = (\mathcal{A}_{\bar{t}-1} \cdots \mathcal{A}_0)\mathcal{N}^{\frac{t-\bar{t}}{q}} V_0 =$
$(\mathcal{A}_{\bar{t}-1} \cdots \mathcal{A}_0)(\mathcal{N} - \lambda_j\, id + \lambda_j\, id)^{\frac{t-\bar{t}}{q}} V_0 =$
$(\mathcal{A}_{\bar{t}-1} \cdots \mathcal{A}_0) \sum_{l=0}^{\frac{t-\bar{t}}{q}} \begin{pmatrix} \frac{t-\bar{t}}{q} \\ l \end{pmatrix} (\mathcal{N} - \lambda_j\, id)^l \lambda_j^{\frac{t-\bar{t}}{q}-l} V_0 =$
$(\mathcal{A}_{\bar{t}-1} \cdots \mathcal{A}_0) \sum_{l=0}^{\min(\frac{t-\bar{t}}{q}, n-1)} \lambda_j^{\frac{t-\bar{t}}{q}-l} \begin{pmatrix} \frac{t-\bar{t}}{q} \\ l \end{pmatrix} (\mathcal{N} - \lambda_j\, id)^l V_0.$ $\square$

Because of proposition 5 one obtains:

Corollary 15 *Let $V_0 \in \mathbb{C}^N$. Decompose $V_0 = \sum_{j=1}^k V_{0;j}$ according to (36). Then*
$\mathcal{G}_{0;V_0}^{(0)}(t) = \sum_{j=1}^k (\mathcal{A}_{\bar{t}-1} \cdots \mathcal{A}_0) \sum_{l=0}^{\min(\frac{t-\bar{t}}{q}, r_j-1)} \lambda_j^{\frac{t-\bar{t}}{q}-l} \begin{pmatrix} \frac{t-\bar{t}}{q} \\ l \end{pmatrix} (\mathcal{N} - \lambda_j\, id)^l V_{0;j}.$ $\diamond$

Proposition 69 *For each $j \in \{1, \ldots, k\}$, let F_i^j $(1 \le i \le \alpha_j)$ be a base of $\mathcal{E}_j^{(r_j)}$. Then, a base for* $\mathrm{SOL}^{(0)}$ *equals*

$(\mathcal{A}_{\bar{t}-1} \cdots \mathcal{A}_0) \sum_{l=0}^{\min(\frac{t-\bar{t}}{q}, r_j-1)} \lambda_j^{\frac{t-\bar{t}}{q}-l} \begin{pmatrix} \frac{t-\bar{t}}{q} \\ l \end{pmatrix} (\mathcal{N} - \lambda_j\, id)^l F_i^j$ $(1 \le j \le k,\ 1 \le i \le \alpha_j)$. $\diamond$

Proof.— By lemma 4, all the given objects belong to $\mathrm{SOL}^{(0)}$. Because $\sum_{j=1}^k \alpha_j$ $N = \dim(\mathrm{SOL}^{(0)})$, the proof is complete if we show that they are linear independent. For this it is sufficient to show the linear independency of their values (vectors from $\mathbb{C}^N$) at $t = 0$. These values equal F_i^j $(1 \le j \le k, 1 \le i \le \alpha_i)$, which are linearly independent. $\square$

Notice that for t large enough the general solution of $\mathrm{SOL}^{(0)}$ equals

$$\sum_{j=1}^k \sum_{i=1}^{\alpha_j} C_i^j (\mathcal{A}_{\bar{t}-1} \cdots \mathcal{A}_0) \sum_{l=0}^{r_j-1} \lambda_j^{\frac{t-\bar{t}}{q}-l} \begin{pmatrix} \frac{t-\bar{t}}{q} \\ l \end{pmatrix} (\mathcal{N} - \lambda_j\, id)^l F_i^j.$$

The above explicit expressions are, in fact, not very explicit. They simplify in the autonomous case. However, the best results for explicit expressions are obtained in subsubsection B.4.7 for the arbitrary order autonomous scalar case.

[109] The Jordan canonical form of $\mathcal{N}$ may be useful to calculate further $(\mathcal{N} - \lambda_j\, id)^l$.

B.3.4 Asymptotic behaviour of solutions of ▶

Theorem 4 *Consider* ▶ *with q–periodic U. One has:* ▶ *has a bounded solution* $\Leftrightarrow$ ▶ *has a q–periodic solution.* $\diamond$

Proof.— ' $\Leftarrow$': Evident. ' $\Rightarrow$': It is equivalent to prove: ▶ has no q–periodic solution $\Rightarrow$ ▶ has no bounded solution. So, suppose ▶ has no q–periodic solution. Because of proposition 60, 0 is not the unique q–periodic solution of ▶$^{(0)}$. So, $\mathrm{SOL}^{(0)} \cap \mathrm{PER}_q$ has positive dimension, say m. By proposition 58, also $\mathrm{SOL}^{(0)}_{\mathrm{ad}} \cap \mathrm{PER}_q$ has dimension m. Let $V^{(1)}, \ldots, V^{(m)}$ be a base of $\mathrm{SOL}^{(0)}_{\mathrm{ad}} \cap \mathrm{PER}_q$. Because of proposition 61 one does not have $< V^{(j)}|U >_q = 0$ $(1 \leq j \leq k)$. Let j_0 so that $< V^{(j_0)}|U >_q \neq 0$, i.e. with $V := V^{(j_0)}$, $\sum_{t=0}^{q-1} < V(t)|U(t) > \neq 0$. Because of proposition 20 one has

$$V(t) \;=\; (\Phi^{(0)}_{q,t+1})^{\star}(V(q-1)) \; (0 \leq t \leq q-1).$$

So, this inequality becomes

$$x := \sum_{s=0}^{q-1} < V(q-1)|\Phi^{(0)}_{q,s+1}(U_s) > \neq \; 0$$

and by taking $t = 2q-1$ one obtains

$$\Phi^{\star}_{q,0}(V(q-1)) = V(q-1).$$

Because of proposition 27 one has $\Phi_{t,0} = \Phi^{(0)}_{t,0} + \sum_{s=0}^{t-1} \Phi^{(0)}_{t,s+1}(U_s)$ $(t \geq 0)$. Because of proposition 47(1) this becomes $\Phi_{t+jq,jq} = \Phi^{(0)}_{t,0} + \sum_{s=0}^{t-1} \Phi^{(0}_{t,s+1}(U_s)$ $(t, j \geq 0)$. For $t = q$ this becomes

$$\Phi_{(j+1)q,jq} = \mathcal{N} + \sum_{s=0}^{q-1} \Phi^{(0}_{q,s+1}(U_s) \; (j \geq 0).$$

Now fix a solution G of ▶. From the above $\Phi_{(j+1)q,jq}(G(jq)) = \mathcal{N}(G(jq)) + \sum_{s=0}^{q-1} \Phi^{(0}_{q,s+1}(U_s)$, that is $G((j+1)q) = \mathcal{N}(G(jq)) + \sum_{s=0}^{q-1} \Phi^{(0)}_{q,s+1}(U_s)$, hence $< V(q-1)|G((j+1)q) > = < V(q-1)|\mathcal{N}(G(jq)) > +x$. From this

$$< V(q-1)|G((j+1)q) > = < V(q-1)|G(jq) > +x \; (j \geq 0).$$

Write this as $y_{j+1} = y_j + x$. One finds $y_j = y_0 + jx$ $(j \geq 0)$. So, we have proven that

$$< V(q-1)|G((j+1)q) > = \; y_0 + jx \; (j \geq 0),$$

which shows that G is not bounded. $\square$

B.3.5 Reducibility

If in ▶$^{(0)}$ one makes the linear substitution of variables $J(t) = L_t(G(t))$, then, as we have seen in proposition 17, one obtains the vector recurrence equation

$$J(t+1) = (L_{t+1} A_t L_t^{-1})(J(t)) \; (t \geq 0).$$

Proposition 70 *If in* ▶$^{(0)}$ *one makes the linear substitution of variables $J(t) = L_t(G(t))$ where $L_{t+q} = L_t$ $(t \geq 0)$, then the vector recurrence equation obtained is again of type* ▶$^{(0)}$, *and its monodromy operator is similar to that of* ▶$^{(0)}$. $\diamond$

Proof.— Because of the invertibility and q−periodicity of the L_t, the equation obtained is of type $\blacktriangleright^{(0)}$. By proposition 65 the monodromy operator of $\blacktriangleright^{(0)}$ equals $\mathcal{A}_{q-1}\cdots\mathcal{A}_0$ and that of the obtained vector recurrence equation equals $(L_q\mathcal{A}_{q-1}L_{q-1}^{-1})(L_{q-1}\mathcal{A}_{q-2}L_{q-2}^{-1})\cdots(L_1\mathcal{A}_0L_0^{-1}) = L_q(\mathcal{A}_{q-1}\cdots\mathcal{A}_0)L_0^{-1} =$ $L_0(\mathcal{A}_{q-1}\cdots\mathcal{A}_0)L_0^{-1}$. □

Proposition 70 makes clear that given two vector recurrence equations of type $\blacktriangleright^{(0)}$ (with the same q), it will not always be possible to transform the one into the other by a q−periodic linear substitution of variables. However, one has:

Theorem 5 *$\blacktriangleright^{(0)}$ can be transformed by a q−periodic linear substitution of variables into an autonomous equation of type $\blacktriangleright^{(0)}$.* ◇

Proof.— Applying proposition 39 to $\Phi_{t,0}^{(0)}$ with $C = \Phi_{q,0}^{(0)}$ (see proposition 47(3)) gives $\Phi_{t,0}^{(0)} = P(t)B^t$ with $P(t+q) = P(t)$ and B invertible. Take $L_t = P(t)^{-1}$. Then $L_{t+1}\mathcal{A}_tL_t^{-1} = B^{t+1}(\Phi_{t+1,0}^{(0)})^{-1}\Phi_{t+1,t}^{(0)}\Phi_{t,0}^{(0)}B^{-t} = B$. □

Theorem 5 is at the heart of the reason why the theory for periodic coefficient equations is so similar to constant coefficient ones. However, to determine the Floquet multipliers one needs to know the Floquet or a monodromy operator. In the first order constant coefficient case the monodromy operator is $\mathcal{A}$ and in the case of periodic coefficients it equals $\mathcal{A}_{q-1}\cdots\mathcal{A}_0$, a much more complicated object. This is one of the differences between constant and periodic coefficient equations.

B.3.6 Autonomous (first order) case

We always shall denote the autonomous linear vector recurrence equation $\triangleright$ by $\underline{\blacktriangleright}$:

$$G(t+1) = \mathcal{A}(G(t)) + U \ (t \geq 0) \qquad \underline{\blacktriangleright},$$

where $\mathcal{A} \in \mathrm{GL}(\mathbb{C}^N)$ and $U \in \mathbb{C}^N$. By proposition 65(1), the monodromy operator of $\underline{\blacktriangleright}^{(0)}$ equals $\mathcal{A}$. Thus the Floquet multipliers of $\underline{\blacktriangleright}^{(0)}$ are nothing else than the eigenvalues of $\mathcal{A}$. Instead of 'Floquet multiplier of $\underline{\blacktriangleright}^{(0)}$' one usually speaks of 'characteristic root of $\underline{\blacktriangleright}^{(0)}$'. So, a 'characteristic root of $\underline{\blacktriangleright}^{(0)}$' is nothing other than an eigenvalue of $\mathcal{T}_1 \restriction \mathrm{SOL}^{(0)}$.

Here are additional (typical) properties for the semi-flows for autonomous equations.

Proposition 71 *Consider $\underline{\blacktriangleright}$. One has:*

1. $\Phi_{t,t'} = \mathcal{F}^{t-t'} \quad (t,t' \geq 0)$;
2. $\Phi_{t,t'} = \Phi_{t-t',0}$ $(t \geq t' \geq 0)$ *and* $\Phi_{t,0}\Phi_{t',0} = \Phi_{t+t',0}$ $(t,t' \geq 0)$;
3. $(\Phi_{t,t_0})_{t\in\mathbb{N}}$ *is a one-parameter semi-group of mappings of* $\mathbb{C}^N$;
4. *For the principal operator solution F_p of $\underline{\blacktriangleright}^{(0)}$ one has $F_p(t+t') = F_p(t)F_p(t')$ $(t,t' \geq 0)$.* ◇

Proof.— 1. This follows from proposition 18(2).

2. Because of 1.

3. $\Phi_{0,0} = id$ and $\Phi_{t+t',0} = \Phi_{t+t',t}\Phi_{t,0} = \Phi_{t',0}\Phi_{t,0}$.

4. Fix t'. Both $F(t+t')$ as $F(t)F(t')$ are operator solutions. Because they are equal for $t = 0$, they are equal. □

Notice that, if G is a solution of $\underline{\blacktriangleright}^{(0)}$, then $G(t) = \mathcal{A}^{t-t_0}(G(t_0))$ $(t_0, t \geq 0)$.

For the generalized eigenspaces belonging to the characteristic root λ_j of $\underline{\blacktriangleright}^{(0)}$ one now has

$$\mathcal{E}_j^{(n)} \;=\; \ker\left(\mathcal{A} - \lambda_j \; id\right)^n \; (n \geq 1).$$

Propositions 67 - 69 and corollary 15 can be simplified in an obvious way. For example proposition 68 can be restated as follows.

Proposition 72 *Suppose each eigenvalue of $\mathcal{A}$ is semi-simple. For $j \in \{1, \ldots, k\}$, let F_i^j $(0 \leq i \leq \alpha_j - 1)$ be a base of the eigenspace of λ_j. Then a base for* $\mathrm{SOL}^{(0)}$ *equals $\lambda_j^t F_i^j$ $(1 \leq j \leq k,\ 0 \leq i \leq \alpha_j - 1)$.* ◇

Here is a new result.

Proposition 73 *For $V_0 \in \mathcal{E}_j^{(n)}$ one has $\mathcal{G}_{0;V_0}^{(0)}(t) \in \mathcal{E}_j^{(n)}$ $(t \geq 0)$.* ◇

Proof.— $(\mathcal{A} - \lambda_j \; id)^n(\mathcal{G}_{0;V_0}(t)) = (\mathcal{A} - \lambda_j \; id)^n(\mathcal{A}^t(V_0)) = \mathcal{A}^t((\mathcal{A} - \lambda_j \; id)^n(V_0)) = \mathcal{A}^t(0) = 0$. □

Also a new result is the following 'stable subspace theorem'.

Proposition 74 *Consider $\underline{\blacktriangleright}^{(0)}$. Let $E := \sum_{j,|\lambda_j|<1}^{\oplus} \mathcal{E}_j^{(r_j)}$. If G is a solution of $\underline{\blacktriangleright}^{(0)}$ with $G(0) \in E$, then $G(t) \in E$ $(t \geq 0)$ and $\lim_{t\to\infty} G(t) = 0$.* ◇

Proof.— Write $G(0) = \sum_j H_j$ with $H_j \in \mathcal{E}_j^{(r_j)}$. Then $G = \mathcal{G}_{0;G(0)}^{(0)} = \mathcal{G}_{0;\sum_j H_j}^{(0)} = \sum_j \mathcal{G}_{0;H_j}^{(0)}$. By proposition 73, $\sum_j \mathcal{G}_{0;H_j}^{(0)}(t) \in \sum_j \mathcal{E}_j^{(r_j)} = E$. Moreover, $\lim_{t\to\infty} G(t) = 0$ by lemma 4. □

Finally, let us determine a particular solution of $\underline{\blacktriangleright}$. Well, from the variation of constants formula (22) we may take for this

$$G(t) := \mathcal{A}^t \sum_{j=0}^{t-1} \mathcal{A}^{-s-1} U.$$

Moreover, in case $id - \mathcal{A}$ is invertible, $(id - \mathcal{A})^{-1}U$ is a particular solution of $\underline{\blacktriangleright}$. This solution even is constant.

B.4 Arbitrary order scalar case

B.4.1 The object

We are going to study an equations of the form

$$\sum_{s=0}^{M} v_t^{(s)} g(t+s) = w_t \quad (t \geq 0) \qquad \mathrm{J},$$

where $w_t, v_t^{(0)}, \ldots, v_t^{(M)} \in \mathbb{C}$ $(t \geq 0)$, no $v_t^{(M)}$ and no $v_t^{(0)}$ is zero, each $v_t^{(s)}$ is $q-$periodic and $g : \mathbb{N} \to \mathbb{C}$. Notice that J is nothing other than ● for $N = 1$. So, all results for ● also apply to J. From the other hand J is a scalar recurrence equation, so we can also consider the with J associated vector recurrence equation < J > and all specific results for ▶ also apply (in some sense) to J.

B.4.2 Monodromy matrix

Proposition 75 *1. The Floquet operator of* $J^{(0)}$ *and of* $< J^{(0)} >$ *are similar by (the linear isomorphism)* $\mathcal{P} : \mathrm{SOL}^{(0)} \to < \mathrm{SOL}^{(0)} >$.

2. The monodromy operator of $J^{(0)}$ *and of* $< J^{(0)} >$ *are equal.* ◇

Proof.— 1. For $g \in \mathrm{SOL}^{(0)}$, both $((\mathcal{P}\mathcal{T}_q)g)(t)$ and $((\mathcal{T}_q\mathcal{P})g)(t)$ equal $(\mathcal{P}g)(t+q)$.

2. By theorem 1 the monodromy operator equals $\Phi^{(0)}_{q,0}$. Now apply proposition 35. □

By 'the monodromy matrix of $J^{(0)}$ we understand the matrix of the monodromy operator of J with respect to the canonical base of $\mathbb{C}^N$. Denote by

$$\mathcal{M}_t = \mathrm{Comp}\,(\frac{v_t^{(M-1)}}{v_t^{(M)}}, \ldots, \frac{v_t^{(0)}}{v_t^{(M)}})$$

the transfer matrices associated with $J^{(0)}$.

Corollary 16 *The monodromy matrix of* $J^{(0)}$ *equals* $\mathcal{M}_{q-1}\cdots\mathcal{M}_0$. ◇

Proof.— By proposition 65(1), $\mathcal{A}_{q-1}\cdots\mathcal{A}_0$, where $\mathcal{A}_t = \mathcal{M}_t \in \mathrm{End}(\mathbb{C}^M)$, is the monodromy operator of $< J^{(0)} >$. This implies the desired result. □

B.4.3 Determination of Floquet multipliers

Because of corollary 16, one has that the characteristic equation of $J^{(0)}$ equals $|\mathcal{M}_{q-1}\cdots\mathcal{M}_1\mathcal{M}_0 - z\ id| = 0$. For the Floquet multipliers $\lambda_1, \ldots, \lambda_k$ of $J^{(0)}$ one has:

Proposition 76 *1.* $\lambda_1\lambda_2\cdots\lambda_k = (-1)^{Mq}\dfrac{v_0^{(0)}\cdots v_{q-1}^{(0)}}{v_0^{(M)}\cdots v_{q-1}^{(M)}}$.

2. $\lambda_1 + \cdots + \lambda_k = \mathrm{Tr}\,(\mathcal{M}_{q-1}\mathcal{M}_{q-2}\cdots\mathcal{M}_0)$.

Proof.— 2 is evident and 1 holds because of $\lambda_1\cdots\lambda_k = |\mathcal{N}|$ and (24). □

It is possible to generalize the above results for ●. However, this is not done in this article.

B.4.4 Explicit expression for the general solution of $J^{(0)}$ in case of semi-simple Floquet-multipliers

In trying to obtain explicit expressions for the general solution of $J^{(0)}$ one can take the route via $< J^{(0)} >$ and the (not so explicit) results in subsubsection B.3.3. Here we use another route.

Consider $J^{(0)}$. By proposition 56, $\mathrm{SOL}^{(0)} = \sum^{\oplus}_{i=1,\ldots,k} \ker(\mathcal{H} \restriction \mathcal{FLCOQ}^{(r_i)}_{q,\lambda_i})$. Because of proposition 51 this becomes

$$\mathrm{SOL}^{(0)} = \sum_{i=1,\ldots,k}^{\oplus} \lambda_i^{t/q} \ker(\mathcal{H}_{\lambda_i} \restriction \mathcal{FLCOQ}^{(r_i)}_{q,1}).$$

Now further suppose that all Floquet-multipliers are semi-simple. Then one has

$$\mathrm{SOL}^{(0)} = \sum_{i=1,\ldots,k}^{\oplus} \lambda_i^{t/q} \ker(\mathcal{H}_{\lambda_i} \restriction \mathcal{FLCOQ}_{q,1})$$

and $\ker(\mathcal{H}_{\lambda_i} \restriction \mathcal{FLCOQ}_{q,1})$ has dimension α_i. Fix a base $y^{(1)},\ldots,y^{(q)}$ of $\mathcal{FLCOQ}_{q,1}$ and consider $[\mathcal{H}_{\lambda_i} \restriction \mathcal{FLCOQ}_{q,1}]^y_y \in M_q(\mathbb{C})$, i.e. the matrix of $\mathcal{H}_{\lambda_i} \restriction \mathcal{FLCOQ}_{q,1}$ with respect to this base. Let $f_{i,j} = \begin{pmatrix} f_{i,j}(1) \\ \vdots \\ f_{i,j}(q) \end{pmatrix} \in \mathbb{C}^q$ $(1 \le j \le \alpha_i)$ be a base of $\ker([\mathcal{H}_{\lambda_i} \restriction \mathcal{FLCOQ}_{q,1}]^y_y)$. Then $\sum_{l=1}^q f_{i,j}(l) y^{(l)}$ $(1 \le j \le \alpha_i)$ is a base of $\ker(\mathcal{H}_{\lambda_i} \restriction \mathcal{FLCOQ}_{q,1})$ and thus $\lambda_i^{t/q} \sum_{l=1}^q f_{i,j}(l) y^{(l)}$ $(1 \le i \le k, 1 \le j \le \alpha_i)$ is a base of $\mathrm{SOL}^{(0)}$. The rest of this subsubsection is devoted to determine $[\mathcal{H}_{\lambda_i} \restriction \mathcal{FLCOQ}_{q,z}]^y_y$ in case $y^{(i)} = \delta^{(i)}$; Here

$$\lambda_i^{t/q} f_{i,j}(\bar{t}) \ (1 \le i \le k, 1 \le j \le \alpha_i)$$

is a base of $\mathrm{SOL}^{(0)}$. In the following we only consider the case $q \ge M$.

Lemma 5 *Suppose $q \ge M$.*

1. $\mathcal{H}\delta^{(l)} = \sum_{j=1}^l v_j^{(l-j)} \delta^{(j)} + \sum_{j=q+l-M}^q z v_j^{(q+l-j)} \delta^{(j)}$ $(1 \le l \le M)$.
2. $\mathcal{H}\delta^{(l)} = \sum_{j=l-M}^l v_j^{(l-j)} \delta^{(j)}$ $(M+1 \le l \le q)$. ◇

Proof.— With lemma 2. 1. $\mathcal{H}\delta^l = ((\sum_{s=0}^{l-1} + \sum_{s=l}^M)(v_t^{(s)} \mathcal{T}_s))\delta^{(l)} = \sum_{s=0}^{l-1} v_t^{(s)} \delta^{(l-s)} + \sum_{s=l}^M z v_t^{(s)} \delta^{(q+l-s)} = \sum_{j=1}^l v_t^{(l-j)} \delta^{(j)} + \sum_{j=q+l-M}^q z v_t^{(q+l-j)} \delta^{(j)} = \sum_{j=1}^l v_j^{(l-j)} \delta^{(j)} + \sum_{j=q+l-M}^q z v_j^{(q+l-j)} \delta^{(j)}$.

2. $\mathcal{H}\delta^l = \sum_{s=0}^M v_t^{(s)} \mathcal{T}_s \delta^l = \sum_{s=0}^M v_t^{(s)} \delta^{l-s} = \sum_{j=l-M}^l v_t^{(l-j)} \delta^j = \sum_{j=l-M}^l v_j^{(l-j)} \delta^j$. □

Corollary 17 *For the matrix $[\mathcal{H} \restriction \mathcal{FLCOQ}_{q,z}]^\delta_\delta = (H_{jl})_{1 \le j \le q, 1 \le l \le q}$ one has in case $q \ge M$*

1. *if $1 \le l \le M$:*
   $H_{jl} = v_j^{(l-j)}$ $(1 \le j \le l-1)$;
   $H_{jl} = v_l^{(0)} + \delta_{Mq} z v_l^{(M)}$ $(j = l)$;
   $H_{jl} = 0$ $(l+1 \le j \le q+l-M-1)$;
   $H_{jl} = z v_j^{(q+l-j)}$ $(q+l-M \le j \le q)$.
2. *if $M+l \le l \le q$:*
   $H_{jl} = 0$ $(1 \le j \le l-M-1)$;
   $H_{jl} = v_j^{(l-j)}$ $(l-M \le j \le l)$;
   $H_{jl} = 0$ $(l+1 \le j \le q)$. ◇

B.4.5 Asymptotic behaviour of solutions of J

The result of theorem 4 also applies to J:

Theorem 6 *Consider J with $q-$periodic w. One has: J has a bounded solution ⇔ J has a $q-$periodic solution.* ◇

Proof.— '⇐': Evident. '⇒': Suppose g is a bounded solution of J. Because of proposition 29(2), $\mathcal{P}g$ is a solution of < J >. $\mathcal{P}g$ is bounded. Because of theorem 4, < J > has a $q-$periodic solution, say G. Because of proposition 29(3), QG is a solution of J. QG is $q-$periodic. □

B.4.6 Second order scalar case

In this subsubsection we always consider J with $M = 2$:

$$v_t^{(2)} g(t+2) + v_t^{(1)} g(t+1) + v_t^{(0)} g(t) = w_t \ (t \geq 0). \quad \bullet$$

Proposition 77 *1. The transfer matrices associated with $\bullet^{(0)}$ are given by*

$$\mathcal{M}_t = \begin{pmatrix} 0 & 1 \\ -\frac{v_t^{(0)}}{v_t^{(2)}} & -\frac{v_t^{(1)}}{v_t^{(2)}} \end{pmatrix} \ (t \geq 0).$$

2. The monodromy matrix of $\bullet^{(0)}$ equals $\mathcal{M}_{q-1} \cdots \mathcal{M}_0$.

3. The characteristic equation for $\bullet^{(0)}$ equals

$$z^2 - (\mathrm{Tr} \prod_{m=0}^{q-1} \begin{pmatrix} 0 & 1 \\ -\frac{v_{q-1-m}^{(0)}}{v_{q-1-m}^{(2)}} & -\frac{v_{q-1-m}^{(1)}}{v_{q-1-m}^{(2)}} \end{pmatrix})z + \frac{v_0^{(0)} v_1^{(0)} \cdots v_{q-1}^{(0)}}{v_0^{(2)} v_1^{(2)} \cdots v_{q-1}^{(2)}} = 0. \ \diamond$$

Proof.— 1. $\mathcal{M}_t = \mathrm{Comp}\,(\frac{v_t^{(1)}}{v_t^{(2)}}, \frac{v_t^{(0)}}{v_t^{(2)}}) = \begin{pmatrix} 0 & 1 \\ -\frac{v_t^{(0)}}{v_t^{(2)}} & -\frac{v_t^{(1)}}{v_t^{(2)}} \end{pmatrix}$.

2. By corollary 16. 3. If A is a linear transformation of a 2−dimensional linear space, then its characteristic polynomial equals $|A - z\ id| = z^2 - \mathrm{Tr}\ (A)z + |A|$. Because of 1 and 2 the desired equation now follows. □

In case $\bullet^{(0)}$ has real coefficients, its characteristic equation also has real coefficients. This implies in this case for the Floquet multipliers λ_1, λ_2 of $\bullet^{(0)}$ that $\lambda_1, \lambda_2 \in \mathbb{R}$ or $\lambda_1 = \overline{\lambda_2}$. (Also see proposition 90.)

Proposition 78 *Consider the Floquet multipliers of $\bullet^{(0)}$ with real coefficients. Define $\Delta_q := \mathrm{Tr} \prod_{m=0}^{q-1} \begin{pmatrix} 0 & 1 \\ -\frac{v_{q-1-m}^{(0)}}{v_{q-1-m}^{(2)}} & -\frac{v_{q-1-m}^{(1)}}{v_{q-1-m}^{(2)}} \end{pmatrix}$ and $D := \frac{v_0^{(0)} \cdots v_{q-1}^{(0)}}{v_0^{(2)} \cdots v_{q-1}^{(2)}}$. Then*

1. *One Floquet multiplier is inside and one is outside $\mathbb{T} \Leftrightarrow |1 + D| < |\Delta_q|$;*
2. *Both Floquet multipliers are inside $\mathbb{T} \Leftrightarrow 1 + D > |\Delta_q|,\ D < 1$;*
3. *Both Floquet multipliers are outside $\mathbb{T} \Leftrightarrow 1 + \frac{1}{D} > |\frac{\Delta_q}{D}|,\ \frac{1}{D} < 1$. $\diamond$*

Proof.— Because of propositions 77(3) and 92. □

Proposition 79 *For $q \geq 3$, the matrix of $H \restriction \mathcal{FLOQ}_{q,z}$ with respect to the base $\delta^{(1)}, \ldots, \delta^{(q)}$ equals*

$$\begin{pmatrix} v_1^{(0)} & v_1^{(1)} & v_1^{(2)} & 0 & \cdots & 0 & 0 & 0 \\ 0 & v_2^{(0)} & v_2^{(1)} & v_2^{(2)} & \cdots & 0 & 0 & 0 \\ 0 & 0 & v_3^{(0)} & v_3^{(1)} & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & v_4^{(0)} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & v_{q-2}^{(0)} & v_{q-2}^{(1)} & v_{q-2}^{(2)} \\ z v_{q-1}^{(2)} & 0 & 0 & 0 & \cdots & 0 & v_{q-1}^{(0)} & v_{q-1}^{(1)} \\ z v_q^{(1)} & z v_q^{(2)} & 0 & 0 & \cdots & 0 & 0 & v_q^{(0)} \end{pmatrix}. \ \diamond$$

Proof.— Because of corollary 17 with $M = 2$. □

Finally, here is an example of a semi-simple Floquet-multiplier that is not simple: Take $q = 2$, $v_t^{(0)} = v_t^{(2)} = 1$ and $v_t^{(1)} = 0$. Then the monodromy matrix equals $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, so -1 is a semi-simple eigenvalue that is not simple.

B.4.7 Autonomous case

We shall denote the autonomous linear scalar recurrence equation J by $\underline{\mathsf{J}}$. Thus:

$$\sum_{s=0}^{M} v^{(s)} g(t+s) = w \quad (t \geq 0) \qquad \underline{\mathsf{J}},$$

where $v^{(s)}, w \in \mathbb{C}$, $v^{(0)} \neq 0$ and $v^{(M)} \neq 0$. We shall see that the characteristic polynomial of $\underline{\mathsf{J}}^{(0)}$ takes a very simple form and shall give a very explicit expression for the general solution of $\underline{\mathsf{J}}$.

A Floquet multiplier of $\underline{\mathsf{J}}^{(0)}$ usually is called 'characteristic root of $\underline{\mathsf{J}}^{(0)}$'. A 'characteristic root of $\underline{\mathsf{J}}^{(0)}$' is thus nothing other than an eigenvalue of the Floquet operator $\mathcal{T}_1 \restriction \mathrm{SOL}^{(0)}$ or, by corollary 16, an eigenvalue of the monodromy matrix

$$\mathcal{M} = \mathrm{Comp}\,(\frac{v^{(M-1)}}{v^{(M)}}, \ldots, \frac{v^{(0)}}{v^{(M)}}). \tag{37}$$

Again, we shall denote the (different) characteristic roots of $\underline{\mathsf{J}}^{(0)}$ by

$$\lambda_1, \ldots, \lambda_k$$

and their respective (algebraic) multiplicities by

$$\alpha_1, \ldots, \alpha_k.$$

Concerning the geometric multiplicities one has:

Proposition 80 *Each characteristic root of $\underline{\mathsf{J}}^{(0)}$ has geometric multiplicity* 1, *so a characteristic root of $\underline{\mathsf{J}}$ is semi-simple if and only if it is simple.* ◇

Proof.— See proposition 88(1). □

Proposition 80 does not hold any more for $\mathsf{J}^{(0)}$. For instance, for $q = 2$ it is possible to have a Floquet multiplier with geometric (and algebraic) multiplicity 2 as we have seen in subsubsection B.4.6.[110]

Proposition 81 *The characteristic polynomial of $\underline{\mathsf{J}}^{(0)}$ equals*
$$(-1)^M (z^M + \frac{v^{(M-1)}}{v^{(M)}} z^{M-1} + \cdots + \frac{v^{(0)}}{v^{(M)}} z^0). \quad \diamond$$

Proof.— This polynomial equals $|\mathcal{M} - zE_M| = (-1)^M |zE_M - \mathcal{M}|$ which by proposition 89 takes the desired form. □

Thus the characteristic equation is equivalent with $v^{(M)} z^M + v^{(M-1)} z^{M-1} + \cdots + v^{(0)} z^0 = 0$. It is this equation that one normally uses in practice. And one thus has:

$$v^{(M)} z^M + \cdots + v^{(0)} z^0 = v^{(M)} (z - \lambda_1)^{\alpha_1} \cdots (z - \lambda_k)^{\alpha_k}. \tag{38}$$

[110] This is (for $q = 2$) connected with the so called 'coexistence problem'. See, for instance, [19].

Proposition 82 *Let λ be a characteristic root of $\underline{J}^{(0)}$. Then*
$|\lambda| \leq \max(1+|\frac{v^{(M-1)}}{v^{(M)}}|, \ldots, 1+|\frac{v^{(1)}}{v^{(M)}}|, |\frac{v^{(0)}}{v^{(M)}}|)$. ◇

Proof.— The characteristic equation is equivalent to $(-1)^M(z^M + \frac{v^{(M-1)}}{v^{(M)}}z^{M-1} + \cdots + \frac{v^{(0)}}{v^{(M)}}z^0) = 0$. Proposition 91(1) implies the desired result. □

It is possible to provide a very explicit expression for the general solution of $\underline{J}^{(0)}$ as we shall see in theorem 7. The with $\underline{J}^{(0)}$ associated recurrence operator equals

$$\mathcal{H} = \sum_{s=0}^{M} v^{(s)} \mathcal{T}_1^s.$$

Lemma 6 *1.* $\ker(\mathcal{T}_1 - \lambda_i\, id)^n \subseteq \mathrm{SOL}^{(0)}$ $(1 \leq i \leq k,\ 1 \leq n \leq \alpha_i)$.

2. $\mathrm{FLOQ}^{(n)}_{1,\lambda_i} = \ker(\mathcal{T}_1 - \lambda_i\, id \restriction \mathrm{SOL}^{(0)})^n \subseteq \mathrm{SOL}^{(0)}$ $(1 \leq i \leq k,\ 1 \leq n \leq \alpha_i)$. ◇

Proof.— 1. Suppose $(\mathcal{T}_1 - \lambda_i\, id)^{\alpha_i} g = 0$. By (38), $\mathcal{H} = \sum_{s=0}^{M} v^{(s)}\mathcal{T}_1^s = v^{(M)}(\mathcal{T}_1 - \lambda_1\, id)^{\alpha_1} \cdots (\mathcal{T}_1 - \lambda_k\, id)^{\alpha_k}$. So, $\mathcal{H}g = v^{(M)}(\prod_{l=1, l\neq i}^{k}(\mathcal{T}_1 - \lambda_l\, id)^{\alpha_l})$ $(\mathcal{T}_1 - \lambda_i\, id)^{\alpha_i - n}(\mathcal{T}_1 - \lambda_i\, id)^n g = 0$. 2. Because of 1. □

Consider the generalized eigenspaces

$$\mathcal{D}_i^{(n)} = \ker(\mathcal{T}_1 - \lambda_i\, id \restriction \mathrm{SOL}^{(0)})^n \quad (1 \leq i \leq k,\ n \geq 1)$$

of $\mathcal{T}_1 \restriction \mathrm{SOL}^{(0)}$.

Proposition 83 *The index of a characteristic root of $\underline{J}^{(0)}$ equals its algebraic multiplicity.* ◇

Proof.— Because of lemma 6(2) and proposition 41(4) one has for $1 \leq n \leq \alpha_i$ that $\dim(\mathcal{D}_i^{(n)}) = \dim\ker((\mathcal{T}_1 - \lambda\, id)^n) = \dim(\mathrm{FLOQ}^{(n)}_{1,\lambda_i}) = n$. Because $r_i \leq \alpha_i = \dim(\mathcal{D}_i^{(r_i)})$ one finds $\alpha_i = r_i$. □

Proposition 84 *Consider $\underline{J}^{(0)}$. One has:*

1. $\mathrm{SOL}^{(0)} = \mathcal{D}_1^{(\alpha_1)} \oplus \cdots \oplus \mathcal{D}_k^{(\alpha_k)} = \mathcal{FLOQ}^{(\alpha_1)}_{1,\lambda_1} \oplus \cdots \oplus \mathcal{FLOQ}^{(\alpha_k)}_{1,\lambda_k}$;

2. *A base of* $\mathrm{SOL}^{(0)}$ *is given by* $t^i\lambda_j^t$ $(1 \leq j \leq k,\ 0 \leq i \leq \alpha_j - 1)$. ◇

Proof.— 1. Because of (32), proposition 83 and lemma 6. 2. Because of 1 and proposition 43(2). □

Proposition 84 implies:

Theorem 7 *The general solution of $\underline{J}^{(0)}$ is* $\sum_{i=1}^{k}\sum_{j=0}^{\alpha_i - 1} C_i^j t^j \lambda_i^t$.[111] ◇

Theorem 2(1) and corollary 12(3) become, because of proposition 80:

Proposition 85 *1.* $\dim(\mathrm{SOL}^{(0)} \cap l^\infty)$ *equals the sum of the algebraic multiplicities of the characteristic roots inside $\mathbb{T}$ plus the number of characteristic roots on $\mathbb{T}$;*

---

[111] If one calls an expression of the form $\lambda^t p(t)$, where p is a polynomial, a 'quasi-polynomial', then this theorem can be restated as follows: Any solution of $\underline{J}^{(0)}$ is a sum of quasi-polynomials of the form $\sum_{i=1}^{k}\lambda_i^t p_i(t)$, where p_i is a polynomial of degree less than α_i.

2. $\dim(\mathrm{SOL}^{(0)} \cap l^\infty) = M \quad \Leftrightarrow \quad$ *all characteristic roots lie outside* $\mathbb{T}$ *and each characteristic root on* $\mathbb{T}$ *is simple.* $\diamond$

Proposition 36 may be used to find a particular solution of $\underline{\mathrm{J}}$. However, the following considerations are more appropriate for it. Let us look for a polynomial solution of $\underline{\mathrm{J}}$ of the form

$$g(t) = ct^n.$$

Inserting this into $\underline{\mathrm{J}}$ gives $\sum_{s=0}^{M} v^{(s)} c(t+s)^n = w$. That is

$$c \sum_{k=0}^{n} t^k \binom{n}{k} \left(\sum_{s=0}^{M} v^{(s)} s^{n-k}\right) = w. \tag{39}$$

This leads to:

Proposition 86 *Consider* $\underline{\mathrm{J}}$. $R := \inf\{r \in \mathbb{N} \mid \sum_{s=0}^{M} v^{(s)} s^r \neq 0\}$. *Then* $0 \leq R \leq M$ *and* $\frac{w}{\sum_{s=0}^{M} v^{(s)} s^R} t^R$ *is a solution of* $\underline{\mathrm{J}}$. $\diamond$

Proof.— With $V := (v^{(0)}, v^{(1)}, \ldots, v^{(M)}) \in \mathbb{C}^{M+1}$ and $W_r := (0^r, 1^r, \ldots, M^r) \in \mathbb{C}^{M+1}$ one has $\sum_{s=0}^{M} v^{(s)} s^r = 0 \quad \Leftrightarrow \; < V | W_r > = 0 \quad \Leftrightarrow \; W_r \in (\mathrm{Vect}(V))^\perp$. $W_0, \ldots, W_M$ are linear independent: Indeed, the matrix $A \in M_{M+1}(\mathbb{C})$ with W_i as i-th row equals $\mathrm{Vand}(0, 1, \ldots, M)$ and has non-zero determinant, So $\mathbb{C}^{M+1} = \mathrm{Vect}(W_0, \ldots, W_M)$. Because $(\mathrm{Vect}(V))^\perp$ has dimension M, one now has $0 \leq R \leq M$. With (39), it follows that $\frac{w}{\sum_{s=0}^{M} v^{(s)} s^R} t^R$ is a solution of $\underline{\mathrm{J}}$. $\square$

C Miscellanea

C.1 Companion matrices

A Vandermonde matrix is a matrix of the form

$$\begin{pmatrix} z_1^0 & z_2^0 & z_3^0 & \cdots & z_n^0 \\ z_1^1 & z_2^1 & z_3^1 & \cdots & z_n^1 \\ \vdots & \vdots & \vdots & & \vdots \\ z_1^{n-1} & z_2^{n-1} & z_3^{n-1} & \cdots & z_n^{n-1} \end{pmatrix},$$

where $z_1, \ldots, z_n \in \mathbb{C}$. We denote this matrix by $\mathrm{Vand}(z_1, \ldots, z_n)$.

Proposition 87 $|\mathrm{Vand}(z_1, \ldots, z_n)| = \prod_{i \leq j < i \leq n} (z_i - z_j)$. $\diamond$

Proof.— For $= 1, 2$ the formula is correct. Now take $n \geq 3$. By reducing the last column in the expression of $|\mathrm{Vand}(z_1, \ldots, z_n)|$ to $1, 0, \ldots, 0$ and then developing to this column, one obtains $|\mathrm{Vand}(z_1, \ldots, z_n)| = (z_n - z_{n-1})(z_n - z_{n-2}) \cdots (z_n - z_1)|\mathrm{Vand}(z_1, \ldots, z_{n-1})|$. (For example, $|\mathrm{Vand}(z_1, z_2, z_3)| = \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ z_1 - z_3 \cdot 1 & z_2 - z_3 \cdot 1 & 0 \\ z_1^2 - z_3 z_1 & z_2^2 - z_3 z_2 & 0 \end{vmatrix} = (z_2 - z_3)(z_1 - z_3)(-1)^{3-1} \begin{vmatrix} 1 & 1 \\ z_1 & z_2 \end{vmatrix} = (z_3 - z_2)(z_3 - z_1)|\mathrm{Vand}(z_1, z_2)|$.) This leads in an obvious way to the desired result. $\square$

A companion matrix is a matrix of the form[112]

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_M & -a_{M-1} & -a_{M-2} & \cdots & -a_1 \end{pmatrix}.$$

We denote this matrix by

$$\mathrm{Comp}\,(a_1, \ldots, a_M).$$

Proposition 88 *Each eigenvalue of a companion matrix has geometric multiplicity 1, thus is semi-simple if and only if it is simple.* $\diamond$

Proof.— Looking to a companion matrix A, we see that $\mathrm{Im}(A - z\ id)$ has dimension at least $M-1$ for each z. So, $\ker(A - z\ id)$ has dimension 0 or 1. Therefore the geometric multiplicity of an eigenvalue equals 1. $\square$

If $p = z^M + p_1 z^{M-1} + \cdots + p_{M-1} z^1 + p_M z^0$ is a (monic) polynomial (in z), one calls the matrix $\mathrm{Comp}\,(p_1, \ldots, p_M)$ the 'companion matrix of p'. One has the following result:

Proposition 89 *For each monic polynomial* $p = z^M + p_1 z^{M-1} + \cdots + p_M z^0$ *in* z *one has* $p = |zE_M - \mathrm{Comp}\,(p_1, \ldots, p_M)|$. $\diamond$

Proof.— By induction. The result is evident for $M = 1$ and $M = 2$. One has $|zE_{M+1} - \mathrm{Comp}\,(p_1, \ldots, p_{M+1})| = (-1)^{M+1} |\mathrm{Comp}\,(p_1, \ldots, p_{M+1}) - zE_{M+1})|$. Expanding the determinant by the first column this becomes
$(-1)^{M+1}(-p_{M+1}(-1)^{M+2} \cdot 1^M - x|\mathrm{Comp}\ (p_1, \ldots, p_M) - zE_M)| = p_{M+1} + z(-1)^M |\mathrm{Comp}\,(p_1, \ldots, p_M) - zE_M)| = p_{M+1} + z(x^M + p_1 z^{M-1} + \cdots + p_{M-1} z^1 + p_M z^0) = x^{M+1} + p_1 z^M + \cdots + p_M z^1 + p_{M+1} z^0$. $\square$

The explicit form of $\mathrm{Comp}\,(a_1, \ldots, a_M)$ easily leads to

$$|\mathrm{Comp}\,(a_1, \ldots, a_M)| = (-1)^M a_M, \tag{40}$$

a result that also follows from proposition 89.

C.2 Location of zeros of polynomials

Here, we collect for our purposes some useful results for the zeros of a polynomial

$$p(z) = p^0 z^M + p_1 z^{M-1} + \cdots + p_{M-1} z^1 + p_M z^0$$

(where $M \geq 1$) with real or complex coefficients and $p_0 \neq 0$.

One type of result concerns the location of the zeros in relation to the real axis (on the axis or off the axis). They are useful in the context of real-valued solutions (and of oscillatory behaviour) and are especially interesting for $M = 2$. Such an important (easy) result is in case p has real coefficients: If z is a zero of p, then $\overline{z}$ is also one. Another type of result concerns the location of the zeros in relation to the complex unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ (inside, on or outside the circle). As we have already seen, they are useful in the context of qualitative properties of solutions; There are many such results.[113] We present a selection of them, useful for our purpose.

[112] For $M = 1$ one has to interpret this as $\mathrm{Comp}\,(a_1) = a_1$.

[113] For application to the differential equations case there are related results for zeros with negative, zero or positive real part.

Proposition 90 *Consider the two zeros z_1, z_2 of $z^2 + bz + c \in \mathbb{R}[z]$. Let z_1, z_2 be the two zeros of p. Then*

1. $z_1, z_2 \in \mathbb{R}$ *or* $z_1 = \overline{z_2}$;
2. $c \neq 0 \Rightarrow z_1 \neq 0, z_2 \neq 0$;
3. *If* $c < 0$, *then* $z_1, z_2 \in \mathbb{R}^\star$;
4. $z_1, z_2 \in \mathbb{T} \Rightarrow |c| = 1$;
5. *If* $c > \frac{b^2}{4}$, *then* $z_1 = \overline{z_2}$ *and* $|z_1| = |z_2| = \sqrt{c}$. $\diamond$

Proof.— This is easy. For example, 4. $z_1 z_2 = c$, so $|c| = |z_1||z_2| = 1 \cdot 1 = 1$. 5. if $c > b^2/4$, then by the square root formula $z_1 = \overline{z_2}$, so $|c| = |z_1||z_2| = |z_1|^2$, thus $|z_1| = |z_2| = \sqrt{c}$. □

Proposition 91 *Consider p with complex coefficients, $p_0 = 1$ and $p_M \neq 0$.*[114] *Then for each zero z of p:*

1. $|z| \leq \max(1 + |p_1|, \dots, 1 + |p_{M-1}|, |p_M|) \leq 1 + \max(|p_1|, \dots, |p_M|)$ *[Cauchy's bound];*
2. $|z| \leq \max(1, |p_1| + \dots + |p_M|) \leq 1 + |p_1| + \dots + |p_M|$ *[Montel's bound];*
3. $|z| \leq \sqrt{1 + |p_1|^2 + \dots + |p_M|^2}$ *[Carmichael and Mason's bound].* $\diamond$

Proof.— See, for instance, [15, pages 316, 317]. □

Proposition 92 *Consider the two zeros of $z^2 + bz + c \in \mathbb{R}[z]$.*

1. *One zero is inside and one is outside* $\mathbb{T} \Leftrightarrow |1 + c| < |b|$. *(If* $\frac{|b| - |1+c|}{1 + |b| + |c|} \geq \theta > 0$, *then (*$0 < \theta < 1$ *and)* $|z_1| < 1 - \frac{\theta}{2}$ *and* $|z_2| > (1 - \frac{\theta}{2})^{-1}$.*)*
2. *Both zeros are inside* $\mathbb{T} \Leftrightarrow 1 + c > |b|$, $c < 1$.
3. *If* $c \neq 0$, *then: all zeros are outside* $\mathbb{T} \Leftrightarrow 1 + \frac{1}{c} > |\frac{b}{c}|$, $\frac{1}{c} < 1$. $\diamond$

Proof.— Let z_1, z_2 be these two zeros. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + bx + c$. Then $f(1)f(-1) = (|1 + c| - |b|)(|1 + c| + |b|)$.

1. $\Leftarrow$: Then $f(1)f(-1) < 0$ which implies (because f is a parabola) that f has exactly one zero in $(-1, 1)$. The other zero has thus absolute value > 1.

$\Rightarrow$: Then $z_1, z_2 \in \mathbb{R}$. If $|1+c| = |b|$ would hold, then $f(1)f(-1) = 0$ and f would have a zero with absolute value 1, which is a contradiction. If $|1+c| > |b|$, then $f(1)$ and $f(1)$ would have the same sign and there would not be a zero of f on $(-1, 1)$ or both zeros would lie there, which is again a contradiction. Thus $|1 + c| < |b|$. (Moreover, because $b \neq 0$, one has $|b| - |1+c| \geq \theta(1+|b|+|c|) \geq \theta(1+|b|) > \theta + \frac{\theta}{2}|b| > \theta(1 - \frac{\theta}{4}) + \frac{\theta}{2}|b|$. Thus $|b|(1 - \frac{\theta}{2}) > |1+c| + \theta(1 - \frac{\theta}{4}) \geq |c + 1 - \theta(1 - \frac{\theta}{4})| = |c + (1 - \frac{\theta}{2})^2|$. Thus $f(1 - \frac{\theta}{2})f(-1 + \frac{\theta}{2}) = (c + (1 - \frac{\theta}{2})^2 + b(1 - \frac{\theta}{2}))(c + (1 - \frac{\theta}{2})^2 - b(1 - \frac{\theta}{2})) < 0$. Thus f has exactly one zero in $(-1 + \frac{\theta}{2}, 1 - \frac{\theta}{2})$. Thus $|z_1| < 1 - \frac{\theta}{2}$.

We now prove that $|z_2| > (1 - \frac{\theta}{2})^{-1}$. If $c = 0$, then $z_1 = 0, z_2 = -b$ and $|b| - 1 \geq \theta(1 + |b|)$. Thus in this case $|z_2| = |b| \geq \frac{\theta+1}{1-\theta} \geq$ (because $0 < \theta < 1$) $(1 - \frac{\theta}{2})^{-1}$. Now suppose $c = 0$. Then $z_1' := 1/z_2, z_2' := 1/z_1$ are the two roots of the

[114] Any polynomial $f(z)$ of degree at least 1 can be written in the form $f(z) = Cz^k g(z)$ where C is a non-zero constant, $g(z) = z^n + p_1 z^{n-1} + \dots + p_{n-1}z + p_n$ (where $n \geq 0$) and $p_n \neq 0$. The zeros of g are the non-zero zeros of f.

quadratic equation $z^2+\frac{b}{c}z+\frac{1}{c}=0$ and $|z_1'|<|z_2'|$. Because $\frac{|\frac{b}{c}|-|1+\frac{1}{c}|}{1+|\frac{b}{c}|+|\frac{1}{c}|}=\frac{|b|-|1+c|}{1+|b|+|c|}\geq\theta$, it follows from the above that $|z_1'|<1-\frac{\theta}{2}$. Thus $|z_2|>(1-\frac{\theta}{2})^{-1}$.)

2. $\Rightarrow$: Then $c\leq|c|=|z_1z_2|<1$. Because of 2 and $|c|<1$ it follows that $1+c=|1+c|\geq|b|$. Thus $1+c\geq b, 1+c\geq -b$. Even $1+c>b, 1+c>-b$ since otherwise $f(1)f(-1)=0$.

$\Leftarrow$: In this case $|c|<1$ because $-1\leq|b|-1<c<1$, thus $|1+c|=1+c>|b|$. Because of 2 one has therefore $|z_1|\leq|z_2|\leq1$ or $|z_2|\geq|z_1|\geq1$. Because $|z_1||z_2|=|c|<1$, one has $|z_1|\leq|z_2|\leq1$. Because $|1+c|>|b|$, one has $f(1)f(-1)>0$. Thus f has on $(-1,1)$ no or two zeros. In the second case $|z_2|<1$. In the first case, the roots are not real and thus, because of 1, $z_1=\overline{z_2}$ and thus $|z_2|=\sqrt{|c|}<1$.

3. Because $(z-\frac{1}{z_1})(z-\frac{1}{z_2})=z^2-\frac{z_1+z_2}{z_1z_2}z+\frac{1}{z_1z_2}=z^2-\frac{-b}{c}z+\frac{1}{c}$, we see that $\frac{1}{z_1},\frac{1}{z_2}$ are the two roots of $z^2+\frac{b}{c}z+\frac{1}{c}=0$. Thus because of 3, $|\frac{1}{z_2}|\leq|\frac{1}{z_1}|<1\Leftrightarrow 1+\frac{1}{c}>\frac{b}{c},\ 1+\frac{1}{c}>-\frac{b}{c},\ \frac{1}{c}<1$. □

A result in the spirit of proposition 92 for the arbitrary order case has been given by Jury and Marden. This can, for instance, be found in [3, page 165].

Floquet multipliers are the zeros of characteristic polynomials. So, one may wish to have results on the location of eigenvalues of matrices in terms of the matrix coefficients. For such results we refer to [15, chapter 6].

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