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A COMPARISON OF PEAKED STORM FUNCTIONS USED IN RAINFALL MODELLING

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C O N T E N T S

1. INTRODUCTION

Functions that describe the distribution of rainfall intensities in a storm have been used to derive interstation correlation relationships analytically.

In order to be able to compare different so-called storm functions, the parameters of these functions have to be chosen in a way that - approximately - the same rainfall situation is simulated.

In this report the triangular and exponential storm type will be compared. Both are assumed to be symmetrical about the storm center with a maximum at the center. Storms with the same diameter, the same maximum and the same volume will provide a basis for further discussions.

2. BASIC EQUATIONS

For an introduction to the subject of determining interstation correlation functions analytically, the reader is referred to STOL (1977a, b). Here the two storm-functions of interest will be presented again.

Let B represent storm diameter and H the maximum amount in the center, then with x to denote the location in the storm we define, for rainfall amounts h as a function of x, respectively the *t r i a n g u l a r* storm type

$$h = {}^{1}f(x) = \frac{2H}{B}x, \qquad 0 \leqq x < \frac{1}{2}B$$

$$h = {}^{2}f(x) = 2H - \frac{2H}{B}x, \quad \frac{1}{2}B \leqq x \leqq B$$

the *e x p o n e n t i a l* storm type

$$h = {}^{1}f(x) = He^{2b(x - \frac{1}{2}B)}, \qquad 0 \leqq x < \frac{1}{2}B$$

$$h = {}^{2}f(x) = He^{2b(\frac{1}{2}B - x)}, \quad \frac{1}{2}B \leqq x \leqq B$$

where b is a parameter.

The mathematical expectation of these storms are

triangular type : $\mu_t = \frac{1}{2}H$

exponential type: $\mu_e = \frac{H}{bB}(1 - e^{-bB})$

It can be proved, STOL (1977c), that with

$$bB = 1.593624 \qquad (1)$$

giving

$$e^{-bB} = 0.203188 \qquad (2)$$

and

$$\frac{1 - e^{-bB}}{bB} = \frac{1}{2}$$

both storm types have equal maximum value in the center, the same storm diameter and equal storm volume $\mu \times B$.

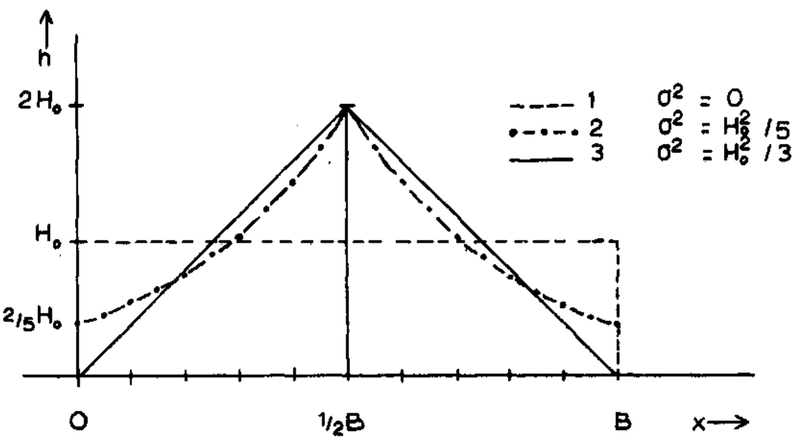

Fig. 1. Graphical representation of three storm models with equal diameter B and equal mean value H_o. They are 1: rectangular type; 2: exponential type; 3: triangular type. Storms are ordered according increasing values of their variance

In Fig. 1 both storm functions are drawn for comparison. The rectangular storm type is given as well. All storms have the same value for their diameter B and the same mean value or expectation $\mu = H_o$. Consequently they have the same storm volume.

3. COMPARISON BETWEEN TRIANGULAR AND EXPONENTIAL STORM TYPE

The correlation function for the triangular and exponential type are complicated expressions. We distinguish between three situations depending on the storm size, namely

Case I: $0 \leq D < \frac{1}{2} B < B$

Case II: $0 < \frac{1}{2} B \leq D \leq B$

Case III: $0 < \frac{1}{2} B < B < D$

where D is interstation distance. We shall use Roman figures for reference purposes. Then we have for the interstation correlation ρ

with

η = expectation of random exposure errors

τ = variance of random or exposure errors

θ_{ab} = correlation between random or exposure errors at locations a and b in the storm, at interstation distance D

p = fraction of dry days

for the *triangular* storm type

$$\rho_I = 1 - \frac{12(L + B)}{B^3} \cdot \frac{2H^2(B - D)\,D^2 + B^3(1 - \theta_{ab})\,\tau^2}{(L + B)\,(H^2 + 12\tau^2) + 3(L + pB)\,(H + 2\eta)^2} \quad (3)$$

$$\rho_{II} = 1 - \frac{4(L + B)}{B^3} \cdot \frac{B^3H^2 - 2H^2(B - D)^3 + 3B^3(1 - \theta_{ab})\,\tau^2}{(L + B)\,(H^2 + 12\tau^2) + 3(L + pB)\,(H + 2\eta)^2} \quad (4)$$

$$\rho_{III} = 1 - 4(L + B) \cdot \frac{H^2 + 3(1 - \theta_{ab})\,\tau^2}{(L + B)\,(H^2 + 12\tau^2) + 3(L + pB)\,(H + 2\eta)^2} \quad (5)$$

and for the *exponential* storm type, with

$$u = 1 - e^{-bB}$$

$$v = 1 + e^{-bB}$$

$$w = e^{-2bD}$$

$$\{\ldots\} = (uv - w - 1)\,(w - 1) - 2bDw^2$$

$$\{---\} = H^2u(bBv - 2u) + 2b^2B^2\tau^2$$

the correlation functions

$$\rho_I = 1 - (L + B) \cdot \frac{bB}{w} \cdot \frac{H^2\{\ldots\} + 2bBw(1 - \theta_{ab})\,\tau^2}{(L + B)\,\{---\} + 2(L + pB)\,(Hu + bB\eta)^2} \quad (6)$$

$$\rho_{II} = 1 - (L + B) \cdot bB \cdot \frac{H^2\{uv - 2bw(B - D)\} + 2bB(1 - \theta_{ab})\,\tau^2}{(L + B)\,\{---\} + 2(L + pB)\,(Hu + bB\eta)^2} \quad (7)$$

$$\rho_{III} = 1 - (L + B) \cdot bB \cdot \frac{H^2uv + 2bB(1 - \theta_{ab})\,\tau^2}{(L + B)\,\{---\} + 2(L + pB)\,(Hu + bB\eta)^2} \quad (8)$$

Although the functions are quite different the results are approximately the same.

This is demonstrated in some graphs (Fig. 2).

For trivial values of some parameters($\eta = \tau = \theta = p = 0$) and $\mu = 0.5$ (which via H cancels in this case) the correlation functions of the rectangular and exponential storm type are given for various values of the storm diameter B between 0.1 and 25. The ecponential storm type produces higher correlations for moderate values of D when B is less than 2. In other cases correlations for the exponential storm type appear to be a little bit smaller.

4. AN APPROXIMATING FORMULA

The correlation function for the exponential storm type is a rather complicated one because of the auxiliary functions u and v that occur in it.

The condition that the exponential storm should have the same volume as comparable rectangular storms, causes that the auxiliary functions u and v can be evaluated nummerically on the basis of equations (1) and (2).

However, decimal fractions do not simplify the results and so it was decided to reduce the decimal fractions by continued fractions that would give the best approximation with smallest values for the numerator and denominator.

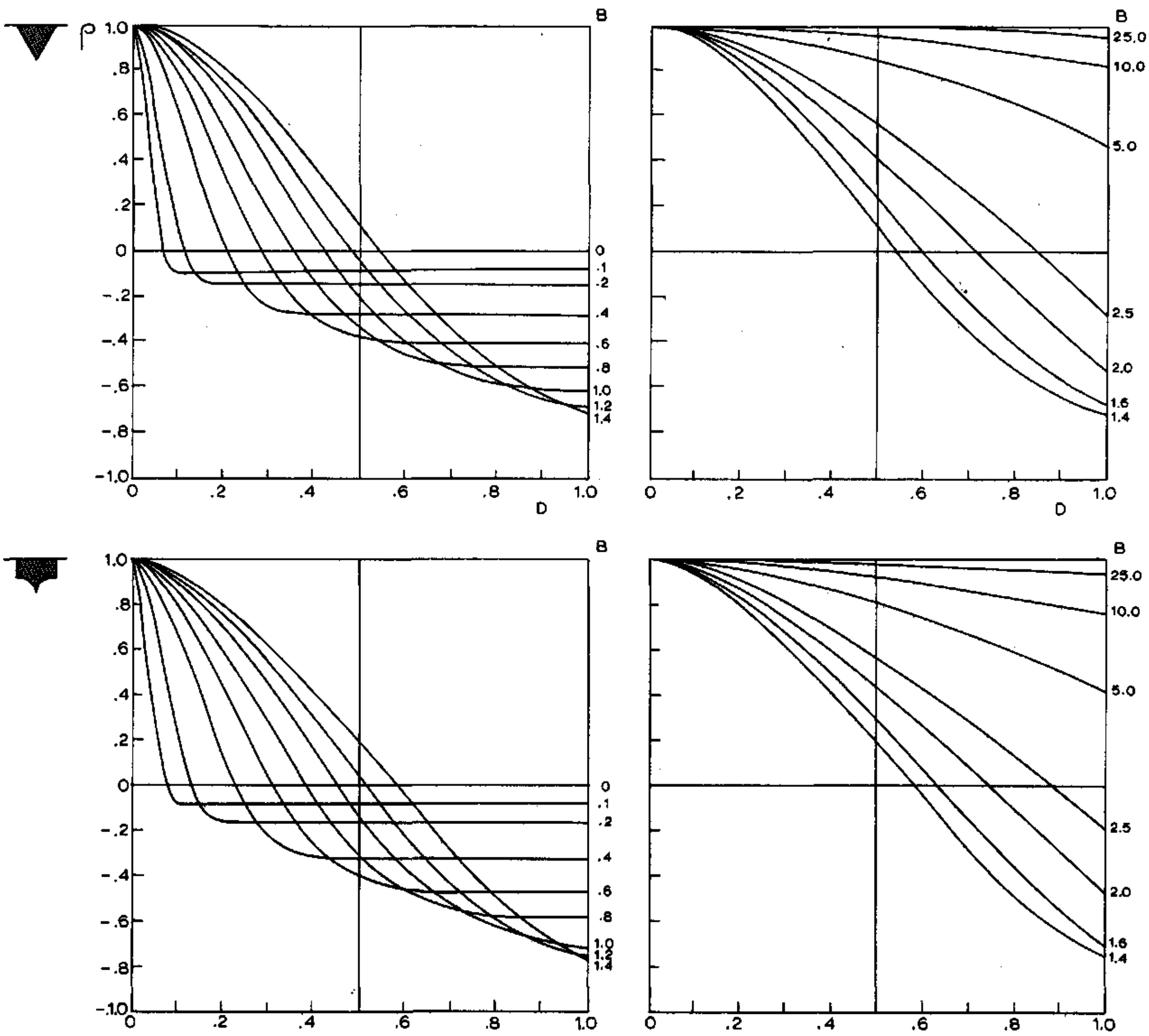

Fig. 2. The correlation function of the triangular (upper row) and exponential storm type (lower row) for various values of the storm diameter B. The graphs illustrate the numerical evaluation of equations (3), (4) and (5) in the upper pair, and of the equations (6), (7) and (8) in the lower pair

In Report (ICW Nota) 1001, Section 11, (STOL, 1977c), it was proved that, using continued fractions, the best solution with small fractions could read

$$b = \frac{8}{5} \cdot \frac{1}{B}$$

which means that

$$bB = \frac{8}{5}$$

and

$$e^{-bB} = \frac{1}{5} \qquad \text{(approximately)}$$

consequently

$$u = \frac{4}{5} \quad \text{and} \quad v = \frac{6}{5}$$

$$uv = \frac{24}{25}$$

and

$$w = \exp\left(-\frac{16}{5} \cdot \frac{D}{B}\right)$$

With these results the denominator of the correlation function becomes

$$(L + B)\left\{H^2 \cdot \frac{4}{5}\left(\frac{8}{5} \cdot \frac{6}{5} - 2 \cdot \frac{4}{5}\right) + 2 \cdot \frac{64}{25} \cdot \tau^2\right\}$$

$$+ 2(L + pB)\left(H \cdot \frac{4}{5} + \frac{8}{5} \cdot \eta\right)^2$$

$$= \frac{32}{125}(L + B)(H^2 + 20\,\tau^2) + \frac{32}{25}(L + pB)(H + 2\eta)^2$$

With the auxiliary functions u and v given numerically we can derive the correlation functions and obtain after some elaborations the following results written such that they easily can be compared with those for the triangular storm type.

Full elaborations are given in the Appendix.

Approximated interstation correlation functions for the exponential storm type thus are

$$\rho_I = 1 - \frac{20(L + B)}{B} \cdot \frac{H^2\{\frac{B}{80w}(1 + 25w)\ (1 - w) - wD\} + B(1 - \theta_{ab})\ \tau^2}{(L + B)\ (H^2 + 20\tau^2) + 5(L + pB)\ (H + 2\eta)^2} \qquad (9)$$

$$\rho_{II} = 1 - \frac{20(L + B)}{B} \cdot \frac{H^2\{\frac{3}{10}B - w(B - D)\} + B(1 - \theta_{ab})\ \tau^2}{(L + B)\ (H^2 + 20\tau^2) + 5(L + pB)\ (H + 2\eta)^2} \qquad (10)$$

$$\rho_{III} = 1 - 20(L + B) \cdot \frac{\frac{3}{10}H^2 + (1 - \theta_{ab})\ \tau^2}{(L + B)\ (H^2 + 20\tau^2) + 5(L + pB)\ (H + 2\eta)^2} \qquad (11)$$

were the use of the auxiliary functions u and v has been got round and the storm parameter b has been replaced by the condition that the storm volume (or the expected mean value) is the same as that for triangular storms with the same maximum H (at the center) and the same diameter B. In Table 1 and 2 numerical values for the exact correlation function and its approximation are given. Discrepancies between both are small. They amount approximately 2% at most.

Table 1 compares storm with trivial (zero) values for the parameters η, τ, θ and p. The storm diameter B is taken equal to 0.10; 0.40; 0.80; 1.20; 2.00 and 10.00 in units of the gaged area length L.

In Table 2 some non-trivial parameter values have been used. Here too the approximating formula give accurate results up to two decimal places at least.

Table 1. Comparison of numerical values of correlation functions $\rho(D)$, rounded to 3 decimal places, for the exponential storm function (left part illustrating numerical results of equations (6), (7) and (8)) and the approximating function (right part illustrating numerical results of equations (9), (10) and (11)) respectively, for several values of B, and expectation $\mu = 0.5$ other parameter values are taken equal to zero

Inter-station distance	Storm diameter B					
D	0.10	0.40	0.80	1.20	2.00	10.00
.0	1.000	1.000	1.000	1.000	1.000	1.000
.05	0.284	0.889	0.956	0.973	0.985	0.997
.10	-0.082	0.671	0.865	0.920	0.957	0.992
.15	-0.082	0.411	0.744	0.845	0.917	0.987
.20		0.132	0.602	0.754	0.867	0.980
.25		-0.088	0.449	0.652	0.810	0.973
.30		-0.212	0.287	0.541	0.745	0.964
.35		-0.278	0.121	0.424	0.675	0.955
.40		-0.311	-0.050	0.303	0.600	0.944
.45		-0.311	-0.202	0.178	0.520	0.933
.50			-0.316	0.051	0.438	0.921
.55			-0.402	-0.079	0.352	0.908
.60			-0.465	-0.211	0.265	0.894
.65			-0.512	-0.333	0.176	0.880
.70			-0.545	-0.435	0.085	0.864
.75			-0.569	-0.518	-0.008	0.848
.80			-0.586	-0.587	-0.101	0.832
.85			-0.586	-0.644	-0.196	0.814
.90				-0.690	-0.291	0.796
.95				-0.728	-0.388	0.777
1.00	-0.082	-0.311	-0.586	-0.758	-0.485	0.758

Exact correlation function for exponential storm functions

Inter-station distance	Storm diameter B					
D	0.10	0.40	0.80	1.20	2.00	10.00
.0	1.000	1.000	1.000	1.000	1.000	1.000
.05	0.286	0.889	0.956	0.974	0.985	0.997
.10	-0.082	0.672	0.866	0.920	0.957	0.992
.15	-0.082	0.412	0.744	0.845	0.917	0.987
.20		0.134	0.603	0.754	0.867	0.980
.25		-0.090	0.449	0.652	0.810	0.973
.30		-0.213	0.288	0.541	0.745	0.964
.35		-0.279	0.123	0.424	0.674	0.955
.40		-0.313	-0.054	0.303	0.599	0.944
.45		-0.313	-0.205	0.178	0.520	0.933
.50			-0.320	0.051	0.437	0.921
.55			-0.405	-0.078	0.352	0.908
.60			-0.468	-0.216	0.264	0.894
.65			-0.515	-0.338	0.175	0.879
.70			-0.548	-0.440	0.084	0.864
.75			-0.572	-0.523	-0.008	0.848
.80			-0.588	-0.592	-0.101	0.831
.85			-0.588	-0.649	-0.196	0.813
.90				-0.695	-0.291	0.795
.95				-0.732	-0.387	0.776
1.00	-0.082	-0.313	-0.588	-0.763	-0.493	0.757

Approximating correlation function for exponential storm functions

Table 2. Comparison of numerical values of correlation functions $\rho(D)$, rounded to 3 decimal places, for the exponential storm function (left part, illustrating numerical results of equations (6), (7) and (8))and the approximating function (right part, illustrating numerical results of equations (9), (10) and (11)) respectively, for some non-trivial parameter values as mentioned in the headings

B	0.50	0.50	0.50	B	0.50	0.50	0.50
μ	10	10	0.5	μ	10	10	0.5
τ	2	5	0	τ	2	5	0
p	0	0	0.45	p	0	0	0.45
Inter-station distance D				Inter-station distance D			
.0	.956	.777	1.000	.0	.956	.776	1.000
.05	.878	.713	.930	.05	.878	.713	.930
.10	.720	.585	.789	.10	.720	.585	.790
.15	.523	.425	.613	.15	.523	.425	.614
.20	.307	.249	.421	.20	.308	.250	.422
.25	.080	.065	.219	.25	.083	.067	.221
.30	-.107	-.087	.052	.30	-.109	-.088	.050
.35	-.225	-.183	-.053	.35	-.227	-.184	-.055
.40	-.298	-.242	-.119	.40	-.299	-.243	-.119
.45	-.341	-.277	-.157	.45	-.343	-.278	-.158
.50	-.366	-.298	-.180	.50	-.368	-.299	-.180
.55	-.366	-.298	-.180	.55	-.368	-.299	-.180
.60				.60			
.65				.65			
.70				.70			
.75				.75			
.80				.80			
.85				.85			
.90				.90			
.95				.95			
1.00	-.366	-.298	-.180	1.00	-.368	-.299	-.180
Exact correlation function for exponential storm functions				Approximating correlation function for exponential storm functions			

5. SIMPLIFIED CORRELATION FUNCTIONS

The most simple form of the correlation function is obtained by taking the following values of the parameters:

$\eta = 0$ (no bias of random errors)

$\tau^2 = 0$ (variance of random errors absent)

$\theta = 0$ (no correlation between random errors)

$p = 0$ (no dry days)

In these cases numerator and denominator can be devided by H^2. This gives for the triangular storm type

$$\rho_I = 1 - \frac{24(L + B)}{B^3} \cdot \frac{(B - D)\, D^3}{4L + B}$$

$$\rho_{II} = 1 - \frac{4(L + B)}{B^3} \cdot \frac{B^3 - 2(B - D)^3}{4L + B}$$

$$\rho_{III} = 1 - 4(L + B) \cdot \frac{1}{4L + B}$$

For the exponential storm type in which the denominator reduces to

$$(L + B)\left[H^2 u(bBv - 2u)\right] + 2LH^2u^2$$

$$= H^2\left[(L + B)\, bBuv - 2Lu^2 - 2Bu^2 + 2Lu^2\right]$$

$$= H^2\left[(L + B)\, bBuv - 2Bu^2\right]$$

$$= uH^2B\{(L + B)\, bv - 2u\}$$

we have

$$\rho_I = 1 - (L + B) \cdot \frac{b}{w} \cdot \frac{\{\ldots\}}{u\{(L + B)\, bv - 2u\}}$$

$$\rho_{II} = 1 - (L + B) \cdot b \cdot \frac{uv - 2bw(B - D)}{u\{(L + B)\, bv - 2u\}}$$

$$\rho_{III} = 1 - (L + B) \cdot b \cdot \frac{v}{(L + B)\, bv - 2u}$$

The numerical evaluation of the auxiliary functions, as carried out in the Appendix, gives for the simplified correlation functions

$$\rho_I = 1 - \frac{20(L + B)}{B} \cdot \frac{\frac{B}{80w}(1 + 25w)\ (1 - w) - wD}{6L + B}$$

$$\rho_{II} = 1 - \frac{20(L + B)}{B} \cdot \frac{\frac{3}{10}B - w(B - D)}{6L + B}$$

$$\rho_{III} = 1 - 20(L + B) \cdot \frac{\frac{3}{10}}{6L + B}$$

$$= 1 - \frac{6(L + B)}{6L + B}$$

Since we have $w = \exp(- \frac{16}{5} \cdot \frac{D}{B})$ then, in good approximation:

for $D = 0$, $w = e^{0} = 1$

$D = \frac{1}{2} B$, $w = e^{-\frac{16}{5} \cdot \frac{1}{2}} = \frac{1}{5}$

$D = B$, $w = e^{-\frac{16}{5}} = (\frac{1}{5})^2 = \frac{1}{25}$

we obtain

$$\rho_I\ (0) = 1$$

$$\rho_I\ (\frac{1}{2}\ B) = 1 - \frac{20(L + B)}{B} \cdot \frac{\frac{B}{16}(1 + 5)\ (\frac{4}{5}) - \frac{1}{10}B}{6L + B}$$

$$= 1 - 20(L + B) \cdot \frac{1/5}{6L + B} = 1 - \frac{4(L + B)}{6L + B}$$

$$\rho_{II}(\frac{1}{2}\ B) = 1 - \frac{20(L + B)}{B} \cdot \frac{\frac{3}{10}B - \frac{1}{5}(\frac{1}{2}\ B)}{6L + B}$$

$$= 1 - 20(L + B) \cdot \frac{1/5}{6L + B} = 1 - \frac{4(L + B)}{6L + B}$$

so $\rho_I(\frac{1}{2}\ B) = \rho_{II}(\frac{1}{2}\ B)$ and it can easily be verified that

$$\rho_{II}(B) = \rho_{III} = 1 - \frac{6(L + B)}{6L + B}$$

We can conclude that the approximating function has the same property of contuinity as the exact correlation function.

Differentiating with respect to D we have

$$\frac{d}{dD}\rho_I = - \frac{20(L + B)}{B} \cdot \frac{1}{6L + B} \cdot \frac{B}{80} \cdot \alpha$$

where

$$\alpha = \frac{16}{5B} \cdot \frac{1}{w} \cdot (1 + 25w)\ (1 - w)$$

$$- \frac{1}{w} \cdot \frac{25.16}{5B} w(1 - w) + \frac{1}{w}(1 + 25w) \cdot \frac{16}{5B} w$$

$$+ (\frac{16}{5B} wD - w) \frac{80}{B}$$

and so $\rho_I'(\frac{1}{2} B)$ gives for α the expression

$$\frac{16}{5B} \cdot 5 \cdot 6 \cdot \frac{4}{5} - \frac{25.16}{B} \cdot \frac{1}{5} \cdot \frac{4}{5} + 5 \cdot 6 \cdot \frac{16}{25B} + (\frac{16}{50} - \frac{1}{5}) \frac{80}{B}$$

since for $D = \frac{1}{2} B$ the auxiliary function $w = \frac{1}{5}$. Simplified we have $\alpha = \frac{800}{25B} + \frac{6}{50} \cdot \frac{80}{B}$

and

$$\frac{B}{80} \cdot \alpha = \frac{10}{25} + \frac{6}{50} = \frac{13}{25}$$

Then we have for ρ_{II}

$$\frac{d}{dD}\rho_{II} = - \frac{20(L + B)}{B} \cdot \frac{1}{6L + B} \cdot \beta$$

where

$$\beta = \frac{16}{5B} w(B - D) + w$$

which for $D = \frac{1}{2} B$ becomes

$$\beta = \frac{16}{5B} \cdot \frac{1}{5} \cdot \frac{1}{2} B + \frac{1}{5}$$

$$= \frac{8}{25} + \frac{1}{5} = \frac{13}{25}$$

This means that also the property of being 'smooth' at $D = \frac{1}{2} B$ (since $\frac{d}{dD} \rho_I \Big|_{\frac{1}{2} B} = \frac{d}{dD} \rho_{II} \Big|_{\frac{1}{2} B} = \frac{-20(L + B)}{B(6L + B)} \cdot \frac{13}{25}$) is preserved by the approximating function.

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CORRELATION FUNCTION FOR THE EXPONENTIAL STORM TYPE APPROXIMATED BY SIMPLE FRACTIONS

In all elaborations we have

$$u = \frac{4}{5}, \quad v = \frac{6}{5}, \quad uv = \frac{24}{25}$$

$$bB = \frac{8}{5}, \quad e^{-bB} = \frac{1}{5}$$

$$w = \exp(-\frac{16}{5} \cdot \frac{D}{B})$$

Case I

The numerator

$$\{\ldots\} = (uv - w - 1)\ (w - 1) - 2bDw^2$$

$$= (\frac{24}{25} - w - 1)\ (w - 1) - \frac{16}{5}\frac{D}{B}\,w^2$$

$$= \frac{1}{25}\{(1 + 25w)\ (1 - w) - 80\,\frac{D}{B}\,w^2\}$$

and

$$2bBw(1 - \theta_{ab})\ \tau^2 = \frac{16}{5}\,w(1 - \theta_{ab})\ \tau^2$$

so, combining these results,

$$\frac{1}{25}\left[H^2\{(1 + 25w)\cdot(1 - w) - 80\,\frac{D}{B}\,w^2\} + 80w(1 - \theta_{ab})\ \tau^2\right]$$

The denominator

We found already

$$\frac{32}{125}\left[(L + B)\ (H^2 + 20\tau^2) + 5(L + pB)\ (H + 2\eta)^2\right]$$

The correlation function

The completed correlation function now reads

$$\rho_I = 1 - (L + B) \cdot \frac{8}{5w} \cdot \frac{\frac{1}{25}\left[\cdots\right]}{\frac{32}{125}\left[---\right]}$$

We multiply numerator and denominator by the factors

$$\frac{20}{B} \cdot \frac{8}{5w} \cdot \frac{125}{32} \cdot \frac{B}{20} \quad \text{and} \quad \frac{125}{32}$$

respectively. The result then is

$$\rho_I = 1 - \frac{20(L + B)}{B} \cdot \frac{H^2\{\frac{B}{80w}(1 + 25w)(1 - w) - wD\} + B(1 - \theta_{ab})\tau^2}{(L + B)(H^2 + 20\tau^2) + 5(L + pB)(H + 2\eta)^2}$$

Case II

The numerator

$$\{uv - 2bw(B - D)\} = \{\frac{24}{25} - \frac{16}{5}\frac{w}{B}(B - D)\}$$

$$= \frac{1}{25B}\{24B - 80w(B - D)\}$$

and

$$2bB(1 - \theta_{ab})\tau^2 = \frac{16}{5}(1 - \theta_{ab})\tau^2$$

so, combining these results,

$$\frac{1}{25B}\left[H^2\{24B - 80w(B - D)\} + 80B(1 - \theta_{ab})\tau^2\right]$$

The denominator

The denominator is the same as the one for Case I.

The correlation function

The completed correlation function now reads

$$\rho_{II} = 1 - (L + B) \cdot \frac{8}{5} \cdot \frac{\frac{1}{25B}\left[\cdots\right]}{\frac{32}{125}\left[---\right]}$$

We multiply numerator and denominator by the factors

$$\frac{20}{B} \cdot \frac{8}{5} \cdot \frac{125}{32} \cdot \frac{B}{20} \quad \text{and} \quad \frac{125}{32}$$

respectively. The result then is

$$\rho_{II} = 1 - \frac{20(L + B)}{B} \cdot \frac{H^2\{\frac{3}{10} B - w(B - D)\} + B(1 - \theta_{ab}) \tau^2}{(L + B)(H^2 + 20\tau^2) + 5(L + pB)(H + 2\eta)^2}$$

Case III

The numerator

$$H^2 uv + 2bB(1 - \theta_{ab}) \tau^2$$

$$= \frac{24}{25} \cdot H^2 + \frac{16}{5} \cdot (1 - \theta_{ab}) \tau^2$$

$$= \frac{16}{5}\left[\frac{3}{10} H^2 + (1 - \theta_{ab}) \tau^2\right]$$

The denominator

The denominator is the same as the one for Case I.

The correlation function

The completed correlation function now reads

$$\rho_{III} = 1 - (L + B) \cdot \frac{8}{5} \cdot \frac{\frac{16}{5}\left[\cdots\right]}{\frac{32}{125}\left[---\right]}$$

We multiply numerator and denominator by the factors

$$20 \cdot \frac{8}{5} \cdot \frac{125}{32} \cdot \frac{1}{20} \quad \text{and} \quad \frac{125}{32}$$

respectively. The result then is

$$\rho_{III} = 1 - 20(L + B) \cdot \frac{\frac{3}{10} H^2 + (1 - \theta_{ab}) \tau^2}{(L + B)(H^2 + 20\tau^2) + 5(L + pB)(H + 2\eta)^2}$$

which completes the elaborations.