Kubelka-Munk equations in vector–matrix forms and the solution for bidirectional vegetative canopy reflectance

J. Chen

The radiation from different directions can be specified by upward and downward radiation vectors, and the interactions of the radiation with a leaf or with a vegetative canopy can be specified by matrices. The Kubelka-Munk equations, which are applicable only to a canopy with horizontal and Lambertian leaves, can then be extended to describe the directional transfer of radiation in a canopy with nonhorizontal, non-Lambertian leaves. In the extended Kubelka-Munk equations, variables are upward and downward radiation vectors, and the coefficients are matrices. The solutions are found from which the bidirectional vegetative canopy reflectance, including azimuthal variations, can be obtained. Simplified and approximate methods are presented for a canopy with leaves without azimuthal preference in order to reduce the execution time.

I. Introduction

The transfer of radiation through a turbid medium, such as the atmosphere or clouds, has been of interest for some time. Recent developments in remote sensing techniques require calculation of bidirectional reflectance patterns of various vegetative canopies. Although the integral equations of Chandrasekhar[1] have been established for more than thirty years, the semianalytic solution is possible only for the simplest phase functions such as that of Rayleigh scatter. For cases of Mie scatter, even numerical solution is difficult.[2]

The interaction of shortwave sun radiation with vegetative canopies has an additional complexity because the scattering elements are now mainly leaves, which are planar, so the bidirectional reflectance of a leaf depends not only on the angle between the incident and exitant directions but also on the orientation of the leaf. In the simplest case, all the leaves of a horizontally homogeneous canopy are assumed to be Lambertian scatterers and orientate horizontally, the directional distribution of radiation within and above the canopy is then a known function. The radiation transfer in such a canopy can therefore be fully described by the vertical variation of the total downward and upward radiation intensities. The Kubelka-Munk equations[3] address this situation. When two boundary conditions—incident radiation above the canopy and the reflectance of the underlying soil surface—are given, the Kubelka-Munk equations can be solved for profiles of the total downward and upward radiation intensities within the canopy, thus the reflectance of the canopy can also be obtained.

If a canopy consists of nonhorizontal leaves, it is no longer a Lambertian reflector as a whole, even though all the leaves are Lambertian scatterers.[4] The simple form of the Kubelka-Munk equations cannot be applied, because the directional reflection and transmission of the radiation have to be taken into account. If azimuthal variations of the radiation are ignored and only the change of the radiation in zenith (or inclination) is considered, the radiation from all directions in a hemisphere can be specified by the radiation intensities from several discretized and contiguous zones which span the whole hemisphere.[4] Goudriaan[4] further divided the whole canopy into several layers and derived a set of equations for these unknown upward and downward radiation components. He solved these equations by iteration, Cooper *et al.*[5] applied the Adding method, developed by van de Hulst[6] under vector–matrix notation, and solved the same transfer problem without referring to the equations.

It was shown by Chen[7] that in vector–matrix notation the equations for radiation transfer derived by Goudriaan[4] can be written as difference equations in vector–matrix forms. In this paper it is shown that these

When this work was done the author was with Agricultural University, Department of Theoretical Production Ecology, Bornsesteeg 65, 6708 PD Wageningen, The Netherlands; he is now with Academia Sinica, Institute of Plant Physiology, 300 Fenglin Road, Shanghai, China.

Received 12 July 1984.
0003-6935/85/030376-07$02.00/0.

difference equations can be derived directly under vector–matrix notation and transformed into differential equations, which are, in fact, extended Kubelka-Munk equations (where the variables are downward and upward radiation vectors and the coefficients are matrices). These equations can then be solved using standard matrix algebra methods. The directional reflectance into different zones of a hemisphere can be directly obtained from the solutions. In this paper it is also shown that the equations and the solutions are also able to account for azimuthal variations, but although analytical solutions are available the resolution in azimuth is restricted by the execution time. A special method is then developed to reduce the execution time for leaf canopies without obvious azimuthal preference, which is the case for most crop canopies.[8] A few illustrative examples are presented to show the feasibility of the theory, while comparison of the results with experimental data is left for future work.

II. Vector–Matrix Representation of Radiation and Its Interaction with a Leaf or a Canopy

For the transfer of radiation in a horizontally homogeneous vegetative canopy, it is convenient to divide the radiation into downward and upward components x and y from and into an upper hemisphere, respectively. The direction from a hemisphere is determined by two variables, inclination i (or zenith) and azimuth j, so that x and y are continuous functions of i and j. If a whole hemisphere is subdivided into several contiguous sectors each with a solid angle $\cos(i)w_iw_j$, where w_i and w_j are, respectively, the inclination and azimuth widths of the sector, and if within each sector the radiance is assumed the same, x and y can be represented by tensors of order two (matrices). The bidirectional reflectance and transmittance of a leaf or a horizontally homogeneous canopy layer can be specified by tensors of order four.[7] An illustration is given in Fig. 1. Two sectors, A and A' with solid angles $\cos(i)w_iw_j$ and $\cos(i')w_{i'}w_{j'}$, respectively, are shown. The radiation flux densities from all sectors in the hemisphere constitute a downward radiation tensor. In Fig. 1 one component of the downward radiation tensor from the direction specified by (i,j) is incident on a horizontal leaf. The leaf reflects radiaton into all the sectors in the hemisphere, these reflected radiation components constitute an upward radiation tensor. In the figure one of these components in the direction specified by (i',j') is shown. For these two fixed incident and exitant directions, the bidirectional reflectance of the leaf is denoted by $r(i',j',i,j)$. These bidirectional reflectance coefficients for all different values of i', j', i, and j constitute the reflectance tensor of the leaf, which is of order four.

If the azimuthal variation of the radiation is ignored, the sectors shown in Fig. 1 with the same inclinations can be combined into horizontal zones, and the radiation intensities from the relevant sectors can be summed up to form a total intensity of the zone, then the downward and upward radiation can be specified by vectors (tensors of order one), and the bidirectional reflectance and transmittance by matrices (tensors of order two).

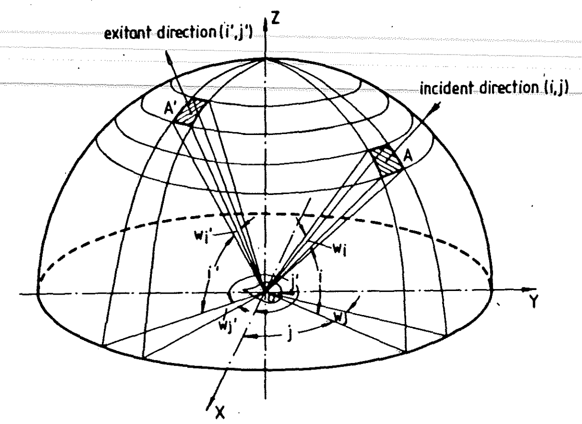

Fig. 1. Vector–matrix representation of the radiation and its interaction with a horizontal leaf.

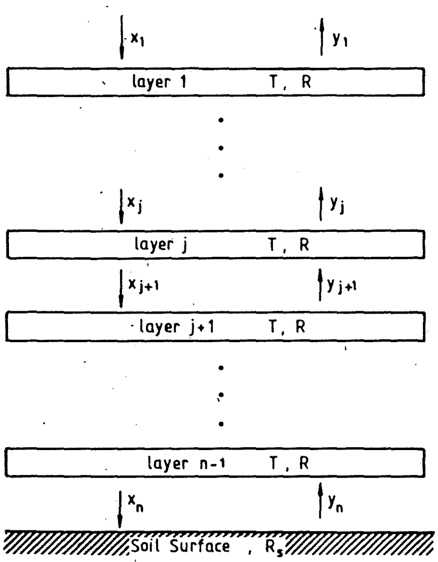

Fig. 2. Downward and upward radiation vectors at different layers. For meaning of the symbols see text.

This situation will be examined first. The radiation flux densities in this paper, following Goudriaan's usage,[4] refer to those in a horizontal plane.

III. Kubelka-Munk Equations in Vector–Matrix Forms

Divide a vegetative canopy into several layers, each having a leaf area index s. Denote the downward and upward radiation vectors above layer j by $\mathbf{x}_j$ and $\mathbf{y}_j$, respectively (Fig. 2). As a downward radiation vector, say $\mathbf{x}_j$, interacts with the layer j, both upward and downward radiation vectors are generated. If there are no other layers above and below the layer j, the generated downward radiation vector $\mathbf{x}_{j+1}$ is $\mathrm{T}\mathbf{x}_j$ and the upward one $\mathbf{y}_j$ is $\mathrm{R}\mathbf{x}_j$, where T and R are, respectively, the transmittance and reflectance matrices of the layer. If the leaf area index s of the layer is very small, the multiple scattering between the leaves within the layer can be ignored. The interception fractions of the radiation from different directions are determined by the projections of the total leaf area in the layer onto the relevant directions and can be denoted by $s\mathrm{M}$, where M is the interception matrix and is diagonal. The penetration fraction then is $\mathrm{I} - s\mathrm{M}$, where I is the identity matrix. The radiation intercepted by the leaves will be scattered either backward or forward, and this interaction can be specified by the backscattering matrix B and the forward scattering matrix F. The transmittance and reflectance matrices T and R can then be obtained as

$$\mathrm{T} = \mathrm{I} - s\mathrm{M} + s\mathrm{F}, \qquad \mathrm{R} = s\mathrm{B}. \tag{1}$$

When there are other layers above and below the layer j, the radiation reflected by the layers $j - 1$ and $j + 1$ have to be taken into account. By referring to Fig. 2, the following equations can be obtained:

$$\mathbf{x}_{j+1} = \mathrm{T}\mathbf{x}_j + \mathrm{R}\mathbf{y}_{j+1}, \qquad \mathbf{y}_j = \mathrm{T}\mathbf{y}_{j+1} + \mathrm{R}\mathbf{x}_j. \tag{2}$$

Substituting Eqs. (1) into Eqs. (2) and rearranging give

$$\begin{aligned}(\mathbf{x}_{j+1} - \mathbf{x}_j)/s &= -(\mathrm{M} - \mathrm{F})\mathbf{x}_j + \mathrm{B}\mathbf{y}_{j+1},\\ (\mathbf{y}_{j+1} - \mathbf{y}_j)/s &= (\mathrm{M} - \mathrm{F})\mathbf{y}_{j+1} - \mathrm{B}\mathbf{x}_j.\end{aligned} \tag{3}$$

As s tends to zero, Eqs. (3) become differential equations:

$$d\mathbf{x}/dl = -(\mathrm{M} - \mathrm{F})\mathbf{x} + \mathrm{B}\mathbf{y}, \qquad d\mathbf{y}/dl = -\mathrm{B}\mathbf{x} + (\mathrm{M} - \mathrm{F})\mathbf{y}, \tag{4}$$

where l is the cumulative leaf area index reckoned from the top of the canopy. The boundary conditions are

$$\mathbf{x}(0) = \mathbf{x}_0, \qquad \mathbf{y}(l_c) = \mathrm{R}_s\mathbf{x}(l_c), \tag{5}$$

where l_c is the total leaf area index of the canopy, $\mathbf{x}_0$ is a known downward radiation vector on the top of the canopy, and R_s is the reflectance matrix of the soil surface.

Compared with the Kubelka-Munk equations,[9]

$$dx/dl = -(l - t)x + ry, \qquad dy/dl = -rx + (l - t)y, \tag{6}$$

where t and r are, respectively, the transmission and reflection coefficients of the leaves, it can be seen that Eqs. (4) are an extended version of the Kubelka-Munk equations. The variables are now the downward and upward radiation vectors, and the coefficients are the interception matrix M, forward scattering matrix F, and backscattering matrix B in place of the scalars l, t, and r, respectively.

By introducing two new variables,

$$\mathbf{u}(l) = \mathbf{x}(l) + \mathbf{y}(l), \qquad \mathbf{v}(l) = \mathbf{x}(l) - \mathbf{y}(l), \tag{7}$$

Eqs. (4) can be written as

$$d\mathbf{u}/dl = -(\mathrm{M} - \mathrm{F} + \mathrm{B})\mathbf{v}, \qquad d\mathbf{v}/dl = -(\mathrm{M} - \mathrm{F} - \mathrm{B})\mathbf{u}. \tag{8}$$

From Eqs. (8),

$$d^2\mathbf{v}/dl^2 = (\mathrm{M} - \mathrm{F} - \mathrm{B})(\mathrm{M} - \mathrm{F} + \mathrm{B})\mathbf{v} = \mathrm{Q}\mathbf{v}. \tag{9}$$

To solve Eq. (9) the matrix Q must first be transformed into a diagonal matrix (it is no longer necessary if using a more efficient method newly developed by van Rootselaar[10]). Computation shows that Q can be diagonalized and is positive definite, so Q can be written as

$$\mathrm{Q} = \mathrm{V}\mathrm{P}^2\mathrm{V}^{-1}, \tag{10}$$

where the matrices P and V can be obtained by using standard software. In terms of Eq. (10), Q^n and $\mathrm{Q}^{1/2}$ can be obtained as $\mathrm{Q}^n = \mathrm{V}(\mathrm{P}^2)^n\mathrm{V}^{-1}$ and $\mathrm{Q}^{1/2} = \mathrm{V}\mathrm{P}\mathrm{V}^{-1}$. A matrix exponential function of the independent variable l, $\exp(\mathrm{Q}^{1/2}l)$, can then be obtained as $\mathrm{V}\exp(\mathrm{P}l)\mathrm{V}^{-1}$. Because P is a diagonal matrix, there is no difficulty in calculating $\exp(\mathrm{P}l)$. It can be verified by substitution that

$$\mathbf{v}_1(l) = \mathrm{V}\exp(\mathrm{P}l)\mathrm{V}^{-1}2\mathbf{c}, \qquad \mathbf{v}_2(l) = \mathrm{V}\exp(-\mathrm{P}l)\mathrm{V}^{-1}2\mathbf{d} \tag{11}$$

are two solutions of Eq. (9). The general solution can be obtained as

$$\mathbf{v}(l) = \mathrm{V}\exp(\mathrm{P}l)\mathrm{V}^{-1}2\mathbf{c} + \mathrm{V}\exp(-\mathrm{P}l)\mathrm{V}^{-1}2\mathbf{d}, \tag{12}$$

where $\mathbf{c}$ and $\mathbf{d}$ are two arbitrary vectors, and $\mathbf{u}$ can be found from Eqs. (8):

$$\begin{aligned}\mathbf{u}(l) = -(\mathrm{M} - \mathrm{F} - \mathrm{B})^{-1}\mathrm{V}\mathrm{P}\mathrm{V}^{-1}[&\mathrm{V}\exp(\mathrm{P}l)\mathrm{V}^{-1}2\mathbf{c}\\ &- \mathrm{V}\exp(-\mathrm{P}l)\mathrm{V}^{-1}2\mathbf{d}].\end{aligned} \tag{13}$$

Since $\mathrm{V}\mathrm{P}\mathrm{V}^{-1} = (\mathrm{V}\mathrm{P}^2\mathrm{V}^{-1})(\mathrm{V}\mathrm{P}^{-1}\mathrm{V}^{-1}) = (\mathrm{M} - \mathrm{F} - \mathrm{B})(\mathrm{M} - \mathrm{F} + \mathrm{B})\mathrm{V}\mathrm{P}^{-1}\mathrm{V}^{-1}$, Eq. (13) can be rewritten as

$$\begin{aligned}\mathbf{u} = -(\mathrm{M} - \mathrm{F} + \mathrm{B})\mathrm{V}\mathrm{P}^{-1}\mathrm{V}^{-1}[&\mathrm{V}\exp(\mathrm{P}l)\mathrm{V}^{-1}2\mathbf{c}\\ &- \mathrm{V}\exp(-\mathrm{P}l)\mathrm{V}^{-1}2\mathbf{d}].\end{aligned} \tag{14}$$

This procedure removes an inversion operation, which is more time-consuming than multiplication, particularly when the matrices are large.

It follows from Eqs. (7), (12), and (14) that

$$\mathbf{x}(l) = -\mathrm{R}_i\mathrm{E}(l)\mathrm{J}\mathbf{c} + \mathrm{E}(-l)\mathrm{J}\mathbf{d}, \qquad \mathbf{y}(l) = -\mathrm{E}(l)\mathrm{J}\mathbf{c} + \mathrm{R}_i\mathrm{E}(-l)\mathrm{J}\mathbf{d}, \tag{15}$$

where

$$\begin{aligned}\mathrm{R}_i = \mathrm{H}\mathrm{J}^{-1}, \qquad \mathrm{H} &= (\mathrm{M} - \mathrm{F} + \mathrm{B})\mathrm{V}\mathrm{P}^{-1}\mathrm{V}^{-1} - \mathrm{I},\\ \mathrm{J} &= (\mathrm{M} - \mathrm{F} + \mathrm{B})\mathrm{V}\mathrm{P}^{-1}\mathrm{V}^{-1} + \mathrm{I},\end{aligned} \tag{16}$$

and $\mathrm{E}(l)$ is a matrix function of l defined as

$$\mathrm{E}(l) = (\mathrm{J}\mathrm{V})\exp(\mathrm{P}l)(\mathrm{J}\mathrm{V})^{-1}. \tag{17}$$

The two constant vectors $\mathbf{c}$ and $\mathbf{d}$ can be determined by the boundary conditions [Eqs. (5)] as

$$\mathbf{c} = -\mathrm{J}^{-1}(\mathrm{I} + \mathrm{C}\mathrm{R}_i)^{-1}\mathrm{C}\mathbf{x}_0, \qquad \mathbf{d} = \mathrm{J}^{-1}[\mathrm{I} - \mathrm{R}_i(\mathrm{I} + \mathrm{C}\mathrm{R}_i)^{-1}\mathrm{C}]\mathbf{x}_0, \tag{18}$$

where the matrix C is defined as

$$\mathrm{C} = \mathrm{E}(-l_c)\mathrm{G}\mathrm{E}(-l_c) \quad \text{with } \mathrm{G} = (\mathrm{R}_s\mathrm{R}_i - \mathrm{I})^{-1}(\mathrm{R}_i - \mathrm{R}_s). \tag{19}$$

Substituting Eqs. (18) into Eqs. (15) yields

$$\mathbf{x}(l) = [\mathrm{R}_i\mathrm{D}(l) + \mathrm{A}(-l)]\mathbf{x}_0, \qquad \mathbf{y}(l) = [\mathrm{D}(l) + \mathrm{R}_i\mathrm{A}(-l)]\mathbf{x}_0, \tag{20}$$

where

$$\mathrm{D}(l) = \mathrm{E}(l)(\mathrm{I} + \mathrm{C}\mathrm{R}_i)^{-1}\mathrm{C}, \qquad \mathrm{A}(-l) = \mathrm{E}(-l)[\mathrm{I} - \mathrm{R}_i(\mathrm{I} + \mathrm{C}\mathrm{R}_i)^{-1}\mathrm{C}]. \tag{21}$$

The zonal transmittance matrix, $\mathrm{T}_{\mathrm{zon}}$, and the zonal reflectance matrix, $\mathrm{R}_{\mathrm{zon}}$, of the whole canopy can be obtained from Eqs. (20) by substituting l_c or zero, respectively, for l:

$$T_{zon} = R_i E(l_c)(I + CR_i)^{-1}C + E(-l_c)[I - R_i(I + CR_i)^{-1}C], \tag{22}$$

$$R_{zon} = R_i + (I - R_i^2)(I + CR_i)^{-1}C. \tag{23}$$

As l_c tends to infinity, $E(-l_c)$ and then C tend to zero, so R_{zon} tends to R_i. R_i is thus the zonal reflectance matrix of a canopy with an infinite leaf area index.

IV. Including Azimuthal Variations

In the general case, if azimuthal variations of the radiations are also of interest, radiation must be represented by a tensor of order two. But the radiation tensor of order two can be represented by an extended vector, if all the components are arranged in one column:

$$\mathbf{x}^* = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)^t, \tag{24}$$

where $\mathbf{x}_j$ ($j = 1$ to m) is the radiation vector for a fixed azimuth j, m is the total number of the intervals in the azimuth, and t denotes transposition.

The forward scattering matrix F, for example, should also be extended to form the matrix F*:

$$F^* = \begin{vmatrix} F_{11} & F_{12} & \ldots & F_{1m} \\ F_{21} & F_{22} & \ldots & F_{2m} \\ \vdots & \vdots & & \vdots \\ F_{m1} & F_{m2} & \ldots & F_{mm} \end{vmatrix}, \tag{25}$$

where $F_{j'j}$ ($j,j' = 1$ to m) is the forward scattering matrix for the incident radiation vector $\mathbf{x}_j$ and the exitant radiation vector $\mathbf{x}_{j'}$. The upward radiation vector **y** and matrices I, M, and B should also be extended in the same way. The Kubelka-Munk equations in vector–matrix forms [Eqs. (4)], the solutions [Eqs. (20)], and the directional transmittance and reflectance of the whole canopy [Eqs. (22) and (23)] can then be used to determine the azimuthal variations.

V. Calculation of Matrices M*, F*, and B* and the Normalization

The coefficients in the generalized Kubelka-Munk equations [Eqs. (4)], the matrices M, F, and B (or their extended forms M*, F*, and B*), must be determined. These basic matrices can be obtained from those of the single leaves and the leaf angular distribution. The backscattering and forward scattering matrices of a horizontal layer containing only one Lambertian leaf with inclination i_L and azimuth j_L can be determined as[7]

$$B_L^*(i',j',i,j) = s(q/\pi)\cos(i')w_{i'}w_{j'}|\cos(a')\cos(a)|\sin^{-1}(i), \tag{26}$$

$$F_L^*(i',j',i,j) = s(q'/\pi)\cos(i')w_{i'}w_{j'}|\cos(a')\cos(a)|\sin^{-1}(i). \tag{27}$$

Here s is the leaf area index, a is the angle between the incident radiation and the normal of the leaf, and a' is that for exitant radiation:

$$\cos(a) = \cos(i)\cos(j)\sin(i_L)\cos(j_L) + \cos(i)\sin(j)\sin(i_L)\sin(j_L) + \sin(i)\cos(i_L), \tag{28}$$

$$\cos(a') = \cos(i')\cos(j')\sin(i_L)\cos(j_L) + \cos(i')\sin(j')\sin(i_L)\sin(j_L) + \sin(i')\cos(i_L), \tag{29}$$

and $q = r$, $q' = t$ when $\cos(a)\cos(a') \geq 0$; $q = t$, $q' = r$ when $\cos(a)\cos(a') < 0$. The notation of $\sin(i)$ and $\cos(i)$ means that the sine and cosine functions for the angle interval i are calculated using a representative angle, e.g., the value in the middle of the interval.

Summing all the backscattering and forward scattering matrices of the leaves with different orientations, weighted by angular distribution, yields the corresponding matrices B* and F*. The diagonal components of the interception matrix M* can be calculated as

$$M^*(i,j,i,j) = O(i,j)/\sin(i), \tag{30}$$

where $O(i,j)$ is the projection of the leaves in a layer with a unit leaf area index onto the direction (i,j).

Because of the discretization the sum of all the components of F* and B* is usually not exactly equal to $s(t + r)$ multiplied by the incident radiation. This means that the conservation of the radiation energy is violated. When the value of $t + r$ is high, the multiple scattering between layers plays an important rôle. The nonconservation of energy in matrices F* and B* will be greatly amplified in the end results. Thus the normalization procedure is not trivial, as noted by Goudriaan[4].

Denote the sum of F_L^* and B_L^* as S_L^*. Now

$$S_L^*(i',j',i,j) = s[(t + r)/\pi] \times \cos(i')w_{i'}w_{j'}|\cos(a')\cos(a)|\sin^{-1}(i). \tag{31}$$

Consider the horizontal leaf first. According to Eqs. (28) and (29), $\cos(a) = \sin(i)$ and $\cos(a') = \sin(i')$ in this case, so Eq. (31) becomes

$$S_L^*(i',j',i,j) = s[(t + r)/\pi]\cos(i')w_{i'}w_{j'}\sin(i'). \tag{32}$$

The normalization condition requires that the sum of $S_L^*(i',j',i,j)$, with respect to i' and j' over the whole upper hemisphere, be exactly equal to $s(t + r)$. After summing $S_L^*(i',j'i,j)$ over the azimuth, this requirement becomes that of the sum of $2\sin(i')\cos(i')w_{i'}$ over all inclination intervals should be exactly equal to unity. This is true, however, only as $w_{i'}$ tends to zero, when the summation becomes an integral of $2\sin(t)\cos(t)dt$ from $t = 0$ to $\pi/2$. But if $w_{i'}$ is replaced by $\sin(w_{i'})$ (the difference between them tends to zero as $w_{i'}$ tends to zero), it can be proved that the normalization condition is fulfilled. In fact, the integral of $2\sin(t)\cos(t)dt$ from the lower boundary b_1 to the upper boundary b_2 of the interval $w_{i'}$ is equal to $2\sin[(b_1 + b_2)/2]\cos[(b_1 + b_2)/2]\sin(b_2 - b_1)$. This expression can be written as $2\sin(i')\cos(i')\sin(w_{i'})$, if the middle point of the interval $w_{i'}$ is used to calculate $\sin(i')$ and $\cos(i')$. Since the sum of $2\sin(i')\cos(i')\sin(w_{i'})$ over all the intervals is the integral of $2\sin(t)\cos(t)dt$ from $t = 0$ to $\pi/2$, which equals unity, the normalization condition will be fulfilled for a horizontal leaf, if $w_{i'}$ in Eqs. (26) and (27) is replaced by $\sin(w_{i'})$. For an inclined leaf the normalization condition can be fulfilled by adjusting $\cos(a')$ according to the following equation:

$$(1/\pi)\sum_{i'=1}^{n}\sum_{j'=1}^{m}\cos(i')\sin(w_{i'})w_{j'}|\cos(a')| = 1. \tag{33}$$

The value of $\cos(a')$ thus obtained is also used for cos(a), which ensures the validity of the reciprocity relation.[7]

VI. Techniques of Reducing Execution Time for a Leaf Canopy Without Azimuthal Preference

As the basic matrices M^*, F^*, B^* and the boundary condition R_s^* have been determined, the bidirectional reflectance pattern of a canopy can be calculated by the analytical solution Eq. (23). It can be seen that multiplication, inversion, and similarity transformation of matrices are involved. The execution time is approximately proportional to the cube of the dimensions of the matrices. If each inclination interval is taken as 10°, the matrices involved in calculating zonal reflectance of the canopy have dimensions of 9 × 9. To account for the azimuthal variations, if the azimuthal interval is also taken as 10°, the relevant extended matrices will have dimensions of 324 × 324. The execution time for calculating the bidirectional reflectance pattern of the canopy will be prohibitively long even though the analytical solution is available. It is desirable, therefore, to develop techniques to reduce the execution time. This is possible for a leaf canopy without obvious azimuthal preference, as is the case for most crops.[8]

For such a canopy, because of the azimuthal symmetry the interception matrix M^* is independent of the azimuth, and the azimuthal dependence of the backscattering and forward scattering matrices F^* and B^* is related only to the difference between the azimuths of incident and exitant directions. Therefore, among the component matrices of an extended matrix only m matrices are distinct. The matrix F^* [Eq. (25)], e.g., has only m distinct matrices $F_{j'j}$. Hence, Eq. (25) becomes

$$F^* = \begin{vmatrix} F_1 & F_2 & F_3 & \cdots & F_m \\ F_2 & F_1 & F_2 & \cdots & F_{m-1} \\ F_3 & F_2 & F_1 & \cdots & F_{m-2} \\ \vdots & \vdots & \vdots & & \vdots \\ F_m & F_{m-1} & F_{m-2} & \cdots & F_1 \end{vmatrix}, \tag{34}$$

where F_k ($k = 1$ to m) equals $F_{|j-j'|+1}$, so that $k = 1$ means that the azimuths of the incident and exitant directions are coincident. Moreover, if m is taken as an even number, and $m/2$ is denoted by m', only $m' + 1$ matrices among the F_k are distinct, because $F_{m'+1+k} = F_{m'+1-k}$ ($k = 1$ to $m' - 1$). Hence, the matrix F^* can be represented as

$$F^* = \begin{vmatrix} F_1 & F_2 & \cdots & F_{m'} & F_{m'+1} & F_{m'} & \cdots & F_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ F_{m'} & F_{m'-1} & \cdots & F_1 & F_2 & F_3 & \cdots & F_{m'+1} \\ F_{m'+1} & F_{m'} & \cdots & F_2 & F_1 & F_2 & \cdots & F_{m'} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ F_2 & F_3 & \cdots & F_{m'+1} & F_{m'} & F_{m'-1} & \cdots & F_1 \end{vmatrix}. \tag{35}$$

When the matrices F_1 to $F_{m'+1}$ are known, the matrix F^* is determined, so F_1 to $F_{m'+1}$ are called elementary matrices of F^*. It can be proved that the product of two such matrices A^* and B^*, C^* retains the same property as A^* and B^*, and the elementary matrices of C^*, C_k can be obtained directly from the elementary matrices of A^* and B^*, A_j and B_j, by

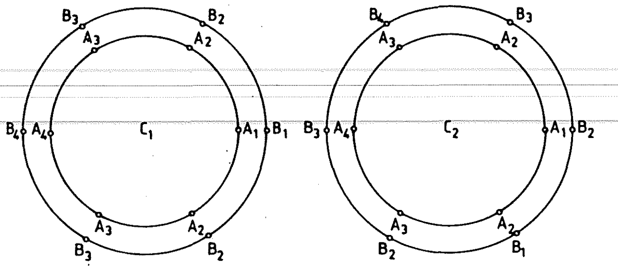

Fig. 3. Scheme of positioning matrices A and B to find their product C.

$$C_k = A_1B_k + \sum_{j=2}^{m'} A_j(B_{|k-j|+1} + B_{m'+1-|m'+2-k-j|}) + A_{m'+1}B_{m'+2-k}. \tag{36}$$

The diagram shown in Fig. 3 is designed for $m = 6$ to derive Eq. (36). The elementary matrices of A^* and B^* are arranged counterclockwise along two circles as $A_1A_2A_3A_4A_3A_2$ and $B_1B_2B_3B_4B_3B_2$. The elementary matrices of the product C_k are the sum of the products of A_k and B_k at the same positions in the circles. For C_1, the A and B matrices have the same subscripts at the same positions. For C_2 the A circle is fixed, while the B circle is turned clockwise one step; for C_3, two steps and so on. It is clear that Eq. (36) greatly reduces storage as well as the computing time.

Unfortunately, no simple method is found to invert such matrices directly from their elementary matrices. But a method exists to reduce the dimension by a factor of 2. Inspecting Eq. (35) shows that the equivalent matrix F^* contains only two different blocks P and Q as

$$F^* = \begin{vmatrix} P & Q \\ Q & P \end{vmatrix}, \tag{37}$$

so that the inverse can be determined by

$$(F^*)^{-1} = \begin{vmatrix} S & G \\ G & S \end{vmatrix}, \tag{38}$$

where

$$S = (P - QP^{-1}Q)^{-1}, \qquad G = -P^{-1}QS. \tag{39}$$

For similarity transformation of matrix Q^* to a diagonal matrix, the same method can be applied. By denoting Q^* in the block form,

$$Q^* = \begin{vmatrix} A & B \\ B & A \end{vmatrix}, \tag{40}$$

Q^* can be rewritten as

$$Q^* = \tfrac{1}{2}\begin{vmatrix} I & I \\ -I & I \end{vmatrix}\begin{vmatrix} A-B & 0 \\ 0 & A+B \end{vmatrix}\begin{vmatrix} I & -I \\ I & I \end{vmatrix}. \tag{41}$$

Computation shows that $A - B$ and $A + B$ can be transformed into diagonal matrices S and G:

$$A - B = VSV^{-1}, \qquad A + B = UGU^{-1}, \tag{42}$$

Q^* can thus be expressed as

$$Q^* = \tfrac{1}{2}\begin{vmatrix} V & U \\ -V & U \end{vmatrix}\begin{vmatrix} S & 0 \\ 0 & G \end{vmatrix}\begin{vmatrix} V^{-1} & -V^{-1} \\ U^{-1} & U^{-1} \end{vmatrix}. \tag{43}$$

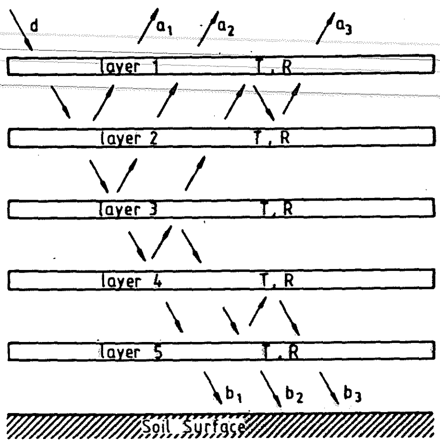

Fig. 4. Components of the reflected and transmitted radiation vectors with different paths.

The validity of Eq. (43) can be verified directly by multiplication.

VII. Approximation Method

Although techniques have been developed to reduce the execution time, the resolution in azimuth is still restricted. It has been shown[7] that the reflected and transmitted radiation vectors of a canopy are composed of an infinite number of component vectors. An illustration is shown in Fig. 4. A radiation vector **d** is incident on the top of the canopy. The radiation vector $\mathbf{a}_1$ is obtained by the interaction of **d** with layers 1 and 2: $\mathbf{a}_1 = \mathrm{TRT}\mathbf{d}$, as shown in the figure. The radiation vectors $\mathbf{a}_2$ and $\mathbf{a}_3$ can be obtained similarly. An infinite number of the component radiation vectors such as $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ constitute a reflected radiation vector from the top of the canopy. The transmitted radiation vector through the bottom of the canopy is similarly composed of an infinite number of the component radiation vectors such as $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$. It can be seen from Fig. 4 that, for establishing either a reflected or a transmitted component radiation vector, there must be an odd number of reflections by the layers. For a leaf canopy without azimuthal preference, the backscattering and forward scattering matrices are composed of the elementary matrices as mentioned above. The formula determining the product of two such matrices [Eq. (36)] ensures that the more the interaction of the radiation vector with the layers takes place, the more the variations of the radiation intensity with azimuth will be smoothed. In fact, little variation is left after threefold interactions. For practical purposes, it is sufficient to consider only the single reflection from different layers to find the contribution to the azimuthal variation of the reflected radiation vectors.

Consider an infinitesimal layer with leaf area index dl at depth l. Assume the azimuth of the incident radiation to be zero. The component reflectance matrix, $d\mathrm{R}^*$, formed by single reflection from the layer dl with no interaction with the other layers, can be calculated as

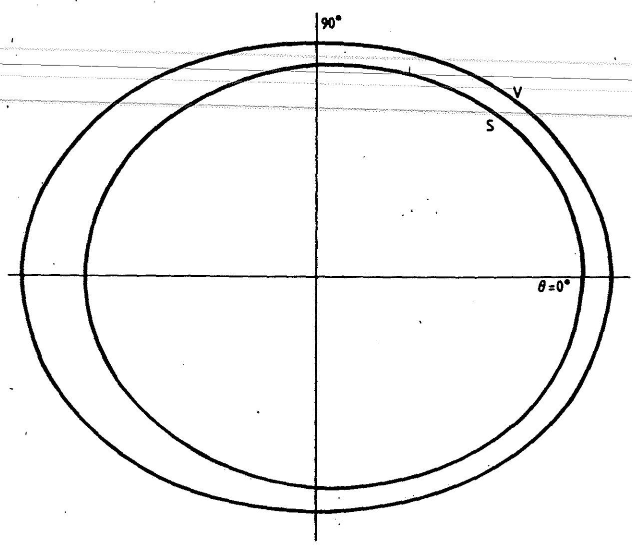

Fig. 5. Azimuthal variations of reflection radiance from a canopy with spherical inclination distribution (S) and with vertical leaves (V). The inclination for incident and exitant directions is 25°.

$$d\mathrm{R}^*(i',j',i,0) = \exp[-\mathrm{M}(i',i')l] \times \exp[-\mathrm{M}(i,i)l]B^*(i',j',i,0)dl, \tag{44}$$

where M is the elementary matrix of the interception matrix M^*. The total contribution of all the layers can be obtained by integration:

$$\mathrm{R}^*(i',j',i,0) = \mathrm{B}^*(i',j',i,0)(1-\exp\{-[\mathrm{M}(i',i') + \mathrm{M}(i,i)]l_c\})/[\mathrm{M}(i',i') + \mathrm{M}(i,i)]. \tag{45}$$

Meanwhile, the total zonal reflectance matrix $\mathrm{R}_{\mathrm{zon}}$ of the canopy can be easily calculated using the analytical solution Eq. (24). The difference between $\mathrm{R}_{\mathrm{zon}}(i',i)$ and the sum of $\mathrm{R}^*(i',j',i,0)(j' = 1 \text{ to } m)$ can be considered evenly distributed over azimuth. The elementary matrices of the reflectance matrix of the canopy thus can be obtained:

$$\mathrm{R}_c^*(i',j',i,0) = \mathrm{R}^*(i',j',i,0) + [\mathrm{R}_{\mathrm{zon}}(i',i) - \sum_{j'=1}^{m} \mathrm{R}^*(i',j',i,0)]/m. \tag{46}$$

The transmittance matrix of the canopy can be treated similarly, except that a directly transmitted part should be added:

$$\mathrm{T}^*(i',j',i,0) = \mathrm{F}^*(i',j',i,0)\{\exp[-\mathrm{M}(i',i')l_c] - \exp[-\mathrm{M}(i,i)l_c]\}/[\mathrm{M}(i,i) - \mathrm{M}(i',i')] + d_{i'i}\exp[-\mathrm{M}(i,i)l_c], \tag{47}$$

where $d_{i'i}$ is equal to unity when $i = i'$, and zero otherwise. The transmittance matrix of the canopy is

$$\mathrm{T}_c^*(i',j',i,0) = \mathrm{T}^*(i',j',i,0) + [\mathrm{T}_{\mathrm{zon}}(i',i) - \sum_{j'=1}^{m} \mathrm{T}^*(i',j',i,0)]/m, \tag{48}$$

where $\mathrm{T}_{\mathrm{zon}}$ is the total zonal transmittance matrix.

VIII. Some Illustrative Results

It is not the purpose of this paper to calculate and discuss the reflectance and transmittance matrices for various kinds of crop canopy, although the method developed is aimed primarily at practical applications.

For total zonal reflectance and transmittance, the results are almost exactly the same as those obtained by Goudriaan,[4] while the execution time is greatly reduced (~1 sec on the computer DEC10). The approximate method is used to calculate the detailed azimuthal variations of the reflected radiance, as shown in Fig. 5. The results are given for vertical leaves and for the leaves with a spherical inclination distribution. The inclinations for incident and exitant directions are 25°. The azimuthal angle interval is 10°, so high resolution is ensured. The execution time is ~10 sec. The azimuthal variations shown in Fig. 5 refer to a canopy with $t = r = 0.4$. Because of the multiple scattering between the layers they are less variable than the results obtained by Ross[11] for a mean leaf.

IX. Discussion

A. Non-Lambertian Scatterers

That leaves are Lambertian scatterers is an oversimplified assumption. It is adopted in this paper merely for convenience of explaining the method. Under vector–matrix notation, it is no longer a restriction. The reflectance and transmittance matrices of a given leaf can be measured experimentally.[12] The basic backscattering and forward scattering matrices of a canopy can be calculated by the formulas given by Chen,[7] and the rest of the procedure remains the same.

B. Applicability to the Atmosphere and Clouds

Although the differential equations and the methods to solve the equations in this paper are developed with special attention to crop canopies, it can be obviously applied to the radiation transfer through the atmosphere or through clouds. The only difference lies in the way of calculating the basic backscattering and forward scattering matrices. In this case, they can be calculated from the phase function of the constituent scattering substances, such as gas molecules, particles, or water droplets, and the knowledge of their size distribution functions. The cumulative leaf area index, of course, should be replaced by the optical depth used conventionally.

C. Superposition of Several Heterogeneous Layers

Sometimes the scattering medium cannot be represented by one layer with uniform properties. For example, a mature rice or wheat crop canopy is better represented by two layers, one corresponding to the ears, and the other to the leaves. For remote sensing, between the crop canopy and the sensors, there is a layer of air, which also scatters radiation. This effect must be included if a more accurate result is demanded. The method developed in this paper can be readily adapted to these cases. The calculation should be started from the lowest layer, and the reflectance matrix of the underlying surface, the soil surface, say, is taken as the boundary condition. The solution of the reflectance matrix of the lowest layer thus obtained can be employed as the boundary condition for the second layer, and so on.

This work was supported by the Ministry of Education and Science of The Netherlands. I gratefully acknowledge this support. I am much indebted to J. Goudriaan, C. T. de Wit, and B. van Rootselaar for many helpful suggestions.

References

1. S. Chandrasekhar, *Radiative Transfer* (Clarendon, Oxford, 1950).
2. G. W. Paltridge and C. M. R. Platt, *Radiative Processes in Meteorology and Climatology* (Elsevier, New York, 1976).
3. W. A. Allen, T. V. Gayle, and A. J. Richardson, "Plant-Canopy Irradiance Specified by the Duntley Equations," J. Opt. Soc. Am. **60**, 372 (1970).
4. J. Goudriaan, *Crop Micrometeorology: A Simulation Study* (Pudoc, Wageningen, 1977).
5. K. Cooper, J. A. Smith, and D. Pitts, "Reflectance of a Vegetation Canopy Using the Adding Method," Appl. Opt. **21**, 4112 (1982).
6. H. C. van de Hulst, *Multiple Light Scattering* (Academic, London, 1980).
7. J. Chen, "The Reciprocity Relation for Reflection and Transmission of Radiation by Crops and Other Plane-Parallel Scattering Media," Remote Sensing Environ. **13**, 475 (1983).
8. C. T. de Wit, *Agricultural Research Report 663* (Pudoc, Wageningen, 1965).
9. W. A. Allen and A. J. Richardson, "Interaction of Light with a Plant Canopy," J. Opt. Soc. Am. **58**, 1023 (1968).
10. B. van Rootselaar, "How to Solve the System $x' = Ax$," Am. Math. Mon. **91**, 10 (1984).
11. J. Ross, *The Radiation Regime and Architecture of Plant Stands* (Dr. W. Junk Publishers, The Hague, 1981).
12. H. T. Breece III and R. A. Holmes, "Bidirectional Scattering Characteristics of Healthy Green Soybean and Corn Leaves *in vivo*," Appl. Opt. **10**, 119 (1971).