THE FALL VELOCITY OF GRAIN PARTICLES

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CONTENTS

1 INTRODUCTION

The fall or settling velocity of grain particles figures prominently in all sediment transport problems and although the concept is straight forward its precise evaluation or calculation is not. For implementation in computer models an adequate describtion of the fall velocity is required. In this note governing equations are presented to predict the fall velocity for laminar and turbulent motion and for the transition region. In addition simple power functions are used to express the relation between the drag coefficient and the particle Reynolds number. The latter relation is needed to obtain a direct solution for the fall velocity.

2 FALL VELOCITY OF A SPHERE

The fall velocity of a grain particle is a function of size, shape and density of the particle and viscosity of the fluid. It also depends on the extent of the fluid in which it falls, on the number of particles falling and on the level of turbulence intensity. Falling under influence of gravity, the particle will reach a constant velocity, the terminal velocity, when the drag force equals the submerged weight of the particle. For spherical particles the submerged weight is expressed as:

$$G = (\rho_S - \rho_W) \cdot \frac{1}{6} \pi D^3 \cdot g \qquad (1)$$

where,

G	= submerged weight	$[M\ L\ T^{-2}]$
ρ_S	= density of sediment	$[M\ L^{-3}]$
ρ_W	= density of water	$[M\ L^{-3}]$
D	= particle diameter	$[L]$
g	= acceleration due to gravity	$[L\ T^{-2}]$

The frictional drag force equals:

$$F_D = C_D \cdot \frac{1}{4} \pi D^2 \cdot \frac{\rho_W w^2}{2} \qquad (2)$$

where,

F_D = drag force [M L T^{-2}]

C_D = drag coefficient [1]

w = particle fall velocity [L T^{-1}]

From Equation 1 and 2 the terminal fall velocity can be calculated as:

$$w^2 = \frac{4}{3} \frac{1}{C_D} g D \frac{\rho_s - \rho_w}{\rho_w} \qquad (3)$$

The value of C_D depends on the Reynolds number of the settling particle:

$$Re = \frac{w \cdot D}{\nu} \qquad (4)$$

where,

Re = Reynolds number [1]

ν = kinematic viscosity [L^2 T^{-1}]

The relation between C_D and Re holds three regions. For laminar flow the frictional resistance is only due to viscous forces and C_D varies inverse proportional to Re.

$$C_D = \frac{24}{Re} \qquad (5)$$

for Re < 0.5 and approximately for up to 1.0. At high Reynolds numbers the flow of water takes place under fully developed turbulent conditions. Compared with the eddying resistance the viscous forces are negligible and C_D is approximately constant, or

$$C_D = 0.40 \qquad (6)$$

for Re > 2000, and up to 10^5.

For the transition zone an exact formula for C_D cannot be given but for Reynolds numbers below 10^4 (including the laminar flow region) a good approximation that can be used is (Huisman, 1973):

$$C_D = \frac{24}{Re} + \frac{3}{\sqrt{Re}} + 0.34 \qquad (7)$$

3 FALL VELOCITY OF GRAINS

So far a single spherical particle is considered in a fluid of infinite extent. For non-spherical particles with regular shape a correlation with the drag coefficient and a shape factor can be obtained. The shape factor is expressed as the ratio of the surface area of a sphere of the same volume as the particle to the actual surface area of the particle, and is also called "degree of true sphericity"

$$\Psi = \frac{A_S}{A_{SN}} \tag{8}$$

where

Ψ	= shape factor	[1]
A_S	= surface area of spherical particle	$[L^2]$
A_{SN}	= surface area of nonspherical particle	$[L^2]$

Graf (1971) shows graphs of the settling velocity for various regular shapes, together with an average for irregular quartz grains (Figure 1).

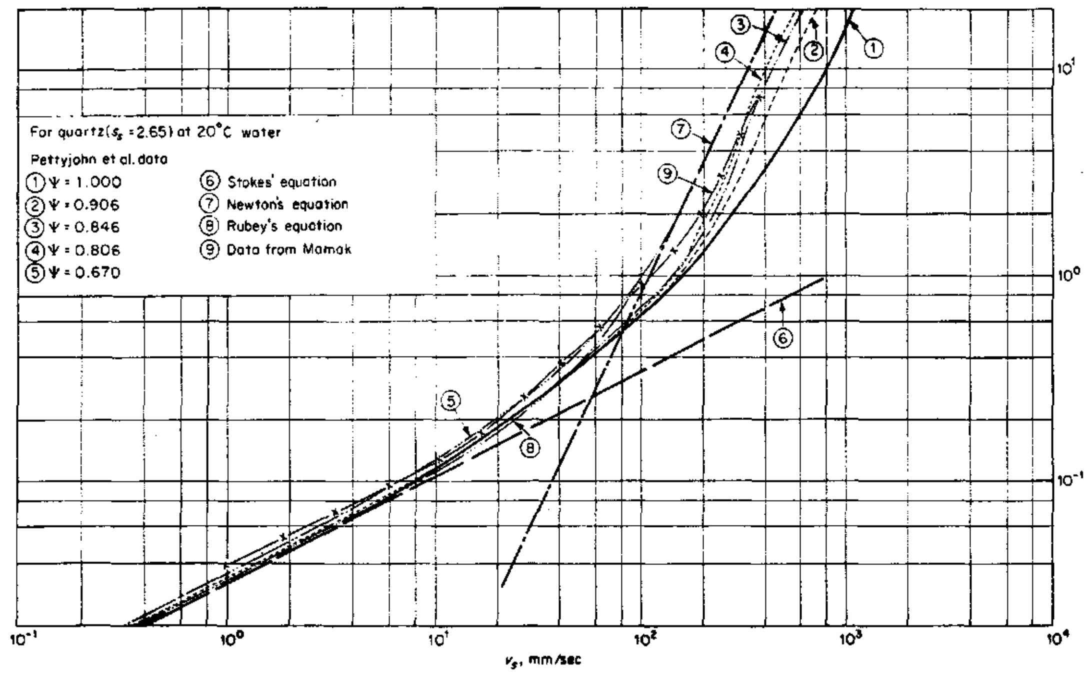

Figure 1 Settling velocity versus particle diameter [After Graf et al. (1966)].

In case of a shape factor of 0.670 (tetahedron) the drag coefficient can be calculated as

$$C_D \approx \frac{28}{Re} \tag{9}$$

for the laminar region and

$$C_D = 2.00 \tag{10}$$

in case of turbulent motion.

Expressions for the fall velocity from which the drag coefficient can be derived are given by Raudkivi (1976) and Van Rijn (1985) for laminar and turbulent motion while Zanke (1977) proposed an equation for the transition zone. Raudkivi also defines the transition zone by a number of points. The equations given by Raudkivi are:

$$w \approx 663\ D^2 \tag{11}$$

where w is in mm/s and D in mm for $D \leq 0.15$ mm, and

$$w \approx 135.5\ D^{1/2} \tag{12}$$

for $D \geq 1.50$ mm (Both equations at 20°C). If meter instead of millimeter is used as unit, and for comparitive and dimensional reasons the relative density, acceleration due to gravity and kinematic viscosity is incorporated, these Equations become:

$$w = \frac{1}{24.42} \frac{\Delta g\ D^2}{\nu} \qquad (D \leq 0.15\ \text{mm}) \tag{13}$$

and

$$w = 1.057\ (\Delta g D)^{1/2} \qquad (D > 1.5\ \text{mm}) \tag{14}$$

respectively, with

$$\Delta = \frac{\rho_s - \rho_w}{\rho_w} \tag{15}$$

where,

Δ = relatively density [1]

From Equation (3) and (13) it can be derived that in this case for laminar motion:

$$C_D = \frac{32.56}{Re} \tag{16}$$

and, similar, for turbulent motion:

$$C_D = 1.193 \tag{17}$$

Van Rijn gives the same formulas as (13) and (14) but uses 18 and 1,1 as coefficients meaning that for laminar motion equation (5) is found and for tubulent motion:

$$C_D = 1.102 \tag{18}$$

The equation proposed bij Zanke and also used by Van Rijn for $0.1 \leq D \leq 1.0$ mm, yields:

$$w = 10 \frac{\nu}{D} \left[\left\{1 + 0.01 \frac{\Delta g D^3}{\nu^2}\right\}^{1/2} - 1\right] \tag{19}$$

Which corresponds to (see also appendix 1):

$$C_D = \frac{4}{3} \left(1 + \frac{20}{Re}\right) \tag{20}$$

This Equation results in a drag coefficient of 26.7/Re for small Reynolds number, a C_D value of 28 for Reynolds number equal to 1, and of 1.333 in case of high Reynolds numbers. This includes a discontinuity at the boundary of the transition region if used in combination with the mentioned equations for laminar and turbulent motion. In this way an inadequate description of the dependancy of the drag coefficient on the Reynolds number is obtained.

4 EQUATIONS FOR DRAG COEFFICIENT AND FALL VELOCITY

To deal with the problem mentioned, for laminar motion at $Re \leq 1$ the equation for the drag coefficient becomes:

$$C_D = \frac{28}{Re} \tag{21}$$

For turbulent motion at Re $\geq$ 250 a constant value of 1.15 is taken:

$$C_D = 1.15 \tag{22}$$

The limit of 250 is found using equation (14) with a particle diameter of 0.15 mm and corresponds to the figures of Graf (1971).
Adapting an equation like (20) but with different coefficients the formula for the transition region would yield:

$$C_D = 1.042 \left(1 + \frac{25.87}{Re}\right) \tag{23}$$

The accuracy of the coefficients is for computational reasons only.

The corresponding equations for the particle fall velocity become for laminar motion (Re $\leq$ 1):

$$w = \frac{1}{21} \frac{\Delta g D^2}{\nu} \tag{24}$$

for turbulent motion (Re $\geq$ 250):

$$w = 1.077 \; (\Delta g D)^{1/2} \tag{25}$$

and for the transition region with 1 $\leq$ Re $\leq$ 250
(see appendix 2):

$$w = \frac{\beta\nu}{2D} \left\{ \left(1 + \frac{4}{\alpha\beta^2} \frac{4}{3} \frac{\Delta g D^3}{\nu^2}\right)^{1/2} - 1 \right\} \tag{26}$$

with α = 1.077 and β = 25.87 this results in:

$$w = 12.94 \frac{\nu}{D} \left\{ \left(1 + 7.399 \cdot 10^{-3} \frac{\Delta g D^3}{\nu^2}\right)^{1/2} - 1 \right\} \tag{27}$$

A comparison of the particle fall velocity at 20°C, calculated according to the different equations is given in table 1.

Region	D mm	w mm/s			
		Raudkivi	Van Rijn	Zanke	equation (27)
laminar	0.01	0.0663	0.0899		0.0771
	0.02	0.2652	0.3598		0.3084
	0.04	1.061	1.439		1.234
	0.06	2.387	3.238		2.775
	0.08	4.243	5.756		4.934
	0.10	6.63	8.994	7.791	7.709
transition	0.15	14.8		16.24	17.25
	0.20	20.4		25.75	26.57
	0.40	42.9		59.27	64.27
	0.60	65.2		83.29	92.00
	0.80	87.6		102.0	113.6
	1.0	110.0	140.0	117.6	131.6
turbulent	1.5	166.0	171.4		165.0
	2.0	190.2	197.9		193.8
	4.0	269.0	279.9		274.1
	6.0	329.5	342.8		335.7
	8.0	380.4	395.9		387.6
	10	425.3	442.6		433.3
	20	601.5	625.9		612.7

Table 1 Comparison of particle fall velocity calculated in different way.

5 POWER FUNCTIONS FOR THE TRANSITION REGION

The results in the transition region are not very satisfactory, because a too high fall velocity is found, meaning that the drag coefficient is underpredicted. To avoid this an Equation like (7) could be tried out, but the disadvantage of this formule is that no direct solution for the fall velocity can be found when it is substituted in Equation (3). A superior method, also suggested by Huisman (1973), is to subdivide the transition region and to approximate the drag coefficient by using formulae of the type:

$$C_D = \frac{\alpha}{Re^\beta} \tag{28}$$

Combining equation (3) and (28) yields:

$$w^2 = \frac{Re^\beta}{\alpha} \frac{4}{3} \Delta gD \tag{29}$$

which is equivalent to:

$$Re^2 = \frac{w^2D^2}{\nu^2} = \frac{Re^\beta}{\alpha} \frac{4}{3} \frac{\Delta gD^3}{\nu^2} \tag{30}$$

or:

$$\alpha \ Re^{2-\beta} = \frac{4}{3} \frac{\Delta gD^3}{\nu^2} \tag{31}$$

so that:

$$w = \frac{\nu}{D} \left(\frac{1}{\alpha} \frac{4}{3} \frac{\Delta gD^3}{\nu^2}\right)^{\frac{1}{2-\beta}} \tag{32}$$

The upper and lower limit for the use of the different formulae can be based on numeric values of the righthand side of equation (31). In this way the choice for the right formula can made in advance while this is not possible if the boundaries are defined by the Reynolds number. The next equations are appropriate to predict the particle fall velocity:

$$C_D = \frac{28}{Re} \qquad \frac{4}{3} \frac{\Delta gD^3}{\nu^2} < 28 \tag{33}$$

$(Re < 1)$

$$C_D = \frac{28}{Re^{2/3}} \qquad 28 \leq \frac{4}{3} \frac{\Delta gD^3}{\nu^2} < 960.1 \tag{34}$$

$(1 \leq Re < 14.17)$

$$C_D = \frac{18}{Re^{1/2}} \qquad 960.1 \leq \frac{4}{3}\frac{\Delta g D^3}{\nu^2} < 6.903 \cdot 10^4 \qquad (35)$$

$$(14.17 \leq Re < 245)$$

$$C_D = 1.15 \qquad \frac{4}{3}\frac{\Delta g D^3}{\nu^2} \geq 6.903 \cdot 10^4 \qquad (36)$$

$$(Re \geq 245)$$

The particle fall velocity calculated according to equation (33) up until (36) is given in Table 2, showing very satisfactory results.

Region	D mm	w mm/s	$\frac{4}{3}\frac{\Delta g D^3}{\nu^2}$ 1	Re 1	C_D 1
laminar	0.01	0.077	$2.159 \cdot 10^{-2}$	$7.709 \cdot 10^{-4}$	$3.632 \cdot 10^{+4}$
	0.02	0.308	$1.727 \cdot 10^{-1}$	$6.168 \cdot 10^{-3}$	$4.540 \cdot 10^{+3}$
	0.04	1.234	1.382	$4.934 \cdot 10^{-2}$	$5.675 \cdot 10^{+2}$
	0.06	2.775	4.663	$1.665 \cdot 10^{-1}$	$1.681 \cdot 10^{+2}$
	0.08	4.934	$1.105 \cdot 10^{+1}$	$3.947 \cdot 10^{-1}$	$7.094 \cdot 10^{+1}$
	0.10	7.709	$2.159 \cdot 10^{+1}$	$7.709 \cdot 10^{-1}$	$3.632 \cdot 10^{+1}$
transition	0.15	13.66	$7.285 \cdot 10^{+1}$	2.049	$1.736 \cdot 10^{+1}$
	0.20	19.57	$1.727 \cdot 10^{+2}$	3.914	$1.127 \cdot 10^{+1}$
	0.40	45.15	$1.382 \cdot 10^{+3}$	$1.806 \cdot 10^{+1}$	4.236
	0.60	67.73	$4.663 \cdot 10^{+3}$	$4.064 \cdot 10^{+1}$	2.824
	0.80	90.30	$1.105 \cdot 10^{+4}$	$7.224 \cdot 10^{+1}$	2.118
	1.0	112.9	$2.159 \cdot 10^{+4}$	$1.129 \cdot 10^{+2}$	1.694
turbulent	1.5	167.8	$7.285 \cdot 10^{+4}$	$2.517 \cdot 10^{+2}$	1.150
	2.0	193.8	$1.727 \cdot 10^{+5}$	$3.875 \cdot 10^{+2}$	1.150

Table 2 Particle fall velocity at 20°C.

The dependancy of the drag coefficient on the Reynolds number according to Equation (33) up until (36) is shown in Figure 2 in comparison to other formulae.

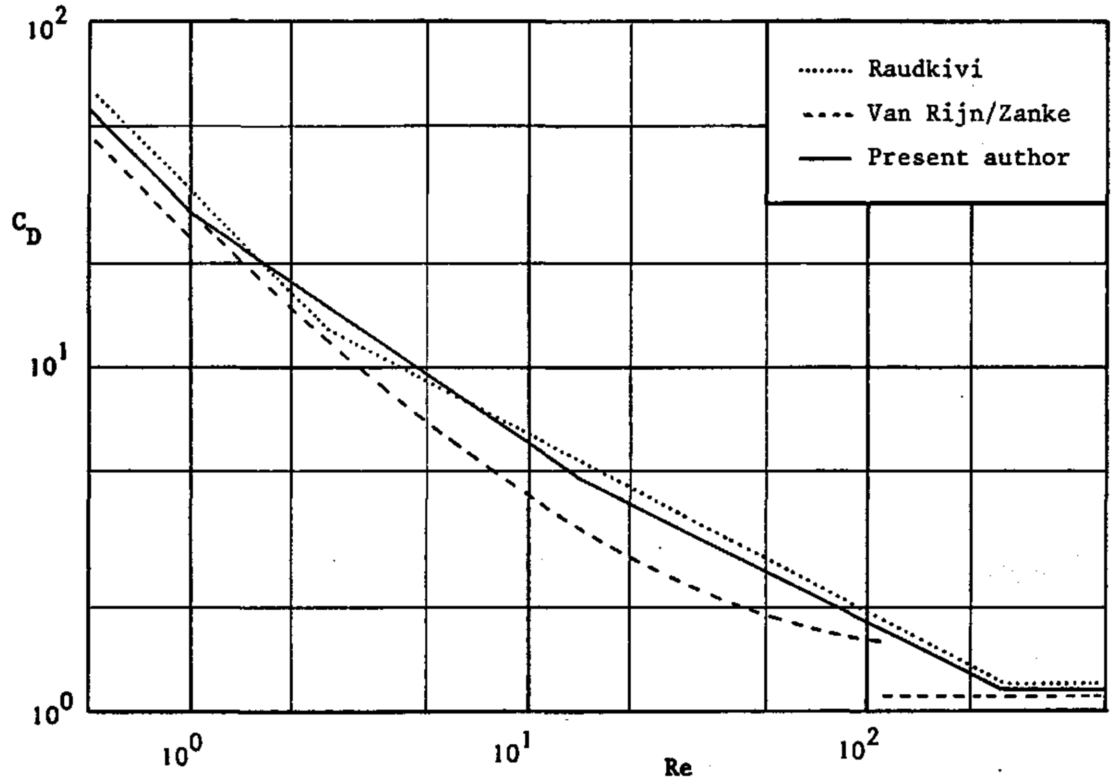

Figure 2 Drag coefficient versus Reynolds number.

6 CONCLUDING REMARK

The foregoing analysis gives an arbitrary solution to the problem of calculating the particle fall velocity. A refinement in the constants and powers in the proposed formulae can be accomplished by a more comprehensive study using additional literature and also experimental data for making a least square fit.

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LIST OF SYMBOLS

A_{sN} =	surface area of nonspherical particle	$[L^2]$
A_s =	surface area of spherical particle	$[L^2]$
C_D -	drag coefficient	[1]
D -	particle diameter	[L]
F_D -	drag force	$[M\ L\ T^{-2}]$
g -	acceleration due to gravity	$[L\ T^{-2}]$
G -	submerged weight	$[M\ L\ T^{-2}]$
Re -	Reynolds number	[1]
w -	particle fall velocity	$[L\ T^{-1}]$
Δ -	relative density	[1]
ν -	kinematic viscosity	$[L^2\ T^{-1}]$
ρ_w -	density of water	$[M\ L^{-3}]$
ρ_s -	density of sediment	$[M\ L^{-3}]$
Ψ -	shape factor	[1]

APPENDIX 1: Derivation of drag coefficient from different formulas for the fall velocity.

Fall velocity equation:

$$w^2 = \frac{1}{C_D} \frac{4}{3} \Delta gD \qquad (1)$$

Laminar

$$w = \frac{1}{\alpha} \frac{\Delta gD^2}{\nu} \qquad (2)$$

(1)&(2) →

$$w = \frac{1}{C_D} \frac{4}{3} \frac{\nu}{\alpha D} \qquad (3)$$

$$C_D = \frac{4}{3} \frac{1}{\alpha Re} \qquad (4)$$

Turbulent motion:

$$w = \beta \ (\Delta gD)^{1/2} \qquad (5)$$

$$\rightarrow \quad w^2 = \beta^2 \ \Delta gD \qquad (6)$$

(1) & (6)

$$\rightarrow \quad C_D = \frac{4}{3} \frac{1}{\beta^2} \qquad (7)$$

Transition region:

$$w = 10 \frac{\nu}{D} \left\{ \left[1 + 0.01 \frac{\Delta gD^3}{\nu^2}\right]^{1/2} - 1 \right\} \qquad (8)$$

(1) & (8)

$$\frac{wD}{\nu} = 10 \left\{ \left[1 + 0.01 \frac{3}{4} w^2 \ C_D \frac{D^2}{\nu^2}\right]^{1/2} - 1 \right\} \qquad (9)$$

$$\rightarrow \quad \frac{1}{10} Re + 1 = \left\{1 + 0.01 \frac{3}{4} C_D \ Re^2\right\}^{1/2}$$

$$\rightarrow \quad 0.01 \ Re^2 + 0.2 \ Re = 0.01 \frac{3}{4} C_D \ Re^2$$

$$\rightarrow \quad Re^2 + 20Re = \frac{3}{4} C_D Re^2$$

$$\rightarrow \quad C_D = \frac{4}{3} \left(1 + \frac{20}{Re}\right) \tag{10}$$

APPENDIX 2: Derivation of the equation for the fall velocity in the transition region.

Equation for the drag coefficient:

$$C_D = \alpha \left(1 + \frac{\beta}{Re}\right) \tag{1}$$

Equation for the fall velocity:

$$w^2 = \frac{1}{C_D} \frac{4}{3} \Delta g D \tag{2}$$

$$(1) \,\&\, (2) \rightarrow w^2 = \frac{1}{\alpha\left(1 + \frac{\beta}{Re}\right)} \frac{4}{3} \Delta g D \tag{3}$$

$$\rightarrow w^2 \left(1 + \frac{\beta\nu}{wD}\right) = \frac{1}{\alpha} \frac{4}{3} \Delta g D \tag{4}$$

$$\rightarrow w^2 + \beta \frac{\nu}{D} w - \frac{1}{\alpha} \frac{4}{3} \Delta g D = 0 \tag{5}$$

The relevant solution of equation (5) yields:

$$w = - \frac{\beta\nu}{2D} + \left\{\left(\frac{\beta\nu}{2D}\right)^2 + \frac{1}{\alpha} \frac{4}{3} \Delta g D\right\}^{1/2} \tag{6}$$

or

$$w = \frac{\beta\nu}{2D} \left\{\left(1 + \frac{4}{\alpha\beta^2} \frac{4}{3} \frac{\Delta g D^3}{\nu^2}\right)^{1/2} - 1\right\} \tag{7}$$