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Sudden Change in Second Order Nonlinear Systems

Slow Passage Through Bifurcation

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Proefschrift
ter verkrijging van de graad van doctor
in de landbouw- en milieuwetenschappen
op gezag van de rector magnificus,
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in het openbaar te verdedigen
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des namiddags te één uur dertig in de Aula
van de Landbouwuniversiteit te Wageningen.

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Stellingen

1. Wiskundige modellen geven een menselijke interpretatie van de werkelijkheid weer en zullen altijd aan fouten onderhevig zijn. Veronderstelde constanten in deze modellen zullen op den duur kleine veranderingen ondergaan, die van grote invloed zijn op het gedrag van het systeem op lange termijn en tot ongewenste gevolgen kunnen leiden.

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   J.C. Maxwell, Science and free will, in: L. Campbell and W. Garnett (eds.), *The Life of James Clerk Maxwell*, Macmillan, London, 1882.

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14. Het maken van een proefschrift kan beschouwd worden als het herkauwen van een reeds lang verteerde maaltijd.

Stellingen behorend bij het proefschrift 'Sudden Change in Second Order Nonlinear Systems; Slow Passage through Bifurcation' van G.J.M. Marée, 19 juni 1995.

Finis coronat opus

To know the Way,
We go the Way;
We do the Way
The way we do,
It's all there in front of you,
But if you try too hard to see it,
You'll only become confused.

I am me,
And you are you,
As you can see,
But when you do
The things that yóu can do,
Then you will find the Way,
And the Way will follow you.

Uit *The Tao of Pooh*
Benjamin Hopf

voor mijn ouders

Voorwoord

De afgelopen vier jaar heb ik een groot deel van mijn tijd besteed aan onderzoek naar plotselinge veranderingen en bifurcaties in dynamische systemen. Een deel van de resultaten van dit onderzoek heb ik in dit proefschrift beschreven. Omdat alle kennis menselijk is, zullen ook de door mij verkregen wetenswaardigheden vermengd zijn met fouten, vooroordelen, wensdromen en verwachtingen. De oplossingen die ik heb aangedragen voor wetenschappelijke problemen zullen weer genoeg stof voor nieuwe problemen opleveren. Met het promotieonderzoek en het maken van dit proefschrift heb ik een zeer prettige en leerzame ervaring opgedaan. Hopelijk wordt iets van mijn enthousiasme en plezier op de lezer overgedragen.

Graag wil ik in dit voorwoord een aantal mensen bedanken die op verschillende manieren hun bijdrage hebben geleverd aan het tot stand komen van dit proefschrift. In het bijzonder gaat mijn dank uit naar Johan Grasman met wie ik vrijwel wekelijks contact heb gehad over de voortgang van mijn onderzoek. Van zijn originele kijk op zaken en zijn inzicht heb ik veel geleerd. Door zijn aanstekelijke enthousiasme voor wetenschappelijk onderzoek, zijn optimistische levensvisie en zijn goede begeleiding heb ik met plezier onderzoek verricht en geleerd artikelen te schrijven. Ferdinand Verhulst hield zich meer op de achtergrond bezig, maar had wel goed in de gaten waar ik mee bezig was. Zowel op wetenschappelijk als op organisatorisch gebied was hij bereid een helpende hand toe te steken. Bovendien hielp hij mij aan zeer interessante wetenschappelijke contacten en bruikbare referenties.

Mijn collega's en oud-collega's bij de vakgroep wiskunde van de Landbouwuniversiteit Wageningen zorgden voor een inspirerende werksfeer door de vele wiskundige en vooral ook de niet-wiskundige activiteiten. Ik denk met veel plezier terug aan de vele discussies en gesprekken die ik met mijn vroegere kamergenoot Jaco van Kooten en met Oscar Buse heb gevoerd. In een goede verstandhouding hebben we gedurende deze vier jaar prettig veel ervaringen kunnen delen. Onno van Herwaarden heeft diverse suggesties gedaan voor de verbetering van de lay-out van het proefschrift. De zaalvoetbalploeg wil ik bedanken voor de uren sportieve afleiding en de vaak erg gezellige "derde helft".

Ook de collega's en oud-collega's van de "groep" toegepaste analyse in Utrecht wil ik bedanken voor een leerzame tijd en voor de welkome ontspanning bij congresbezoeken. Met het programma *DynaPlot* van Igor Hoveijn is een groot aantal van de illustraties gemaakt. Met Taoufik Bakri heb ik meerdere malen over problemen kunnen discussiëren.

Gedurende mijn promotietijd zorgde de studievereniging A-Eskwadraat voor enige aangename uitspattingen. Bovendien zorgde de jaarclub TumTum met aanhang van Unitas SR voor gezellige belevenissen. Bij Arie Hol en Iris Calon kon ik altijd terecht zowel met een lach als met een traan. Een groep studievrienden mag hier ook niet onvernoemd blijven.

Marijke, mijn ouders en broers en zussen en de familie van Marijke hebben mij steeds steun en vertrouwen gegeven en er voor gezorgd dat ik, buiten werktijd, de wiskunde soms even op het tweede plan kon zetten. De omgang met Marijke en haar belangstelling voor mijn bezigheden zijn van grote invloed geweest op de laatste twee jaar van mijn promotieonderzoek.

De Stichting Mathematisch Centrum (NWO Contract Nr. 611-306-526) heeft de verschillende reizen die ik maakte in het kader van mijn onderzoek gefinancierd.

Joris Marée
Wageningen, 12 juni 1995

De vragen zijn voor de wetenschap het brood des levens, niet de antwoorden. W. de Sitter

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Finis coronat opus

Contents

Knowledge is proud that he has learned so much; wisdom is humble that he knows no more. W. Cowper, *The task.*

Chapter 1

General Introduction

Introduction

Nonlinear bifurcation problems with slowly varying parameters appear in several areas of practical interest. In mathematical studies of bifurcation problems it is usually assumed that the bifurcation or control parameter is independent of time. However, in many experiments that are mathematically modelled as bifurcation problems, either the bifurcation parameter can be deliberately varied by the experimentalist or this parameter changes by itself in time and may lead to an undesirable response of the system.

In this study second order nonlinear differential equations with a slowly varying parameter are analyzed. The jump and bifurcation phenomena that are investigated belong to the class of differential equations of the following type:

$$\frac{d^2u}{dt^2} = G(u, \frac{du}{dt}, \alpha(\varepsilon t)) \ , \ \ 0 < \varepsilon << 1 \ . \tag{1}$$

By a slow change of the parameter a nonlinear system of differential equations may suddenly switch from one limit solution to a next one. For certain values of the bifurcation parameter α in (1) a sudden transition will occur, because the equilibrium solution disappears or becomes unstable. In order to describe the bifurcation local asymptotic approximations are constructed. Nonlinear transition layers are necessary at the critical times at which the parameter slowly varies through a critical bifurcation value. Local scaling analysis may yield the first or second Painlevé equation as generic equation. The fact that the recent theory of Painlevé transcendents can be applied to asymptotic approximation techniques is one of the major results of this research. It also appears to be possible to prove the validity of the matched asymptotic approximations for a large class of problems. Moreover, one can predict accurately the behaviour of the system after passage of the bifurcation point depending on the initial state and the values of the parameters.

Background

Since the system parameter slowly changes in time, the equilibrium solutions of the differential equations will also slowly vary. The general solution of the problem can be approximated asymptotically by using perturbation techniques like averaging, matched asymptotic expansions, and boundary layer approximations. With the aid of averaging the behaviour of solutions is approximated on a large time-scale. The method of averaging already dates from the eighteenth century (Lagrange). Around 1920 the Dutch scientist Balthasar van der Pol advocated the use of this method for equations that arise from electronic circuit theorems. Until that time the main field of its application had been celestial mechanics.

Solutions of second order differential equations with a slowly varying parameter can be described by oscillations with a slowly varying amplitude and phase. By averaging over a certain fixed time period a simplified system of differential equations for these state variables is obtained. The method has been used in many fields of applications, because the idea of averaging is very natural. In 1928 the first proof of asymptotic validity was given by Fatou. Krylov, Bogoliubov and Mitropolsky further developed this theory. In 1985 Sanders and Verhulst reviewed the theory and its applications and added new results.

The problems that are studied in this thesis can be transformed to perturbed oscillating systems with a slowly varying frequency:

$$\frac{d^2x}{dt^2} + \omega^2(\varepsilon t)x = \mu g(x, \frac{dx}{dt}, \varepsilon) \quad , \quad 0 < \varepsilon, \mu << 1 \ . \tag{2}$$

When for $t \geq 0$ the inequalities $0 < a < \omega(\varepsilon t) < b$ and $|d\omega/d(\varepsilon t)| < c$ hold with a, b and c arbitrary constants independent of ε, then the ratio of the energy of the system and its frequency (the "action" of a harmonic oscillator) is conserved with accuracy $O(\varepsilon)$ on the time-scale $1/\varepsilon$. When we study differential equations for which the coefficients ω slowly vary in time, $\omega = \omega(\varepsilon t)$, we call the quantities that are conserved mathematically adiabatic invariants. The classical theory of adiabatic invariants is an important tool for the study of the evolution of dynamical systems on the long time-scale. The idea was first introduced by Ehrenfest in 1916. A new formulation was given by Born and Fock in 1928. Meanwhile, Kneser gave a first mathematical proof of first order adiabatic invariance in 1924. Only much later the importance of the theory of adiabatic invariance was generally seen when the acceleration of cosmic rays was investigated in the beginning of the fifties. Lenard (1959) and later Arnold (1963) showed the relation of the adiabatic invariance with the classical perturbation theory. Since that time the theory also proved its usefulness in the physics of particles (the motion of a charged particle in an electromagnetic field) and in celestial mechanics (the evolution of solar systems). Moreover, in special circumstances, sharper (exponential) estimates of the invariance were obtained.

For the class of systems that are investigated in this thesis the slowly varying frequency has a singularity ($\omega = 0$) that coincides with the bifurcation or jump point. With the aid of an approximation theorem -an extended averaging theorem- it is proven that an asymptotic approximation remains valid within a certain time region. With the aid of an extension theorem, that was proven by Eckhaus in 1979, it becomes clear that there exists an overlap between the regions where the inner and outer solutions are valid. In the case of a second order jump phenomenon and for the slow passage through a pitchfork bifurcation, the transition problem reduces to a nonlinear ordinary differential equation of which the solutions have no other movable singularities than poles; an equation with the "Painlevé property".

These equations were subject to extensive investigations in the late nineteenth century. In the initial stage the accent was on first order differential equations. One of the most important contributions of Painlevé in this field is the development of simple methods that are applicable to higher order differential equations. In particular, these methods allow for a complete analysis of the equation

$$\frac{d^2y}{dx^2} = f(y, \frac{dy}{dx}, x) \tag{3}$$

with f a rational function in dy/dx and y, and with coefficients that are analytic in x. A fundamental result, formulated by Painlevé in 1902 and Gambier in 1910, is that there exist fifty classes of equations of the form (3) of which the solutions do not have movable critical points. It turned out that only six of them can not be reduced to linear equations, i.e. their general integral can not be expressed in terms of known special functions. These equations are called the six Painlevé equations and their corresponding solutions Painlevé transcendents or Painlevé functions. In the beginning of the sixties there was a renewed interest in Painlevé transcendents when the theory of solitons, the "inverse scattering" techniques, and the theory of infinitely dimensional nonlinear integrable systems were developed. Consequently, these Painlevé transcendents appeared in many branches of physics and applied mathematics. Moreover, the Painlevé property appeared to be closely related with integrability. The linear techniques that were developed in order to solve integrable nonlinear partial differential equations appeared to be very useful to study the properties of Painlevé transcendents, in particular their asymptotic behaviour.

Connection formulas reflect relations between the behaviour of solutions of ordinary differential equations in different singular points. For Painlevé transcendents this means the behaviour in different fixed singularities, like 0, $+\infty$, $-\infty$. The isomonodromy method in the modern theory of Painlevé equations was introduced by Flaschka and Newell (1980), and Ueno, Jimbo and Miwa (1981). Solving a Cauchy problem for a given Painlevé equation appeared to be equivalent to solving the inverse monodromy problem for an associated system of linear ordinary differential equations with rational coefficients, the "λ-equation". The method can be considered as a nonlinear analogue of the method of Laplace to solve linear ordinary differential equations. It yields the following results:

- A complete description of all types of asymptotic behaviour for all solutions, also in the complex plane (nonlinear Stokes phenomena).
- The description of explicit connection formulas for the asymptotics in different domains.
- The distribution of the poles and the zeros for all solutions.

The idea of the asymptotic approximation is based on the fact that the "monodromy data" of the λ-equation are a complete set of first integrals for the associated Painlevé equation. These data can be evaluated asymptotically and yield connection formulas for the Painlevé transcendents. Its, Fokas and Kapaev (1994) give a rigorous justification for the asymptotic results that are obtained with the isomonodromy method.

For example, with the second Painlevé equation $u_{xx} - xu - 2u^3 = 0$ the following two-dimensional linear system can be associated:

$$\frac{d\psi}{d\lambda} = (-(4i\lambda^2 + ix + 2iu^2)\sigma_3 - 4u\lambda\sigma_2 - 2w\sigma_1)\psi \tag{4}$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5}$$

x a real parameter, and u and w complex parameters. The monodromy data for this system do not depend on x if and only if u and w are smooth functions of x such that $w = u_x$ and $u(x)$ satisfies $u_{xx} - xu - 2u^3 = 0$. With the aid of the isomonodromy theory a connection can be found between the behaviour of solutions of the second Painlevé equation for $z \to -\infty$ and the behaviour for $z \to +\infty$. The result completely corresponds with the behaviour of the "outer" solutions when they approach the bifurcation point.

Research objectives of this study

In this study slow oscillations around parameter-dependent equilibrium solutions are analyzed. The parameter slowly varies through a critical value which corresponds with a jump or with a bifurcation phenomenon. The aim of this thesis is to develop a mathematical theory for this class of differential equations with a slowly varying parameter, which is, typical for a perturbation approach, qualitative as well as quantitative.

We analyze the three representatives of the class of nonlinear second order problems in which the system parameter varies with a constant (slow) velocity.

These problems correspond qualitatively with oscillating systems with a slowly varying frequency. A critical value of the parameter occurs when the frequency becomes zero. At the critical times significant degenerations are computed that describe the sudden transition behaviour of the system. Asymptotic solutions are constructed in and outside the transition region. The proof of the validity of the local asymptotic expansions is an essential part of the problem that has been tackled in this thesis. The matching procedure involves a limit process expansion in which a common overlap is demonstrated in the "inner" and "outer" regions. The matching procedure provides a global picture of the behaviour of the system. The matching problem is a major topic in singular perturbation theory. With the aid of extended averaging techniques, the extension theorem of Eckhaus (1979) and a thorough analysis of the transition equation it is our objective to obtain a global insight in the theory of nonlinear second order differential equations. The problems of this study can be tackled by using the method of multiple scales (more specifically two-timing) with a fast and a slow time. At the end of the 1970's a group of mathematicians in Strassbourg, the non-standard analysts, developed a new approach to study multiple-timescale problems, based on the use of infinitesimals.

The three representatives of second order jump and bifurcation phenomena are the jump transition, the pitchfork bifurcation, and the transcritical bifurcation. For simplicity, the analysis has sometimes been carried out only for a prototype system, but the generalization is straightforward. Our objective is to predict as accurately as possible the moment of sudden change and the behaviour of the system after the critical jump or bifurcation moment has been passed. We also consider the periodic and state dependent passage through a pitchfork bifurcation. In this case a Poincaré map is studied and a sensitive dependence on the initial conditions, a criterion for chaos, can be observed. From a dynamical point of view the Poincaré map exhibits a very exciting behaviour.

Outline of the thesis

This thesis is a compilation of four different articles, published in or submitted to international scientific journals and intended as independent papers. This thesis can be seen as a continuation of the work of Haberman (1979), who dealt with first and second order jump and transition phenomena. In contrast with Haberman we apply averaging techniques instead of the method of eliminating secular terms, and study dissipative systems. Moreover, attention is focused on the validity of our approximations.

In this section a summary is presented of each chapter of the thesis. For each theoretical case that has been analyzed an example of a mechanical system has been given that corresponds qualitatively with the jump or bifurcation problem that has been taken in consideration.

In chapter 2 a bistable dynamical system has been analyzed that corresponds qualitatively with the Euler arc from mechanical physics (see e.g. Stoker (1950)). On a large time-scale the system exhibits a damped oscillation until the moment that the stiffness has decreased in such a way that the arc collapses. The jump phenomenon is described by the first Painlevé equation. It appears not to be possible to derive the exact moment of snap-through. With numerical methods an expression is obtained for the upper and lower limit of the expected moment of snap-through.

In chapter 3 a second order pitchfork bifurcation problem is studied that corresponds qualitatively with the analogue of an elastic column. In the vicinity of a certain critical value of the bifurcation parameter a transition occurs from a stable linear equilibrium to a parabolic equilibrium curve, whereas the originally stable equilibrium becomes unstable. The pitchfork bifurcation is described by the second Painlevé equation. An analytic research of this transition equation provides information about the required matching procedures. There exists a connection between the slowly oscillating solutions before and after the bifurcation. It is possible to predict which stable branch of the parabolic equilibrium curve the solution will approach after bifurcation depending on the initial conditions. Moreover, the behaviour of the solution beyond the bifurcation point can again be described by averaging methods.

In chapter 4 we consider the dynamics of a class of second order differential equations with a slowly varying forcing, which periodically crosses a critical value corresponding to a pitchfork bifurcation. A prototype of such a system is the nonlinear Mathieu equation. Mechanical examples are a simple pendulum attached to a rotating rigid frame, and a particle moving on a smooth, rotating circular wire. A Poincaré map for one forcing period is constructed. Depending on the initial state and the values of the parameters this Poincaré map exhibits (quasi) periodicity or chaos. Chaos in the system means that the sequence of (upper and lower) branches that the system will follow after each passage of the bifurcation point, is irregular and there is a sensitive dependence on the initial conditions. Lyapunov exponents describe the structure of the attractor. This study is an extension of the work of Coppola and Rand (1990), and Bridge and Rand (1992).

In chapter 5 a second order transcritical bifurcation problem is analyzed that corresponds qualitatively with the dynamical behaviour of a harmonic current-carrying conductor restrained by strings and subjected to a magnetic field. Two linear equilibrium solutions cross and their stabilities exchange at a certain critical value of the bifurcation parameter. Depending on the matching conditions, solutions of the nonlinear transition equation either algebraically grow (corresponding to a transition from one stable equilibrium to another), exponentially decay (corresponding to the transition from a stable equilibrium to an unstable equilibrium), or explode. The chance of an explosion becomes larger when the amplitude of the original oscillation is larger. Depending on the initial conditions and the values of the parameters a separation condition is obtained, which provides a prediction whether or not an explosion will take place after passage of the bifurcation point.

Finally, at the end of the thesis a summary of the results is given and some general conclusions evolving from this thesis are stated.

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Chapter 2

Sudden collapse of a second order nonlinear system with a slowly varying parameter[1]

Abstract

This paper deals with systems of differential equations with a slowly varying parameter. The parameter slowly varies through a critical value corresponding to a jump phenomenon. Fluctuations around a slowly varying equilibrium are considered and, with the aid of averaging, an asymptotic approximation is obtained. At a certain moment this approximation breaks down and a nonlinear transition layer approximation is necessary. In such a domain the solution exhibits a turning point behaviour. In this paper an estimate of the jump moment is made for a simple bistable system.

1. Introduction

In this paper we consider a bistable dynamical system. An example of a bistable mechanical system is the Euler arc that is sketched in figure 1. The aim of this study is to estimate the moment of collapsing (snap-through) of the arc in case the stiffness slowly decreases with time. We assume the behaviour of the lateral spring force to be given by $f(x) = s(x^3 - x)$ with s a slowly varying parameter. Furthermore, we assume the load F to be constant and the damping, which is not drawn in the figure, to be linear. Stoker (1950) presents more mechanical examples.

In order to concentrate on the essentials of this problem we consider a slowly varying dynamical system of the following form:

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + s(x^2 - 1)x = -F \quad , \tag{1.1a}$$

$$\frac{ds}{dt} = -\varepsilon \quad . \tag{1.1b}$$

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[1] by G.J.M. Marée; published in *Int. J. Non-linear Mechanics* 28, p. 409-426, 1993.

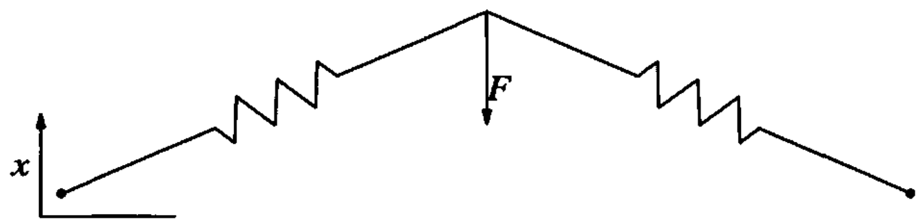


Figure 1 *The Euler arc.*

For the purpose of describing chaotic motion of periodically driven bistable mechanical systems, this differential equation has been studied earlier by Holmes (1979) and Rudowski and Szemplińska-Stupnicka (1987). A bistable system similar to the Euler arc has been introduced by Ghil and Childress (1987). It describes the forces on the lithosphere by the weight of the polar ice mass. Besides the reliability of mechanical systems and climatological changes other fields of possible applications are astronomy, spacecraft technology and cell membrane biology. The problem is usually treated from a purely static point of view in which the critical buckling load is defined as the load for which the arc collapses. Here we want to stress the dynamical side of the problem of oscillating springs with a slowly decreasing stiffness. Thus, we treat the problem dynamically instead of statically.

Presently, the force F is assumed to be constant and the slowly decreasing parameter s denotes the "stiffness" of the system. The initial value $s(0)$ is chosen larger than a certain critical value s_c for which the system exhibits a strongly nonlinear behaviour. We will discuss the manner in which then an abrupt jump takes place when s is approaching s_c. For the damping we set

$$k = \kappa\varepsilon^{1/2} \tag{1.2}$$

so that the effect of the damping is in the same time-scale as that of the slowly varying parameter s. In figure 2 the behaviour of the system is illustrated.

For s fixed, larger than s_c, the system (1.1) has three equilibria, two of them are stable and one is unstable. Because s is slowly decreasing these equilibria are slowly varying in time. The solution in the neighbourhood of one of the stable slowly varying equilibria (the one that approaches the unstable equilibrium as s tends to s_c) is approximated by a harmonic oscillation. A two-time-scales approximation, that is valid in a large time interval of length $O(\varepsilon^{-1})$, is obtained with averaging methods and a first prediction of the moment that the arc collapses is made. In order to improve the prediction a local approximation, being a transition layer, has to be made. Local scaling analysis yields as approximating differential equation the first Painlevé equation, a generic equation for this type of problems. In section 4 this Painlevé equation is analyzed.

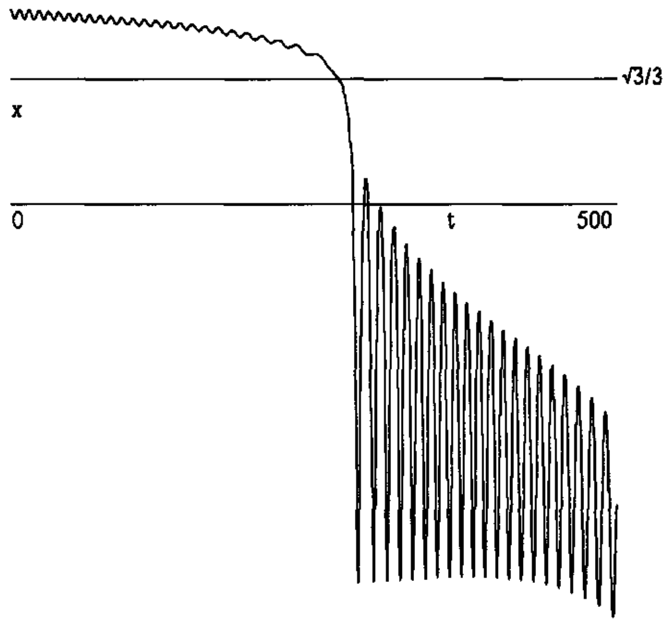


Figure 2 *Numerical simulation of system (1.1) with $k=\sqrt{\varepsilon}/4$, $F = 0.1$, $\varepsilon = 0.0009$, $s(0) = 0.5$, $x(0) = 0.9$ and $x'(0) = 0$.*

Because the transitional equation cannot be solved analytically, a better approximation of the "collapsing time" can only be obtained with the aid of numerical procedures. Depending on the initial values $x(0)$ and $x'(0)$ and the parameter ε we compute a time interval in which the solution certainly will snap through for ε sufficiently small. The resolutions of the Painlevé equation explode at a finite value of the local time variable. Then the transition layer equation is not valid anymore and a next local approximation should be made. However, the aim of this study is to get a good prediction of the moment that the system snaps through, so this second local approximation will not be treated. Snap-through occurs near the critical s-value based on the slow scale. This means that the error in the prediction of the moment of snap-through is of the order $O(\varepsilon^{-1/5})$. From local analysis strict bounds in this latter time-scale can be given.

In an advanced stage of the investigation of the problem, a study of Haberman (1979) was brought to the attention of the author. There is a resemblance in his approach and ours, but there are also differences. Haberman studies systems without damping and uses the method of eliminating secular terms, while we apply averaging. Moreover, there are obvious discrepancies in the computed values of coefficients of some asymptotic series. The principal difference is that in our investigation the attention is focused on the problem of predicting the moment of snap-through. It turns out that the local behaviour, governed by the first Painlevé transcendent, depends through matching on the initial values.

In section 2 we consider the different equilibrium states of the system when the parameter s is fixed. Moreover, the jump phenomenon is illustrated. In section 3 the solution in the neighbourhood of the slowly varying stable equilibrium is approximated by averaging for an interval in which the solution is sufficiently bounded away from the jump point. From this asymptotic expansion we can compute a first approximation of the time that the system collapses. In section 4 we analyze the transition layer equation, describing the jump, and obtain matching conditions for this local asymptotic solution. In section 5 we predict the collapsing time for different values of ε, $x(0)$ and $x'(0)$ and obtain different time intervals which should contain the moment of snap-through. A comparison with numerical solutions is made. In section 6 we consider the class of jump phenomena of which the present problem is a special case, and in section 7 we make some concluding remarks and mention related nonlinear systems that require a further investigation in the direction the present problem has been analyzed.

2. The reduced system with fixed parameter

Substitution of $\varepsilon = 0$ in (1.1) yields the reduced system

$$\frac{d^2x}{dt^2} + s(x^2 - 1)x = -F \ , \qquad (2.1a)$$

$$\frac{ds}{dt} = 0 \qquad (2.1b)$$

or

$$\frac{dx}{dt} = y \ , \qquad (2.2a)$$

$$\frac{dy}{dt} = -s(x^2 - 1)x - F \ . \qquad (2.2b)$$

For fixed s larger than the critical value

$$s_c = \frac{3}{2}F\sqrt{3} \qquad (2.3)$$

the reduced system (2.2) has three equilibria (for which $y = 0$):

$$x_1 = \frac{2}{3}\sqrt{3}\cos[\frac{1}{3}\arccos(\frac{-3F\sqrt{3}}{2s})] \ ,$$

$$x_0 = \frac{2}{3}\sqrt{3}\cos[\frac{1}{3}\arccos(\frac{-3F\sqrt{3}}{2s}) + \frac{4}{3}\pi] \ , \tag{2.4}$$

$$x_{-1} = \frac{2}{3}\sqrt{3}\cos[\frac{1}{3}\arccos(\frac{-3F\sqrt{3}}{2s}) + \frac{2}{3}\pi] \ .$$

Linearization of (2.2) at the equilibrium $(x_0 \, , \, 0)$ yields a system with an equilibrium being a saddle point. Thus, for the nonlinear system the equilibrium is a saddle point as well. For the other equilibria $(x_i \, , \, 0)$, $i = \pm 1$, a centre point is found after linearization. Using the lemma of Morse (see Verhulst, 1990) it can be proven that for the nonlinear system (2.2) these points are centre points as well. When $s = s_c$, x_1 equals x_0 and nonlinear analysis shows that the system has a degenerated equilibrium point $x_1 = x_0$ and a centre point x_{-1}. Finally, for s fixed smaller than s_c, only one equilibrium is left: the one corresponding with the stable lower branch of system (2.1).

A glance at the situation for a slowly decreasing stiffness parameter s, with $s(0) > s_c$, shows that we first have three slowly varying equilibria; two of them are stable and one is unstable. When the unstable branch and the stable upper branch of the limit solution coalesce the system will suddenly jump from the one stable branch to the other. This jump phenomenon is illustrated in figure 3 and figure 4. For $\varepsilon = 0$ the energy integral of the system (1.1) equals

$$E = \frac{1}{2}y^2 + \frac{1}{4}sx^4 - \frac{1}{2}sx^2 + Fx \ . \tag{2.5}$$

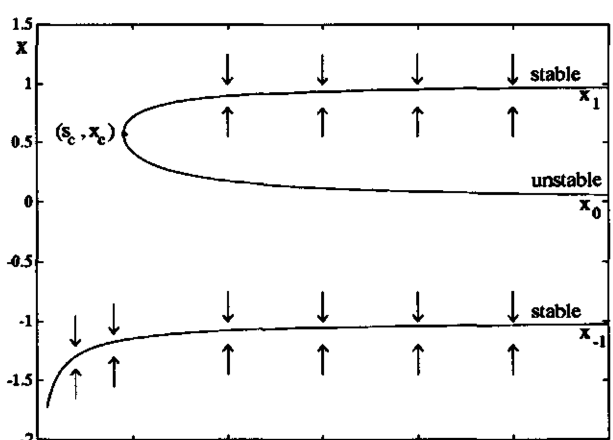


Figure 3 *The branches of the limit solution for different values of s.*

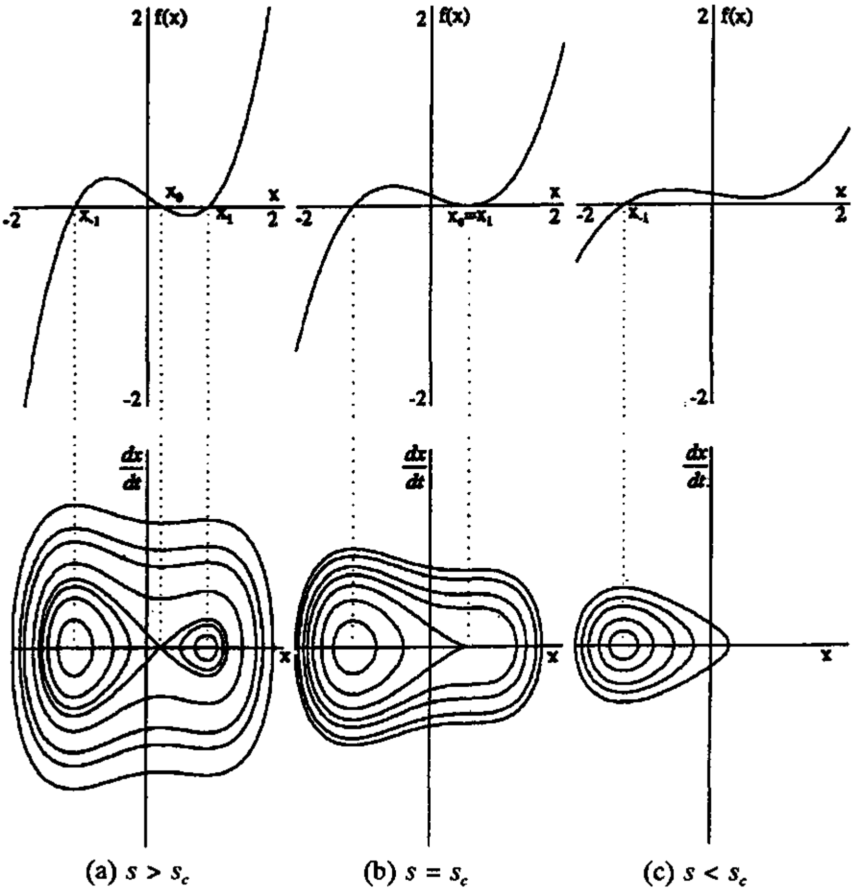


Figure 4 *The graph of the function $f(x) = s(x^2 - 1)x + F$ and the phase portrait of (2.2) for different values of s.*

3. An asymptotic expansion for the time interval before the jump

Because s is slowly decreasing in time, the three equilibria that were found for $s > s_c$ as solutions of the reduced system ($\varepsilon = 0$) are usually not solutions of the full system (1.1), but they will also vary in time. To analyze the slowly varying system it is convenient to write (1.1) as

$$\frac{d^2x}{dt^2} = \varepsilon^2 \frac{d^2x}{ds^2} = \kappa\varepsilon^{3/2}\frac{dx}{ds} + f(x, s) \tag{3.1}$$

with

$$f(x, s) = -F - s(x^2 - 1)x. \tag{3.2}$$

If we assume the derivatives in (3.1) to be small, we can obtain a so-called slowly varying equilibrium solution for $s > s_c$ by perturbing the dependent variable x around one of the stable reduced solutions $x_1(s)$. Because $f(x_1, s) = 0$, we can use the Taylor series of $f(x, s)$ around $x = x_1(s)$ which yields for (3.1):

$$\varepsilon^2\frac{d^2x}{ds^2} = \kappa\varepsilon^{3/2}\frac{dx}{ds} + (x - x_1)\frac{\partial f}{\partial x}\Big|_{x_1} + \frac{(x - x_1)^2}{2}\frac{\partial^2 f}{\partial x^2}\Big|_{x_1} + \frac{(x - x_1)^3}{6}\frac{\partial^3 f}{\partial x^3}\Big|_{x_1} . \tag{3.3}$$

In this way we obtain an asymptotic expansion for a slowly varying equilibrium solution of (1.1):

$$x_{1sv} = x_1(s) + \varepsilon^{3/2}\left(\frac{\kappa F}{s^3(3x_1^2 - 1)^2}\right) + \varepsilon^2\frac{F(2s(3x_1^2 - 1)^2 + 6x_1F)}{s^3(3x_1^2 - 1)^4} + \ldots \quad . \tag{3.4}$$

For a solution that holds in a $\sqrt{\varepsilon}$-neighbourhood of this slowly varying solution we write:

$$x(t) = x_{1sv}(s) + \sqrt{\varepsilon}\, v(t) \quad . \tag{3.5}$$

Furthermore, we take as initial values

$$x(0) = x_1(s(0)) + \sqrt{\varepsilon}\, x_0 \quad , \tag{3.6a}$$

$$\frac{dx}{dt}(0) = \frac{dx_1}{dt}(s(0)) + \sqrt{\varepsilon}\, v_0 \quad , \tag{3.6b}$$

$$s(0) = s_0 > \frac{3}{2}F\sqrt{3} \quad . \tag{3.6c}$$

Substitution in (1.1) yields:

$$\frac{d^2v}{dt^2} + \omega^2(s)v = \sqrt{\varepsilon}\,(-\kappa\frac{dv}{dt} - 3sv^2x_{1sv}(s)) - \varepsilon sv^3 \,, \tag{3.7a}$$

$$\frac{ds}{dt} = -\varepsilon \tag{3.7b}$$

with

$$\omega^2(s) = s(3x_{1sv}^2 - 1). \tag{3.8}$$

The reason why a $\sqrt{\varepsilon}$-neighbourhood of x_{1sv} and a damping of order $O(\sqrt{\varepsilon})$ are chosen is the following. There are three terms in equation (3.7a) which are responsible for the slow variation of the solution. These are the term with the angular velocity $\omega(s)$, the damping term and the (leading) nonlinear term. The first term has an effect on the time-scale of $O(\varepsilon^{-1})$. For $x = x_{1sv} + \varepsilon^{\mu}v$ and $k = \kappa\varepsilon^{\rho}$ the nonlinearity causes a slow modulation of the solution on a time-scale of $O(\varepsilon^{-2\mu})$ and the damping causes a modulation on the time-scale of $O(\varepsilon^{-\mu-\rho})$. Of particular interest is the significant case where the three time-scales balance, that is for $\rho = \mu = 1/2$.

For a nonlinear oscillator it is useful to apply the transformation

$$\frac{dv}{dt} = \omega(s)w \,, \tag{3.9a}$$

$$\frac{dw}{dt} = \frac{1}{\omega(s)}\frac{d^2v}{dt^2} + \frac{\varepsilon}{\omega(s)}\frac{d\omega}{ds}w \,. \tag{3.9b}$$

After introducing polar coordinates

$$v = r\cos\phi \,,\; w = -r\sin\phi \tag{3.10a}$$

with initial values

$$v(0) = r_0\cos\phi_0 \,,\; w(0) = -r_0\sin\phi_0 \tag{3.10b}$$

we obtain the following system

$$\frac{dr}{dt} = \sqrt{\varepsilon}\,(-\kappa r\sin^2\phi + 3sr^2\cos^2\phi\frac{x_{1sv}\sin\phi}{\omega(s)})$$

$$+ \varepsilon(\frac{sr^3\cos^3\phi\sin\phi}{\omega(s)} + \frac{r}{\omega(s)}\frac{d\omega}{ds}\sin^2\phi) \ , \tag{3.11a}$$

$$\frac{d\phi}{dt} = \omega(s) + \sqrt{\varepsilon}\,(-\kappa\cos\phi\sin\phi + 3sr\cos^3\phi\frac{x_{1sv}}{\omega(s)})$$

$$+ \varepsilon(\frac{sr^2\cos^4\phi}{\omega(s)} + \frac{\cos\phi\sin\phi}{\omega(s)}\frac{d\omega}{ds}) \ , \tag{3.11b}$$

$$\frac{ds}{dt} = -\varepsilon \ . \tag{3.11c}$$

To this system we apply second order averaging as formulated by Sanders and Verhulst (1987). It is noted that the above system does not satisfy all conditions stated in their theorem. In particular the $\sqrt{\varepsilon}$-term of (3.11a) does not vanish after averaging. Since the effect of this term dies out at a $O(1/\sqrt{\varepsilon})$ time-scale, it can be proven that for the above system second order averaging applies at a $O(1/\varepsilon)$ time-scale. The averaged system is then of the form:

$$\frac{dr_a}{dt} = \sqrt{\varepsilon}\,(-\frac{1}{2}\kappa r_a) + \varepsilon(\frac{r_a}{2\omega(s)}\frac{d\omega}{ds}) \ , \tag{3.12a}$$

$$\frac{d\phi_a}{dt} = \omega(s) + \varepsilon(\frac{3sr_a^2}{8\omega(s)} - \frac{15r_a^2s^2x_{1sv}^2}{4\omega^3(s)} - \frac{1}{8}\frac{\kappa^2}{\omega}) \tag{3.12b}$$

with solution

$$r_a(t) = r_0\omega^{1/2}(s(0))\exp(-\kappa t\sqrt{\varepsilon}/2)\omega^{-1/2}(s) \ , \tag{3.13a}$$

$$\phi_a(t) = \frac{-\int_{s(0)}^{s}\omega(s)ds}{\varepsilon} - \int_{s(0)}^{s}(\frac{3sr_a^2}{8\omega(s)} - \frac{15r_a^2s^2x_{1sv}^2}{4\omega^3(s)})ds + \frac{1}{8}\kappa^2\int_{s(0)}^{s}\frac{1}{\omega(s)}ds + \phi_0 \ . \tag{3.13b}$$

This yields an asymptotic expansion of the solution in the neighbourhood of the slowly varying stable equilibrium $x_{1sv}(s)$:

$$x(t) = \frac{2}{3}\sqrt{3}\cos[\frac{1}{3}\arccos(\frac{-3F\sqrt{3}}{2s})] + \tag{3.14}$$

$$\sqrt{\varepsilon}\, r_0 \omega^{\frac{1}{2}}(s(0))\exp(-\kappa\sqrt{\varepsilon}\,\frac{t}{2})\omega^{-\frac{1}{2}}(s)\cos\{\frac{-\int_{s(0)}^{s(t)}\omega(s)ds}{\varepsilon} - \int_{s(0)}^{s(t)}(\frac{3sr_a^2}{8\omega(s)} - \frac{15r_a^2 s^2 x_{1sv}^2}{4\omega^3(s)})ds + \phi_0 + \frac{1}{8}\kappa^2\int_{s(0)}^{s(t)}\frac{1}{\omega(s)}ds$$

This asymptotic expansion reflects the slow oscillation of the solution around the slowly varying equilibrium on a time-scale of order $t = O(\varepsilon^{-1})$. For $s = 3F\sqrt{3}/2$ the stable reduced solution $x_1(s)$ and the unstable one, $x_0(s)$, coalesce. Then the above approximation of a solution near the slowly varying equilibrium $x_{1sv}(s)$ does not hold anymore. The solution will rapidly tend towards the other stable reduced solution $x_{-1}(s)$. It is remarked that then the angular velocity $\omega(s)$ tends to zero. So there is a boundary layer behaviour in the neighbourhood of $3F\sqrt{3}/2$. Because s is explicitly given as a function of t ($s = s(0) - \varepsilon t$), we obtain a first approximation of the collapsing time:

$$t_0 = \frac{s(0) - \frac{3}{2}F\sqrt{3}}{\varepsilon} \ . \tag{3.15}$$

4. The jump described by the transition layer equation

In order to obtain matching conditions for the local asymptotic solution describing the jump, we determine the asymptotic development of x when s is in the neighbourhood of the critical value $s_c = 3F\sqrt{3}/2$. Near $s = s_c$ the reduced solution $x_1(s)$ is nearly $x_1(3F\sqrt{3}/2) = \sqrt{3}/3$. Near $s = 3F\sqrt{3}/2$ and $x = \sqrt{3}/3$ we can assume

$$\begin{aligned} f(x,s) &= -s(x^2 - 1)x - F \\ &= -\frac{9}{2}F((x-\frac{1}{3}\sqrt{3}) - \frac{2}{9}\sqrt{\frac{\sqrt{3}}{F}}\,(s-\frac{3}{2}F\sqrt{3})^{1/2} + \ldots)((x-\frac{1}{3}\sqrt{3}) + \frac{2}{9}\sqrt{\frac{\sqrt{3}}{F}}\,(s-\frac{3}{2}F\sqrt{3})^{1/2} + \ldots). \end{aligned} \tag{4.1}$$

The stable reduced upper branch solution now has the following form:

$$x_1(s) = \frac{1}{3}\sqrt{3} + \frac{2}{9}\sqrt{\frac{\sqrt{3}}{F}}\,(s - \frac{3}{2}F\sqrt{3})^{1/2} + \ldots \ . \tag{4.2}$$

We set

$$s = \frac{3}{2}F\sqrt{3} - \varepsilon^{\nu}\sigma \quad , \tag{4.3}$$

which yields the following approximations:

$$x_1(s) = \frac{1}{3}\sqrt{3} + \frac{2}{9}\sqrt{\frac{\sqrt{3}}{F}}\,(-\sigma)^{1/2}\varepsilon^{\frac{1}{2}\nu} + O(\varepsilon^{\nu}) \quad , \tag{4.4a}$$

$$x_{1s\nu}(s) = \frac{1}{3}\sqrt{3} + \frac{2}{9}\sqrt{\frac{\sqrt{3}}{F}}\,(-\sigma)^{1/2}\varepsilon^{\frac{1}{2}\nu} + \frac{1}{36F}(-\sigma)^{-2}\varepsilon^{2-2\nu} + \frac{\kappa}{18F|\sigma|}\varepsilon^{\frac{3}{2}-\nu} + \ldots , \tag{4.4b}$$

$$\omega^2(s) = (4F\sqrt{3})^{1/2}(-\sigma)^{1/2}\varepsilon^{\frac{\nu}{2}} , \tag{4.4c}$$

$$r_a = r_0\omega^{1/2}(s(0))(4F\sqrt{3})^{-1/8}(-\sigma)^{-1/8}\varepsilon^{-\frac{\nu}{8}}\exp\left(\frac{-\kappa\varepsilon^{1/2}t_0}{2}\right) + \ldots , \tag{4.4d}$$

$$\begin{aligned}\phi_a = &-\frac{4}{5}(4F\sqrt{3})^{1/4}(-\sigma)^{5/4}\varepsilon^{\frac{5}{4}\nu-1} + \frac{45}{64}\sqrt{3}Fr_0^2\omega(s(0))(\ln|\sigma| + \nu\ln\varepsilon)\exp(-\kappa\sqrt{\varepsilon}\,t_0) \\ &- \frac{9}{8}(4F\sqrt{3})^{-1/2}(-\sigma)^{1/2}F\sqrt{3}\,r_0^2\omega(s(0))\varepsilon^{\frac{1}{2}\nu} + \psi_0 - \frac{1}{6}\kappa^2(4F\sqrt{3})^{-\frac{1}{4}}(-\sigma)^{\frac{3}{4}}\varepsilon^{\frac{3}{4}\nu} + \ldots ,\end{aligned} \tag{4.4e}$$

where ψ_0 is a constant determined by the initial conditions.
Consequently, near the jump point the solution behaves asymptotically as

$$\begin{aligned}x \sim\ &\frac{1}{3}\sqrt{3} + \frac{2}{9}\sqrt{\frac{\sqrt{3}}{F}}\,(-\sigma)^{1/2}\varepsilon^{\nu/2} + \frac{1}{36F}(-\sigma)^{-2}\varepsilon^{2-2\nu} + \ldots + \frac{\kappa}{18F|\sigma|}\varepsilon^{\frac{3}{2}-\nu} \\ &+ \exp(-\kappa t_0\sqrt{\varepsilon}/2)\omega^{\frac{1}{2}}(s(0))(4F\sqrt{3})^{-1/8}(-\sigma)^{-\frac{1}{8}}\varepsilon^{\frac{1}{2}-\frac{1}{8}\nu}\cos\{-\frac{4}{5}(4F\sqrt{3})^{\frac{1}{4}}(-\sigma)^{\frac{5}{4}}\varepsilon^{\frac{5}{4}\nu-1} \\ &+ \frac{45}{64}\sqrt{3}Fr_0^2\omega(s(0))(\ln|\sigma| + \nu\ln\varepsilon)\exp(-\kappa\sqrt{\varepsilon}\,t_0) + \psi_0 - \frac{1}{6}\kappa^2(4F\sqrt{3})^{-\frac{1}{4}}(-\sigma)^{\frac{3}{4}}\varepsilon^{\frac{3}{4}\nu}\} .\end{aligned} \tag{4.5}$$

From this expression it is seen that the outer expansion breaks down if $\nu/2 = 1/2 - \nu/8 = 2 - 2\nu$; that is for $\nu = 4/5$. It implies that the transition layer (inner) equation follows from the scaling

$$x = \frac{1}{3}\sqrt{3} + \varepsilon^{\frac{2}{5}} y(\sigma) . \tag{4.6}$$

Another way to obtain this equation is based on the analysis of significant degenerations of the differential equation. We put then

$$s = \frac{3}{2}F\sqrt{3} - \varepsilon^{\nu}\sigma \ , \ x = \frac{1}{3}\sqrt{3} + \varepsilon^{\mu}y(\sigma) . \tag{4.7}$$

Substitution in (1.1) yields

$$\frac{d^2y}{d\sigma^2} = -\kappa\frac{dy}{d\sigma}\varepsilon^{\nu - \frac{1}{2}} - \frac{9}{2}Fy^2\varepsilon^{\mu + 2\nu - 2} - \frac{3}{2}F\sqrt{3}\,\varepsilon^{2\nu + 2\mu - 2}y^3$$

$$- \frac{2}{9}\sqrt{3}\,\sigma\varepsilon^{3\nu - \mu - 2} + \sqrt{3}\,y^2\sigma\varepsilon^{\mu + 3\nu - 2} + \sigma y^3\varepsilon^{2\mu + 3\nu - 2} . \tag{4.8}$$

For an arbitrary value of ν and μ most likely one term of the equation will be of leading order of magnitude. For specific values of ν and μ three terms may be jointly the leading in order of magnitude. It is expected that the solution of the equation formed by these terms represents the local asymptotic behaviour to the full extent. Thus, letting $\varepsilon \to 0$ we obtain a degenerated equation which is significant for approximating asymptotically the solution.

There is a significant degeneration for $\mu = 2/5$, $\nu = 4/5$ of the following form:

$$\frac{d^2y}{d\sigma^2} = -\frac{9}{2}Fy^2 - \frac{2}{9}\sqrt{3}\,\sigma \ . \tag{4.9}$$

The solution of this transition layer equation describes the jump transition from a slowly varying equilibrium solution to a slowly varying periodic nonlinear wave oscillating around the new stable equilibrium (see also Haberman (1979)). This equation can be transformed to the first Painlevé equation, defined by the irreducible equation

$$\frac{d^2w}{dz^2} = 6w^2 + z , \tag{4.10}$$

see Ince (1956) and Painlevé (1900).

At the time that a jump is expected equation (4.9) holds. Its solution must match the outer solution, as given by (4.5) with $\nu = 4/5$:

$$
\begin{aligned}
y \sim\ & \frac{2}{9}\sqrt{\frac{\sqrt{3}}{F}}\,(-\sigma)^{1/2} + \dots \\
& + \exp\left(-\frac{1}{2}\kappa\sqrt{\varepsilon}\,t_0\right) r_0 \omega^{\frac{1}{2}}(s(0))(4F\sqrt{3})^{-\frac{1}{8}}(-\sigma)^{-\frac{1}{8}}\cos\{-\frac{4}{5}(4F\sqrt{3})^{\frac{1}{4}}(-\sigma)^{\frac{5}{4}} \\
& + \frac{45}{64}\sqrt{3}\,F r_0^2 \omega(s(0))(\ln|\sigma| + \frac{4}{5}\ln\varepsilon)\exp(-\kappa\sqrt{\varepsilon}\,t_0) + \psi_0\}
\end{aligned}
\tag{4.11}
$$

as $\sigma \to -\infty$.

This asymptotic condition is automatically fulfilled by (4.9). However, all solutions to (4.9) explode towards $-\infty$ at a finite value of σ. In particular, for the first Painlevé transcendent, the type of singularity is

$$
y \sim \frac{-4}{3F(\sigma - \sigma_0)^2}
\tag{4.12}
$$

with σ_0, the time of the singularity, depending on the specific asymptotic condition as $\sigma \to -\infty$, that means it depends on r_0 and ψ_0 and ε in such a way that ε has an influence both on the amplitude and the angle of the harmonic matching solution. With numerical methods we can compute the time that can be identified as the jump point at which the solution shows a turning point behaviour. If one considers the transformed time variable

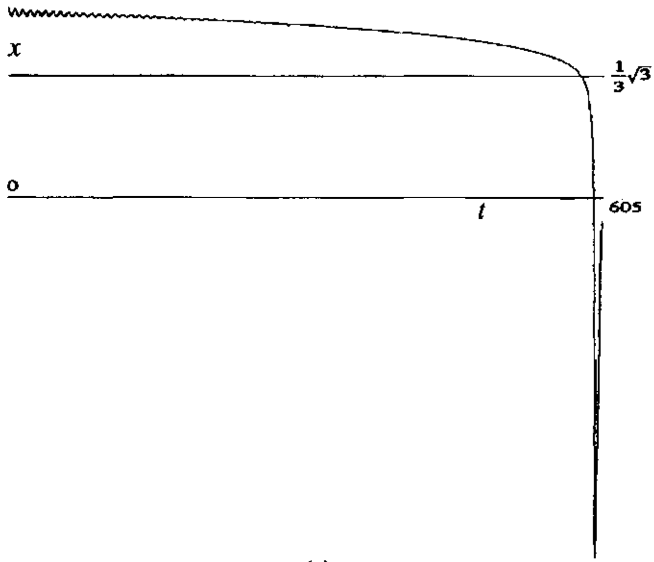
$$
\tau = \frac{-\int_{s(0)}^{s(t)} \omega(s)\,ds}{\varepsilon},
\tag{4.13}
$$

then the outer expansion (3.14) is valid on a time-scale of order $\tau = O(\varepsilon^{-1})$ and a jump takes place on a time-scale of order $\tau = O(1)$. Furthermore, the adiabatic invariant of the system without damping has the following form:

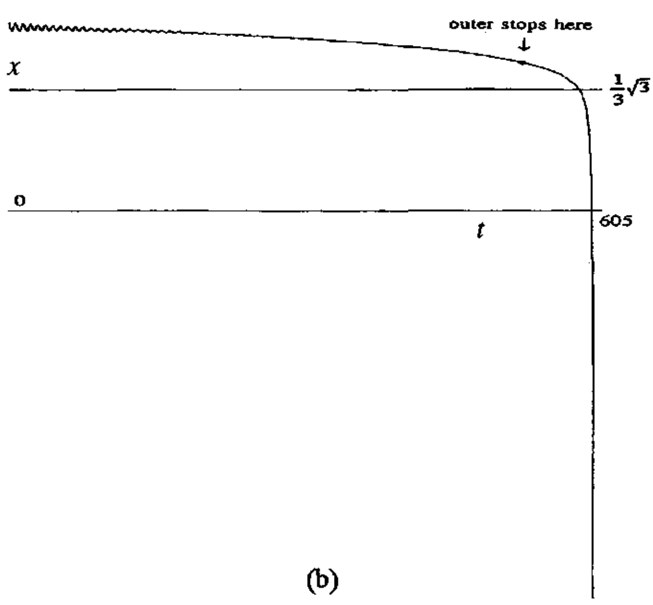
$$
\frac{\omega(s)(x - x_{1sv})^2}{\varepsilon} + \frac{\left(\frac{dx}{dt} - \frac{dx_{1sv}}{dt}\right)^2}{\varepsilon\omega(s)} = \text{constant}
\tag{4.14}
$$

on a time-scale of order $t = O(\varepsilon^{-1})$.

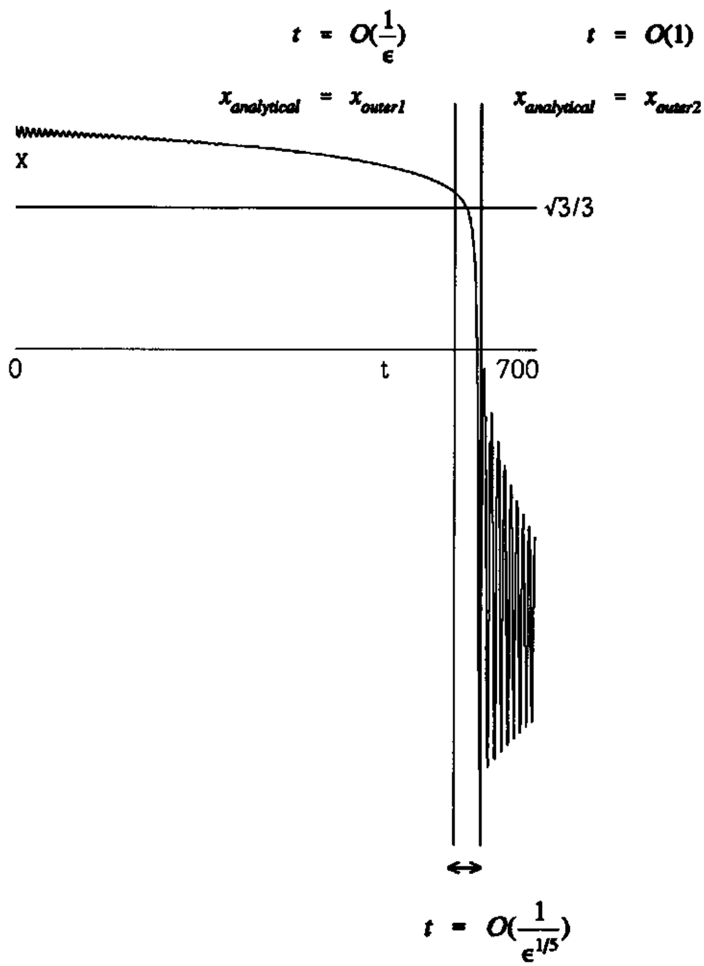
In figure 5 we illustrate the various analytical approximations of sections 3 and 4 with their domain of validity. In the same figure a numerical solution of the original equation is drawn for comparison. There is no notable difference between this numerical solution (a) and the composite analytical solution (c).



(a)



(b)



(c)

(a) *The numerical solution of the original equation*

(b) *The outer asymptotic expansion x_{outer} and the boundary layer solution x_{inner} together in one figure*

(c) *The composite analytical solution: $x_{analytical} = x_{outer} + x_{inner} - x_{match}$*

Figure 5 *Various analytical approximations obtained in sections 3 and 4 and a numerical solution of equation (1.1) with $x(0) = 0.9$, $x'(0) = 0$, $s(0) = 0.5$, $F = 0.1$, $k = 0.02$ and $\varepsilon = 0.0004$.*

5. Bounds for the zero of the solution of the transitional equation with initial conditions given by matching

With numerical methods we can compute the zero of equation (4.9) that can be identified as the jump point. For the initial conditions we can use the asymptotic expansion (4.11) with σ a large negative number. In the limit for $\sigma \to -\infty$ the asymptotic solution and the solution of the significant degeneration are the same:

$$s_{SD} \sim x_A \sim \frac{1}{3}\sqrt{3} + \varepsilon^{\frac{2}{5}}\left(\frac{2}{9}\sqrt{\frac{\sqrt{3}}{F}}\,(-\sigma)^{\frac{1}{2}}\right) \tag{5.1a}$$

and

$$\frac{dx_{SD}}{d\sigma} \sim \frac{dx_A}{d\sigma} \sim -\,\varepsilon^{\frac{2}{5}} r_0 \omega^{\frac{1}{2}}(s(0))(4F\sqrt{3})^{\frac{1}{8}}(-\sigma)^{\frac{1}{8}} \exp\left(-\frac{1}{2}\kappa\sqrt{\varepsilon}\,t_0\right)\cos\Big\{-\frac{4}{5}(4F\sqrt{3})^{\frac{1}{4}}(-\sigma)^{\frac{5}{4}}$$

$$+ \frac{45}{64}\sqrt{3}\,F r_0^2 \omega(s(0))\left(\ln|\sigma| + \frac{4}{5}\ln\varepsilon\right)\exp(-\kappa\sqrt{\varepsilon}\,t_0) + \psi_0\Big\}. \tag{5.1b}$$

In this mixed asymptotical-numerical way we obtain a better approximation of the collapsing time, namely:

$$t_1 = \frac{t^*}{\varepsilon} + \frac{\sigma^*}{\varepsilon^{1/5}} \tag{5.2}$$

with

$$t^* = s(0) - \frac{3}{2}F\sqrt{3} \tag{5.3}$$

and σ^* the approximation of the zero of the generic equation

$$\frac{d^2y}{d\sigma^2} = -\frac{9}{2}Fy^2 - \frac{2}{9}\sqrt{3}\,\sigma \tag{5.4}$$

with well-chosen initial conditions. We remark that the initial values of (5.4) still depend on ε, see (4.11). Consequently, we cannot construct an asymptotic solution: for each ε a numerical integration has to be made. To come as close as possible to our goal of approximating asymptotically the moment of snap-through we construct a time interval that contains the jump point if ε is taken sufficiently small.

After the transformation

$$y = F^{-3/5}w \ , \ \sigma = F^{-1/5}z \ , \tag{5.5}$$

the transition equation becomes independent of F:

$$\frac{d^2w}{dz^2} = -\frac{9}{2}w^2 - \frac{2}{9}\sqrt{3}z \ . \tag{5.6}$$

First we consider the special case $r_0 = 0$. We then obtain a specific solution of the Painlevé transcendent (5.6): the one that matches (4.11) with $y = F^{-3/5}w$, $\sigma = F^{-1/5}z$, and $r_0 = 0$. It is noted that (4.11) is then independent of ε (and ψ_0). For $z << -1$ this solution has an asymptotic series of the form

$$y(z) = \sum_{n=1}^{\infty} c_n(-z)^{\frac{1}{2} - \frac{5}{2}(n-1)} \tag{5.7}$$

with c_n satisfying a recurrent relation:

$$c_1 = \frac{2}{9}3^{1/4} \ , \ c_2 = \frac{1}{36} \tag{5.8a}$$

and for $n \geq 3$:

$$c_{n-1}\left(\frac{1}{2} - \frac{5}{2}(n-2)\right)\left(-\frac{1}{2} - \frac{5}{2}(n-2)\right) = \begin{cases} -9\left(\sum_{i=1}^{\frac{1}{2}(n-1)} c_i c_{n-i+1} + c_{\frac{1}{2}(n+1)}\right) \ , & n \text{ odd} \\ -9\left(2\sum_{i=1}^{\frac{1}{2}n} c_i c_{n-i+1}\right) & , n \text{ even} \ . \end{cases} \tag{5.8b}$$

This specific solution has a zero z_0 that can be considered as a natural constant for the Painlevé transcendent (5.6) which has a solution that takes the form (5.7) for $z << -1$.

For $r_0 > 0$ we construct a time interval $(z_{min} \ , z_{max})$ for the zero of (5.6) satisfying (4.11) with $y = F^{-3/5}w$, $\sigma = F^{-1/5}z$ and with arbitrary ψ_0 and ε taken sufficiently small. We expect the time interval

$$\left(t_0 + \frac{z_{min} F^{-1/5}}{\varepsilon^{1/5}},\ t_0 + \frac{z_{max} F^{-1/5}}{\varepsilon^{1/5}}\right) = \left(t_0 + \frac{\sigma_{min}}{\varepsilon^{1/5}},\ t_0 + \frac{\sigma_{max}}{\varepsilon^{1/5}}\right)\ , \tag{5.9}$$

with t_0 as in (3.15), to be a good approximation interval of the collapsing time of the full system (1.1). It is assumed that $\kappa > 0$. This means that for $\varepsilon \to 0$ the length of the interval $(\sigma_{min}\ , \sigma_{max})$ will tend to zero, see (4.11). From this formula and (5.5) we conclude that the bounds of $(z_{min}\ , z_{max})$ depend on

$$\rho_0 = r_0 \omega^{\frac{1}{2}}(s(0))(4F\sqrt{3})^{-\frac{1}{8}} \exp\left(-\frac{1}{2}\kappa\sqrt{\varepsilon}\, t_0\right) F^{1/2} << 1 \tag{5.10}$$

with t_0 as in (3.15). Using the numerical solution of the Painlevé transcendent it is found that for $\rho_0 << 1$

$$z_0 = 0.64,\ z_{min} \approx z_0 - 5.1\rho_0,\ z_{max} \approx z_0 + 4.5\rho_0\ . \tag{5.11}$$

From (5.11) and (5.5) we obtain an approximation interval $(\sigma_{min}\ , \sigma_{max})$ for the zero of (5.4) satisfying (4.11) with arbitrary ψ_0 and with ε taken sufficiently small:

$$\sigma_0 = F^{1/5} z_0,\ \sigma_{min} \approx \sigma_0 - 5.1F^{1/5}\rho_0,\ \sigma_{max} \approx \sigma_0 + 4.5F^{1/5}\rho_0\ . \tag{5.12}$$

In figure 6 the situation is illustrated for a special case.

We verify the result by integrating numerically the full differential equation (1.1) with $k = \sqrt{\varepsilon}$ and $F = 0.1$ and register the moment t_c the solution takes the value $x = \sqrt{3}/3$. The moment of collapse t_c is transformed to σ time-scale:

$$\sigma_c = (t_c - t_0)\varepsilon^{\frac{1}{5}} \tag{5.13}$$

with t_0 as in (3.15). We choose as initial values $s_0 = 0.5$, $r_0 = 1$ and take ψ_0 randomly from a uniform distribution in $[0, 2\pi]$. In figure 7 we give the bounds σ_{min} and σ_{max} derived from the asymptotic solution (4.11) and the collapse times σ_c obtained from numerical integration of the full differential equation (1.1). x_1 is the solution of the reduced system ($\varepsilon = 0$), as defined in (2.3), with initial values

$$x_1(0) = \frac{2}{3}\sqrt{3}\cos[\frac{1}{3}\arccos(\frac{-3F\sqrt{3}}{2s(0)})] \ , \tag{5.14a}$$

$$\frac{dx_1}{dt}(0) = x_1'(0) = \frac{-\varepsilon F}{s(0)^2(3x_1(0)^2 - 1)} \ . \tag{5.14b}$$

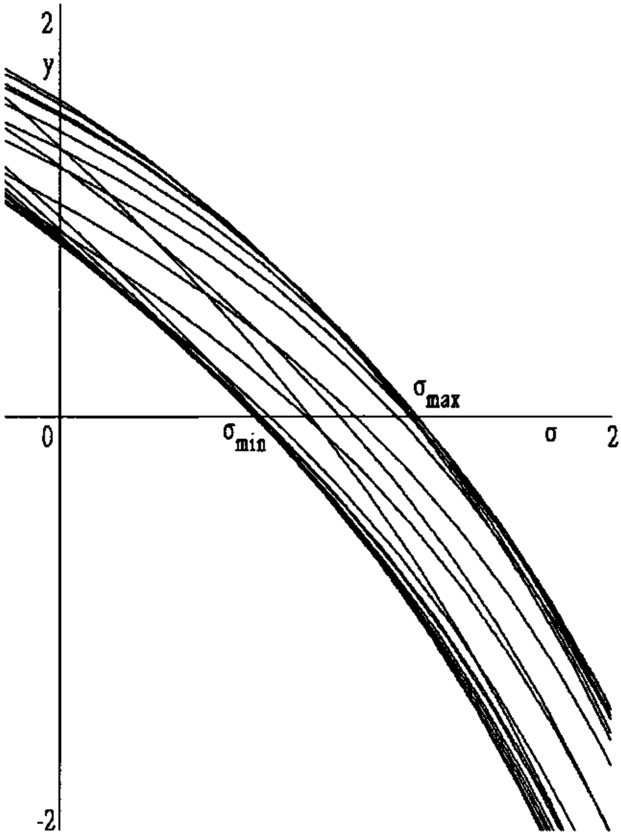


Figure 6 *The interval (σ_{min} , σ_{max}) for the zero of (5.4) with F = 0.1 satisfying (4.11) with r_0 = 2, ε = 0.0025, κ = 1 and arbitrary ψ_0.*

It can be concluded from figure 7 that the obtained approximation interval for the moment of snap-through seems to be a reasonably good prediction interval, even for ε not small compared with F, $s(0)$ and $x(0)$. The fact that not all numerically obtained collapse times are contained in this prediction interval arises from neglecting the higher order terms of (3.4). Incorporation of these terms would cause a small upward shift of the bounds such that all points are within the approximation interval.

In the case without damping ($\kappa = 0$), studied by Haberman, a change in ε has the same influence as a change in ψ_0 for the position of the zero of (5.4), like for $\kappa > 0$, but now the amplitude of the matching condition is independent of ε:

$$\rho_0 = r_0 \omega^{\frac{1}{2}} (s(0)) (4F\sqrt{3})^{-\frac{1}{8}} F^{\frac{1}{2}} \ . \tag{5.15}$$

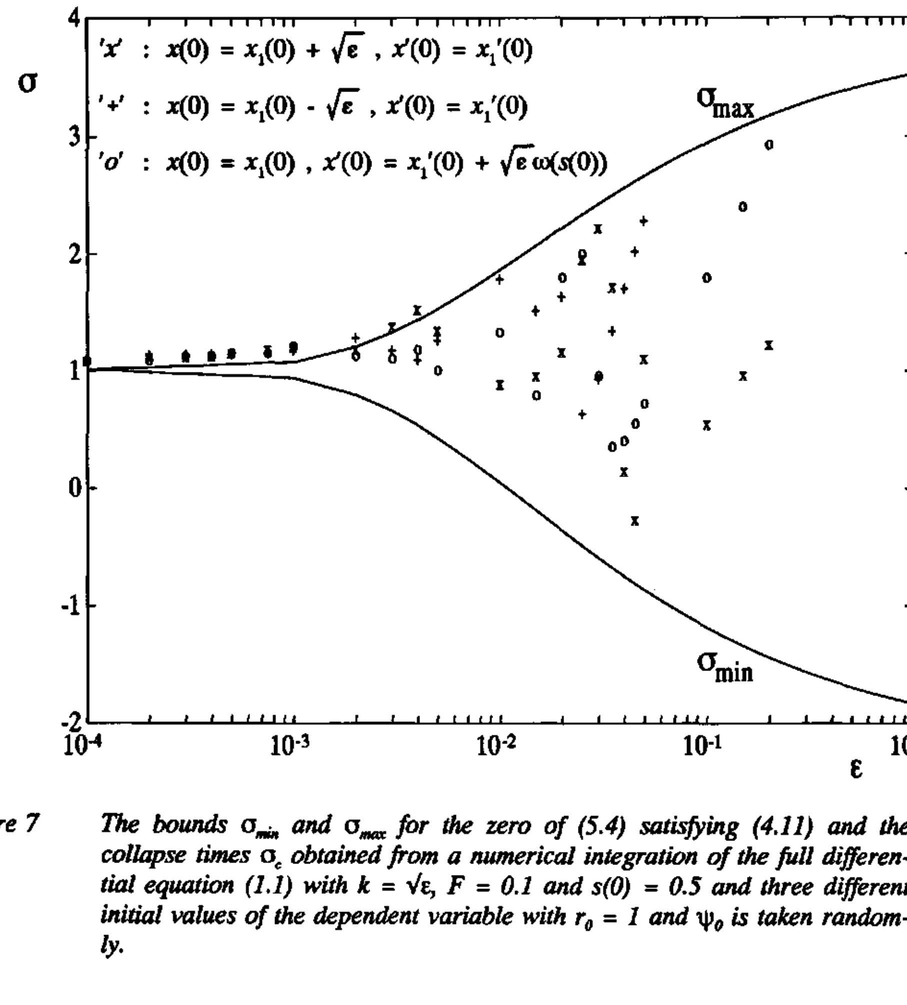
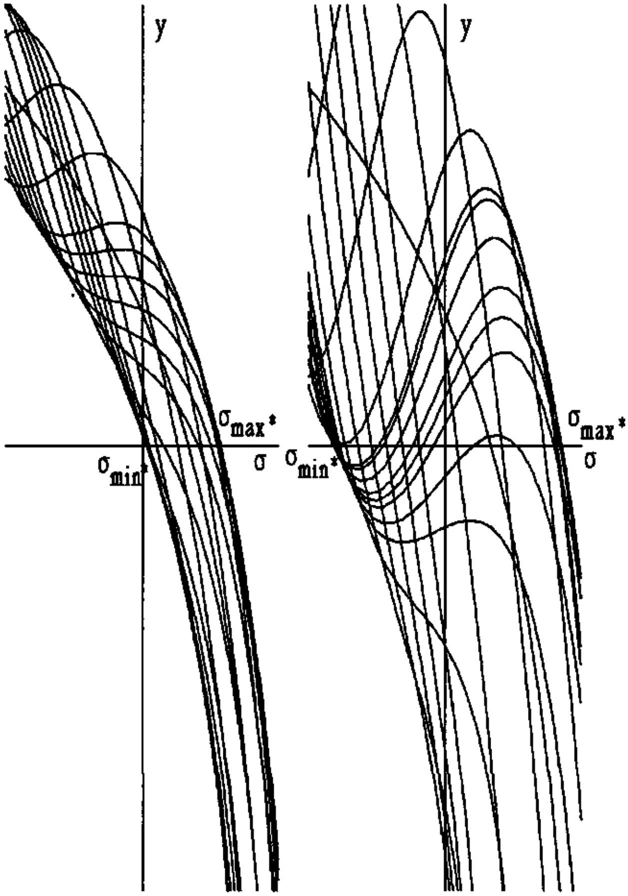


Figure 7 *The bounds σ_{min} and σ_{max} for the zero of (5.4) satisfying (4.11) and the collapse times σ_c obtained from a numerical integration of the full differential equation (1.1) with $k = \sqrt{\varepsilon}$, $F = 0.1$ and $s(0) = 0.5$ and three different initial values of the dependent variable with $r_0 = 1$ and ψ_0 is taken randomly.*

In fact, the bounds for the zero of (5.4) only depend on r_0 in this case. Again, we can compute a time interval in which the collapse will take place. For small r_0 a time interval for the zero of (5.4) is found. If we enlarge r_0, however, solutions of (5.4) can contain more than one zero. This situation is illustrated in figure 8.



(a) $r_0 = 0.5$ *(b)* $r_0 = 1.5$

Figure 8 *The interval* $(\sigma_{min^*}, \sigma_{max^*})$ *for the zero of (5.4) with* $F = 0.1$ *satisfying (4.11) with* $\kappa = 0$ *and arbitrary* ψ_0 *and* ε.

As a criterium for the moment that the system will collapse we take as a minimum for the value of σ^* in (5.2) the first time the solution of (5.4) passes zero for arbitrary ψ_0 and ε, and as a maximum the last time that this may happen when (4.11) is satisfied. In figure 9 for different values of r_0 the values of σ_{min^*} and σ_{max^*} are computed. The crosses in this figure indicate the numerically obtained collapse times transformed to σ time-scale as in (5.9) when the full differential equation (1.1) is integrated with $k = 0$, $\varepsilon = 0.0025$, $F = 0.1$, $s(0) = 0.5$, $x(0) = x_1(0) + \sqrt{\varepsilon} r_0$ and $x'(0) = x_1'(0)$ with $x_1(0)$ and $x_1'(0)$ as defined in (5.10).

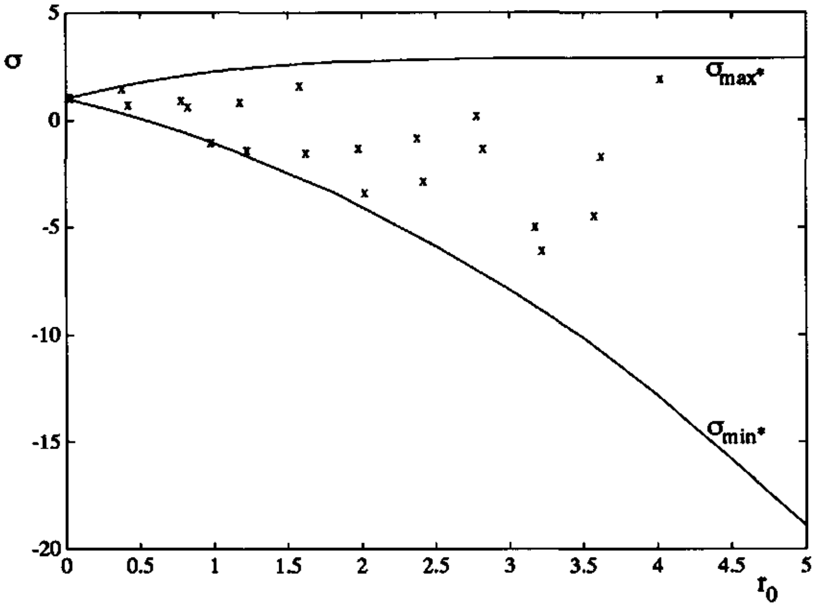


Figure 9 *Changes in the bounds for the zero of (5.4) with $F = 0.1$ satisfying (4.11) with $\kappa = 0$ when r_0 is varied (so the case without damping). Also some numerically obtained collapse times are indicated.*

When the moment of snap-through is again defined by the time t at which $x(t) = \sqrt{3}/3$, it can indeed be shown that we have again obtained a good prediction interval of the collapse time.

In figure 10 we present a complete picture of the change in the solution of system (1.1) for certain values of the initial conditions and the parameters. Also the predicted time interval of the jump is illustrated. As can be seen from this figure there is a large time interval of length $O(\varepsilon^{-1})$ in which the solution is slowly oscillating where one does not expect a sudden change. Then, on a relatively short time-scale of length $O(\varepsilon^{-1/5})$, the system suddenly snaps through and a jump takes place. It is shown that the moment of this unexpected sudden change can be very well predicted by the analysis described in this study.

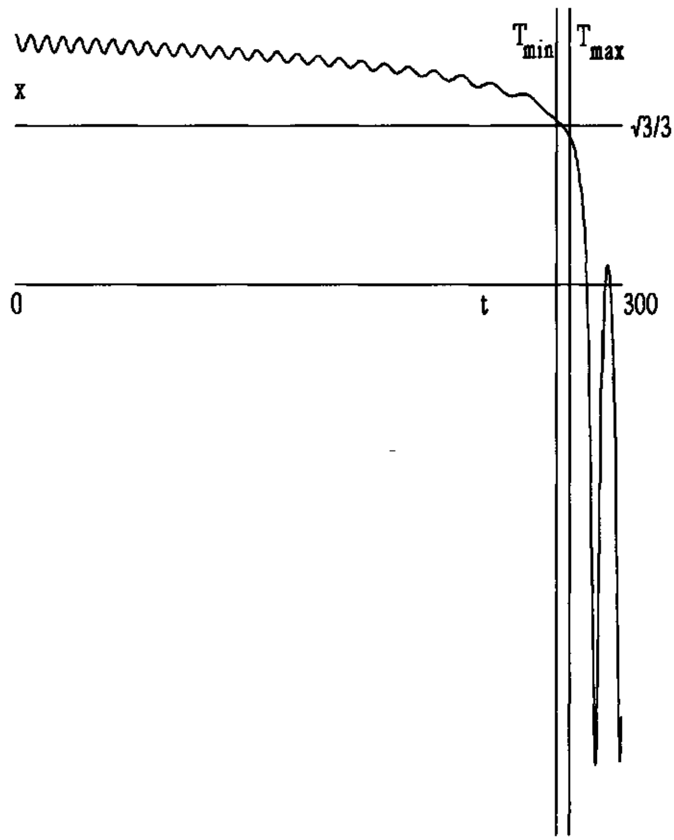
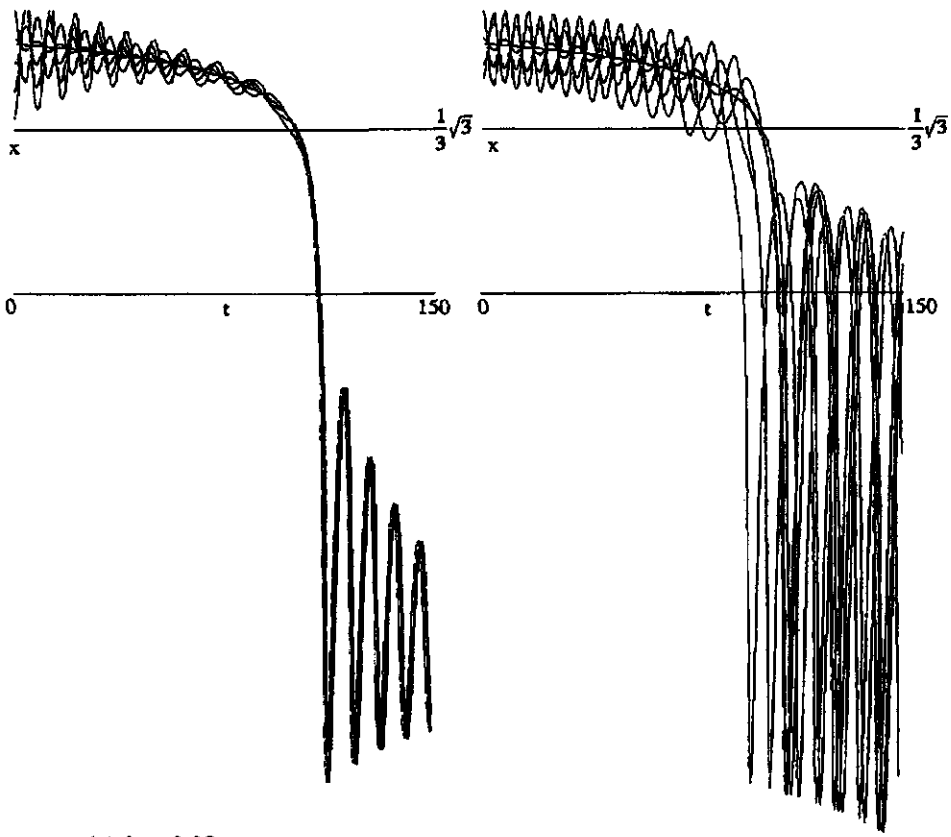


Figure 10 *Numerical solution of system (1.1) and the asymptotic bounds for the approximation of the moment of snap-through for the parameter values* $F = 0.1$, $k = 0.01$, $\varepsilon = 0.0009$ *and initial values* $x(0) = 0.909$, $x'(0) = 0$ *and* $s(0) = 0.5$.

Finally, in figure 11, the behaviour of the full system (1.1) is considered for various values of $x(0)$, both in the case with damping $k = \sqrt{\varepsilon}$ and in the case without damping.



(a) k = 0.05 *(b) k = 0*

Figure 11 *Solutions of equation (1.1) for various values of x(0) with x'(0)=0, F = 0.1, ε = 0.0025 and s(0) = 0.5.*

6. The general case of second order jump phenomena with damping

Many vibration problems can be described by a second order nonlinear differential equation dependent on a parameter s:

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} = H(x, s) \ . \tag{6.1}$$

We have considered a system with damping of order $\sqrt{\varepsilon}$ and the parameter s slowly varying in time: $s = s(\varepsilon t)$. For fixed s the linear stability of an equilibrium solution

$x_E(s)$ is determined by the linearization of (6.1). A critical value s_c of s occurs if

$$\frac{\partial H}{\partial x}(x_E(s_c),\ s_c) = 0\ . \tag{6.2}$$

For this value of s a stable and an unstable equilibrium solution of (6.1), with s fixed, coalesce. In our case we obtain:

$$\frac{dx_E}{ds}(s_c) = \infty\ . \tag{6.3}$$

In the neighbourhood of $x = x_E(s_c)$, $s = s_c$ we assume H to have the following form:

$$H(x,s) = \alpha_{20}(x - x_E(s_c))^2 - \sigma^2(s - s_c) + \ldots$$

$$= \alpha_{11}(s - s_c) + \alpha_{20}(x - x_E(s_c))^2 + \alpha_{11}(s - s_c)(x - x_E(s_c)) + \alpha_{02}(s - s_c)^2\ , \tag{6.4}$$

where $\alpha_{nm} = \dfrac{1}{n!m!}(\dfrac{\partial}{\partial x})^n(\dfrac{\partial}{\partial s})^m H(x_E(s_c),\ s_c)$.

We demand $\alpha_{01} > 0$ (in order that a parabolic curve exists for $s > s_c$) and $\alpha_{20} < 0$ (so that $x - x_E(s_c) \sim \pm\sigma(s - s_c)^{1/2}$ is stable). As one can see $\sigma = (-\alpha_{01}/\alpha_{20})^{1/2}$. A first prediction of the jump moment on a time-scale of order $O(\varepsilon^{-1})$ can now implicitly be obtained:

$$s(\varepsilon t_c) = s_c\ . \tag{6.5}$$

In the same way as in section 3 we can obtain an asymptotic expansion for a slowly varying equilibrium solution of (6.1) and consider $\sqrt{\varepsilon}$-perturbations of this solution. We obtain a slowly varying oscillator approaching a turning point as the frequency tends to zero. For the jump transition problem this frequency has a square root singularity. With the aid of averaging techniques we obtain an asymptotic expansion of the solution in the neighbourhood of the slowly varying stable equilibrium. This expansion breaks down when $\varepsilon t = O(\varepsilon^{4/5})$. Matching implies that the local inner equation follows from the scaling:

$$x = x_E(s_c) + \varepsilon^{\frac{2}{5}}y(\sigma)\ ,\ s = s_c + \varepsilon^{\frac{4}{5}}\sigma\ . \tag{6.6}$$

Making these scale changes we find that (6.1) transforms into:

$$\frac{d^2y}{d\sigma^2} = \alpha_{01}\frac{ds}{d(\varepsilon t)}(\varepsilon t_c)\sigma + \alpha_{20}y^2 + O(\varepsilon^{\frac{2}{5}}) \,. \tag{6.7}$$

This equation must be solved with the matching condition following from the asymptotic expansion for the outer solution. The specific solution of this equation when we consider no perturbation around the slowly varying equilibrium solution has a zero σ^* that can be considered as a natural constant. In the same way as in section 5 we can now construct an approximation interval on a time-scale of order $t = O(\varepsilon^{-1/5})$ for the collapsing time of the full system (6.1). The bounds of this interval depend on the amplitude of the oscillatory matching perturbation in a similar way as in section 5.

7. Some remarks

As the problem we formulated contains a small parameter, its solution is approximated asymptotically using perturbation techniques. It turned out to be necessary to apply both averaging and boundary layer methods. The boundary layer solution describes a rapid change of the system. The phenomenon of instability of a mechanical system known as snap-through is a typical example of such a sudden change. The aim of this study was to predict the moment of snap-through as accurately as possible. This moment is situated within the boundary layer domain, where the dynamics is governed by a Painlevé transcendent. This behaviour is typical for a class of instability problems. Because of the complexity of this irreducible local dynamics, it is not possible to produce an asymptotic expression for the moment of snap-through. However, we were in a position to construct an interval that contains the moment if ε is taken sufficiently small. We considered an $O(\sqrt{\varepsilon})$-vicinity of the slowly varying equilibrium solution and a damping of order $O(\sqrt{\varepsilon})$ because then slow changes of amplitude and phase are due to both terms with variable coefficients and nonlinearities and to the damping term. The jump takes place on a relatively short time-scale of length $O(\varepsilon^{-1/5})$ in which the time-scale of a transformed time variable τ (see (4.13)) has length $O(1)$. Generically for a system of type (6.1) the critical equilibrium curve is locally a parabola and the breakdown occurs for $\varepsilon t = O(\varepsilon^{4/5})$, which implies that the local inner equation follows from the scaling $x = x_{\text{critical}} + \varepsilon^{2/5}y$.

The oscillating physical system with a slowly decreasing parameter that we considered in this study was a special case, that stands for a class of jump problems where the slow decrease of the parameter is only dependent on time. The influence of the state variable on the change of the slowly varying parameter will be a goal for further research. Our aim is to get a better knowledge of systems of the form:

$$\frac{dx}{dt} = f(x, p) \,, \tag{7.1a}$$

$$\frac{dp}{dt} = \varepsilon g(x, p) \tag{7.1b}$$

with $x = (x_1, \ldots, x_n)$ a vector of state variables and $p = (p_1, \ldots, p_m)$ a vector of parameters. We are especially interested in forecasting a moment of sudden change. Other systems could be considered for which there would be more jumps or a jump back to the original equilibrium solution. Another interesting topic is the influence of stochastical effects on the oscillating system (see also Grasman, 1990). Furthermore, we can analyze the validity of the formal approximations we obtained by using perturbation theorems, as there are the approximation theorem and the extension theorem. For a discussion and proofs of these theorems we refer to Eckhaus (1979). In Grasman (1987) the proof of validity for a worked-out model problem can be found.

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Chapter 3

Slow passage through a pitchfork bifurcation[2]

Abstract

This paper deals with a class of second order differential equations with a slowly varying bifurcation parameter. The parameter slowly varies through a critical value corresponding to a transition from a stable equilibrium to one of the two stable branches of an intersecting parabolic curve. The local transition behaviour is described by the second Painlevé transcendent. In this study we predict which branch will be followed after passage of the bifurcation point given the initial state. For that purpose use is made of averaging methods and of asymptotic matching techniques connecting local solutions valid before, during, and after the transition.

1. Introduction

We consider a second order nonlinear differential equation depending on a parameter F:

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} = G(x, F) . \tag{1.1}$$

The damping is assumed to be fixed and small, $k = \kappa\varepsilon$, while the parameter F slowly varies in time. We have chosen a damping of order $O(\varepsilon)$, because then the effect of the damping is of order $O(1)$ on a time-scale $O(1/\varepsilon)$. In literature the effect of the dissipation is also often $O(\varepsilon)$. The matched asymptotic results of this study can be extended to a damping of order $O(\varepsilon^\alpha)$, $\alpha > 1/3$. In this study we analyze a class of bifurcation problems represented by the prototype system:

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} = x(F - 2x^2) , \tag{1.2a}$$

$$\frac{dF}{dt} = \varepsilon \tag{1.2b}$$

[2] by G.J.M. Marée; to appear in *SIAM J. Appl. Math.*

The initial value $F(0)$ is chosen smaller than a certain critical value F_c for which the system exhibits a bifurcation. An example of a mechanical system that corresponds qualitatively with system (1.2) is the analogue of an elastic column, sketched in figure 1 (see also Stoker (1950)).

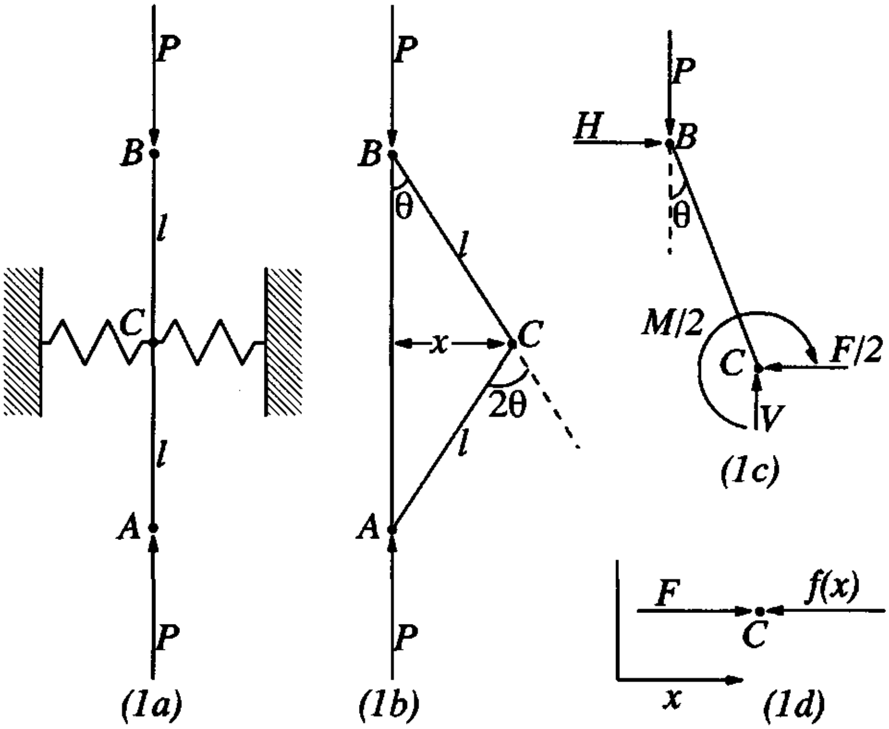


Figure 1 *Analogue of an elastic column.*

If a slender straight elastic rod is subjected at its ends to compressive forces along the axis of the rod, the unbended equilibrium position is stable if the compressive forces are kept under a certain critical value. Beyond this value the column bends or buckles. We simplify the problem by considering an elastic system with only one degree of freedom. Two rods are connected by a hinge at point C and are both free to slide along a vertical line at their other ends A and B. The hinge C is assumed to carry a particle of mass m. At the ends A and B forces P act along the vertical line through A and B. At point C springs are placed producing a sidewise-restoring force $f(x)$ depending upon the displacement x. Moreover, a restoring moment M acts upon the hinge. This moment is proportional to the angle 2θ between the two rods (see figure 1b). In this way the bending stiffness of a continuous elastic column is simulated. Neglecting the damping and the mass of the rod, we obtain the following equations:

- Equilibrium of forces in the vertical direction:

$$P = V\,. \tag{1.3a}$$

- Equation of moments about point B:

$$-\frac{M}{2} + Vl\sin\theta - \frac{Fl}{2}\cos\theta = 0\,. \tag{1.3b}$$

- Equation of motion of the mass m:

$$m\frac{d^2x}{dt^2} = F - f(x)\,. \tag{1.3c}$$

For the lateral spring force we take $f(x) = \alpha x + \beta x^3$, with α and β both positive constants, while for the restoring moment M we have $M = 2K_1\theta$ with K_1 a constant. Assuming that the sidewise displacement x is sufficiently small so that terms with powers of x/l higher than three can be ignored, we arrive at the following equation:

$$m\frac{d^2x}{dt^2} + (\alpha + 2K_1/l^2 - 2P/l)x + (\beta + 4K_1/3l^4 - P/l^3)x^3 = 0\,. \tag{1.4}$$

One solution is given by $x = 0$. The question is whether or not this solution is stable when P is increased. If P is smaller than the critical value P_{crit} a disturbance of the equilibrium position results in a small oscillation. We aim to predict the side to which the column will bend after P has become larger than P_{crit}, while the oscillation is not yet damped out.

For F fixed, smaller than F_c, the system (1.2) has one stable equilibrium. On a large time-scale the system shows a damped oscillation until F has reached F_c. The parameter F slowly passes F_c corresponding to a transition from a simple stable equilibrium to one of the stable equilibria positioned on a parabolic curve: for F fixed, larger than F_c, the system has three equilibria, two of them are stable and one is unstable. Because F is slowly increasing, the two stable equilibria are slowly varying in time. The solution in the neighbourhood of the stable slowly varying equilibria is approximated by a harmonic oscillation. For F outside a certain ε-neighbourhood of F_c asymptotic approximations are obtained with the aid of averaging methods. In order to describe the bifurcation a local approximation, being a transition layer, has to be made. Local scaling analysis yields as approximating differential equation the second Painlevé equation. In order to connect local solutions that are valid before, during, and after the transition, asymptotic matching techniques have to be used and an extension theorem formulated by Eckhaus (1979) has to be applied.

It is remarked that existence and uniqueness of solutions of (1.2) is guaranteed for $t \in [-M/\varepsilon, M/\varepsilon]$ with M an arbitrary large positive number independent of ε. Such a proof of existence and uniqueness can be based on the energy integral of the system, see also Chillingworth (1976, p.187-188). In lemma 5.1 we will use the same method to prove the validity of the approximation near the bifurcation point.

Haberman (1979) also studied this type of nonlinear differential equations. He studies systems without damping and uses the method of eliminating secular terms. In a similar way we apply averaging to eliminate secular terms and, in contrast with Haberman, we study systems with damping. There are obvious discrepancies in the computed values of coefficients of some asymptotic series. More relevant is that we prove the validity of the matched asymptotic approximations and predict the state of the system after passing the bifurcation point from the initial state. This result almost looks like predicting the outcome of flipping a coin. Neishtadt (1987, 1988) and Baer, Erneux and Rinzel (1989) concentrate on the slow passage through a Hopf bifurcation from a stable steady state to a stable time-periodic solution and demonstrate that this case is quite different from a steady bifurcation or limit point. The transition from an oscillatory solution to a steady state has been investigated by Holden and Erneux (1993).

In this study the potential takes the form of a double well potential after bifurcation. We assume that the trajectory is close to the equilibrium before bifurcation. Therefore, it has directly been captured in either well after bifurcation. This means that the unperturbed separatrix, that appears after passing the bifurcation point, will not be crossed. The problem of the crossing of a separatrix by nonlinear oscillations that correspond to a slowly varying potential which remains double-welled, has been analyzed by Bourland and Haberman (1990, 1994), Neishtadt (1986, 1993), and Henrard (1993). In the asymptotic limit ($\varepsilon \to 0$) the capture problem becomes probabilistic and predictions become pointless in view of the extreme sensitivity to initial data. The work of Bourland, Haberman and Kath (1991) contains formal averaging formulas similar to those of this study. These are only valid away from the bifurcation point. Denier and Grimshaw (1988) also study nonlinear differential equations with a slowly varying parameter. They find that, depending on the initial amplitudes, the solutions of the transition equations are either asymptotically equivalent to the bifurcated solutions or develop algebraic singularities at some positive time.

In section 2 we consider the different equilibrium states of the system when the parameter F is fixed. In section 3 the solution is approximated by averaging for an interval in which the solution is sufficiently bounded away below the bifurcation point, whereas in section 4 we obtain an asymptotic approximation for the solution beyond the critical point. Moreover, it is shown that these approximations remain valid outside an $\varepsilon^{2/3}$-neighbourhood of the critical point $F = F_c$. In section 5 we analyze the transition layer equation and obtain matching conditions for this local asymptotic solution. Furthermore, it is proven that the transition layer solution is a local approximation of the exact solution. This local solution has an overlap with the other (outer) approximations. In section 6 we formulate an important result which connects the integration constants in the averaged asymptotic solution below criticality with those in the one above criticality. In this way we obtain a remarkable result: one can find asymptotic approximations for predicting the behaviour of solutions after passing the bifurcation point at the basis of the initial values. In section 7 we consider the general class of bifurcation problems of which the present problem is a special case. Finally, in section 8, we make some concluding remarks.

2. The reduced system with fixed parameter

Substitution of $\varepsilon = 0$ in (1.2) yields the reduced system

$$\frac{d^2x}{dt^2} - x(F - 2x^2) = 0 \,, \tag{2.1a}$$

$$\frac{dF}{dt} = 0 \,. \tag{2.1b}$$

For F fixed and smaller than the critical value

$$F_c = 0 \tag{2.2}$$

the reduced system (2.2) has one equilibrium (for which $dx/dt = 0$):

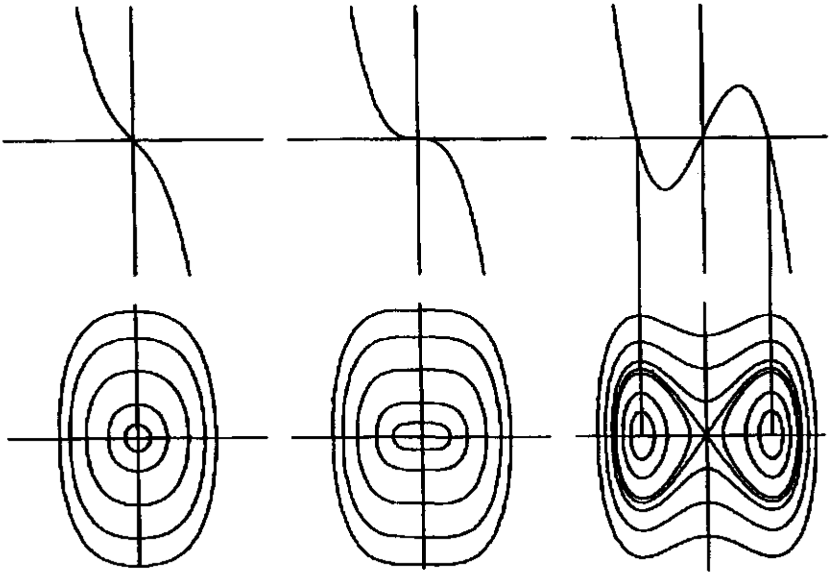
$$x_0 = 0 \,. \tag{2.3}$$

Linearization at $(x_0, 0)$ yields a system with an equilibrium being a centre point. Using the lemma of Morse (see Verhulst, 1990b) it is clear that for the nonlinear system (2.1a) this point is also a centre point. When $F = F_c$ the unique equilibrium $x = x_c$ is still stable. For $F > F_c$ it becomes an unstable point of saddle point type. Moreover the system then exhibits two "supercritical" stable equilibria (for which $y = 0$):

$$x_{-1} = -\sqrt{\frac{F}{2}} \;,\; x_1 = \sqrt{\frac{F}{2}} \;. \tag{2.4}$$

Haberman (1979) calls the transition from a stable line to a parabolic arc, as F passes F_c, a "parabolic bifurcation". In literature this phenomenon has commonly been called a pitchfork bifurcation. In figures 2 and 3 this pitchfork bifurcation is illustrated. For $\varepsilon = 0$ the energy integral of the system (1.2) equals

$$E = \tfrac{1}{2}y^2 - \tfrac{1}{2}Fx^2 + \tfrac{1}{2}x^4 \,. \tag{2.5}$$



(a) $F < F_c$ *(b)* $F = F_c$ *(c)* $F > F_c$

Figure 2 *The graph of the function $f(x) = x(F - 2x^2)$ in three cases of fixed F and the corresponding phase portrait of (2.1a) for different values of F.*

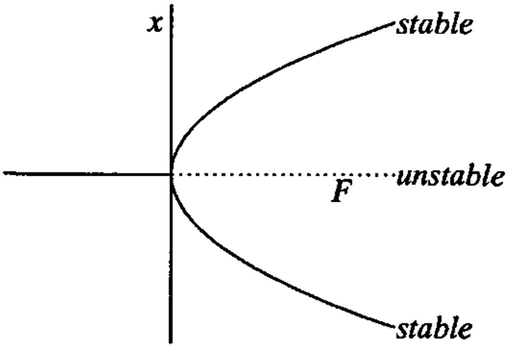


Figure 3 *The branches of the limit solution as a function of F.*

3. Asymptotic expansion valid before passage of the bifurcation point

We construct an asymptotic solution to the initial value problem (1.2) valid outside a certain ε-neighbourhood of $F = 0$, corresponding to initial conditions close to the outer equilibrium solution $x = x_0 = 0$. We consider perturbations of the equilibrium of the form

$$x(t) = \sqrt{\varepsilon}\, u(t) \, . \tag{3.1}$$

Furthermore, we take as initial values

$$x(0) = \sqrt{\varepsilon}\, a \, , \tag{3.2a}$$

$$\frac{dx}{dt}(0) = \sqrt{\varepsilon}\, b \, , \tag{3.2b}$$

$$F(0) = F_0 < F_c = 0. \tag{3.2c}$$

Substitution in (1.2a) yields

$$\frac{d^2u}{dt^2} + \omega^2(F)u = -\kappa\varepsilon\frac{du}{dt} - 2\varepsilon u^3 \tag{3.3}$$

with

$$\omega^2(F) = -F \, . \tag{3.4}$$

The solution remains for a certain time in a $\sqrt{\varepsilon}$-neighbourhood of $x_0 = 0$. Using the transformations

$$u = \exp(-\frac{1}{2}\kappa\varepsilon t)w := r_1 w \, , \tag{3.5a}$$

$$w = \omega^{-\frac{1}{2}}(F)y \, , \tag{3.5b}$$

$$\frac{dy}{dt} = \omega(F)v, \tag{3.6a}$$

$$\frac{dv}{dt} = \frac{1}{\omega(F)}\frac{d^2y}{dt^2} - \frac{\varepsilon}{\omega(F)}\frac{d\omega}{dF}v \, , \tag{3.6b}$$

introducing polar coordinates

$$y = r \sin \varphi \;, \; v = r \cos \varphi \tag{3.7a}$$

with initial values

$$y(0) = r_0 \sin \varphi_0 \;, \; u(0) = r_0 \cos \varphi_0 \;, \tag{3.7b}$$

transforming the time-scale

$$\tau = \int_{F_0}^{F} \frac{\omega(F)dF}{\varepsilon} \quad \text{or} \quad F = -\left((-F_0)^{3/2} - \frac{3\varepsilon}{2}\tau\right)^{2/3} , \tag{3.8a,b}$$

eliminating F, and setting

$$\psi = \varphi - \tau \;, \tag{3.9}$$

we finally arrive at the following initial value problem

$$\frac{dr_1}{d\tau} = \frac{-\kappa\varepsilon}{2\omega(\tau,\varepsilon)} r_1 \;, \; \omega(\tau,\varepsilon) = \left((-F_0)^{3/2} - \frac{3\varepsilon}{2}\tau\right)^{1/3} \;, \; r_1(0) = 1, \tag{3.10a}$$

$$\frac{dr}{d\tau} = \frac{\varepsilon}{\omega^3(\tau,\varepsilon)}\left(-2r_1^2 r^3 \sin^3(\psi+\tau)\cos(\psi+\tau) + \frac{1}{4}\kappa^2 r\varepsilon\omega(\tau,\varepsilon)\sin(\psi+\tau)\cos(\psi+\tau)\right)$$

$$+ \frac{1}{\omega^2(\tau,\varepsilon)}\left(\frac{d\omega}{d\tau}\right)^2\left(-\frac{3}{4}r\sin(\psi+\tau)\cos(\psi+\tau)\right) + \frac{1}{2}\omega^{-1}(\tau,\varepsilon)\frac{d^2\omega}{d\tau^2} r\sin(\psi+\tau)\cos(\psi+\tau), \tag{3.10b}$$

$$\frac{d\psi}{d\tau} = \frac{\varepsilon}{\omega^3(\tau,\varepsilon)}\left(2r_1^2 r^2 \sin^4(\psi+\tau) - \frac{1}{4}\kappa^2\varepsilon\omega(\tau,\varepsilon)\sin^2(\psi+\tau)\right)$$

$$+ \frac{1}{\omega^2(\tau,\varepsilon)}\left(\frac{d\omega}{d\tau}\right)^2\left(\frac{3}{4}\sin^2(\psi+\tau)\right) - \frac{1}{2}\omega^{-1}(\tau,\varepsilon)\frac{d^2\omega}{d\tau^2}\sin^2(\psi+\tau). \tag{3.10c}$$

From (3.8b) we conclude that $\omega(\tau,\varepsilon) = 0$ for $\tau = \tau_0$ with

$$\tau_0 = \frac{2}{3\varepsilon}(-F_0)^{3/2} \;. \tag{3.11}$$

With the aid of the lemma of Gronwall we will prove an approximation theorem for $\tau \in [0, \tau_0 - \delta^{-1}(\varepsilon)]$ -so for $F \in [F_0, -(3\varepsilon\delta^{-1}(\varepsilon)/2)^{2/3}]$- with $\delta(\varepsilon) = o(1)$ an asymptotic order function.

Lemma 3.1 (Gronwall) Suppose for $t_0 \le t \le t_0 + a$ that

$$\varphi(t) \le \delta_2(t - t_0) + \delta_1 \int_{t_0}^{t} \varphi(s)ds + \delta_3 \tag{3.12}$$

with $\varphi(t)$ continuous and $\varphi(t) \ge 0$ for $t_0 \le t \le t_0 + a$, where δ_1, δ_2 and δ_3 are constants with $\delta_1 > 0$, $\delta_2 \ge 0$ and $\delta_3 \ge 0$. Then for $t_0 \le t \le t_0 + a$

$$\varphi(t) \le \left(\frac{\delta_2}{\delta_1} + \delta_3\right) \exp(\delta_1(t - t_0)) - \frac{\delta_2}{\delta_1} . \tag{3.13}$$

A proof of this Lemma is given by Sanders and Verhulst (1987).

We now state an approximation theorem:

Theorem 3.2 Consider the initial value problem

$$\frac{dr_1}{d\tau} = \frac{-\kappa\varepsilon}{2\omega(\tau,\varepsilon)} r_1 \; , \; \omega(\tau,\varepsilon) = \left((-F_0)^{3/2} - \frac{3\varepsilon}{2}\tau\right)^{1/3} \; , \; r_1(0) = 1 \; , \tag{3.14a}$$

$$\frac{dr}{d\tau} = \frac{\varepsilon}{\omega^3(\tau,\varepsilon)}(g_1(r_1,r,\psi,\tau) + \varepsilon\omega(\tau,\varepsilon)h_1(r,\psi,\tau)) + \frac{1}{\omega^2(\tau,\varepsilon)}\left(\frac{d\omega}{d\tau}\right)^2 m_1(r,\psi,\tau)$$

$$+ \frac{1}{\omega(\tau,\varepsilon)} \frac{d^2\omega}{d\tau^2} l_1(r,\psi,\tau) \; , \; r(0) = r_0 \; , \tag{3.14b}$$

$$\frac{d\psi}{d\tau} = \frac{\varepsilon}{\omega^3(\tau,\varepsilon)}(g_2(r_1,r,\psi,\tau) + \varepsilon\omega(\tau,\varepsilon)h_2(r,\psi,\tau)) + \frac{1}{\omega^2(\tau,\varepsilon)}\left(\frac{d\omega}{d\tau}\right)^2 m_2(r,\psi,\tau)$$

$$+ \frac{1}{\omega(\tau,\varepsilon)} \frac{d^2\omega}{d\tau^2} l_2(r,\psi,\tau) \; , \; \psi(0) = \varphi_0 \tag{3.14c}$$

for $0 \le \tau < \tau_0$.

Suppose

a) The vector functions g, h, m, l, ∇g, and ∇h are continuous in τ and in $x = (r_1, r, \psi)'$ or $y = (r, \psi)'$ and bounded by a constant M; independent of ε.

b) m and l are Lipschitz-continuous in $y = (r, \psi)'$ with Lipschitz constant L.

c) g and h are T-periodic in τ with averages $g°$ and $h°$.

Let (r_1, r, ψ) be the solution of this system. Moreover, let (r_{1a}, r_a, ψ_a) be the solution of the τ-averaged system

$$\frac{dr_{1a}}{d\tau} = \frac{-\kappa\varepsilon}{2\omega(\tau,\varepsilon)} r_{1a} \;,\; r_{1a}(0) = 1, \tag{3.15a}$$

$$\frac{dr_a}{d\tau} = \frac{\varepsilon}{\omega^3(\tau,\varepsilon)}(g_1^0(r_{1a},r_a,\psi_a) + \varepsilon\omega(\tau,\varepsilon)h_1^0(r_a,\psi_a)) \;,\; r_a(0) = r_0 \;, \tag{3.15b}$$

$$\frac{d\psi_a}{d\tau} = \frac{\varepsilon}{\omega^3(\tau,\varepsilon)}(g_2^0(r_{1a},r_a,\psi_a) + \varepsilon\omega(\tau,\varepsilon)h_2^0(r_a,\psi_a)) \;,\; \psi_a(0) = \varphi_0 \;. \tag{3.15c}$$

Let $\delta(\varepsilon)$ be an asymptotic order function satisfying $\delta(\varepsilon) = o(1)$ as well as $\varepsilon = O(\delta(\varepsilon))$ then for $\tau \in [0, \tau_0 - \delta^{-1}(\varepsilon)]$

$$r_1(\tau) = r_{1a}(\tau),\; r(\tau) = r_a(\tau) + o(1),\; \psi(\tau) = \psi_a(\tau) + o(1). \tag{3.16}$$

Remark 1: for a vectorfunction $f \in \mathbb{R}^n$ with components f_i, $i = 1 \ldots n$, we will use the norm $\|f\| = \sum_{i=1}^{n} |f_i|$.

Remark 2: the time-scale of τ_0, that has been determined as a zero of ω, is of order $O(1/\varepsilon)$. $\delta(\varepsilon)$ is such that $\tau_0 - \delta^{-1}(\varepsilon) \geq 0$.

Proof We immediately obtain

$$r_{1a}(\tau) = r_1(\tau) = \exp\{\frac{\kappa}{2}\left((-F_0)^{3/2} - \frac{3\varepsilon}{2}\tau\right)^{2/3} + \frac{\kappa}{2}F_0\} \;, \tag{3.17}$$

so $$0 \leq r_{1a}(\tau) \leq 1 \quad \text{for} \quad 0 \leq \tau < \tau_0 \;. \tag{3.18}$$

Furthermore, for $\tau \in [0, \tau_0 - \delta^{-1}(\varepsilon)]$

$$(-F_0)^{-1/2} \le \omega^{-1}(\tau,\varepsilon) \le \left(\frac{3}{2}\varepsilon\delta^{-1}(\varepsilon)\right)^{-1/3}, \quad \frac{\varepsilon}{2}(F_0)^{-1} \le \left|\frac{d\omega}{d\tau}\right| \le \frac{\varepsilon}{2}\left(\frac{3}{2}\varepsilon\delta^{-1}(\varepsilon)\right)^{-2/3}. \tag{3.19}$$

We define

$$x = (r_1, r, \psi)' = (x_1, x_2)' \text{ with } x_1 = r_1 \text{ and } x_2 = (r, \psi)' \tag{3.20}$$

and introduce

$$u^1(x,\tau) = \int_0^\tau [g(x,s) - g^0(x) + \varepsilon\omega(s,\varepsilon)(h(x_2,s) - h^0(x_2))]ds. \tag{3.21}$$

This integral is bounded: $\|u^1(x, \tau)\| \le 4MT$. Now a "near-identity" transformation is applied

$$x_2(\tau) = z_2(\tau) + \frac{\varepsilon}{\omega^3(\tau,\varepsilon)}u^1(z(\tau),\tau). \tag{3.22}$$

This is a "near-identity" as $\dfrac{\varepsilon}{\omega^3(\tau,\varepsilon)} = O(\delta(\varepsilon))$ for $\tau \in [0, \tau_0 - \delta^{-1}(\varepsilon)]$. Differentiating (3.22) we obtain with (3.15) and with the aid of (3.21)

$$\left[E + \frac{\varepsilon}{\omega^3(\tau,\varepsilon)}\frac{\partial}{\partial z_2}u^1(z(\tau),\tau)\right]\frac{dz_2}{d\tau} = \frac{\varepsilon}{\omega^3(\tau,\varepsilon)}(g^\circ(z) + \varepsilon\omega(\tau,\varepsilon)h^\circ(z)) + R \tag{3.23}$$

with R a vector function of z, τ and ε. This remainder term R is estimated as follows:

$$R = O(\delta^2(\varepsilon)) \,. \tag{3.24}$$

The matrix E in (3.23) is the unity matrix. Since both $\partial u^1/\partial z_2$ and u^1 are bounded, we can invert

$$\left[E + \frac{\varepsilon}{\omega^3(\tau,\varepsilon)}\frac{\partial}{\partial z_2}u^1(z(\tau),\tau)\right]^{-1} = E - \frac{\varepsilon}{\omega^3(\tau,\varepsilon)}\frac{\partial}{\partial z_2}u^1(z(\tau),\tau) + O\left(\frac{\varepsilon^2}{\omega^6(\tau,\varepsilon)}\right). \tag{3.25}$$

So, from (3.23), we obtain the following equation for z_2:

$$\frac{dz_2}{d\tau} = \delta_1(\varepsilon)(g^\circ(z) + \varepsilon\omega(\tau,\varepsilon)h^\circ(z_2)) + \delta_2(\varepsilon) \tag{3.26a}$$

with

$$\delta_1(\varepsilon) = \varepsilon\omega^{-3}(\tau,\varepsilon) = O(\delta(\varepsilon)) \text{ and } \delta_2(\varepsilon) = O(\delta^2(\varepsilon)) = o(\delta_1(\varepsilon)) . \tag{3.26b}$$

The solution of the averaged system

$$\frac{dx_{2a}}{d\tau} = \frac{\varepsilon}{\omega^3(\tau,\varepsilon)}(g^\circ(x_a) + \varepsilon\omega(\tau,\varepsilon)h^\circ(x_{2a})),\ x_{2a}(0) = z_2(0) , \tag{3.27}$$

approximates the solution of (3.26) for $\tau \in [0, \tau_0 - \delta^{-1}(\varepsilon)]$ in the following way

$$z_2(\tau) = x_{2a}(\tau) + o(1), \tag{3.28}$$

for

$$\begin{aligned} \|z_2(\tau) - x_{2a}(\tau)\| &= \|\int_0^\tau \frac{dz_2}{d\tau}d\tau - \int_0^\tau \frac{dx_{2a}}{d\tau}d\tau\| \\ &\le \|\delta_1(\varepsilon)\int_0^\tau (g^\circ(z) - g^\circ(x_a) + \varepsilon\omega(\tau,\varepsilon)(h^\circ(z_2) - h^\circ(x_{2a})))d\tau\| + \delta_2\varepsilon\tau \\ &\le 2L\delta_1(\varepsilon)\tau\|z - x_a\| + \delta_2\varepsilon\tau \quad (z_1 = x_{1a}). \end{aligned} \tag{3.29}$$

Application of the Lemma of Gronwall now completes the proof

$$\begin{aligned} \|z_2(\tau) - x_{2a}(\tau)\| &\le \frac{\delta_2}{2L\delta_1}\exp(2L\delta_1\tau) = O(\delta(\varepsilon)) \\ &= o(1) \text{ for } \tau \in [0, \tau_0 - \delta^{-1}(\varepsilon)] \quad \square . \end{aligned} \tag{3.30}$$

This yields an asymptotic expansion of the solution of (1.2) in the neighbourhood of x_0 for $F < 0$ and $\varepsilon^{2/3} = o(F)$:

Corollary 3.3 For $F_0 < F < 0$ and $\varepsilon^{2/3} = o(F)$ the solution of (1.2) has the following expansion $x(t) = \sqrt{\varepsilon}\, r_a r_{1a} \omega^{-\frac{1}{2}}(F) \sin(\tau + \psi_a) + o(\sqrt{\varepsilon}\,\omega^{-\frac{1}{2}}(F))$ (3.31) with (r_{1a}, r_a, ψ_a) the solution of the system

$$\frac{dr_{1a}}{d\tau} = \frac{-\kappa\varepsilon}{2\omega(\tau,\varepsilon)}r_{1a}\ ,\ r_{1a}(0) = 1, \tag{3.32a}$$

$$\frac{dr_a}{d\tau} = 0\ ,\ r_a(0) = r_0, \tag{3.32b}$$

$$\frac{d\psi_a}{d\tau} = \frac{\varepsilon}{\omega^3(\tau,\varepsilon)}\left(\frac{3}{4}r_{1a}^2r_a^2 - \frac{1}{8}\kappa^2\omega(\tau,\varepsilon)\varepsilon\right)\ ,\ \psi_a(0) = \varphi_0 \tag{3.32c}$$

with $\omega(\tau,\varepsilon)$ as defined in (3.14a).

Proof Problems arise when the angular velocity $\omega(F)$ tends to zero which are tied up with boundary layer behaviour in the neighbourhood of $F = 0$. An approximation for the bifurcation time is therefore: $t_0 = -F_0\varepsilon^{-1}$. Because (3.10) is a special case of the initial value problem stated in theorem 3.2, the averaged system (3.32) is an $o(1)$-approximation of the complete system (3.10) for $\tau \in [0, \tau_0 - \delta^{-1}(\varepsilon)]$ with $\delta(\varepsilon) = o(1)$ and $\delta(\varepsilon) = O(\varepsilon)$. The result is summarized as follows: (3.32), being an approximation of the solution of (1.2) is valid when $F_0 < F < 0$ and $\varepsilon^{2/3} = o(F)$.

4. Asymptotic expansion valid after passage of the bifurcation point

We analyze the solution to the initial value problem (1.2) for $F \geq F_c + \delta$ (F_c=0) close to one of the two stable outer equilibrium solutions $x = x_{\pm 1}$. Because of the symmetry of the problem it is sufficient to consider the perturbation of one of the equilibria of the reduced system: we choose $x = x_1 = \sqrt{(F/2)}$. To analyze a "slowly varying equilibrium solution" $x_{1sv}(F)$ we rewrite (1.2):

$$\frac{d^2x}{dt^2} = \varepsilon^2\frac{d^2x}{dF^2} = -\kappa\varepsilon^2\frac{dx}{dF} + g(x,F) \tag{4.1}$$

with $\quad g(x,F) = x(F - 2x^2).$ (4.2)

Assuming that the derivatives in (4.1) are small, we obtain a "slowly varying equilibrium solution" for $F > F_c = 0$ by perturbing the dependent variable x at $x_1(F) = \sqrt{(F/2)}$. Because $g(x_1,F) = 0$, we can use the Taylor series of $g(x,F)$ at $x = x_1(F)$.

This changes (4.1) into:

$$\varepsilon^2 \frac{d^2x}{dF^2} = -\kappa\varepsilon^2 \frac{dx}{dF} + (x - x_1)\frac{\partial g}{\partial x}\Big|_{x_1} + \frac{(x - x_1)^2}{2}\frac{\partial^2 g}{\partial x^2}\Big|_{x_1} + \frac{(x - x_1)^3}{6}\frac{\partial^3 g}{\partial x^3}\Big|_{x_1} . \tag{4.3}$$

In this way we obtain an asymptotic expansion for the "slowly varying equilibrium solution" of (4.1):

$$x_{1sv}(F) = \sqrt{\frac{F}{2}} - \frac{1}{16}\kappa\varepsilon^2\left(\frac{F}{2}\right)^{-3/2} + \frac{1}{64}\varepsilon^2\left(\frac{F}{2}\right)^{-5/2} + \ldots\ldots . \tag{4.4}$$

From theorem 9.1 formulated by Verhulst (1990b) it follows that this is the asymptotic expansion of a solution. For a solution that holds in a $\sqrt{\varepsilon}$-neighbourhood of this slowly varying solution we write:

$$x(t) = x_{1sv}(F) + \sqrt{\varepsilon}\, u(t). \tag{4.5}$$

Furthermore, we assume that the solution takes the following values for $t = t_1 = -F_0\varepsilon^{-1} + M\varepsilon^{-1}$ with M an ε-independent positive constant.

$$x(t_1) = x_1(F_1) + \sqrt{\varepsilon}\, a_1 , \tag{4.6a}$$

$$\frac{dx}{dt}(t_1) = \frac{dx_1}{dt}(F_1) + \sqrt{\varepsilon}\, b_1 , \tag{4.6b}$$

$$F(t_1) = F_1 = \varepsilon t_1 , \tag{4.6c}$$

so the solution is at $t = t_1$ "far away" from the bifurcation time $t = -F_0\varepsilon^{-1}$. Substitution in (1.2) yields

$$\frac{d^2u}{dt^2} + \omega_1^2(F)u = -6\sqrt{\varepsilon}\, u^2 x_{1sv}(F) - \kappa\varepsilon\frac{du}{dt} - 2\varepsilon u^3, \tag{4.7a}$$

$$\frac{dF}{dt} = \varepsilon \tag{4.7b}$$

with $\quad \omega_1^2(F) = 6x_{1sv}^2 - F. \qquad (4.8)$

In the same way as in section 3 we now carry out the transformations

$$u = \exp(-\frac{1}{2}\kappa\,\varepsilon\, t)w \;:=\; r_1^* w\ , \tag{4.9a}$$

$$w = \omega_1^{-\frac{1}{2}}(F)y\ , \tag{4.9b}$$

$$\frac{dy}{dt} = \omega_1(F)v\ , \tag{4.9c}$$

$$\frac{dv}{dt} = \frac{1}{\omega_1^2(F)}\frac{d^2y}{dt^2} - \frac{\varepsilon}{\omega_1(F)}\frac{d\omega_1}{dF}v \tag{4.9d}$$

and introduce polar coordinates

$$y = r^*\cos\,\varphi^*\ ,\ v = -r^*\sin\,\varphi^*\ . \tag{4.10a}$$

with values at t_1

$$y(t_1) = r_0^*\cos\,\varphi_0^*,\ u(t_1) = -r_0^*\sin\,\varphi_0^*. \tag{4.10b}$$

Next, we make the transformation

$$\tau = \int_F^{F_1}\frac{\omega_1(F)dF}{\varepsilon} \tag{4.11}$$

and set

$$\psi^* = \varphi^* + \tau, \tag{4.12}$$

so we finally arrive at the following system

$$\frac{dr_1^*}{d\tau} = \frac{\kappa\varepsilon}{2\omega_1(F)}r_1^*\ , \qquad r_1^*(t_1) = \exp(-\frac{1}{2}\kappa\,\varepsilon\, t_1)\ , \tag{4.13a}$$

$$\frac{dF}{d\tau} = \frac{-\varepsilon}{\omega_1(F)}\ , \qquad F(t_1) = F_1\ , \tag{4.13b}$$

$$\begin{aligned}
\frac{dr^*}{d\tau} = &-\sqrt{\varepsilon}\,(6\omega_1^{-5/2}(F)r_1^*(r^*)^2\cos^2(\psi^* + \tau)\sin(\psi^* + \tau)x_{1sv}(F)) \\
&- 2\varepsilon\omega_1^{-3}(F)(r_1)^{*2}(r^*)^3\cos^3(\psi^* + \tau)\sin(\psi^* + \tau) \\
&- \varepsilon^2(-\frac{1}{4}\kappa^2\omega_1^{-2}(F)r^*\cos(\psi^* + \tau)\sin(\psi^* + \tau) \\
&+ \frac{3}{4}\omega_1^{-4}(F)r^*\cos(\psi^* + \tau)\sin(\psi^* + \tau)(\frac{d\omega_1}{dF})^2 \\
&+ \frac{1}{2}\omega_1^{-3}(F)r^*\cos(\psi^* + \tau)\sin(\psi^* + \tau)\frac{d^2\omega_1}{dF^2})\ ,\ r^*(t_1) = r_0^*,
\end{aligned} \tag{4.13c}$$

$$\begin{aligned}
\frac{d\psi^*}{d\tau} = &-\sqrt{\varepsilon}\,(6\omega_1^{-5/2}(F)r_1^*r^*\cos^3(\psi^* + \tau)x_{1sv}(F)) \\
&- 2\varepsilon\omega_1^{-3}(F)(r_1^*)^2(r^*)^2\cos^4(\psi^* + \tau) - \varepsilon^2(-\frac{1}{4}\kappa^2\omega_1^{-2}(F)\cos^2(\psi^* + \tau) \\
&+ \frac{3}{4}\omega_1^{-4}(F)\cos^2(\psi^* + \tau)(\frac{d\omega_1}{dF})^2 + \frac{1}{2}\omega_1^{-3}(F)\cos^2(\psi^* + \tau)\frac{d^2\omega_1}{dF^2})\ ,\ \psi^*(t_1) = \varphi_0^*.
\end{aligned} \tag{4.13d}$$

We will consider

$$0 < \delta_1(\varepsilon) \le F < F_1 = O(1) \tag{4.14a}$$

with $\varepsilon^{2/3} = o(\delta_1(\varepsilon))$. (4.14b)

We now obtain the following approximations

$$x_{1sv}(F) = \sqrt{\frac{F}{2}} + O(\varepsilon^2 F^{-5/2})\ , \tag{4.15a}$$

$$\omega_1(F) = \sqrt{2F}\,(1 + O(\varepsilon^2 F^{-3}))\ , \tag{4.15b}$$

$$\omega_1^{-a}(F) = (2F)^{-a/2}(1 + O(\varepsilon^2 F^{-3})),\ a \in \mathbb{R}\ , \tag{4.15c}$$

$$\tau = \frac{1}{\varepsilon}(\frac{1}{3}(2F)^{3/2} + O\,(\varepsilon^2 F^{-3/2}))\ , \tag{4.15d}$$

so $$\frac{1}{\delta_1^*(\varepsilon)} \le \tau \le \frac{1}{\delta_2^*(\varepsilon)} \text{ with } \delta_1^*(\varepsilon) = o(1) \text{ and } \delta_2^*(\varepsilon) = O(\varepsilon). \tag{4.15e}$$

System (4.13) then transforms into

$$\frac{dr_1^*}{d\tau} = \frac{\kappa\varepsilon}{2\sqrt{2F}} r_1^* + O(\varepsilon^3 F^{-7/2}), \tag{4.16a}$$

$$\frac{dF}{d\tau} = \frac{-\varepsilon}{\sqrt{2F}} + O(\varepsilon^3 F^{-7/2}), \tag{4.16b}$$

$$\begin{aligned}\frac{dr^*}{d\tau} = {} & \frac{-\sqrt{\varepsilon}\,2^{-7/4}}{F^{3/4}}(6r_1^*(r^*)^2\cos^2(\psi^*+\tau)\sin(\psi^*+\tau)) - \frac{\varepsilon}{(2F)^{3/2}}(2(r_1^*)^2(r^*)^3\cos^3(\psi^*+\tau)\sin(\psi^*+\tau)) \\ & - \frac{\varepsilon^2}{2F}(-\frac{1}{4}\kappa^2 r^*\cos(\psi^*+\tau)\sin(\psi^*+\tau)) + O(\varepsilon^2 F^{-3}) ,\end{aligned} \tag{4.16c}$$

$$\begin{aligned}\frac{d\psi^*}{d\tau} = {} & - \frac{\sqrt{\varepsilon}\,2^{-7/4}}{F^{3/4}}(6r_1^* r^*\cos^3(\psi^*+\tau)) - \frac{\varepsilon}{(2F)^{3/2}}(2(r_1^*)^2(r^*)^2\cos^4(\psi^*+\tau)) \\ & - \frac{\varepsilon^2}{2F}\left(-\frac{1}{4}\kappa^2\cos^2(\psi^*+\tau)\right) + O(\varepsilon^2 F^{-3}) .\end{aligned} \tag{4.16d}$$

It follows from $r_1^* = \exp\{\frac{-\kappa\varepsilon}{2}t\}$, that $0 \le r_1^* \le 1$ for $t \in [-F_0\varepsilon^{-1}, t_1]$. (4.17)

With the aid of the Lemma of Gronwall we will now prove a "second order" approximation theorem; we will obtain an estimate of $O(\varepsilon F^{-3/2})$ taking into account both the $O(\varepsilon^{1/2}F^{-3/4})$ terms and the $O(\varepsilon F^{-3/2})$ terms of (4.16). Note the difference with the theorem of the previous section, due to the fact that we are now near a nontrivial equilibrium branch.

Theorem 4.1 Consider the initial value problems

$$\frac{dx}{d\tau} = \frac{\sqrt{\varepsilon}}{F^{3/4}} f(x,\tau) f_1(r_1) + \frac{\varepsilon}{F^{3/2}} g(x,\tau) g_1(r_1) + \frac{\varepsilon^2}{F} h(x,\tau) h_1(r_1) + O(\varepsilon^2 F^{-3}) \,,\ x(\tau_0) = x_0 \tag{4.18}$$

and

$$\frac{dx_a}{d\tau} = \frac{\varepsilon}{F^{3/2}} f^{10}(x_a,\tau) f_1(r_1) + \frac{\varepsilon}{F^{3/2}} g^0(x_a,\tau) g_1(r_1) + \frac{\varepsilon^2}{F} h^0(x_a,\tau) h_1(r_1) \,,\ x_a(\tau_0) = x_0 \tag{4.19}$$

with f, g, h: $\mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}^2$, $\tau \in [\delta_1^{*-1}(\varepsilon), \delta_2^{*-1}(\varepsilon)]$ with $\delta_1^*(\varepsilon) = o(1)$ and $\delta_2^*(\varepsilon) = O(\varepsilon)$ so that $F \in [\delta_1(\varepsilon), \delta_2(\varepsilon)]$ with $\varepsilon^{2/3} = o(\delta_1(\varepsilon))$ and $\delta_2(\varepsilon) = O(1)$, and $\varepsilon \in (0, \varepsilon_0]$;
Furthermore,

$$f^1(x,\tau) = \nabla f(x,\tau) u^1(x,\tau) f_1(r_1) \tag{4.20}$$

and

$$u^1(x,\tau) = \int_{\tau_0}^{\tau} f(x,s)ds - \frac{1}{T}\int_{\tau_0}^{\tau_0+T}\int_{\tau_0}^{\tau} f(x,s)ds d\tau. \tag{4.21}$$

Suppose
a) f has a Lipschitz-continuous first derivative in x, the vector-functions f_1, g_1, h_1, g, $F^{1/2}h$, ∇g, $\nabla F^{1/2}h$, ∇f and $\nabla^2 f$ are continuous in the variables and bounded by a constant M, independent of ε, for $\tau \in [\delta_1^{*-1}, \delta_2^{*-1}(\varepsilon)]$;
b) f, g and h are T-periodic in τ, averages f^0, g^0 and h^0 (f^1 has average f^{10}). Moreover $f^0 = 0$.
Then $x(\tau) = x_a(\tau) + o(1)$ for $\tau \in [\delta_1^{*-1}(\varepsilon), \delta_2^{*-1}(\varepsilon)]$ or $F \in [\delta_1(\varepsilon), \delta_2(\varepsilon)]$.

Remark: The $O(\varepsilon^{1/2}F^{-3/4})$- term did not appear in section 3. Since $f^0 = 0$, we now have to apply second order averaging (see also Sanders and Verhulst (1987)).

Proof Define $y(\tau)$ by

$$x(\tau) = y(\tau) + \varepsilon^{1/2}F^{-3/4}\, u^1(y(\tau),\tau) f_1(r_1) \tag{4.22}$$

with

$$u^1(y(\tau),\tau) = \int_0^{\tau} f(y,s)ds - \frac{1}{T}\int_0^{T}\int_0^{\tau} f(y,s)ds d\tau. \tag{4.23}$$

(f, g and h are T-periodic in τ). Substitution in the differential equation (4.18) produces for y

$$\frac{dy}{d\tau} = \left(\frac{\sqrt{\varepsilon}\, f^{1/2}(r_1)}{F^{3/4}}\right)^2 (\nabla f(y,\tau) u^1(y,\tau) f_1(r_1) + g(y,\tau) g_1(r_1) + \varepsilon F^{1/2} h(y,\tau) h_1(r_1)) \tag{4.24}$$
$$+ O(\varepsilon^{3/2}F^{-9/4}).$$

We make the following estimate

$$\|x(\tau) - x_a(\tau)\| \leq \|x(\tau) - y(\tau)\| + \|y(\tau) - x_a(\tau)\|. \tag{4.25}$$

For $\tau \in [\delta_1^{*-1}(\varepsilon), \delta_2^{*-1}(\varepsilon)]$, $u^1(y(\tau), \tau)$ is bounded, so that

$$\|x(\tau) - y(\tau)\| = O(\varepsilon^{1/2}F^{-3/4}) = o(1). \tag{4.26}$$

We estimate $\|y(\tau) - x_a(\tau)\|$ in the same way as $\|z_2(\tau) - x_{2a}(\tau)\|$ in theorem 3.2 and obtain for $\tau \in [\delta_1^{*-1}(\varepsilon), \delta_2^{*-1}(\varepsilon)]$

$$\|y(\tau) - x_a(\tau)\| = O(\varepsilon^{1/2}F^{-3/4}) = o(1). \tag{4.27}$$

This completes the proof of the theorem.

Corollary 4.2 For $0 < \delta_1(\varepsilon) \leq F \leq F_1$ with $\varepsilon^{2/3} = o(\delta_1(\varepsilon))$ and $F_1 = O(1)$ - so for $\delta_3(\varepsilon) \leq \omega_1(F) \leq \delta_4(\varepsilon)$ with $\varepsilon^{1/3} = o(\delta_3(\varepsilon))$ and $\delta_4(\varepsilon) = O(1)$ (see (4.15b)) - the solution of (1.2) has the following expansion

$$x(t) = \sqrt{\frac{F}{2}} + \sqrt{\varepsilon}\, r_a^* e^{\frac{-\kappa\varepsilon t}{2}} \omega_1^{-1/2}(F) \cos(\tau + \psi_a^*) + o(\sqrt{\varepsilon}\, \omega_1^{-1/2}(F)) , \tag{4.28}$$

with (r_a^*, ψ_a^*) the solution of the system

$$\frac{dr_a^*}{d\tau} = 0 \ , \ r_a^*(t_1) = r_0^*, \tag{4.29a}$$

$$\frac{d\psi_a^*}{d\tau} = -\varepsilon \left(\frac{\varepsilon\kappa^2}{16F} + \frac{3e^{-\kappa\varepsilon t}(r_a^*)^2}{4(2F)^{3/2}} - \frac{15e^{-\kappa\varepsilon t}(r_a^*)^2}{4(2F)^{3/2}} \right) , \ \psi_a^*(t_1) = \varphi_0^*. \tag{4.29b}$$

Proof Because (4.13 c,d) is a special case of the initial value problem stated in theorem 4.1, the averaged system (4.29) is an $o(1)$-approximation of the complete system (4.13 c,d) for $\tau \in [\delta_1^{*-1}(\varepsilon), \delta_2^{*-1}(\varepsilon)]$ with $\delta_1^*(\varepsilon) = o(1)$ and $\delta_2^*(\varepsilon) = O(\varepsilon)$. It follows that for the solution of (1.2) approximation (4.28) holds when $0<\delta_1(\varepsilon) \leq F \leq F_1$ with $\varepsilon^{2/3} = o(\delta_1(\varepsilon))$ and $F_1 = O(1)$. In fact, from (4.13b) it follows that

$$\tau = \left(- \frac{1}{3}(2F)^{3/2} + \frac{1}{3}(2F_1)^{3/2}\right) \varepsilon^{-1}. \tag{4.30}$$

On account of the symmetry of the problem we can also obtain a "slowly varying equilibrium solution" x_{-1sv} (F) for $F > F_c$ by perturbing the dependent variable x at $x_{-1}(F) = -\sqrt{(F/2)}$. By reflecting (4.28) with respect to the z-axis an asymptotic expansion of the solution of (1.2) in the neighbourhood of $x_{-1sv}(F)$ is obtained valid for $t \in [t_c + \delta_5^{-1}(\varepsilon), t_c + \delta_6^{-1}(\varepsilon)]$ with $t_c = -F_0\varepsilon^{-1}$, $\delta_5(\varepsilon) = o(\varepsilon^{1/3})$, and $\delta_6(\varepsilon) = O(\varepsilon)$. For $F = F_c = 0$ the stable reduced solutions $x_{\pm1}(F)$ and the unstable one, $x_0(F)$, coalesce. Then approximation (4.28) does not hold anymore. It is remarked that then the angular velocity $\omega_1(F)$ tends to zero.

Finally, we remark that in the case without damping ($\kappa = 0$), (4.7a) is a Hamiltonian system. Bosley and Kevorkian (1992) consider transient resonance in very slowly varying oscillatory Hamiltonian systems for which the leading order frequency of the reduced system makes a continuous slow passage through zero. After the transformations $v = r\cos\varphi$, $dv/dt = -\omega_1(F)r\sin\varphi$ and $p = r^2\omega_1(F)$ we obtain from (4.7)

$$\begin{aligned}\frac{dp}{d\tau} &= \sqrt{\varepsilon}\,(12p^{3/2}\omega_1^{-3/2}(\varepsilon t)x_{1sv}(\varepsilon t)\cos^2\varphi\sin\varphi)\\ &+ \varepsilon(4p^2\omega_1^{-2}(\varepsilon t)\cos^3\varphi\sin\varphi - 2p\omega_1^{-1}(\varepsilon t)\frac{d\omega_1}{d(\varepsilon t)}\sin^2\varphi + p\omega_1^{-1}\frac{d\omega_1}{d(\varepsilon t)})\ ,\end{aligned} \tag{4.31a}$$

$$\begin{aligned}\frac{d\varphi}{dt} &= \omega_1(\varepsilon t) + \sqrt{\varepsilon}\,(6p^{1/2}\omega_1^{-3/2}(\varepsilon t)x_{1sv}(\varepsilon t)\cos^3\varphi)\\ &+ \varepsilon(2p\omega_1^{-2}(\varepsilon t)\cos^4\varphi - \omega_1^{-1}(\varepsilon t)\frac{d\omega_1}{d(\varepsilon t)}\cos\varphi\sin\varphi)\ ,\end{aligned} \tag{4.31b}$$

which is a Hamiltonian system with a Hamiltonian of the following form:

$$\begin{aligned}H &= \omega_1(\varepsilon t)p + \sqrt{\varepsilon}\,(3p^{3/2}\omega_1^{-3/2}(\varepsilon t)x_{1sv}(\varepsilon t)\cos\varphi + p^{3/2}\omega_1^{-3/2}(\varepsilon t)x_{1sv}(\varepsilon t)\cos3\varphi)\\ &+ \varepsilon(\frac{3}{8}p^2\omega_1^{-2}(\varepsilon t) + \frac{1}{2}p^2\omega_1^{-2}(\varepsilon t)\cos2\varphi + \frac{1}{8}p^2\omega_1^{-2}(\varepsilon t)\cos4\varphi).\end{aligned} \tag{4.32}$$

5. The transition layer equation and matching conditions

In order to obtain matching conditions for the local asymptotic solution describing the pitchfork bifurcation we determine the asymptotic development of x when F is in the neighbourhood of $F_c = 0$. Near $F = 0$ the reduced solutions $x_0(F)$ and $x_{\pm1}(F)$ are close to zero. Setting

$$t = -\frac{F_0}{\varepsilon} + \varepsilon^{\nu-1}z \text{ or } F = \varepsilon^{\nu}z \tag{5.1a,b}$$

we obtain the following approximations for the F-dependent terms in the asymptotic expansions (3.31) and (4.28):

$$r_1 = r_1^* = e^{\frac{1}{2}\kappa F_0} + \ldots , \tag{5.2a}$$

$$\omega_0^2(F) = -z\varepsilon^{\nu}, \tag{5.2b}$$

$$x_{1sv}(F) = (\frac{z}{2})^{\frac{1}{2}}\varepsilon^{\frac{1}{2}\nu} + \frac{1}{64}(\frac{z}{2})^{-\frac{5}{2}}\varepsilon^{2-\frac{5}{2}\nu} + \ldots + \frac{\kappa}{16}(\frac{z}{2})^{-\frac{3}{2}}\varepsilon^{2-\frac{3}{2}\nu} + \ldots , \tag{5.2c}$$

$$\omega_1^2(F) = 2z\varepsilon^{\nu} + \frac{3}{16}(\frac{z}{2})^{-2}\varepsilon^{2-2\nu} + \ldots , \tag{5.2d}$$

$$r_a^* = r_0^* , \; r_a = r_0 , \tag{5.2e,f}$$

$$\varphi_a = \frac{2}{3}(-z)^{\frac{3}{2}}\varepsilon^{\frac{3}{2}\nu-1} + \frac{3}{4}r_0^2\, e^{\kappa F_0}(\ln(-z) + \nu\ln\varepsilon) + \psi_0 + \ldots , \tag{5.2g}$$

$$\varphi_a^* = \frac{2\sqrt{2}}{3}(z)^{\frac{3}{2}}\varepsilon^{\frac{3}{2}\nu-1} - \frac{3}{2}(r_0^*)^2e^{\kappa F_0}(\ln(z) + \nu\ln\varepsilon) + \psi_0^* + \ldots , \tag{5.2h}$$

where ψ_0 is a constant determined by the initial conditions. Consequently, near the bifurcation point the outer solutions (3.31) and (4.28) behave asymptotically as

$$x = r_0e^{\frac{\kappa F_0}{2}}(-z)^{-\frac{1}{4}}\varepsilon^{\frac{1}{2}-\frac{1}{4}\nu}\sin\{\frac{2}{3}(-z)^{\frac{3}{2}}\varepsilon^{\frac{3}{2}\nu-1} + \frac{3}{4}r_0^2\, e^{\kappa F_0}(\ln(-z) + \nu\ln\varepsilon) + \psi_0 + \ldots\} + \ldots ,$$

$$\text{if } z \to -\infty, \tag{5.3a}$$

$$\begin{aligned} x = {} & (\frac{z}{2})\varepsilon^{\frac{\nu}{2}} + \frac{1}{64}(\frac{z}{2})^{-\frac{5}{2}}\varepsilon^{2-\frac{5}{2}\nu} + \ldots + \frac{\kappa}{16}(\frac{z}{2})^{-\frac{3}{2}}\varepsilon^{2-\frac{3}{2}\nu} + \ldots \\ & + r_0^*e^{\frac{\kappa F_0}{2}}(2z\varepsilon^{\nu} + \frac{3}{16}(\frac{z}{2})^{-2}\varepsilon^{2-2\nu} + \ldots)^{-\frac{1}{4}}\varepsilon^{\frac{1}{2}}\cos\{\frac{2\sqrt{2}}{3}z^{\frac{3}{2}}\varepsilon^{\frac{3}{2}\nu-1} - \frac{3}{2}(r_0^*)^2e^{\kappa F_0}(\ln(z) + \nu\ln\varepsilon) \\ & + \psi_0^* + \ldots\} + \ldots , \text{ if } z \to \infty \end{aligned} \tag{5.3b}$$

with the dots standing for higher order ε-terms or for terms that are $o(|z|^{-1/4})$. It is remarked that, in contrast with Haberman (1979), the terms $-3r_1^{*2}r^2\omega^{-3}(-F)$ and $-15r_1^{*2}r^{*2}x^2_{1sv}\omega_1^{-3}(F)$ in the integral of (4.13d) are equally important as $\omega_1(F) \to 0$. As we will see both terms contribute to the matching condition for the pitchfork bifurcation, because both terms contribute to the logarithmic terms in (5.3b).

From the expansion (5.3) it is seen that the outer expansion breaks down if $\nu = 2/3$. It implies that the transition layer (inner) equation follows from the scaling

$$x = \varepsilon^{\frac{1}{3}}y(z). \tag{5.4}$$

An other way to obtain this equation is based on the analysis of significant degenerations of the differential equation. We put then

$$F = \varepsilon^{\nu}z \;;\; x = \varepsilon^{\mu}y(z). \tag{5.5}$$

There is a significant degeneration for $\mu = 1/3$, $\nu = 2/3$ of the following form:

$$\frac{d^2y}{dz^2} = yz - 2y^3. \tag{5.6}$$

This equation, a nonlinear extension of the Airy equation, also occurs in a number of problems of quantum field theory and in the theory of nonlinear evolution equations. It is the second Painlevé equation being one of the six canonical Painlevé equations of the form

$$\frac{d^2y}{dz^2} = R(z, y, \frac{dy}{dz}), \tag{5.7}$$

where R is rational in y and dy/dz, and analytic in z. The first integral of these Painlevé equations has no nonstationary critical points (the Painlevé property), see Painlevé (1900), and can not be reduced to linear equations by local transformations (i.e. their first integral can not be expressed in terms of known special functions, see also Levi and Winternitz (1991)).

The correct scaling for the transition layer equation also follows from the time region where according to theorems 3.2 and 4.1 the outer solutions (3.31) and (4.28) are not valid. In sections 3 and 4 we have seen that the averaging procedures can not be applied anymore if $F = O(\varepsilon^{2/3})$ (or $t = -F_0\varepsilon^{-1} + O(\varepsilon^{-1/3})$). In this region a new local approximation, satisfying the significant degeneration (5.6), is constructed and its validity is proven. Its validity domain may be extended backward and forward overlapping the other (outer) approximations, so that integration constants follow from matching with these approximations.

We will first prove that the transition layer solution is a local approximation of the solution of the complete equation. For that purpose we will first prove the following lemma:

Lemma 5.1 The initial-value problem

$$\frac{d^2y_p}{dz^2} = y_p z - 2y_p^3 \ , \ y_p(-M) = a, \ \frac{dy_p}{dz}(-M) = b \tag{5.8}$$

with $M > 0$ arbitrary large, has a bounded real solution for $-M \leq z \leq M$ for any $M > 0$.

Proof From (5.8) it follows that

$$\left(\frac{dy_p}{dz}\right)^2 = zy_p^2 - \int y_p^2 + C_1 - y_p^4 \tag{5.9}$$

with C_1 a constant. Certainly, $(dy_p/dz)^2 \leq My_p^2 + C_2$, with C_2 a positive constant. Thus,

$$\frac{dy_p}{dz} \leq \sqrt{My_p^2 + C_2}. \tag{5.10}$$

So, $\int(y_p^2 + C_3)^{-1/2}dy_p \leq C_4$ with $C_3 \geq 0$ and $C_4 \geq 0$. Finally, we obtain

$$\log|y_p + \sqrt{y_p^2 + C_3}| \leq C_4 \ . \tag{5.11}$$

Thus, y_p and dy_p/dz are bounded □

In fact, it is known (see e.g. Levi and Winternitz (1991)) that the only singularities that any solution of Painlevé II can have for finite z are poles. However, poles cannot occur as a behaviour of the type $y_p(z) \sim c(z - z_0)^n$, $c \in \mathbf{R}$, $n \in \mathbf{Z}$, is excluded in (5.6) because of the sign of the nonlinear term. We can now prove the following approximation theorem:

Theorem 5.2 Consider the system

$$\begin{pmatrix} \dfrac{dx}{dz} \\ \dfrac{dy}{dz} \end{pmatrix} = \begin{pmatrix} y \\ zx - x^3 \end{pmatrix} + \varepsilon^{1/3}\begin{pmatrix} 0 \\ az^2 + bz \end{pmatrix} + \varepsilon^{2/3}\begin{pmatrix} 0 \\ y + cz \end{pmatrix} + \varepsilon\begin{pmatrix} 0 \\ dz^3 \end{pmatrix} \tag{5.12}$$

or

$$\frac{du}{dz} = f_0(z,u) + \varepsilon^{1/3}R(z,u,\varepsilon), \tag{5.13}$$

with $u(-M) = \eta$, $z \in [-M, M]$, $0 < \varepsilon \le \varepsilon_0$, and u a 2-dimensional vector function. Assume

a) $R(z,u,\varepsilon)$ is continuous in z, u and ε and Lipschitz-continuous in u.

b) $f_0(z,u)$ is continuous in z and u and continuously differentiable in z.

Let $u_p = (x_p, y_p)$ be the solution of

$$\frac{du_p}{dz} = f_0(z,u_p),\ u_p(-M) = \eta\ . \tag{5.14}$$

Then, $\|u(z) - u_p(z)\| \le k\varepsilon^{1/3}$, k a constant, for $|z| < M$.

Proof From (5.13) and (5.14) it follows that

$$u(z) - u_p(z) = \int_{-M}^{z}[f_0(\tau,u(\tau)) - f_0(\tau,u_p(\tau))]d\tau + \varepsilon^{1/3}\int_{-M}^{z}R(\tau,u(\tau),\varepsilon)d\tau. \tag{5.15}$$

We have

$$\begin{aligned}\|R(\tau,u(\tau),\varepsilon)\| &\le \|R(\tau,u(\tau),\varepsilon) - R(\tau,u_p(\tau),\varepsilon)\| + \|R(\tau,u_p(\tau),\varepsilon)\| \\ &\le L_1\|u(\tau) - u_p(\tau)\| + M_1\ ,\end{aligned} \tag{5.16}$$

because of the Lipschitz-continuity of R, the boundedness of u_p (see Lemma 5.1), and $z \in [-M, M]$. Using the Lipschitz-continuity of f_0 we now obtain from (5.15) and (5.16)

$$\|u(z) - u_p(z)\| \le \int_{-M}^{z}L\|u(\tau) - u_p(\tau)\|d\tau + \varepsilon^{1/3}\int_{-M}^{z}[L_1\|u(\tau) - u_p(\tau)\| + M_1]d\tau. \tag{5.17}$$

Application of the Lemma of Gronwall yields the inequality

$$\|u(z) - u_p(z)\| \leq \varepsilon^{1/3} \frac{M_1}{L + \varepsilon^{1/3}L_1}\exp(L(z + M)) - \varepsilon^{1/3}\frac{M_1}{L + \varepsilon^{1/3}L_1}. \tag{5.18}$$

So it follows that $u(z) - u_p(z) = O(\varepsilon^{1/3})$ on the time-scale 1. From this it follows that we are allowed to say

$$u(z) = u_p(z) + \varepsilon^{1/3}\varphi(z,\varepsilon) . \tag{5.19}$$

By using the same techniques, integral equations and the Lemma of Gronwall, we can also show that substitution of

$$u^*(z) = u_p(z) + \varepsilon^{1/3}u_1(z) + \varepsilon^{2/3}u_2(z) + \varepsilon u_3(z) \tag{5.20}$$

yields an $O(\varepsilon^{4/3})$-approximation for the solution of (5.13) for $|z| < M$.

From theorem 5.2 the following corollary immediately follows:

Corollary 5.3 The solution of the significant degeneration (5.6) is an $O(\varepsilon^{1/3})$-approximation of the solution of (1.2) for $t \in [-F_0\varepsilon^{-1} - M\varepsilon^{-1/3}, -F_0\varepsilon^{-1} + M\varepsilon^{-1/3}]$ or $F \in [-M\varepsilon^{2/3}, M\varepsilon^{2/3}]$ with M an arbitrary positive constant.

The extension theorem (Eckhaus, 1979) states that (when ε gets smaller) we may extend the interval during which the approximations are valid, possibly at the cost of accuracy. We now give the precise formulation of this theorem.

Theorem 5.4 (extension theorem) Let for $-M \leq z \leq M$ with M arbitrary but fixed and ε-independent,

$$|y_1(z,\varepsilon) - y_2(z,\varepsilon)| = o(1). \tag{5.21}$$

Then order functions $\delta_e(\varepsilon)$, $\delta_{1e}(\varepsilon) = o(1)$ exist such that

$$|y_1(z,\varepsilon) - y_2(z,\varepsilon)| = O(\delta_{1e}) \tag{5.22}$$

for $-\delta_e^{-1}(\varepsilon) \leq z \leq \delta_e^{-1}(\varepsilon)$.

For a discussion and proof of the extension theorem we refer to Eckhaus (1979). From the approximation theorem 5.2 and the extension theorem it follows that the domain of validity of the local Painlevé approximation can be extended forward and backward to $F \in [-\varepsilon^{2/3}\delta_e^{-1}(\varepsilon), \varepsilon^{2/3}\delta_e^{-1}(\varepsilon)]$ with $\delta_e(\varepsilon) = o(1)$. Thus overlap with the domains, where outer approximations are valid, is ensured and the integration constants can be matched.
At the time that a pitchfork bifurcation is expected eq. (5.6) holds. Its solution must match the outer solution as given by (5.3b) with $\nu = 2/3$:

$$y \sim \left(\frac{z}{2}\right)^{\frac{1}{2}} + r_0^* e^{\frac{\kappa F_0}{2}}(2z)^{-1/4}\cos\left\{\frac{1}{3}(2z)^{\frac{3}{2}} - \frac{3}{2}(r_0^*)^2 e^{\kappa F_0}\ln(z) + \xi_0^*\right\} + o(z^{-1/4}) \text{ as } z \to \infty, \tag{5.23}$$

with ξ_0^* depending on the initial conditions and on $\ln(\varepsilon)$. This is the parabolic matching condition after passage of the bifurcation point when the solution remains near the stable outer solution $x = x_1$. When the solution remains near the other stable outer equilibrium solution $x = x_{-1}$ this matching condition is obtained by reflecting (5.23) with respect to the z-axis. The asymptotic condition is automatically fulfilled by (5.6).

5.1 Two special solutions of the second Painlevé transcendent

When $r_0^* = 0$ we obtain a specific solution of the Painlevé transcendent: the one that matches (5.3b) with $r_0^* = 0$ and that reflects the asymptotic behaviour of the slowly varying equilibrium solution $x_{1sv}(F)$ when the transition layer is approached. It is noted that (5.3b) is then independent of ε (and ψ_0^*). For $z >> 1$ this solution has an asymptotic series of the form

$$y(z) = \sum_{n=1}^{\infty} c_n\left(\frac{z}{2}\right)^{\frac{1}{2} - 3(n-1)}, \tag{5.24}$$

with c_n satisfying a recurrence relation:

$$c_1 = 1,\ c_2 = \frac{1}{64} \tag{5.25a}$$

and for $n \geq 3$

$$\frac{1}{4}c_n\left(\frac{7}{2} - 3n\right)\left(\frac{5}{2} - 3n\right) = \begin{cases} 2c_{n+1} - \sum\limits_{i\geq 1,\, j\geq 1,\, i+2j=n+3} 6c_ic_j^2 - 2c_{(n+3)/3}\,, & n \equiv 0 \ (\mathrm{mod}3) \\ 2c_{n+1} - \sum\limits_{i\geq 1,\, j\geq 1,\, i+2j=n+3} 6c_ic_j^2\,, & n \not\equiv 0 \ (\mathrm{mod}3) \end{cases}$$

(5.25b)

This solution is illustrated in figure 4.

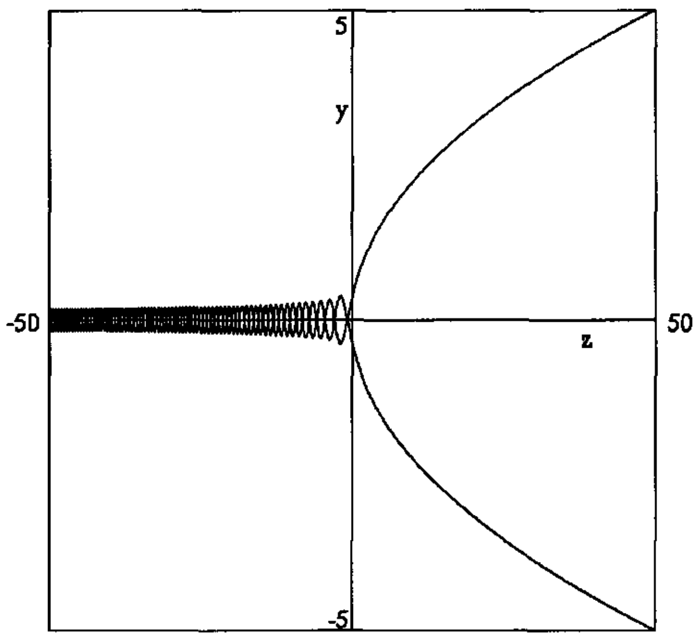


Figure 4 *Numerical approximation of the solutions of (5.6) that match (5.3b) with $r_0^* = 0$ or match with the reflection of (5.3b) with respect to the z-axis with $r_0^* = 0$.*

In the same way, the solution of (5.6) must match the outer solution as given by (5.3a) with $\nu = 2/3$:

$$y \sim r_0 e^{\frac{\kappa F_0}{2}}(-z)^{-1/4} \sin \left\{\frac{2}{3}(-z)^{\frac{3}{2}} + \frac{3}{4}r_0^2 e^{\kappa F_0}\ln(-z) + \zeta_0\right\} + o((-z)^{1/4}) \text{ as } z \to -\infty \,.$$

(5.26)

Again, the asymptotic condition is automatically fulfilled by (5.6). In fact, the equation is satisfied by a series, which has the form (5.26) for $z \to -\infty$:

$$y(z) = \sum_{n=1}^{\infty} \{a_{2n-1,2n-1}\sin((2n-1)(\psi+\psi_0))$$
$$+ \sum_{m=1}^{n-1} [b_{2m-1,2n-1}\sin((2m-1)(\psi+\psi_0)) + c_{2m-1,2n-1}\cos((2m-1)(\psi+\psi_0))]\}(-z)^{-\frac{6n+5}{4}} \tag{5.27}$$

with

$$\psi = \frac{2}{3}(-z)^{\frac{3}{2}} + \frac{3}{4}a_{1,1}^2\ln(-z)\ . \tag{5.28}$$

The two arbitrary parameters are $a_{1,1}$ and ψ_0; the other parameters can be obtained from $a_{1,1}$ by a recurrent relation. In figure 5 we illustrate the various analytical approximations of sections 3, 4 and 5 with their domain of validity. As we have shown the inner and outer approximations overlap. In the next section we will show the connection between the parameters $(r_0 e^{\frac{-\kappa F_0}{2}}, \zeta_0)$ and $(r_0^* e^{\frac{-\kappa F_0}{2}}, \zeta_0^*)$.

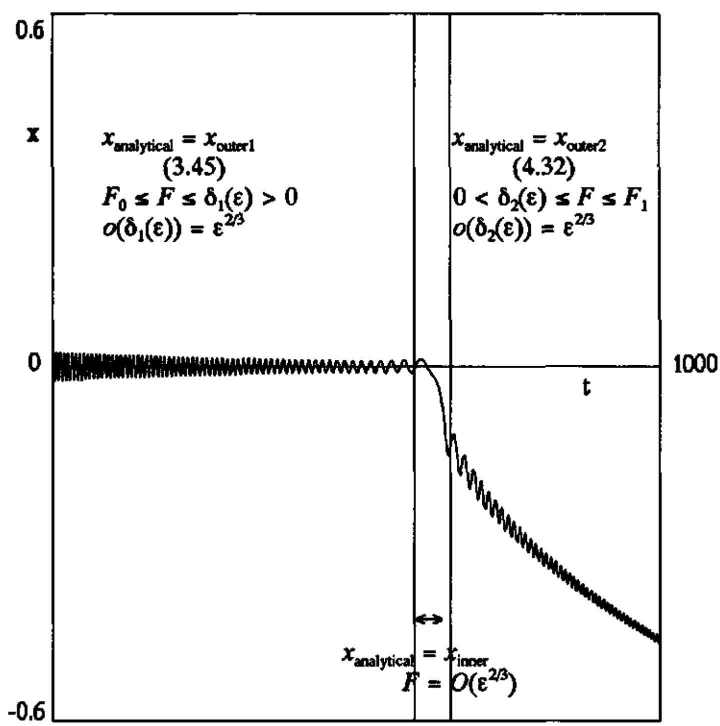


Figure 5 *The composite analytical solution: $x_{analytical} = x_{outer} + x_{inner} - x_{match}$ for (1.2) with $x(0) = 0.04$, $x'(0) = 0$, $F(0) = -1$, $k = 0.005$ and $\varepsilon = 0.0016$.*

6. The prediction of the behaviour of solutions after passing the bifurcation point at the basis of the initial values

If the state of a mechanical system at a certain moment is known, we wish to predict its future behaviour. As we have illustrated in figure 6 most solutions of (1.2) will grow polynomially after passing the pitchfork bifurcation point. From the initial values of the original system (1.2) with $F(0) = F_0 < F_c = 0$, it will be deduced which of the two stable branches will be followed after passage of the bifurcation point.

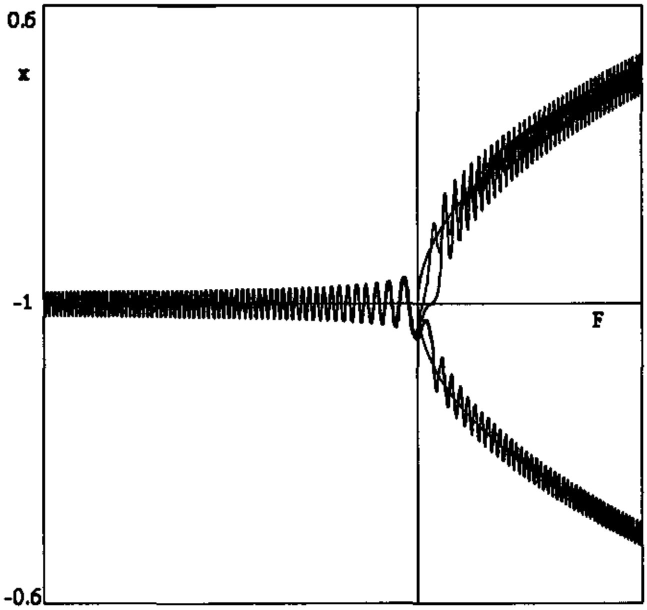


Figure 6 *Solutions of eq. (1.2) for various sets of initial values with $F(0) = -1 < 0$, $\varepsilon = 0.0025$ and $k = 0$.*

In section 5 we derived the following asymptotic behaviour of $y(z)$ satisfying $d^2y/dz^2 = yz - 2y^3$ for $z << -1$

$$y(z) = \gamma(-z)^{-1/4} \sin \{\frac{2}{3}(-z)^{\frac{3}{2}} + \frac{3}{4}\gamma^2 \ln(-z) + \zeta_0\} + o((-z)^{-1/4}) , \tag{6.1a}$$

where

$$\gamma = r_0 e^{\frac{\kappa F_0}{2}}, \tag{6.1b}$$

$$F_0 < F_c = 0 \ , \ \omega_0(F) = \sqrt{-F} , \tag{6.1c}$$

$$r_0 = \sqrt{\frac{\omega_0(F_0)x^2(0)}{\varepsilon} + \frac{(\frac{dx}{dt}(0))^2}{\varepsilon\omega_0(F_0)}} \ , \tag{6.1d}$$

$$\zeta_0 = \frac{-2}{3\varepsilon}(-F_0)^{3/2} - \frac{3}{4}r_0^2\ln|F_0| + \frac{1}{2}\gamma^2\ln\varepsilon + \frac{3}{4}\kappa e^{\kappa F_0}r_0^2\int_{F_0}^{0} e^{-\kappa F}\ln|F|dF - \varphi_0 + \pi, \tag{6.1e}$$

and for $z >> 1$

$$y(z) = \pm\sqrt{\frac{z}{2}} \pm \beta(2z)^{-1/4} \cos\ \{\frac{2\sqrt{2}}{3}z^{3/2} - \frac{3}{2}\beta^2\ln(z) + \zeta_0^*\} + o(z^{-1/4})\ , \tag{6.2a}$$

where

$$\beta = r_0^* e^{\frac{\kappa F_0}{2}}\ , \tag{6.2b}$$

$$F_1 > F_c = 0\ ,\ \omega_1(F) \sim \sqrt{2F}\ \text{ for } \varepsilon^{2/3} = o(F)\ , \tag{6.2c}$$

$$r_0^* = \sqrt{\frac{\omega_1(F_1)x^2(t_1)}{\varepsilon} + \frac{(\frac{dx}{dt}(t_1))^2}{\varepsilon\omega_1(F_1)}}\ , \tag{6.2d}$$

$$\zeta_0^* = \frac{-2\sqrt{2}}{3\varepsilon}(F_1)^{3/2} + \frac{3}{2}\beta^2 e^{-\kappa F_1}\ln(F_1) - \beta^2\ln\varepsilon - \frac{3}{2}\kappa\beta^2\int_{F_1}^{0} e^{-\kappa F}\ln|F|dF + \varphi_0^*\ . \tag{6.2e}$$

The properties of the real solutions of (5.6) follow from the properties of the general pure imaginary solutions of

$$\frac{d^2u}{ds^2} - su - 2u^3 = 0. \tag{6.3}$$

In Its, Fokas and Kapaev (1994) an important result concerning this differential equation is stated; there is a connection between the behaviour of solutions for $s \to -\infty$ with the behaviour for $s \to \infty$. Moreover, they discuss the connection between a method of Deift and Zhou (1993), who obtain a rigorous derivation of the connection problem for the second Painlevé equation, and an approach which was proposed by Kitaev (1989). We now give the formulation of the result.

Theorem 6.1 Let $u(s)$ be an arbitrary solution of eq. (6.3). Then the following assertions hold for $u(s)$:

a) $u(s)$ is smooth for every $s \in \mathbb{R}$ and has the following asymptotics as $s \to -\infty$:

$$u(s) = i\alpha(-s)^{-1/4} \sin \{\frac{2}{3}(-s)^{\frac{3}{2}} + \frac{3}{4}\alpha^2 \ln(-s) + \varphi\} + o((-s)^{-1/4}), \tag{6.4}$$

where the numbers $\alpha > 0$ and $0 \leq \varphi < 2\pi$ may be arbitrary and are parameters of the solution $u(s)$.

b) If the parameters α and φ of the solution $u(s)$ are connected by the relation

$$\varphi = \frac{3}{2}\alpha^2 \ln 2 - \frac{\pi}{4} - \arg\Gamma(i\frac{\alpha^2}{2}) + \varepsilon\pi \ (\mathrm{mod} 2\pi) \ , \ \varepsilon = 0,1, \tag{6.5}$$

then as $s \to +\infty$ the solution $u(s)$ decreases exponentially:

$$u(s) = \frac{ia}{2\sqrt{\pi}} s^{-1/4} \exp(-2s^{3/2}/3)(1 + o(1)) \ , \tag{6.6}$$

where $a^2 = \exp(\pi\alpha^2) - 1$ and $\operatorname{sgn} a = 2(\frac{1}{2} - \varepsilon)$.

c) If (6.5) fails to hold (general position), then as $s \to +\infty$ the solution $u(s)$ grows polynomially:

$$u(s) = \pm i\frac{\sqrt{s}}{2} \pm i(2s)^{-1/4} \rho \cos\{\frac{2\sqrt{2}}{3} s^{3/2} - \frac{3}{2}\rho^2 \ln s + \theta\} + o(s^{-1/4}). \tag{6.7}$$

d) In the asymptotics (6.7) all values of $\rho > 0$ and $0 \leq \theta < 2\pi$ are possible; these quantities characterize the solution $u(s)$ uniquely. The parameters ρ, θ and the choice of the sign in (6.7) are explicitly determined from the parameters α and φ:

$$\rho^2 = \frac{1}{\pi} \ln \frac{1 + |p|^2}{2 - |\mathrm{Im} p|} \ , \tag{6.8a}$$

$$\theta = -\frac{3\pi}{4} - \frac{7}{2}\rho^2\ln 2 + \arg\Gamma(i\rho^2) + \arg(1 + p^2), \tag{6.8b}$$

where

$$p = (\exp(\pi\alpha^2) - 1)^{1/2}\exp\{i\frac{3}{2}\alpha^2\ln 2 - i\frac{\pi}{4} - i\arg\Gamma(\frac{i\alpha^2}{2}) - i\varphi\} \tag{6.8c}$$

and the upper sign in (6.7) is taken if Im$p < 0$.

For a proof of this theorem we refer to Its, Fokas and Kapaev (1994). The result is important for the analysis of a large class of physical problems. It confirms the asymptotic results we obtained for the real solutions of (5.6). Moreover, it connects the integration constants in the asymptotic solution for $z \to -\infty$ with those in the one for $z \to +\infty$. Furthermore, "separating" solutions, that follow the unstable branch beyond the bifurcation point, are singled out. A proof of the completeness of the asymptotic description of the solution of the differential equation is given in Its, Fokas and Kapaev (1994). In order to prove this theorem the method of isomonodromy deformations as formulated by Flaschka and Newell (1980) has been used. This method, as well as Laplace's method in the linear theory, allows us to compute explicitly connection formulas for Painlevé equations.

With the aid of theorem 6.1 and (6.1) we can describe all solutions of (1.2) depending on the initial values and the value of ε; we can predict the branch that is followed after passing the bifurcation point as well as the type of behaviour the solution exhibits. In terms of our system we have obtained in (6.5) an "angle-amplitude relation" for a solution of (1.2) that separates the solutions following the stable upper branch from the ones that take the stable lower branch after passage of the bifurcation point. The quantity φ, stated in (6.5), is discontinuous in $\alpha = 0$. However, this is the case when we will always stay on the equilibrium solution $x = 0$ of (1.2) that becomes unstable for $t > 0$. For α small the quantity φ tends to $\pi/4 + \varepsilon\pi$, $\varepsilon = 0,1$. In the case of a system with damping ($k \neq 0$) the amplitude of the oscillation around zero is small. In that case solutions will follow the stable upper branch when the phase ζ_0 (see (6.1a)) is approximately in the interval $<\pi/4 + 2n\pi, 5\pi/4 + 2n\pi>$, $n \in \mathbb{N}$. In figure 7 the separating solution has been sketched in the γ-ζ_0 plane. Numerical experiments confirm these results. Furthermore, in figure 8, we have illustrated in the phase plane of the original system (1.2) which branch solutions will follow after passing the bifurcation point for fixed ε and F_0 depending on the initial values.

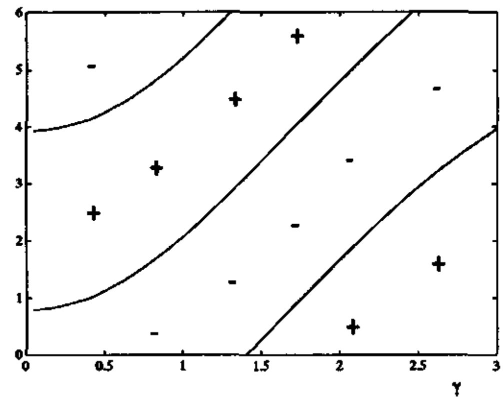


Figure 7 *The branch followed after passage of the bifurcation point depending on γ and ζ_0. A '+' denotes the stable upper branch and a '-' the stable lower branch. In the separating case that is represented by the solid lines, solutions of (1.2) will approach the unstable branch beyond the bifurcation point.*

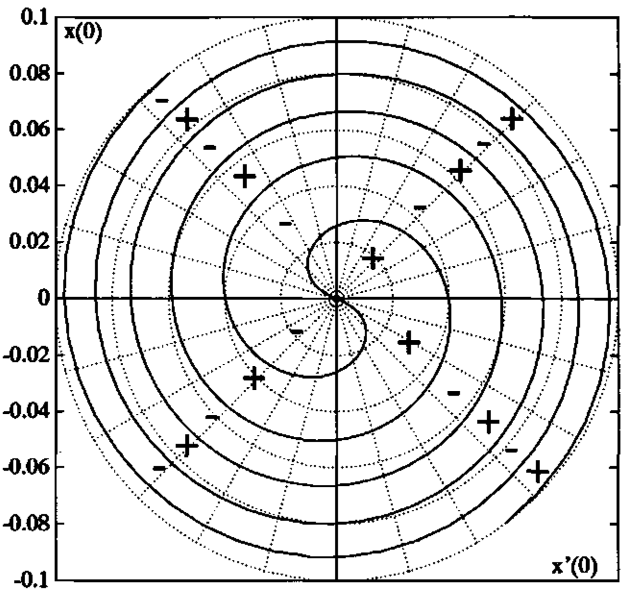


Figure 8 *The branch that will be approached sketched in the $x'(0)$-$x(0)$ plane of (1.2) for $\varepsilon = 0.0025$, $F_0 = -1$ and $k = 0$. The solid lines reflect approximations of the initial values for which the unstable branch will be approached after bifurcation. A '+' denotes that the stable upper branch will be followed after bifurcation, a '-' denotes the stable lower branch.*

7. The general case of second order pitchfork bifurcations with damping

We consider the class of mechanical problems that can be described by the second order nonlinear differential equation

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} = G(x, F) \tag{7.1}$$

from which the parameter F slowly varies in time: $F = F(\varepsilon t)$. The damping is assumed to be small: $k = \kappa\varepsilon$. Solution curves of (7.1) are defined on a time-scale $O(1/\varepsilon)$ if the following condition is satisfied:

$$\int_{-\frac{M}{\varepsilon}}^{\frac{M}{\varepsilon}} G(x, \varepsilon t)\, dt \leq M_1 x^2 + M_2 \quad \text{with } M, M_1, M_2\ \varepsilon\text{-independent positive constants.} \tag{7.2}$$

For fixed F the linear stability of an equilibrium solution $x_E(F)$ is determined by the linearization of (7.1). A critical value F_c of F occurs if

$$\frac{\partial G}{\partial x}(x_E(F_c), F_c) = 0. \tag{7.3}$$

While $\dfrac{\partial^2 G}{\partial x \partial F} \neq 0$ this value separates stable from unstable solutions. In this study we have that at the critical value also

$$\frac{\partial G}{\partial F}(x_E(F_c), F_c) = 0\ . \tag{7.4}$$

In the neighbourhood of $x = x_E(F_c)$, $F = F_c$ we assume G to have the following form:

$$\begin{aligned} G(x,F) &= \alpha_{30}((x - x_E(F_c)) - \beta_1(F - F_c) + \ldots)((x - x_E(F_c))^2 - \sigma^2(F - F_c) + \ldots) \\ &= \alpha_{11}(x - x_E(F_c))(F - F_c) + \alpha_{02}(F - F_c)^2 + \alpha_{30}(x - x_E(F_c))^3 \\ &\quad + \alpha_{21}(x - x_E(F_c))^2(F - F_c) + \alpha_{12}(x - x_E(F_c))(F - F_c)^2 + \alpha_{03}(F - F_c)^3 + \ldots , \end{aligned}$$

$$\text{where } \alpha_{nm} = \frac{1}{n!m!}\left(\frac{\partial}{\partial x}\right)^n\left(\frac{\partial}{\partial F}\right)^m G(x_E(F_c),F_c). \tag{7.5}$$

This phenomenon has been called a pitchfork bifurcation. This case is characterized by $\alpha_{20} = 0$. Note that $\alpha_{30} < 0$ (so that $G(x, F) < 0$ for x sufficiently large) and $\alpha_{11} > 0$ (so that a parabolic curve exists for $F > F_c$). In this study $\alpha_{21} = -\beta_1\alpha_{30}$ and $\alpha_{11} = -\sigma^2\alpha_{30} > 0$. The moment of pitchfork bifurcation is approximated by t_c satisfying

$$F(\varepsilon t_c) = F_c \; . \tag{7.6}$$

In the same way as in sections 3 and 4 we can obtain an asymptotic expansion for a slowly varying equilibrium solution of (7.1) and consider $\sqrt{\varepsilon}$-perturbations of this solution. We obtain a slowly varying oscillator approaching a turning point as the frequency $\omega(F)$ tends to zero for $F \to F_c$. We will see that two different cases must be analyzed; the transition from the parabolic arc to the straight line curve as F decreases through F_c and the opposite case which occurs as F is increased through F_c. With the aid of averaging techniques we obtain asymptotic expansions of the solutions in the neighbourhood of the slowly varying equilibria. The expansions break down when $\varepsilon t = O(\varepsilon^{2/3})$. Matching implies that the local inner equation follows from the scaling:

$$x = x_E(F_c) + \varepsilon^{1/3}y(z),\; F = F_c + \varepsilon^{2/3}z \; . \tag{7.7}$$

Making these scale changes we find that (7.1) transforms into:

$$\frac{d^2y}{dz^2} = \alpha_{11}F_c'zy + \alpha_{30}y^3 + O(\varepsilon^{1/6}) \; , \tag{7.8}$$

with

$$F_c' = \frac{dF}{d(\varepsilon t)}\,(\varepsilon t_c) \; . \tag{7.9}$$

This equation must be solved with the matching conditions following from the asymptotic expansions for the outer solutions. We obtain the following asymptotic conditions:

For $F \uparrow F_c$,

$$\omega^2(F) \to \sigma^2\alpha_{30}(F - F_c)F_c', \tag{7.10a}$$

$$w \sim (-\sigma^2\alpha_{30}F_c')^{-1/4}\rho_0(-z)^{-1/4}\sin\{\frac{2}{3}(\alpha_{11}F_c')^{1/2}(-z)^{3/2} + \frac{3\rho_0^2}{8\sigma^2F_c'}\ln(-z) + \psi_0\} \text{ as } z \to -\infty \tag{7.10b}$$

and for $F \downarrow F_c$,

$$\omega^2(F) \to -2\sigma^2\alpha_{30}(F - F_c)F_c' , \tag{7.11a}$$

$$w \sim \left(\frac{-\alpha_{11}F_c'}{\alpha_{30}}z\right)^{1/2} + (-2\sigma^2\alpha_{30}F_c')^{-1/4}\rho_0^* z^{-1/4} \cos\{\frac{2\sqrt{2}}{3}(\alpha_{11}F_c')^{1/2}z^{3/2} - \frac{3(\rho_0^*)^2}{4\sigma^2F_c'}\ln(z) + \psi_0^*\}$$

$$\text{as } z \to +\infty \tag{7.11b}$$

with ρ_0, ρ_0^*, ψ_0 and ψ_0^* depending on the initial values, on the parameter ε, and on t_c, the bifurcation moment.

In the region in which $F = O(\varepsilon^{2/3})$ a new local approximation is constructed obeying the significant degeneration

$$\frac{d^2y}{dz^2} = \alpha_{11}F_c'zy + \alpha_{30}y^3 \tag{7.12}$$

and its validity is proven in the same way as in section 5. Again the validity domain can be extended forward and backward using the extension theorem of Eckhaus (1979). Overlap with the outer approximations is ensured so that integration constants follow from matching with these approximations. With the result of Its, Fokas and Kapaev (1994) the parameters $(\alpha, \varphi) = ((2\sigma^2F_c')^{-1/2}\rho_0, \psi_0)$ and $(\rho, \theta) = ((2\sigma^2F_c')\rho_0^*, \psi_0^*)$ are connected. In fact, using the transformations

$$y = i\left(\frac{-2}{\alpha_{30}}\right)^{1/2}(\alpha_{11}F_c')^{1/3}w,\ z = (\alpha_{11}F_c')^{-1/3}s, \tag{7.13}$$

we obtain

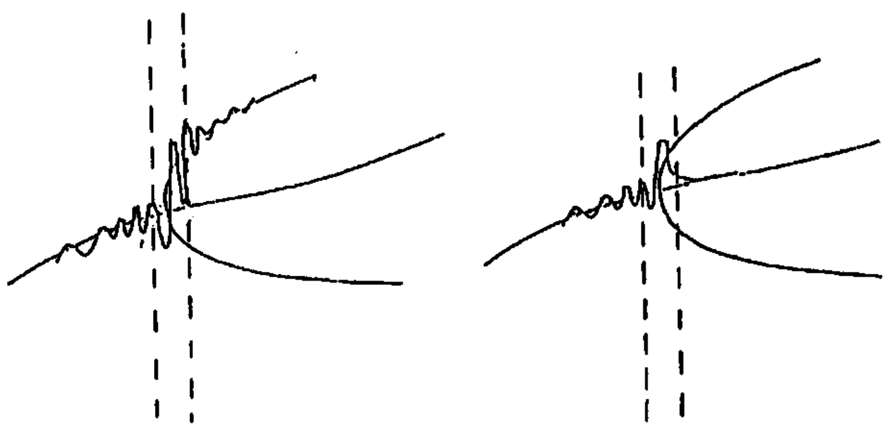
$$\frac{d^2w}{ds^2} = sw + 2w^3. \tag{7.14}$$

In this way the asymptotic validity of formal expansions of (7.1) is proven. Moreover, if the initial values and the small parameter ε are known, we can predict the branch approached and the behaviour of the solution after bifurcation. We emphasize that, in contrast with Haberman (1979), the terms

$-\frac{1}{4}\rho^2\omega^{-1}(F)\frac{\partial^3\sigma}{\partial x^3}|_{x_{sve}}$ and $-\frac{5}{12}\rho^2\omega^{-3}(F)\left(\frac{\partial^2\sigma}{\partial x^2}|_{x_{sve}}\right)^2$ which appear after averaging bifurcation, we have $|5\omega^{-2}(F)(\partial^2\sigma/\partial x^2|_{x=ve})^2| << |3\partial^3 G/\partial x^3|_{x=ve}|$ for $F \uparrow F_c$. This is the reason that we have to apply second order averaging to approximate the solution beyond the bifurcation point, whereas first order averaging satisfies to approximate the solution before this point.

8. Conclusions

In this paper we have analyzed a second order bifurcating system with a slowly varying parameter. We considered the case of a pitchfork bifurcation for which the leading order transition layer equation is the second Painlevé transcendent. Solutions of this equation either exponentially decay, corresponding to the transition to the unstable slowly varying equilibrium beyond the bifurcation point, or algebraically grow, corresponding to the transition from the equilibrium that turned unstable to one or the other of the two stable branches of the parabolic curve. The transitions are illustrated in figure 9.



(9a) (9b)

Figure 9 *For a second order pitchfork bifurcation the second Painlevé transcendent provides the transition between two stable slowly varying equilibria (9a) or the rare transition to the unstable state (9b).*

The solution of the problem we formulated is approximated asymptotically using perturbation techniques. The use of both averaging and boundary layer methods turned out to be necessary. The aim of this study was to predict which branch would be followed after passage of the bifurcation point given the initial state of the system. With the aid of averaging methods the dynamics on the large time-scale is described. Afterwards, the validity of the asymptotic approximation is investigated. It is proven that the different local solutions overlap. An analytical study of the transition layer equation produces the required information on the matching of the different locally valid asymptotic approximations. Its, Fokas and Kapaev (1994) show that there is a connection between the constants of the slowly oscillating solutions that are valid before and after passage of the bifurcation point. Using this result we can predict which stable branch of the parabolic equilibrium curve will be followed by the solution after passage of the bifurcation point given the initial values "far away" from the bifurcation moment. Moreover, the behaviour of the solution beyond the bifurcation point can again be described by averaging methods. The bifurcation takes place on a relatively small time-scale of length $\varepsilon t = O(\varepsilon^{2/3})$. In this study the slow increase of the parameter depends only on time. The influence of the state variable on the change of the slowly varying parameter will be a goal for further research.

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Chapter 4

Slow periodic crossing of a pitchfork bifurcation in an oscillating system[3]

Abstract

A study is made of the dynamics of oscillating systems with a slowly varying parameter. A slowly varying forcing periodically crosses a critical value corresponding to a pitchfork bifurcation. The instantaneous phase portrait exhibits a centre when the forcing does not exceed the critical value, and a saddle and two centres with an associated double homoclinic loop separatrix beyond this value. The aim of this study is to construct a Poincaré map in order to describe the dynamics of the system as it repeatedly crosses the bifurcation point. For that purpose averaging methods and asymptotic matching techniques connecting local solutions are applied. Given the initial state and the values of the parameters the properties of the Poincaré map can be studied. Both sensitive dependence on initial conditions and (quasi) periodicity are observed. Moreover, Lyapunov exponents are computed. The asymptotic expressions for the Poincaré map are compared with numerical solutions of the full system that includes small damping.

1. Introduction

In this study we consider the dynamics of a class of second order differential equations that describe damped oscillating systems with a slowly varying forcing function. The slow forcing periodically crosses a critical value. The phase portrait of the system changes qualitatively with time. Below the critical value this system exhibits a centre point in case the damping is neglected. On the other side of the bifurcation point a symmetric pair of centre points appears, while the original centre point has changed into a saddle point with an associated double homoclinic loop separatrix. In the reversed case we have a subcritical pitchfork bifurcation (see also Guckenheimer and Holmes (1983)). The local transition behaviour is described by the second Painlevé transcendent. A prototype of system exhibiting this kind of behaviour is the nonlinear Mathieu equation. We will consider this system in which

[3] by G.J.M. Marée; submitted to *Nonlinear Dynamics*

a small damping is allowed (turning the centre points into stable spiral points):

$$\frac{d^2x}{dt^2} + k\varepsilon\frac{dx}{dt} - x\cos(\varepsilon t) + 2x^3 = 0, \; 0 < \varepsilon << 1. \tag{1.1}$$

The bifurcation parameter slowly varies and periodically crosses a critical value corresponding to a pitchfork bifurcation. During half the forcing cycle the separatrices slowly grow and then slowly shrink, which implies that the two stable symmetric branches first turn away from the unstable equilibrium and then return until they disappear. At that moment the unstable equilibrium becomes stable.

The aim of this study is to construct a Poincaré map for one forcing period. This map approximates the dynamics of system (1.1) by monitoring the state of a motion once per forcing cycle. It is possible to predict which stable branch will be followed after each passage of the bifurcation point on a time-scale $O(1/\varepsilon)$, given the initial state and the values of the parameters. The initial values and the damping are chosen such that a motion outside the homoclinic separatrix loop does not occur. For the construction of the Poincaré map averaging methods and asymptotic matching techniques connecting local solutions are applied. Moreover, we use an important result concerning the second Painlevé equation stating that there is a connection between the integration constants of the asymptotic solution of the transition equation below criticality with the one above criticality (Its, Fokas and Kapaev, 1994).

The Poincaré map shows a complex dynamical behaviour. Depending on the initial state and the values of the parameters (quasi) periodic and chaotic orbits can be observed. We analyze the Lyapunov exponents that describe the structure of the "attractor" of the orbits. If one of these exponents is positive then the system has a strange attractor. The fractal dimension of this strange attractor follows from the conjecture of Kaplan and Yorke, see e.g. Lichtenberg and Lieberman (1983). Chaos in the system means that the sequence of selected lower and upper branches is irregular and exhibits sensitive dependence on initial conditions. Since the separatrix periodically disappears, Melnikov's method can not be applied to this investigation.

In Bridge and Rand (1992) and Coppola and Rand (1990b) the same system without damping is studied. It is shown that this model is qualitatively equivalent to familiar mechanical systems such as a dynamically buckled beam or a rotating plane pendulum. They obtain an asymptotic approximation by elliptic averaging and connect this solution to a separatrix boundary layer solution by point matching (patching). By introduction of a transition solution (the second Painlevé transcendent) the asymptotic matching technique as described by Eckhaus (1979) can be applied. Our approach is extended to dissipative systems. Moreover, we prove the validity of the matched asymptotic approximations on a time-scale $O(1/\varepsilon)$, and therefore also the validity of the approximating Poincaré map.

In section 2 we formulate the model equation and describe the qualitative nature of its solution. In section 3 the solution is approximated for intervals in which the solution is sufficiently bounded away above or below the bifurcation point. Moreover, the transition layer equation is analyzed and matching conditions for this local asymptotic solution are derived. In section 4 we apply these matching conditions to the local asymptotic expansions and show that there is an overlap between the local (inner) solution of the second Painlevé equation and the other (outer) approximations. With this information a Poincaré map for one period of the forcing is constructed in section 5. This map facilitates the analysis of the complex dynamics of the original system. In section 6 we predict the behaviour of the system at the basis of the initial values and the parameter values. In section 7 results, obtained from the analytical expressions for the map, are compared with the numerical solution of the full system.

2. The system and its qualitative behaviour

We consider two examples of mechanical systems that exhibit the dynamical behaviour as described by eq. (1.1). First we discuss the pendulum of figure 1a. It is attached to a rotating rigid frame and its deflection is measured by the angle θ with respect to the vertical.

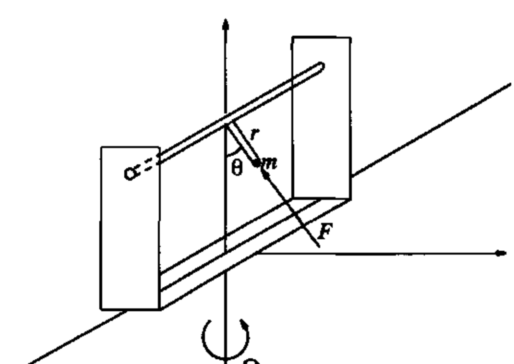


Fig.1a Simple pendulum attached to a rotating rigid frame.

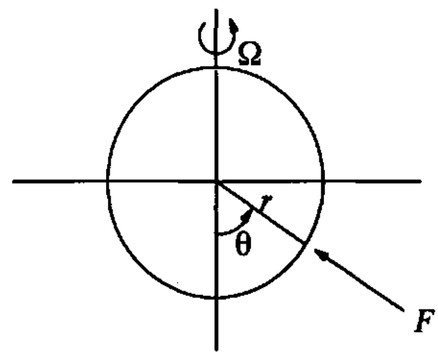


Fig. 1b A particle moving on a smooth, rotating circular wire.

This rigid frame is forced to rotate about a vertical axis at an angular velocity $\Omega(t)$. While the frame rotates, the simple pendulum oscillates. We assume that the pendulum consists of a mass m attached to a hinged weightless rod of length r. The forces acting on the mass m are the centrifugal force $m\Omega^2 r\sin\theta$, the gravitational force mg, and the reaction force F. We take moments about the centre of the circle along which the mass moves and equate their sum to the rate of change of the angular momentum of the particle about this centre. We obtain the following equation for the system without friction:

$$\frac{d^2\theta}{d\eta^2} + (1 - \Lambda\cos\theta)\sin\theta = 0 \tag{2.1}$$

where $\Lambda = \Omega^2((g/r)^{-1/2}\eta)r/g$ and the new independent variable is $\eta = (g/r)^{1/2}t$. For fixed Λ this system possesses one equilibrium if Λ is smaller than 1 and three equilibria if Λ is larger than 1. When we use the following modulations

$$\Lambda = 1 + \varepsilon\cos(\varepsilon^{3/2}\eta),\ \theta = 2\varepsilon^{1/2}x,\ \eta = \varepsilon^{-1/2}s \tag{2.2}$$

and expand for small ε, then eq. (2.1) becomes

$$\frac{d^2x}{ds^2} - x\cos(\varepsilon s) + 2x^3 + \dots = 0, \tag{2.3}$$

where the dots represent terms of order ε. When we omit these higher order terms and assume a small damping of order ε in the original system (2.1) we obtain an equation of the form (1.1). The dynamics of this pendulum is equivalent to that of the motion of a particle on a smooth, rotating circular wire, as shown in figure 1b.

In system (1.1) we define

$$G(t) = \cos(\varepsilon t). \tag{2.4}$$

For G fixed and smaller than the critical value

$$G_c = 0 \tag{2.5}$$

the system without friction contains one centre at the origin. When $G = G_c$ this unique equilibrium is still stable. For $G > G_c$ it becomes an unstable point of saddle point type. Moreover, the system then exhibits two centres at $(\pm\sqrt{(G/2)}, 0)$ as well as a double homoclinic loop separatrix. The transition from a stable line to a parabolic curve as G increases through G_c, is called a supercritical pitchfork bifurcation, whereas the transition in the reverse direction is a subcritical pitchfork bifurcation (see also Guckenheimer and Holmes (1983)). For G fixed the energy integral of system (1.1) without friction is given by the Hamiltonian

$$H = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \frac{1}{2}Gx^2 + \frac{1}{2}x^4\ . \tag{2.6}$$

In figures 2 and 3 the pitchfork bifurcation is illustrated.

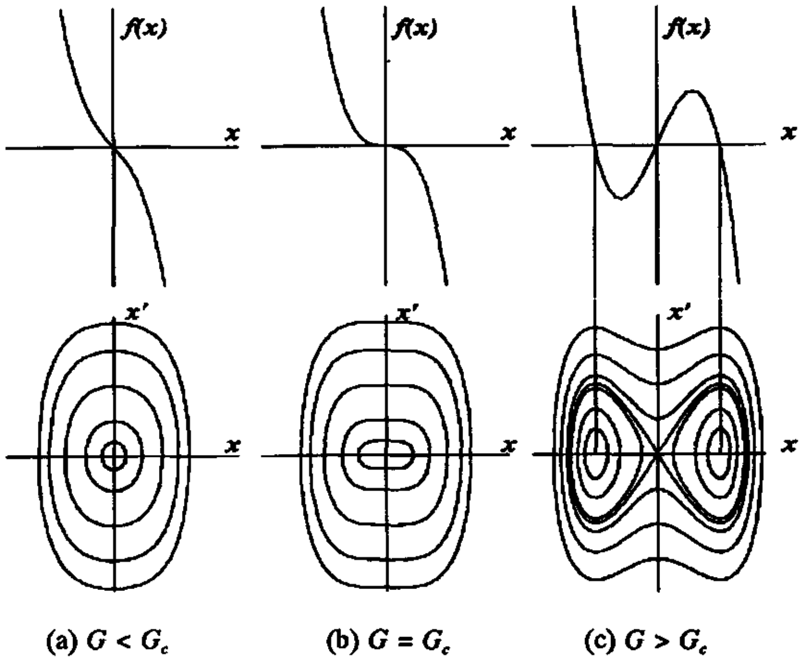


Figure 2 *The graph of the function $f(x) = x(G - 2x^2)$ and the phase portrait of $d^2x/dt^2 - x(G - 2x^2) = 0$ for different values of G.*

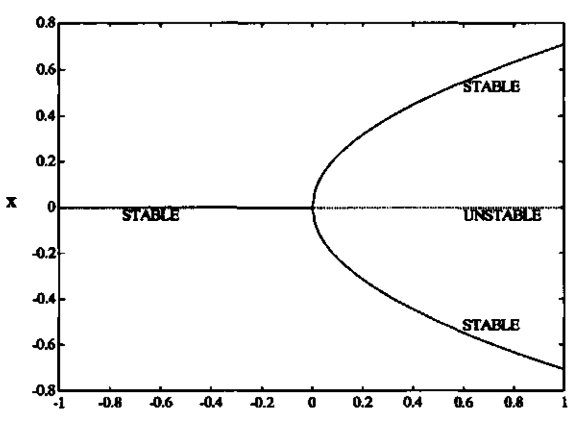
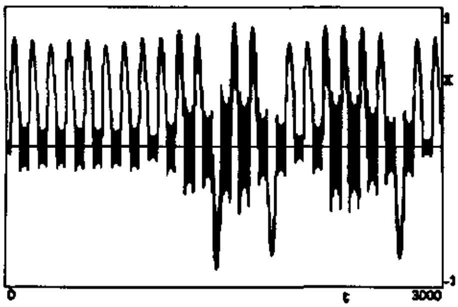
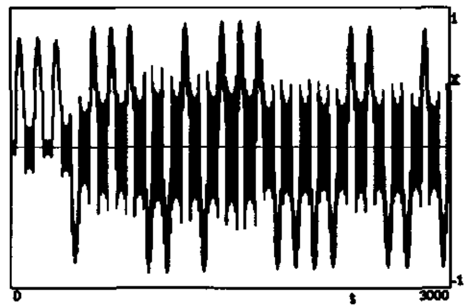


Figure 3 *The branches of the limit solution for different values of G.*

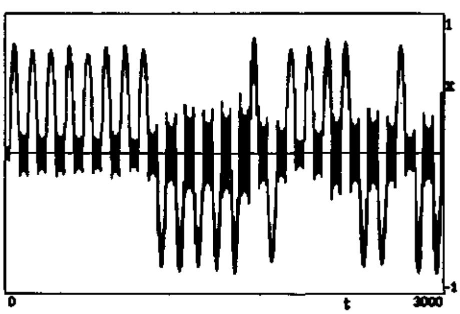
In figure 4 numerical solutions of the complete system (1.1) are given. It shows the different aspects of the dynamic behaviour of the system. The impact of the damping and the sensitive dependence on the initial conditions are shown.



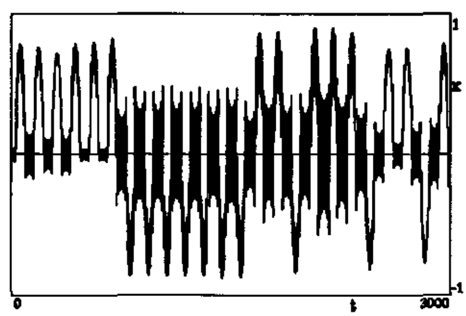
(a) $k = 0.1$, $\varepsilon=0.05$, $x(t_0) = 0.04$



(b) $k = 0$, $\varepsilon=0.05$, $x(t_0) = 0.04$



(c) $k = 0.1$, $\varepsilon=0.05$, $x(t_0) = 0.039$



(d) $k = 0$, $\varepsilon=0.05$, $x(t_0) = 0.039$

Figure 4 *Numerical solutions of eq. (1.1) with $x'(t_0) = 0$ and $\cos(\varepsilon t_0) = -1$.*

3. Local asymptotic expansions

In this section we construct local solutions to the initial value problem

$$\frac{d^2x}{dt^2} + k\varepsilon\frac{dx}{dt} - G(t)x + 2x^3 = 0,\ G(t) = \cos(\varepsilon t),\ 0 < \varepsilon << 1 \tag{3.1}$$

for different values of G and initial values near different stable branches. Note that $G(t)$ is 2π-periodic in $\tau = \varepsilon t$. First we will consider the initial value problem for an interval with $G(t) < 0$ and bounded away from zero, corresponding to initial conditions close to the outer equilibrium solution $x = 0$. Next, for $G > 0$ solutions to initial value problems are analyzed that are close to one of the two stable outer equilibrium solutions $x = \pm\sqrt{(G/2)}$. Finally, we obtain matching conditions for the local asymptotic solution describing the pitchfork bifurcation and show that the transition layer solution is a local approximation of the exact solution to the complete system.

3.1 Asymptotic expansion of the solution below the bifurcation point

We consider perturbations of the equilibrium of the form

$$x(t) = \sqrt{\varepsilon}\, v(t) \tag{3.2}$$

and take as initial values for a certain time t_0

$$G(t_0) = -1,\ x(t_0) = \sqrt{\varepsilon}\, x_1,\ \frac{dx}{dt}(t_0) = \sqrt{\varepsilon}\, v_1\ . \tag{3.3}$$

Substitution in (3.1) yields

$$\frac{d^2v}{dt^2} + \omega_1^2(\tau)v = -k\varepsilon\frac{dv}{dt} - 2\varepsilon v^2 \tag{3.4}$$

with

$$\tau = \varepsilon t,\ \omega_1(\tau) = \sqrt{-\cos\tau} = \sqrt{-G}\ . \tag{3.5}$$

After the transformations

$$v = \exp(-k\varepsilon t/2)w := rw \tag{3.6}$$

and

$$w = \omega_1^{-1/2}(\tau)y \tag{3.7}$$

we obtain the following system

$$\frac{d^2y}{dt^2} + \omega_1^2(\tau)y = \varepsilon\,(-2\omega_1^{-1}(\tau)r^2y^3 + \omega_1^{-1}(\tau)\frac{dy}{dt}\frac{d\omega_1}{d\tau}) + \varepsilon^2(\frac{1}{4}k^2y - \frac{3}{4}\omega_1^{-2}(\tau)y(\frac{d\omega_1}{d\tau})^2 + \frac{1}{2}\omega_1^{-1}(\tau)y\frac{d^2\omega_1}{d\tau^2})\;, \tag{3.8a}$$

$$\frac{dr}{dt} = -\frac{1}{2}k\varepsilon r\;, \tag{3.8b}$$

with

$$\frac{d\omega_1}{d\tau} = \frac{\sin\tau}{2\sqrt{-\cos\tau}}\;, \tag{3.9a}$$

$$\frac{d^2\omega_1}{d\tau^2} = -\frac{1}{2}\sqrt{-\cos\tau} - \frac{\sin^2\tau}{4\sqrt{-\cos\tau}}\;, \tag{3.9b}$$

and

$$G = \cos\tau < 0. \tag{3.9c}$$

We now carry out the transformation

$$\frac{dy}{dt} = \omega_1(\tau)u\;, \tag{3.10a}$$

$$\frac{du}{dt} = \frac{1}{\omega_1(\tau)}\frac{d^2y}{dt^2} - \frac{\varepsilon}{\omega_1(\tau)}\frac{d\omega_1}{d\tau}u \tag{3.10b}$$

and introduce polar coordinates

$$y = r_1\sin\phi_1,\;\; u = r_1\cos\phi_1 \tag{3.11a}$$

with values at t_0

$$y(t_0) = r_{10}\sin\phi_{10},\; u(t_0) = r_{10}\cos\phi_{10}. \tag{3.11b}$$

Next, we make the transformation

$$\tau_1 = \int_{\varepsilon t_0}^{\tau} \frac{\omega_1(\tau) d\tau}{\varepsilon} = \frac{1}{2} B_{\tau - \varepsilon t_0}(\frac{1}{2}, \frac{3}{4}) \tag{3.12}$$

with $B_x(a,b)$ the incomplete Beta function as defined in Abramowitz and Stegun (1964).
Now we set

$$\psi_1 = \phi_1 - \tau_1 \, , \tag{3.13}$$

so we finally arrive at the following initial value problem with $\tau_1 = 0$ for $t = t_0$:

$$\frac{dr}{d\tau_1} = \frac{-k\varepsilon}{2\omega_1(\tau)} \, , \quad r(0) = \exp(-k\varepsilon t_0/2) \, , \tag{3.14a}$$

$$\begin{aligned} \frac{dr_1}{d\tau_1} = & \frac{\varepsilon}{\omega_1^3(\tau)}\left(- 2r^2 r_1^3 \sin^3(\psi_1+\tau_1)\cos(\psi_1+\tau_1) + \frac{1}{4}k^2 \varepsilon r_1 \omega_1(\tau)\sin(\psi_1+\tau_1)\cos(\psi_1+\tau_1)\right) \\ & + \frac{\varepsilon^2}{\omega_1^4(\tau)}\left(- \frac{3}{4} r_1 (\frac{d\omega_1}{d\tau})^2 \sin(\psi_1+\tau_1)\cos(\psi_1+\tau_1) + \frac{1}{2}\omega_1(\tau) r_1 \frac{d^2\omega_1}{d\tau^2}\sin(\psi_1+\tau_1)\cos(\psi_1+\tau_1)\right) , \\ & r_1(0) = r_{10} \end{aligned} \tag{3.14b}$$

$$\begin{aligned} \frac{d\psi_1}{d\tau_1} = & \frac{\varepsilon}{\omega_1^3(\tau)}\left(2r^2 r_1^2 \sin^4(\psi_1+\tau_1) - \frac{1}{4}k^2 \varepsilon \omega_1(\tau)\sin^2(\psi_1+\tau_1)\right) \\ & + \frac{\varepsilon^2}{\omega_1^4(\tau)}\left(\frac{3}{4}(\frac{d\omega_1}{d\tau})^2 \sin^2(\psi_1+\tau_1) - \frac{1}{2}\omega_1(\tau)\frac{d^2\omega_1}{d\tau^2}\sin^2(\psi_1+\tau_1)\right) , \ \psi_1(0) = \phi_{10} \, . \end{aligned} \tag{3.14c}$$

From (3.5) and (3.12) we conclude that $\omega_1(\tau) = 0$ for $\tau = \tau^*$ or $\tau_1 = \tau_1^*$ with

$$\tau^* = \varepsilon t_0 + \frac{\pi}{2}, \quad \tau_1^* = \frac{B(\frac{1}{2},\frac{3}{4})}{2\varepsilon} \quad , \tag{3.15}$$

where $B(z,w)$ is the Beta function as defined in Abramowitz and Stegun (1964). In

Marée (1995) an approximation theorem has been proven for $\tau_1 \in [0, \frac{1}{2}B(\frac{1}{2},\frac{3}{4})\varepsilon^{-1} - \delta_1^{-1}(\varepsilon)]$ - so for $\tau \in [\varepsilon t_0, \varepsilon t_0 + \frac{\pi}{2} - \varepsilon^{2/3}\delta_2^{-2/3}(\varepsilon)]$ or $G \in [-1, -\varepsilon^{2/3}\delta_3^{-2/3}(\varepsilon)]$ - with δ_1, δ_2 and δ_3 positive asymptotic order functions of order $o(1)$. This theorem establishes the validity of an asymptotic expansion of the solution of (3.1) in the neighbourhood of $x_0 = 0$ for $G = \cos\tau < 0$ and $\varepsilon^{2/3} = o(G)$. For the present problem the theorem is stated as follows.

Theorem 3.1 For $-1 < G < 0$ and $\varepsilon^{2/3} = o(G)$ the solution of (3.1) has the following expansion

$$x(t) = \sqrt{\varepsilon}\, r_a r_{1a} \omega_1^{-1/2}(\tau)\sin(\tau_1 + \psi_{1a}) + o(\sqrt{\varepsilon}\,\omega_1^{-1/2}(\tau)) \tag{3.16}$$

with (r_{1a}, ψ_{1a}, r_a) the solution of the system with initial values at $\tau_1 = 0$

$$\frac{dr_{1a}}{d\tau_1} = 0 \quad , \quad r_{1a}(0) = r_{10} \quad , \tag{3.17a}$$

$$\frac{d\psi_{1a}}{d\tau_1} = \frac{\varepsilon}{\omega_1^3(\tau)}\left(\frac{3}{4}r_a^2 r_{1a}^2 - \frac{1}{8}k^2\varepsilon\omega_1(\tau)\right) \quad , \quad \psi_{1a}(0) = \phi_{10} \quad , \tag{3.17b}$$

$$\frac{dr_a}{d\tau_1} = \frac{-k\varepsilon r_a}{2\omega_1(\tau)} \quad , \quad r_a(0) = \exp(-k\varepsilon t_0/2) \quad . \tag{3.17c}$$

□

Problems arise when the angular velocity $\omega_1(\tau)$ tends to zero. So there is a boundary layer behaviour in the neighbourhood of $G = 0$. When G crosses zero from below a supercritical pitchfork bifurcation takes place. Approximations for the supercritical times are therefore $t = (3\pi/2 + 2\pi n)/\varepsilon$, $n \in \mathbb{N}$.

Remark The critical time τ_1^* as defined in (3.15) is obtained by computing

$$I = \int_0^{\pi/2} \sqrt{\cos(s)}\, ds \; . \tag{3.18}$$

It can be shown that

$$I = \frac{1}{2}B(\frac{1}{2},\frac{3}{4}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{4})}{2\Gamma(\frac{5}{4})} = \frac{2\sqrt{2}\cdot\pi\sqrt{\pi}}{\Gamma(\frac{1}{4})^2} = 1.1981\ , \tag{3.19a}$$

or alternatively

$$I = 2\sqrt{2}E(\frac{1}{2}) - \sqrt{2}K(\frac{1}{2}) = 1.1981 \tag{3.19b}$$

with Γ the gamma function and K and E the complete elliptic integrals of respectively the first and second kind.

3.2 Asymptotic expansion of the solution above the bifurcation point

We analyze the solution to the initial value problem (3.1) for $G \geq \delta(\varepsilon) > 0$ close to one of the two stable outer equilibrium solutions $x_{\pm 1} = \pm\sqrt{(G/2)}$. Because of the symmetry of the problem it is sufficient to consider the perturbations of one of the equilibria of the undamped system with fixed forcing. We choose $x = x_1 = \sqrt{(G/2)}$. We rewrite (3.1) in order to analyze a slowly varying equilibrium solution $x_{1sv}(\tau)$:

$$\frac{d^2x}{dt^2} = \varepsilon^2\frac{d^2x}{d\tau^2} = -k\varepsilon^2\,\frac{dx}{d\tau} + g(x,\tau) \tag{3.20}$$

with

$$\tau = \varepsilon t \text{ and } g(x,\tau) = x(\cos\tau - 2x^2). \tag{3.21}$$

Assuming that the derivatives in (3.20) are small, we obtain a slowly varying equilibrium solution for $G = \cos\tau > 0$ by perturbing the dependent variable at $x_1(\tau) = \sqrt{(\cos\tau/2)}$. Because $g(x_1,\tau) = 0$, we can use the Taylor series of $g(x,\tau)$ at $x = x_1(\tau)$. In this way we obtain an asymptotic expansion for the slowly varying equilibrium solution of (3.20):

$$x_{1sv}(\tau) = \sqrt{\frac{\cos\tau}{2}} + \varepsilon^2\left(\frac{1}{16}k\,\sin\tau\,\left(\frac{\cos\tau}{2}\right)^{-3/2} + \frac{1}{2}\left(\frac{\cos\tau}{2}\right)^{1/2} + \frac{1}{64}\,\sin^2\tau\left(\frac{\cos\tau}{2}\right)^{-5/2}\right) + \dots\ . \tag{3.22}$$

For a solution that holds in a $\sqrt{\varepsilon}$-neighbourhood of this slowly varying solution we write:

$$x(t) = x_{1sv}(\tau) + \sqrt{\varepsilon}\, v(t). \tag{3.23}$$

Furthermore, we assume that the solution takes the following values for $t = t_1 = t_0 + \dfrac{\pi}{\varepsilon}$ - so for $\tau_0 = \varepsilon t_0 + \pi$ - with t_0 as defined in section 3.1

$$x(t_1) = x_1(\tau_0) + \sqrt{\varepsilon}\, x_{10}, \tag{3.24a}$$

$$\frac{dx}{dt}(t_1) = \frac{dx_1}{dt}(\tau_0) + \sqrt{\varepsilon}\, y_{10}, \tag{3.24b}$$

$$G(t_1) = 1 \ . \tag{3.24c}$$

Substitution in (3.1) yields

$$\frac{d^2v}{dt^2} + \omega_2^2(\tau)v = -\, 6\sqrt{\varepsilon}\, v^2 x_{1sv}(\tau) - \varepsilon\Big(k\frac{dv}{dt} + 2v^3\Big) \tag{3.25}$$

with

$$\omega_2^2(\tau) = 6x_{1sv}^2(\tau) - \cos\tau. \tag{3.26}$$

After the transformation

$$v = \exp(-k\varepsilon t/2)w := rw \tag{3.27}$$

and

$$w = \omega_2^{-1/2}(\tau)y \tag{3.28}$$

we obtain the following system

$$\frac{d^2y}{dt^2} + \omega_2^2(\tau)y = \sqrt{\varepsilon}\,(-\,6\omega_2^{-1/2}(\tau)ry^2x_{1rv}) + \varepsilon(-2\omega_2^{-1}(\tau)r^2y^3 + \omega_2^{-1}(\tau)\frac{dy}{dt}\,\frac{d\omega_2}{dt}) \\ + \varepsilon^2(\frac{1}{4}k^2y - \frac{3}{4}\omega_2^{-2}(\tau)y(\frac{d\omega}{d\tau})^2 + \frac{1}{2}\omega_2^{-1}(\tau)y\frac{d^2\omega_2}{d\tau^2})\ , \tag{3.29a}$$

$$\frac{dr}{dt} = -\,\frac{1}{2}k\varepsilon r. \tag{3.29b}$$

In the same way as in section 3.1 we now carry out the transformation

$$\frac{dy}{dt} = \omega_2(\tau)u\ , \tag{3.30a}$$

$$\frac{du}{dt} = \frac{1}{\omega_2^2(\tau)}\frac{d^2y}{dt^2} - \frac{\varepsilon}{\omega_2(\tau)}\frac{d\omega_2}{d\tau}u \tag{3.30b}$$

and introduce polar coordinates

$$y = r_2\cos\phi_2,\ u = -r_2\sin\phi_2 \tag{3.31a}$$

with values at t_1

$$y(t_1) = r_{20}\cos\phi_{20},\ u(t_1) = -r_{20}\sin\phi_{20}\ . \tag{3.31b}$$

Next, we make the transformation

$$\tau_2 = \int_{\tau}^{\varepsilon t_1}\frac{\omega_2(\tau)}{\varepsilon}\,d\tau \tag{3.32}$$

and set

$$\psi_2 = \phi_2 + \tau_2\ . \tag{3.33}$$

Thus, we finally arrive at the following system

$$\frac{dr}{d\tau_2} = \frac{k\varepsilon r}{2\omega_2(\tau)}\ ,\quad r_1(\varepsilon t_1) = \exp(-k\varepsilon t_1/2)\ , \tag{3.34a}$$

$$\frac{dr_2}{d\tau_2} = -\sqrt{\varepsilon}\,(6\omega_2^{-5/2}(\tau)rr_2^2\cos^2(\psi_2-\tau_2)\sin(\psi_2-\tau_2)x_{1sv}(\tau))$$
$$-\ \varepsilon\omega_2^{-3}(\tau)(-\frac{1}{4}k^2\varepsilon\omega_2(\tau)r_2\cos(\psi_2-\tau_2)\sin(\psi_2-\tau_2) + 2r^2r_2^3\cos^3(\psi_2-\tau_2)\sin(\psi_2-\tau_2))$$
$$-\ \varepsilon^2\omega_2^{-4}(\tau)(\frac{3}{4}r_2\cos(\psi_2-\tau_2)\sin(\psi_2-\tau_2)(\frac{d\omega_2}{d\tau})^2 + \frac{1}{2}\omega_2(\tau)r_2\cos(\psi_2-\tau_2)\sin(\psi_2-\tau_2)\frac{d^2\omega_2}{d\tau^2}) ,$$
$$r_2(\varepsilon t_1) = r_{20}. \qquad (3.34b)$$

$$\frac{d\psi_2}{d\tau_2} = -\sqrt{\varepsilon}\,(6\omega_2^{-5/2}(\tau)rr_2^2\cos^3(\psi_2-\tau_2)x_{1sv}(\tau))$$
$$-\ \varepsilon(-\frac{1}{4}k^2\varepsilon\omega_2^{-2}(\tau)\cos^2(\psi_2-\tau_2) + 2\omega_2^{-3}(\tau)r^2r_2^2\cos^4(\psi_2-\tau_2))$$
$$-\ \varepsilon^2(\frac{3}{4}\omega_2^{-4}(\tau)\cos^2(\psi_2-\tau_2)(\frac{d\omega_2}{d\tau})^2 + \frac{1}{2}\omega_2^{-3}(\tau)\cos^2(\psi_2-\tau_2)\frac{d^2\omega_2}{d\tau^2}) ,$$
$$\psi_2(\varepsilon t_1) = \phi_{20}. \qquad (3.34c)$$

We consider

$$0 < \delta_1(\varepsilon) \leq G = \cos\tau \leq 1 \qquad (3.35a)$$

with

$$\varepsilon^{2/3} = o(\delta_1(\varepsilon)) \qquad (3.35b)$$

and obtain the following approximations

$$x_{1sv}(\tau) = \sqrt{\cos\tau/2} + O(\varepsilon^2(\cos\tau)^{-5/2}) , \qquad (3.36a)$$

$$\omega_2(\tau) = \sqrt{2\cos\tau}\,(1 + O(\varepsilon^2(\cos\tau)^{-3})). \qquad (3.36b)$$

In Marée (1995) a "second order" approximation theorem has been proven. An estimate of $O(\varepsilon(\cos\tau)^{-3/2})$ has been obtained taking into account both the $O(\varepsilon^{1/2}(\cos\tau)^{-3/4})$ terms and the $O(\varepsilon(\cos\tau)^{-3/2})$ terms of (3.34). Note the difference with section 3.1, due to the fact that we are now near a nontrivial equilibrium branch. Moreover, we remark that (3.35) is valid when

$$\tau_2 \in [0, \sqrt{2}\frac{B(\frac{1}{2},\frac{3}{4})}{\varepsilon} - \delta_2^{-1}(\varepsilon)] \quad \text{with} \quad \delta_2 = o(1). \qquad (3.37)$$

We now formulate the following approximation result.

Theorem 3.2 For $0 < \delta_1(\varepsilon) \le G = \cos\tau \le 1$ with $\varepsilon^{2/3} = o(\delta_1(\varepsilon))$ - so for $\delta_3(\varepsilon) \le \omega_2(\tau) \le \delta_4(\varepsilon)$ with $\varepsilon^{1/3} = o(\delta_3(\varepsilon))$ and $\delta_4(\varepsilon) = O(1)$ - the solution of (3.1) has the following expansion

$$x(t) = \sqrt{\cos\tau/2} + \sqrt{\varepsilon}\, r_{2a}\exp(-k\varepsilon t/2)\omega_2^{-1/2}(\tau)\cos(\psi_{2a} - \tau_2) + o(\sqrt{\varepsilon}\omega_2^{-1/2}(\tau)) \ , \quad (3.38)$$

with (r_{2a}, ψ_{2a}) the solution of the system

$$\frac{dr_{2a}}{d\tau_2} = 0 \ , \quad r_{2a}(t_1) = r_{20} \ , \quad (3.39a)$$

$$\frac{d\psi_{2a}}{d\tau_2} = -\varepsilon\left(\frac{-k^2\varepsilon}{16\cos\tau} + \frac{3e^{-k\varepsilon t}r_{2a}^2}{4(2\cos\tau)^{3/2}} - \frac{15e^{-k\varepsilon t}r_{2a}^2}{4(2\cos\tau)^{3/2}}\right) \ , \ \psi_{2a}(t_1) = \phi_{20} \ . \quad (3.39b)$$

□

Finally we remark that in the case without damping (3.25) is a Hamiltonian system. After the transformations

$$v = r\cos\phi, \ \frac{dv}{dt} = -\omega_2(\tau)r\sin\phi, \text{ and } p = r^2\omega_2(\tau) \quad (3.40a)$$

we obtain a Hamiltonian of the following form

$$\begin{aligned} H = {} & \omega_2(\tau)p + \sqrt{\varepsilon}(3p^{3/2}\omega_2^{-3/2}(\tau)x_{1sv}(\tau)\cos\phi + p^{3/2}\omega_2^{-3/2}(\tau)x_{1sv}(\tau)\cos3\phi) \\ & + \varepsilon(\frac{3}{8}p^2\omega_2^{-2}(\tau) + \frac{1}{2}p^2\omega_2^{-2}(\tau)\cos2\phi + \frac{1}{8}p^2\omega_2^{-2}(\tau)\cos4\phi - \frac{1}{2}p^2\omega_2^{-1}(\tau)\frac{d\omega_2}{d\tau}\sin2\phi). \end{aligned}$$

(3.40b)

3.3 The transition layer equation and matching conditions

In order to obtain matching conditions for the local asymptotic solution describing the pitchfork bifurcation we determine the asymptotic development of x when G is in the neighbourhood of $G_0 = 0$. We assumed that near $G = \cos\tau = 0$ and $x = 0$

$$g(x,\tau) = x(\cos\tau - 2x^2) = -\frac{1}{2}x(x - \sqrt{\cos\tau/2})(x + \sqrt{\cos\tau/2}) \ . \quad (3.41)$$

Setting

$$t = t_0 + \frac{\pi}{2\varepsilon} + \varepsilon^{\nu-1}z \quad \text{or} \quad G = \varepsilon^\nu z \tag{3.42a,b}$$

we obtain the following approximations for the τ-dependent terms in the asymptotic expansions (3.16) and (3.38):

$$r = \exp(-\frac{1}{2}k\varepsilon(t_0+\frac{\pi}{2\varepsilon})) + \dots , \tag{3.43a}$$

$$\omega_1^2(\tau) = -z\varepsilon^\nu , \tag{3.43b}$$

$$x_{1a}(\tau) = (\frac{z}{2})^{1/2}\varepsilon^{\nu/2} + \frac{1}{64}(\frac{z}{2})^{-5/2}\varepsilon^{2-5\nu/2} + \dots + \frac{k}{16}(\frac{z}{2})^{-3/2}\varepsilon^{2-3\nu/2} + \dots , \tag{3.43c}$$

$$\omega_2^2(\tau) = 2z\varepsilon^\nu + \frac{3}{16}(\frac{z}{2})^{-2}\varepsilon^{2-2\nu} + \dots , \tag{3.43d}$$

$$r_{1a} = r_{10}, \quad r_{2a} = r_{20} , \tag{3.43e,f}$$

$$\tau_1 = \int_{\varepsilon t_0}^{\varepsilon t_0 + \pi/2 + \varepsilon^\nu z} \frac{\sqrt{-\cos\tau}}{\varepsilon}d\tau = \frac{B(\frac{1}{2},\frac{3}{4})}{2\varepsilon} - \frac{2}{3}(-z)^{3/2}\varepsilon^{3\nu/2-1} + \dots , \tag{3.43g}$$

$$\tau_2 = \int_{\varepsilon t_0 + \pi/2 + \varepsilon^\nu z}^{\varepsilon t_1} \frac{\sqrt{2\cos\tau}\,d\tau}{\varepsilon} + \dots = \frac{\sqrt{2}B(\frac{1}{2},\frac{3}{4})}{2\varepsilon} - \frac{2\sqrt{2}}{3}(z)^{3/2}\varepsilon^{3\nu/2-1} + \dots , \tag{3.43h}$$

$$\psi_{1a} = \int_{\varepsilon t_0}^{\varepsilon t_0 + \pi/2 + \varepsilon^\nu z}\left(\frac{-3r_{10}^2\exp(-k\tau)}{4\cos\tau} - \frac{k^2\varepsilon}{8\sqrt{-\cos\tau}}\right)d\tau$$

$$= -\frac{3}{4}r_{10}^2e^{-k\varepsilon t_0}(e^{-k\pi/2}(\ln(-z) + \ln(\frac{\varepsilon^\nu}{2})) - k\int_0^{\pi/2}\ln(\frac{1+\sin y}{\cos y})e^{-ky}dy) - \frac{\varepsilon k^2\sqrt{2}K(\frac{1}{2})}{8} + \phi_{10} + \dots , \tag{3.43i}$$

$$\psi_{2a} = \int_{\varepsilon t_0 + \pi/2 + \varepsilon^\nu z}^{\varepsilon t_1}\left(\frac{-3r_{20}^2\exp(-k\tau)}{2\cos\tau} + \frac{k^2\varepsilon\sqrt{2}}{16\sqrt{\cos\tau}}\right)d\tau$$

$$= -\frac{3}{2}r_{20}^2e^{-k\varepsilon t_0}(e^{-k\pi/2}(\ln(z) + \ln(\frac{\varepsilon^\nu}{2})) + k\int_{\pi/2}^{\pi}\ln(\frac{\sin y - 1}{\cos y})e^{-ky}dy) + \frac{\varepsilon k^2K(\frac{1}{2})}{8} + \phi_{20} + \dots \tag{3.43j}$$

with K the complete elliptic integral of the first kind. Consequently, near the bifurcation point the outer solutions (3.16) and (3.38) behave asymptotically as

$$x \sim r_{10} e^{-k\varepsilon(t_0 + \pi/(2\varepsilon))/2}(-z)^{-1/4}\varepsilon^{1/2 - \nu/4}\sin\Big(\frac{2}{3}(-z)^{3/2}\varepsilon^{3\nu/2 - 1} + \frac{3}{4}r_{10}^2 e^{-k\varepsilon t_0}\Big(e^{-k\pi/2}(\ln(-z) + \nu \ln \varepsilon) - k\int_0^{\pi/2}\ln\Big(\frac{1 + \sin y}{\cos y}\Big)e^{-ky}dy\Big) + \psi_{10} + \ldots\Big) + \ldots \quad \text{if } z \to -\infty , \tag{3.44a}$$

$$x \sim \Big(\frac{z}{2}\Big)^{1/2}\varepsilon^{\nu/2} + \frac{1}{64}\Big(\frac{z}{2}\Big)^{-5/2}\varepsilon^{2 - 5\nu/2} + \ldots + \frac{k}{16}\Big(\frac{z}{2}\Big)^{-3/2}\varepsilon^{3/2 - 3\nu/2} + \ldots + r_{20} e^{-k\varepsilon(t_0 + \pi/(2\varepsilon))/2}\Big(2z\varepsilon^{\nu} + \frac{3}{16}\Big(\frac{z}{2}\Big)^{-2}\varepsilon^{2 - 2\nu} + \ldots\Big)^{-1/4}\varepsilon^{1/2}\cos\Big(\frac{2\sqrt{2}}{3}(z)^{3/2}\varepsilon^{3\nu/2 - 1} - \frac{3}{2}r_{20}^2 e^{-k\varepsilon t_0}\Big(e^{-k\pi/2}(\ln(z) + \nu \ln \varepsilon) + k\int_{\pi/2}^{\pi}\ln\Big(\frac{\sin y - 1}{\cos y}\Big)e^{-ky}dy\Big) + \psi_{20} + \ldots\Big) + \ldots \text{ if } z \to \infty , \tag{3.44b}$$

where ψ_{10} and ψ_{20} are constants determined by the initial conditions and the dots stand for higher order ε-terms or for terms that are $o(|z|^{-1/4})$.

From the expansion (3.44) it is seen that the outer expansion breaks down if $\nu = 2/3$. It implies that the transition layer (inner) equation follows from the scaling

$$x = \varepsilon^{1/3}y(z) . \tag{3.45}$$

There is a significant degeneration of the differential equation (3.1) that describes the transition behaviour for $G = \varepsilon^{2/3}z$ and $x = \varepsilon^{1/3}y(z)$. This degeneration is represented by the second Painlevé equation:

$$\frac{d^2y}{dz^2} = yz - 2y^3 . \tag{3.46}$$

We refer to Marée (1995) for more details about this equation.

4. Matching conditions for the local asymptotic expansions and their interrelations

At the time that a pitchfork bifurcation is expected eq. (3.46) holds. Its solution must match the outer solutions as given by (3.44a) and (3.44b) with $\nu = 2/3$

$$y \sim \gamma(-z)^{-1/4}\sin\left\{\frac{2}{3}(-z)^{3/2} + \frac{3}{4}\gamma^2\ln(-z) + \xi_{10}\right\} + o((-z)^{-1/4}) \text{ as } z \to -\infty , \tag{4.1a}$$

$$y \sim (\frac{z}{2})^{1/2} + \beta(2z)^{-1/4}\cos\left\{\frac{2\sqrt{2}}{3}z^{3/2} - \frac{3}{2}\beta^2\ln(z) + \xi_{20}\right\} + o(z^{-1/4}) \text{ as } z \to \infty , \tag{4.1b}$$

where

$$\gamma = r_{10}\exp(-k\varepsilon(t_0 + \pi/(2\varepsilon))/2) , \tag{4.1c}$$

$$\beta = r_{20}\exp(-k\varepsilon(t_0 + \pi/(2\varepsilon))/2) , \tag{4.1d}$$

$$\xi_{10} = \frac{-B(\frac{1}{2},\frac{3}{4})}{2\varepsilon} + \frac{3}{4}r_{10}^2e^{-k\varepsilon t_0}(e^{-k\pi/2}(\ln\frac{\varepsilon^{2/3}}{2}) - k\int_0^{\pi/2}\ln(\frac{1 + \sin y}{\cos y})e^{-ky}dy) - \varphi_{10} + \pi , \tag{4.1e}$$

$$\xi_{20} = \frac{-\sqrt{2}B(\frac{1}{2},\frac{3}{4})}{2\varepsilon} - \frac{3}{2}r_{20}^2e^{-k\varepsilon t_0}(e^{-k\pi/2}(\ln\frac{\varepsilon^{2/3}}{2}) + k\int_{\pi/2}^{\pi}\ln(\frac{\sin y - 1}{\cos y})e^{-ky}dy) + \varphi_{20} . \tag{4.1f}$$

The pitchfork matching condition after passage of the bifurcation point when the solution remains near the stable outer solution $x = x_1$, is represented by (4.1b). When the solution remains near the other stable outer equilibrium solution $x = x_{-1} = -\sqrt{(G/2)}$ this matching condition is obtained by reflecting (4.1b) with respect to the z-axis. The asymptotic conditions are automatically fulfilled by (3.46).

From an approximation theorem formulated by Marée (1995, theorem 5.2) and the extension theorem (Eckhaus, 1979) it follows that the domain of validity of the local Painlevé approximation can be extended forward and backward to $G \in [-\varepsilon^{2/3}\delta_e^{-1}(\varepsilon), \varepsilon^{2/3}\delta_e^{-1}(\varepsilon)]$ with $\delta_e(\varepsilon) = o(1)$. Thus overlap with the domains, where outer approximations are valid, is ensured and so the integration constants can be matched. In Its, Fokas and Kapaev (1994) it is shown that there is a connection between the behaviour of solutions of the second Painlevé transcendent (3.46) for $z \to -\infty$ with the behaviour for $z \to \infty$. Their result can be formulated as follows:

Theorem 4.1 Let $y(z)$ be an arbitrary solution of eq. (3.46). Then the following assertions hold for $y(z)$:

a) $y(z)$ is smooth for every $z \in \mathbf{R}$ and has the following asymptotics as $z \to -\infty$:

$$y(z) = \gamma(-z)^{-1/4}\sin\{\frac{2}{3}(-z)^{3/2} + \frac{3}{4}\gamma^2\ln(-z) + \xi_{10}\} + o((-z)^{-1/4}) \tag{4.2}$$

where the numbers $\gamma > 0$ and $0 \leq \xi_{10} < 2\pi$ may be arbitrary and are parameters of the solution $y(z)$.

b) If the parameters γ and ξ_{10} of the solution $y(z)$ are connected by the relation

$$\xi_{10} = \frac{3}{2}\gamma^2\ln 2 - \frac{\pi}{4} - \arg\Gamma(i\frac{\gamma^2}{2}) + \delta\pi(\mathrm{mod}\,2\pi), \quad \delta = 0,\ 1\ , \tag{4.3}$$

then as $z \to +\infty$ the solution $y(z)$ decreases exponentially:

$$y(z) = \frac{a}{2\sqrt{\pi}}z^{-1/4}\exp(-2z^{3/2}/3)(1 + o(1))\ , \tag{4.4}$$

where $a^2 = \exp(\pi\gamma^2) - 1$ and $\mathrm{sign}(a) = 2(½ - \delta)$.

c) If (4.3) fails to hold (the general case), then as $z \to +\infty$ the solution grows polynomially:

$$y(z) = \pm\sqrt{z/2} \pm (2z)^{-1/4}\beta\cos\left\{\frac{2\sqrt{2}}{3}z^{3/2} - \frac{3}{2}\beta^2\ln z + \xi_{20}\right\} + o(z^{-1/4}). \tag{4.5}$$

d) In the asymptotics (4.5) all values of $\beta > 0$ and $0 \leq \xi_{20} < 2\pi$ are possible; these quantities characterize the solution $y(z)$ uniquely. The parameters β, ξ_{20} and the choice of the sign in (4.5) are uniquely determined from the parameters γ and ξ_{10}

$$\beta^2 = \frac{1}{\pi}\ln\frac{1 + |p|^2}{2|\mathrm{Im}\,p|}\quad , \tag{4.6a}$$

$$\xi_{20} = -\frac{3\pi}{4} - \frac{7}{2}\beta^2\ln 2 + \arg\Gamma(i\beta^2) + \arg(1 + p^2)\ , \tag{4.6b}$$

where

$$p = (\exp(\pi\gamma^2) - 1)^{1/2}\exp(i\frac{3}{2}\gamma^2 \ln 2 - i\frac{\pi}{4} - i\arg(\frac{i\gamma^2}{2}) - i\xi_{10}) \tag{4.6c}$$

and the upper sign in (4.5) is taken if Im $p < 0$.

□

We refer to Its, Fokas and Kapaev (1994) for a proof of this theorem and for the asymptotic description of the solution of the second Painlevé equation. This result confirms our asymptotic results and connects the integration constants in the asymptotic solution for $z \to -\infty$ with those in the one for $z \to +\infty$. Moreover, separating solutions, that follow the unstable branch beyond the bifurcation point, are singled out. We have now illustrated that the integration constants of the local solutions before, during and after crossing the supercritical bifurcation point can be connected. Because of symmetry reasons and since the map, as formulated in theorem 4.1, is invertible, we can now construct a Poincaré map for one forcing period $2\pi/\varepsilon$ that connects the integration constants of a solution on a time t before crossing the supercritical bifurcation point with those on a time $t + 2\pi/\varepsilon$. The initial conditions of the original system (3.1) are chosen such that motions will be in a particular well after passage of the bifurcation point. From the analysis of sections 3 and 4 it follows that motion outside the homoclinic loop separatrix does not occur after each passage of the bifurcation point on a time-scale $O(1/\varepsilon)$.

5. The construction of a Poincaré map

In this section we construct a Poincaré map for one forcing period. With this map we can predict the complex dynamics of the system. The theorems on the composed asymptotic solutions prove the validity of these approximations on a time-scale $O(1/\varepsilon)$, so the predictions are also valid on this time-scale. In order to construct this Poincaré map we will consider four maps that connect the integration constants of the local asymptotic solutions valid at the begin-point and end-point of four different time intervals of $\tau = \varepsilon t$:

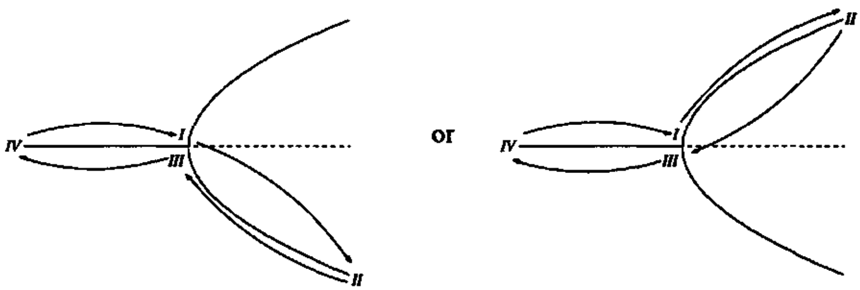
$$I\text{: } \tau \in [-\frac{\pi}{2} + 2\pi n - M\varepsilon^{2/3}, -\frac{\pi}{2} + 2\pi n + M\varepsilon^{2/3}]\ ,\ n \in Z, \tag{5.1a}$$

$$II\text{: } \tau \in [-\frac{\pi}{2} + 2\pi n + M\varepsilon^{2/3}, \frac{\pi}{2} + 2\pi n - M\varepsilon^{2/3}]\ , \tag{5.1b}$$

$$III\text{: } \tau \in [\frac{\pi}{2} + 2\pi n - M\varepsilon^{2/3}, \frac{\pi}{2} + 2\pi n + M\varepsilon^{2/3}]\ , \tag{5.1c}$$

$$IV: \tau \in [\frac{\pi}{2} + 2\pi n + M\varepsilon^{2/3}, \frac{3\pi}{2} + 2\pi n - M\varepsilon^{2/3}] , \tag{5.1d}$$

with M an arbitrary positive constant. In section 4 it has been proven that the different local asymptotic solutions can be matched and the integration constants can be connected. Two parameters, the damping parameter k and the small parameter ε, have to be taken into account. The composition of the four different maps yields the Poincaré map. Region I and III are the transition regions. Region II represents the outer region above the critical point, whereas region IV represents the outer region below the critical point. It has been shown in section 4 that the matching conditions of (3.46) are given by (4.1a) and (4.1b) where γ, ξ_{10}, β and ξ_{20} are integration constants. With the aid of the results of section 3 we obtain connection maps for the local integration constants. We take as initial values $\gamma = \gamma(0)$ and $\xi_{10} = \xi_{10}(0)$, and assume that this is a generic choice meaning that the separating condition (4.3) is not satisfied. In figure 5 the construction of the Poincaré map is sketched.



(5a) (5b)

Figure 5 *With the aid of four time intervals the Poincaré map is constructed.*

Then the maps are as follows:

Region I: $\quad f_1(\gamma(0), \xi_{10}(0)) \rightarrow (\beta(0), \xi_{20}(0), \sigma(0)),$ (5.2a)

where

$$\beta^2(0) = \frac{1}{\pi}\ln\frac{1 + |p|^2}{2|\mathrm{Im}p|} , \tag{5.2b}$$

$$\xi_{20}(0) = -\frac{3\pi}{4} - \frac{7}{2}\beta^2(0)\ln 2 + \arg\Gamma(i\beta^2(0)) + \arg(1 + p^2), \tag{5.2c}$$

$$\sigma(0) = -\mathrm{sign}(\mathrm{Imp}), \tag{5.2d}$$

with

$$p = (\exp(\pi\gamma^2(0)) - 1)^{1/2}\exp(i\frac{3}{2}\gamma^2(0)\ln 2 - i\frac{\pi}{4} - i\arg\Gamma\left(\frac{i\gamma^2(0)}{2}\right) - i\xi_{10}(0)). \tag{5.2e}$$

The sign of σ determines which branch will be approached after passage of the supercritical bifurcation point. If $\sigma = 1$ it will be the upper branch, while the lower branch will be followed for $\sigma = -1$. In the asymptotics both for $\sigma = 1$ and for $\sigma = -1$ all values of $\beta(0) > 0$ and $0 \le \xi_{20}(0) < 2\pi$ are possible.

Region II: $\quad f_2(\beta(0), \xi_{20}(0)) \rightarrow (\beta_1(0), \xi_{21}(0))$, (5.3a)

where

$$\beta_1(0) = \beta(0)g_1(k) , \tag{5.3b}$$

$$\xi_{21}(0) = -\frac{\sqrt{2}B(\frac{1}{2},\frac{3}{4})}{\varepsilon} - \frac{3}{2}\beta^2(0)g_2(k) + g_3(k) - \xi_{20}(0) , \tag{5.3c}$$

with

$$g_1(k) = \exp(\frac{-k\pi}{2}) , \tag{5.3d}$$

$$g_2(k) = \ln(\frac{\varepsilon^{2/3}}{2})(\exp(-k\pi) + 1) - k\int_0^\pi \ln(\frac{1 - \cos s}{\sin s})\exp(-ks)ds \ , \ g_3(k) = o(1). \tag{5.3e}$$

Region III: $\quad f_3(\beta_1(0), \xi_{21}(0), \sigma(0)) \rightarrow (\gamma_1(0), \xi_{11}(0))$, (5.4a)

where

$$\gamma_1(0) = (\frac{\ln(1 + |p_1|^2)}{\pi})^{1/2} , \tag{5.4b}$$

$$\xi_{11}(0) = \frac{3}{2}\gamma_1^2(0)\ln 2 - \frac{\pi}{4} - \arg\Gamma\left(\frac{i\gamma_1^2(0)}{2}\right) - \arg p_1 , \tag{5.4c}$$

with

$$|p_1| = \{2\exp(2\pi\beta_1^2(0))\text{-}1\text{-}2\exp(\pi\beta_1^2(0))\sqrt{\exp(2\pi\beta_1^2(0))\text{-}1}\,\cos(-\frac{3\pi}{4} - \frac{7}{2}\beta_1^2(0)\ln 2 + \arg\Gamma(i\beta_1^2(0)) + \xi_{21}(0))\}^{1/2}\,, \tag{5.4d}$$

$$s = \mathrm{sign}(\sin(-\frac{3\pi}{4} - \frac{7}{2}\beta_1^2(0)\ln 2 + \arg\Gamma(i\beta_1^2(0)) + \xi_{21}(0))), \tag{5.4e}$$

$$\arg p_1 = \frac{1}{2}\pi(s\sigma(0) + 1) - s\sigma(0)\arcsin\left(-\sigma(0)\frac{1 + |p_1|^2}{2|p_1|\exp(\pi\beta_1^2(0))}\right). \tag{5.4f}$$

The map f_3 is the inverse map of f_1.

Region IV: $f_4(\gamma_1(0), \xi_{11}(0)) \rightarrow (\gamma(1), \xi_{10}(1))$, (5.5a)

where

$$\gamma(1) = \gamma_1(0)g_1(k)\,, \tag{5.5b}$$

$$\xi_{10}(1) = \pi + \frac{3}{4}\gamma_1^2(0)g_2(k) - \frac{B(\frac{1}{2},\frac{3}{4})}{\varepsilon} + \sqrt{2}g_3(k) - \xi_{11}(0)\,, \tag{5.5c}$$

with $g_1(k)$, $g_2(k)$ and $g_3(k)$ as defined in (5.3d) and (5.3e).

We have now obtained a Poincaré map for one forcing period

$$P(\gamma(0), \xi_{10}(0)) := f_4 \circ f_3 \circ f_2 \circ f_1(\gamma(0), \xi_{10}(0)) = (\gamma(1), \xi_{10}(1))\,. \tag{5.6}$$

This is a two-dimensional map that contains the parameters k and ε. With the aid of this map the complex dynamics of system (3.1) can be analyzed. Depending on the initial conditions and the values of the parameters it is possible to predict which stable branch will be followed after crossing the supercritical bifurcation point in the course of each forcing period. In the next section the dynamics of the Poincaré map (5.6) is considered.

Remark It is possible to consider the Poincaré map for different values of the parameter k as an order function in ε. Let

$$k = \kappa\varepsilon^\alpha \quad , \alpha \geq -\frac{1}{2}\,, \tag{5.7}$$

then $g_1(k) = \exp(-k\varepsilon^{\alpha}/2)$ and for the functions $g_2(k)$ and $g_3(k)$, defined in (5.3e), we obtain

$$\alpha = -\frac{1}{2} : g_2(k) = \ln(\kappa\varepsilon^{\frac{1}{6}}) + \gamma + o(1)\ ,\ g_3(k) = \frac{\kappa^2 K(\frac{1}{2})}{4}\ , \tag{5.8a}$$

$$-\frac{1}{2} < \alpha < 0 : g_2(k) = \ln(\kappa\varepsilon^{\frac{2}{3}+\alpha}) + \gamma + o(1)\ ,\ g_3(k) = o(1)\ , \tag{5.8b}$$

$$\alpha = 0 : g_2(k) = \ln(\frac{\varepsilon^{\frac{2}{3}}}{2})(e^{-\kappa\pi} + 1) - \kappa\int_0^{\pi}\ln(\frac{1 - \cos s}{\sin s})e^{-\kappa s}ds + o(1)\ ,\ g_3(k) = o(1)\ , \tag{5.8c}$$

$$\alpha > 0 : g_2(k) = 2\ln(\frac{\varepsilon^{\frac{2}{3}}}{2}) + o(1)\ ,\ g_3(k) = o(1)\ . \tag{5.8d}$$

6. Analysis of the Poincaré map

The Poincaré map, that has been constructed in section 5, has the following form for m forcing periods, $m \in \mathbf{N}$,

$$P^{m}(\gamma(0), \xi_{10}(0)) = (\gamma(m), \xi_{10}(m))\ . \tag{6.1}$$

It follows that

$$(\beta(m), \xi_{20}(m), \sigma(m)) = f_1(\gamma(m), \xi_{10}(m)) \tag{6.2}$$

with f_1 as defined in (5.2). The complex dynamics of system (3.1) may be characterized by monitoring the position of a motion once per forcing cycle. With the aid of the Poincaré map it can be predicted on a time-scale $O(1/\varepsilon)$ which stable branch, the upper (U) or the lower (L), will be followed after crossing the bifurcation point $\cos(\varepsilon t) = 0$ from below after each forcing cycle. This is determined by the sign of $\sigma(m)$. This $\sigma(m)$, $m \geq 0$, is defined in (6.2). In this way we obtain a symbol sequence of U's ($\sigma(m) = 1$) and L's ($\sigma(m) = -1$). Moreover, Lyapunov exponents of the two-dimensional Poincaré map can be computed. These exponents describe the structure of the attractor of the orbits. The system exhibits a strange attractor if one of these exponents is positive. The conjecture of Kaplan and Yorke provides the value of the fractal dimension of this attractor. The Poincaré map can exhibit sensitive dependence on initial conditions, a criterion for chaos.

We now introduce the following definitions

$$x_i = (\gamma(i-1), \xi_{10}(i-1))^t, \quad x_{i+1} = P(x_i) \ (P \text{ as defined in (5.6)}) , \tag{6.3a}$$

$$M(x) = \frac{\partial P}{\partial x}(x) \quad \text{(the Jacobian of } P). \tag{6.3b}$$

We define the two (possibly complex) eigenvalues $\lambda_1(n)$, $\lambda_2(n)$ with $|\lambda_1(n)| \geq |\lambda_2(n)|$ of the matrix

$$A_n = [M(x_n)M(x_{n-1})...M(x_1)]^{1/n} . \tag{6.4}$$

The Lyapunov exponents are then given by

$$v_j(x_1) = \lim_{n \to \infty} \ln|\lambda_j(n)| , \quad j = 1,2 . \tag{6.5}$$

An other way to obtain these Lyapunov exponents is by introducing the eigenvalues λ_{1i}, λ_{2i} of the M matrix $M(x_i)$ with $|\lambda_{1i}| \geq |\lambda_{2i}|$, from which we obtain the Lyapunov exponents

$$v_j(x_1) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln|\lambda_{ji}|, \quad j = 1,2 . \tag{6.6}$$

Roughly speaking, the Lyapunov exponents of a given trajectory characterize the mean exponential rate of divergence of trajectories surrounding it. The mean exponential rate of growth of the phase space area is defined by

$$v(x_1) = v_1(x_1) + v_2(x_1) . \tag{6.7}$$

For a measure-preserving flow, of which system (3.1) without damping is a special case, we see that

$$v_1(x_1) + v_2(x_1) = 0, \tag{6.8}$$

while for a dissipative system (system (3.1) with the damping parameter k not equal to zero) this sum must be negative. Note that for the starting point x_1 of an orbit arbitrarily chosen, the Lyapunov exponents can be different when the parameters are fixed. Their sum, however, is a fixed value in that case. We now first consider system (3.1) without damping ($k = 0$) and, next, we examine the dissipative case ($k \neq 0$).

6.1 The non-dissipative case

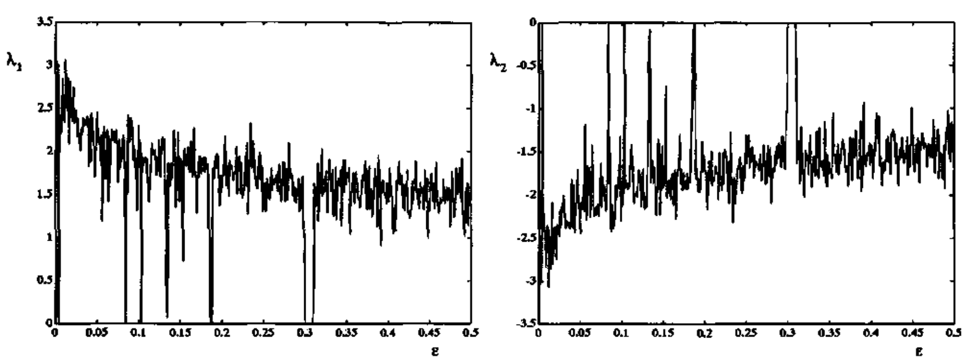
In the case without damping eq. (3.1) describes the dynamics of a slowly varying Hamiltonian system. Due to the absence of damping Hamiltonian chaos will occur, see e.g. Lichtenberg and Lieberman (1983). The Poincaré map is area-preserving and, therefore, there are two possibilities for the Lyapunov exponents of this map

I: $\nu_1(x_1) > 0 > \nu_2(x_1) = -\nu_1(x_1)$, (6.9a)

II: $\nu_1(x_1) = \nu_2(x_1) = 0$. (6.9b)

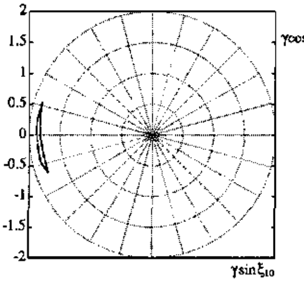
In the first case the orbits are chaotic and there exists a chaotic attractor of dimension 2. In the second case the KAM-theorem (see e.g. Ott (1993)) provides an answer to the question in which way orbits will behave: with probability 1 the attractor is quasiperiodic of dimension 1. For fixed values of the parameters the Poincaré map exhibits both order and chaos depending on the initial state of an evolution. We observe quasiperiodic orbits with closed curves belonging to them and chaotic orbits; it is a matter of "order in a sea of chaos".

We illustrate the situation with the help of some figures. In figure 6 the Lyapunov exponents are shown for a certain (fixed) initial state for different values of ε. As can be seen from this figure the sum of the Lyapunov exponents is equal to zero. From this figure it can be concluded that the system exhibits a quasiperiodic orbit for the initial state $\xi_{10}(0) = \pi$, $\gamma(0) = 1.875$ when $\varepsilon = 0.3$. In figure 7 it is shown that for $\varepsilon = 0.3$ the phase portrait of $(\gamma\sin\xi_{10}, \gamma\cos\xi_{10})$ exhibits besides chaotic orbits also quasiperiodic orbits that form dense closed curves. Moreover, it is illustrated that the modulus of the eigenvalues of A_n (see (6.4)) tends to one when the number of iterations is enlarged in the quasiperiodic case. In the chaotic case the modulus of the largest eigenvalue will always remain larger than one.

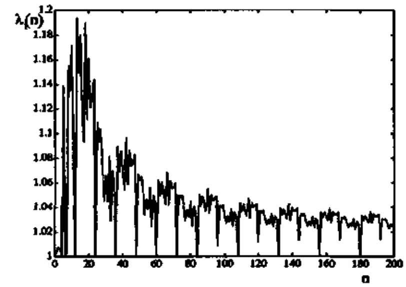


(6a) (6b)

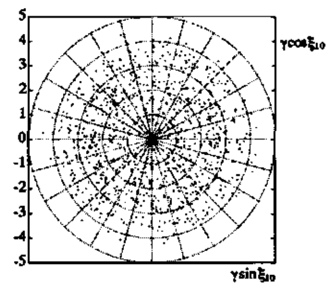
Figure 6 *The first (6a) and second (6b) Lyapunov exponent for the Poincaré map in the non-dissipative case. The (fixed) initial state is* $x_1 = (\gamma(0), \xi_{10}(0))' = (1.875, \pi)'$.



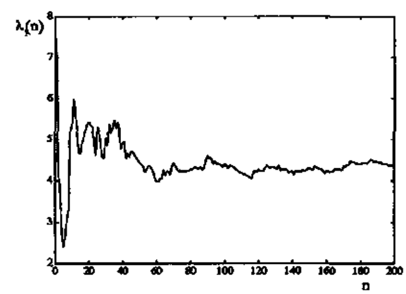
(7a)



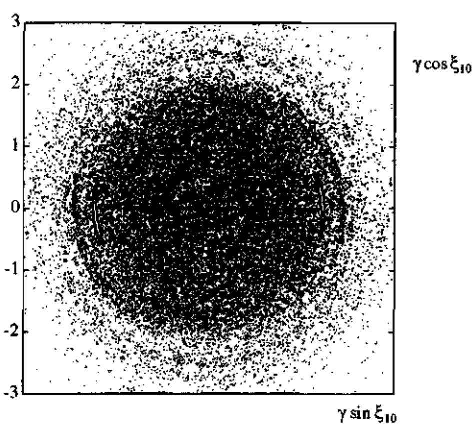
(7b)



(7c)



(7d)



(7e)

Figure 7 *For $\varepsilon = 0.3$ both a quasi-periodic orbit (7a) with $x_1 = (1.875,\pi)'$ and a chaotic orbit (7c) with $x_1 = (1.875,\pi/2)'$ are possible. The modulus of the largest eigenvalue of the matrix A_n tends to one for $x_1 = (1.875,\pi)'$ (7b) and for $x_1 = (1.875,\pi/2)'$ it remains larger than one. The phase portrait (7e) of $(\gamma\sin\xi_{10}\ ,\ \gamma\cos\xi_{10})$ shows that the Poincaré map exhibits quasi-periodic orbits and chaotic orbits.*

6.2 The case with damping

When the damping parameter k in eq. (3.1) is strictly positive the system is dissipative. This means for the sum of the Lyapunov exponents for the Poincaré map:

$$\nu_1(x_1) + \nu_2(x_2) < 0, \tag{6.10}$$

so the volume (area) in phase space decreases under the mapping. Similar to the frictionless case we can distinguish two possibilities:

$$\text{I:} \quad 0 \geq \nu_1(x_1) \geq \nu_2(x_1),\ \nu_1(x_1) \neq \nu_2(x_1) \text{ if } \nu_1(x_1) = 0, \tag{6.11a}$$
$$\text{II:} \quad \nu_1(x_1) > 0 > \nu_2(x_1). \tag{6.11b}$$

In the first case the system possesses a periodic attractor. In the second case the attractor is chaotic. Following Lichtenberg and Lieberman (1983) we compute the fractal dimension d of this chaotic attractor with the aid of the conjecture of Kaplan and Yorke:

$$d = 1 - \frac{\nu_1(x_1)}{\nu_2(x_1)} . \tag{6.12}$$

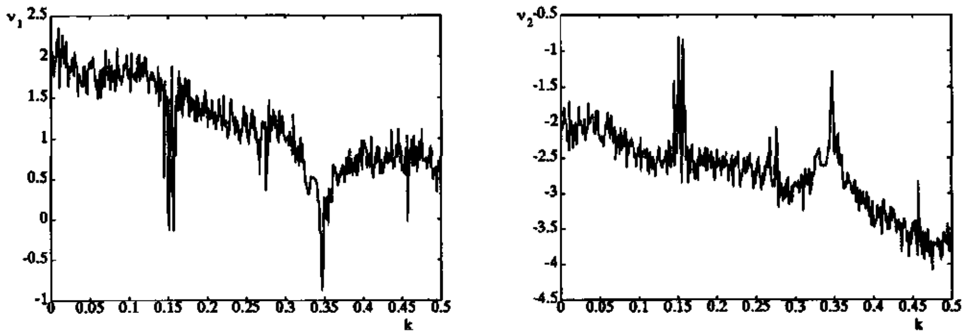
The situation is again illustrated with the aid of some figures. In all figures illustrating the dissipative case the parameter ε is chosen $\varepsilon = 0.1$. The Lyapunov exponents for a certain (fixed) initial state for different values of k are depicted in figure 8. The values that the state variables of the Poincaré map will take after a certain number of iterations for different values of k, are shown in figure 9. In figure 10 it is seen that the rate at which the volume in phase space decreases, becomes smaller when k decreases and ε is fixed. Due to the damping, that is of order $O(k\varepsilon)$, and the forcing period, that is equal to $2\pi/\varepsilon$, we obtain for the phase space volume (area) decreasing rate:

$$\nu_1(x_1) + \nu_2(x_2) = O(-2k\pi) . \tag{6.13}$$

This rate is only dependent on the value of the parameter k, so it does not depend on the initial state of the system. In figure 11 the dimension of the attractor of orbits with a certain initial state is computed as a function of k. Finally, in figure 12, it is shown that for $k = 0.05$ and $\varepsilon = 0.1$ the system exhibits a chaotic attractor. The fractal dimension of this attractor -obtained with the aid of the conjecture of Kaplan and Yorke- is 1.8623.

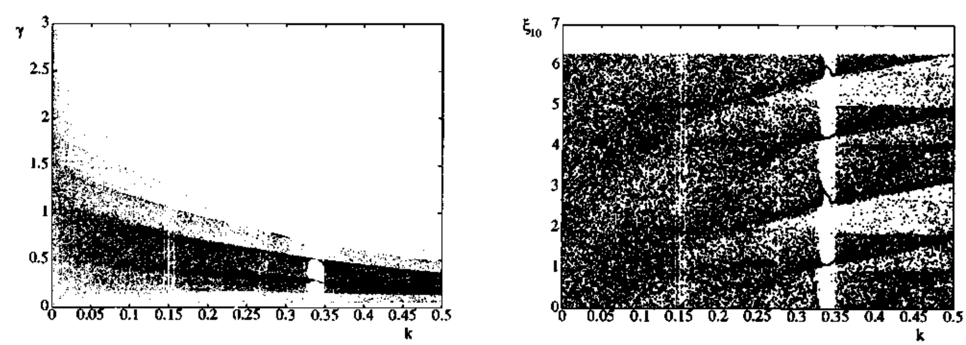
Remark When $k = \kappa\varepsilon^\alpha$, $\alpha \geq -0.5$, the phase space volume (area) decreasing rate does also depend on ε:

$$\nu_1(x_1) + \nu_2(x_2) = O(-k\pi\varepsilon^\alpha) . \tag{6.14}$$



(8a) (8b)

Figure 8 *The first ($v_1(x_1)$) and second ($v_2(x_1)$) Lyapunov exponent for the Poincaré map in the dissipative case with $\varepsilon = 0.1$. The fixed initial state is $x_1 = (1.875, \pi)'$.*



(9a) (9b)

Figure 9 *The limit values that the state variables of the mapping, γ and ξ_{10}, will take for different values of k when $\varepsilon = 0.1$ and $x_1 = (1.875, \pi)'$.*

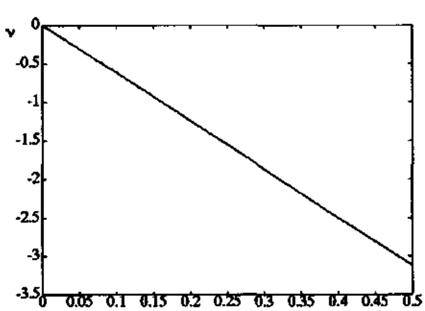


Figure 10 *The rate $v_1(x_1) + v_2(x_1)$ at which the phase space area decreases for $\varepsilon = 0.1$ when k is varied. This rate is independent of the initial state x_1.*

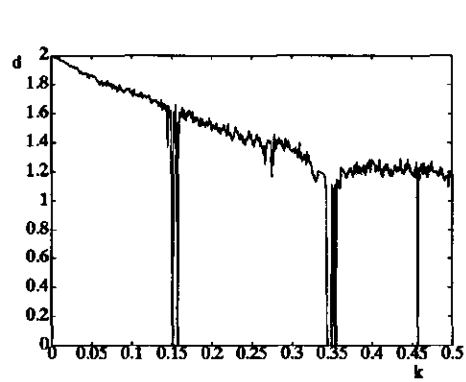


Figure 11 *The (fractal) dimension of the attractor for $\varepsilon = 0.1$ and $x_1 = (1.875,\pi)^t$ for different values of ε. This dimension is obtained with the aid of the conjecture of Kaplan and Yorke. Remark that there is indeed a chaotic attractor when the first Lyapunov exponent is larger than zero. (Compare this figure with fig. 8a).*

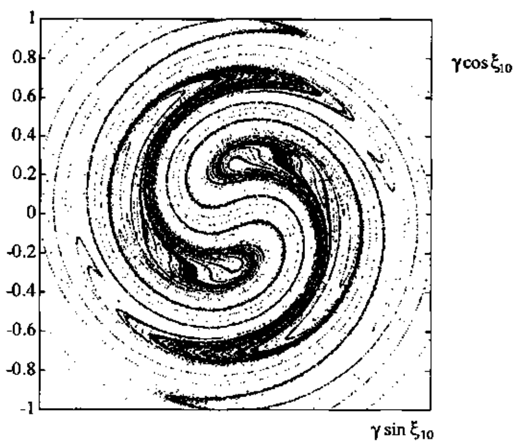


Figure 12 *For $k = 0.05$ and $\varepsilon = 0.1$ for $x_1 = (1.875,\pi)^t$ chaos appears in the Poincaré map. The phase portrait of $(\gamma\sin\xi_{10}, \gamma\cos\xi_{10})$ is illustrated.*

Remark The computation of the Jacobian M of the Poincaré map requires a considerable amount of analytical calculations as well as computing time necessary for the evaluation of the Lyapunov exponents from the product of Jacobians of a sufficiently large number of points of an orbit. For the computation of the Jacobian we refer to the appendix.

7. A comparison between the analytical results and numerical simulations

In this section we compare results of numerical simulations with the analytical approximations derived in the foregoing sections. The simulations have been carried out for the system described in eq. (3.1) by following the path of the state (x, x') for initial states $(x(t_0), x'(t_0))$. We assume for the initial forcing $G(t_0) = -1$. We consider both the conservative and the dissipative case. In order to compare the results we take a Poincaré section when the forcing term $G(t) = -1$. The Poincaré map constructed in section 5 needs a little adjustment. The analytical way in which we obtained this map, however, does not change. The integration constants of the solution at time points where $G(t) = -1$ yield a value for the parameter σ that determines which stable branch will be followed after crossing the bifurcation point. In table 1 we give results for one of the integration constants after some periods when $G(t) = -1$. Furthermore, it is described with the aid of the parameter σ which branch will be followed after the next crossing of the bifurcation point $G(t) = 0$ from below.

The analytical approximations are in reasonably good accordance with the numerical simulation results after a small number of forcing periods. However, when the number of forcing periods is increased, the approximations can deviate enormously. For the initial state that is chosen in table 1a the analytical and numerical approximation yield a quasiperiodic solution. When a periodic attractor arises in the dissipative case for both approximations there is a good agreement between the numerical and analytical results even for $t \to \infty$. This is the case for the initial state chosen in table 1b. The parameter k is deliberately chosen somewhat larger than in previous discussions, so that it can be seen that a periodic attractor arises in this case. When the approximations yield chaotic solutions, they will deviate after a few number of iterations due to the sensitive dependence on computational errors and on initial conditions. Differences between the numerical and analytical results can be due to omitting higher order terms in the calculations of the analytical expressions, to numerical computational errors, and to the sensitive dependence on initial conditions. When ε becomes smaller, the analytically obtained asymptotic solution will yield a better approximation.

	$x_a(t_0+i\frac{2\pi}{\varepsilon})$	$x_a'(t_0+i\frac{2\pi}{\varepsilon})$	$x_n(t_0+i\frac{2\pi}{\varepsilon})$	$x_n'(t_0+i\frac{2\pi}{\varepsilon})$	$\sigma(i)$
$i = 0$	0.0000	-0.1184	0.0000	-0.1184	-1(L)
$i = 1$	-0.0026	0.1185	-0.0107	0.1186	1(U)
$i = 2$	0.0011	-0.1186	0.0028	-0.1195	-1(L)
$i = 3$	0.0022	0.1186	0.0141	0.1187	1(U)
$i = 4$	-0.0019	-0.1184	-0.0208	-0.1174	-1(L)
$i = 5$	-0.0014	0.1184	0.0112	0.1181	1(U)

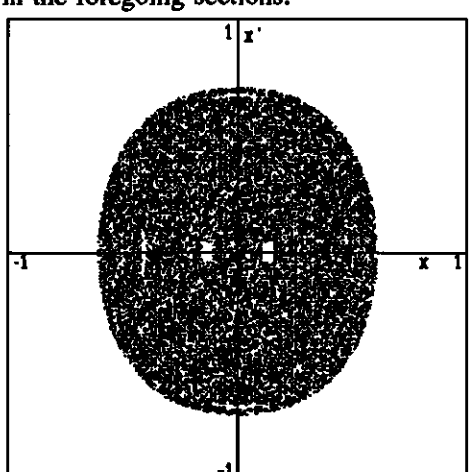
(1a) $k = 0$, $\varepsilon = 0.004$, $(x(t_0), x'(t_0), G(t_0)) = (0, -0.1184, -1)$

	$x_a(t_0+i\frac{2\pi}{\varepsilon})$	$x_a'(t_0+i\frac{2\pi}{\varepsilon})$	$x_n(t_0+i\frac{2\pi}{\varepsilon})$	$x_n'(t_0+i\frac{2\pi}{\varepsilon})$	$\sigma(i)$
$i = 0$	0.0400	0.0000	0.0400	0.0000	1(U)
$i = 1$	0.0398	-0.0427	0.0465	-0.0454	1(U)
$i = 2$	0.0429	-0.0488	0.0480	-0.0457	1(U)
$i = 3$	0.0433	-0.0490	0.0472	-0.0456	1(U)
$i = 4$	0.0432	-0.0490	0.0476	-0.0456	1(U)
$i = 5$	0.0432	-0.0490	0.0474	-0.0456	1(U)

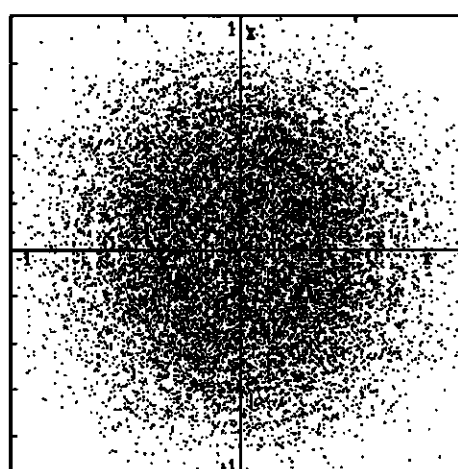
(1b) $k = 1.25$, $\varepsilon = 0.1$, $(x(t_0), x'(t_0), G(t_0)) = (0.04, 0, -1)$

Table 1 *The integration constants $x(t_0 + 2\pi i/\varepsilon)$ and $x'(t_0 + 2\pi i/\varepsilon)$ of the solutions after i forcing periods. Moreover, it is indicated with $\sigma(i)$ which stable branch (U=Upper, L=Lower) will be followed after crossing the bifurcation point from below i + 1 times. The subscripts n and a denote respectively numerical and analytical approximations.*

In the figures 13 and 14 we compare the numerically obtained Poincaré surface of section of the intersection of trajectories $(x(t), x'(t))$ with the surface of section $t = t_0 + 2n\pi/\varepsilon$, $n \in \mathbb{Z}$, with the Poincaré map obtained by analytical methods in the foregoing sections.



(13a)



(13b)

Figure 13 *Poincaré map of the x–x' plane for one forcing period of eq.(3.1) with $k = 0$ and $\varepsilon = 0.084$. The maps are obtained by numerical integration of eq.(3.1) (13a) and by numerical calculations for the analytical expressions (13b). Note the familiar KAM-patterns of islands of order in a sea of chaos.*

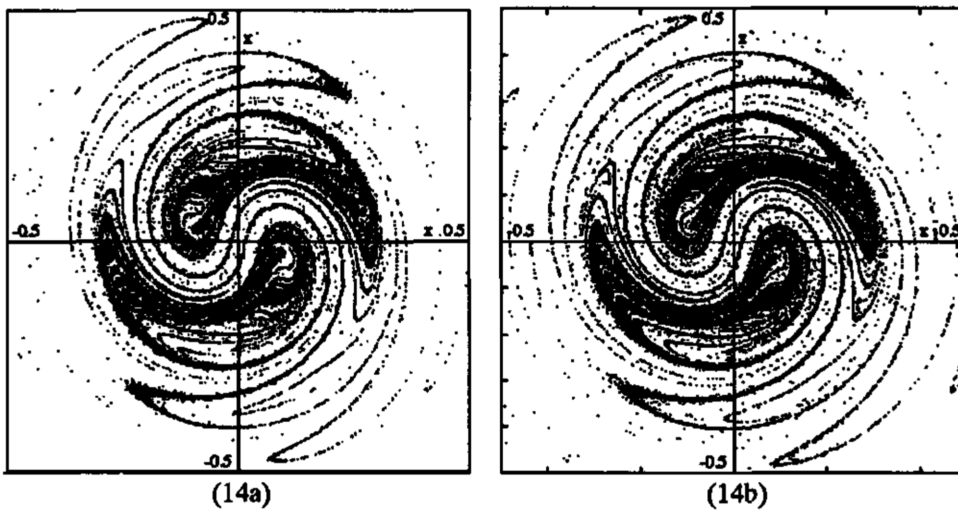


Figure 14 *Poincaré map of the x–x' plane for one forcing period of eq. (3.1) with $k = 0.1$ and $\varepsilon = 0.1$. Again the result of numerical simulations (14a) is compared with the map obtained by using analytical methods (14b).*

At first sight there does not seem to be a good agreement between the numerical and analytical results in the case without damping. However, figure 13 illustrates the behaviour of solutions of eq. (3.1) for $t \to \infty$. For a certain number of iterations of the Poincaré map the analytical results correspond with the numerical results. Chaotic effects and numerical computational errors cause differences between the approximations that become larger when time increases. Both approximations, however, yield a similar structure of quasiperiodic orbits (e.g. if $(x(0), x'(0), G(0)) = (0.15, 0, -1)$ a quasiperiodic attractor appears in both cases).

Although ε is not chosen very small, the analytic approximation seems to be in accordance with the numerical simulation results in the dissipative case of figure 14. The structure of the attractor is very similar for both approximations. The sequence of stable branches which will be followed after passage of the bifurcation point, however, is different for the two approximations.

In Bridge and Rand (1992) the complex dynamical behaviour in the class of systems, of which the problem of this study is a prototype, has been identified as coming from several sources: effects of stretching along the unstable manifold of the saddle for a forcing above the critical value, quasiperiodic rotation, separatrix crossing, and Hamiltonian chaos. Consequently, by analyzing the Poincaré map for one forcing period, the results are only satisfactory for a certain number of iterations of the mapping. When ε is small the effect of omitting higher order terms in the calculations of the analytical expressions is not felt. For ε large, however, these higher order terms will produce a large effect, which also explains the difference between the approximations. In many cases, however, we can gain a better insight in the structure of the attractor of the Poincaré map, when using the analytical results.

8. Conclusions

In this paper we have analyzed a second order system with a slowly varying parameter. We considered the case of repeated passage of a pitchfork bifurcation. By using perturbation techniques the solution of the problem is approximated asymptotically. In order to obtain local asymptotic expansions both averaging and boundary layer methods turned out to be necessary. The dynamics on the large time-scale where the solution oscillates around a (slowly varying) stable equilibrium solution is described with the aid of averaging methods. The approximation of the local outer solution above the bifurcation point requires "second order averaging", whereas "first order averaging" is sufficient to approximate the local outer solution below the bifurcation point. The validity of the asymptotic expansions is investigated and it is proven that the different local solutions overlap. The required information on the matching of the different locally valid asymptotic approximations is produced by an analytical study of the transition layer equation, which is the second Painlevé equation. There is a bijective connection between the integration constants of the slowly oscillating solutions that are valid before and after passage of the bifurcation point. Using the results of averaging methods, matching techniques and local connection formulas we can construct a Poincaré map for one forcing period that approximates the dynamics of the system.

The characteristics of the two-dimensional Poincaré map are analyzed by considering Lyapunov exponents and using the KAM-theorem in order to draw conclusions about the evolution of orbits. In the non-dissipative case for each value of the parameter there may be sensitive dependence on initial conditions, a criterion for chaos. This result confirms the results of Bridge and Rand (1992) and of Coppola and Rand (1990b). Both in the non-dissipative and in the dissipative case for fixed values of the parameters (quasi) periodic orbits, corresponding to closed curves in the phase portrait, and chaotic orbits are possible. With the aid of the Poincaré map the behaviour of the system can be predicted on a time-scale $O(1/\varepsilon)$ given the initial state and the values of the parameters. It is possible to construct symbol sequences of U's (Upper branch) and L's (Lower branch) of the branch that will be followed after each crossing of the bifurcation point from below. The periodic bifurcation takes place on a relatively small time-scale of length
$\varepsilon t = O(\varepsilon^{2/3})$.

It is noted that there is a good correspondence between the results obtained by numerical simulation of the system and the analytical results based on the Poincaré map. After a large number of forcing periods, however, this agreement may disappear due to sensitive dependence, numerical computational errors and the omission of higher order terms in calculating the analytical expressions. The analytical methods that are used in this study, are applicable to a wide class of (mechanical) problems for which the transition layer equation around the bifurcation points is described by the same second Painlevé transcendent. Consequently, the asymptotic matching conditions will have the same form as the conditions we have obtained in this study.

Acknowledgements

The author would like to thank Arjen Doelman for bringing to his attention the problem of this paper, and Johan Grasman for the discussions concerning the subject treated in this paper and for his remarks on the text.

Appendix

In this appendix we give the details of the computation of the Jacobian of the Poincaré map constructed in this study. We use the same notation as in sections 5 and 6. Furthermore, we define

$$p = |p|\exp(i\xi) , \quad p_1 = |p_1|\exp(i\xi_1) . \tag{A1}$$

In order to obtain the Jacobian we use the following results

$$\arg\Gamma(1 + iy) = \arg\Gamma(iy) + \frac{1}{2}\pi , \quad y>0 , \tag{A2}$$

$$\arg\Gamma(iy) = \mathrm{Im}(\ln\Gamma(iy)) , \tag{A3}$$

$$\frac{d}{dy}\mathrm{Im}(\ln\Gamma(1 + \frac{iy^2}{2})) = y\mathrm{Re}\psi(1 + \frac{iy^2}{2}), \tag{A4}$$

with $\psi(z)$ the Psi (Digamma) function. These results can be derived from the expressions 6.1.28, 6.1.44 and 6.3.1 in Abramowitz and Stegun (1964). It is now possible to compute partial derivatives for the transformations that have been carried out to construct a Poincaré map:

$$\begin{pmatrix} \dfrac{\partial |p|(j)}{\partial \gamma(j)} & \dfrac{\partial |p|(j)}{\partial \xi_{10}(j)} \\ \dfrac{\partial \xi(j)}{\partial \gamma(j)} & \dfrac{\partial \xi(j)}{\partial \xi_{10}(j)} \end{pmatrix} = \begin{pmatrix} \pi\gamma(j)\exp(\pi\gamma^2(j))(\exp(\pi\gamma^2(j)) - 1)^{-1/2} & 0 \\ 3\gamma(j)\ln 2 - \gamma(j)\mathrm{Re}\psi(1 + \dfrac{i\gamma^2(j)}{2}) & -1 \end{pmatrix}, \tag{A5}$$

$$\begin{pmatrix} \dfrac{\partial \beta(j)}{\partial \gamma(j)} & \dfrac{\partial \beta(j)}{\partial \xi_{10}(j)} \\ \dfrac{\partial \xi_{20}(j)}{\partial \gamma(j)} & \dfrac{\partial \xi_{20}(j)}{\partial \xi_{10}(j)} \end{pmatrix} \tag{A6}$$

with

$$\frac{\partial \beta(j)}{\partial \gamma(j)} = \frac{\gamma(j)}{2\beta(j)}\left(\frac{\exp(\pi\gamma^2(j)) - 2}{\exp(\pi\gamma^2(j)) - 1} - \frac{3\ln 2 - \mathrm{Re}\psi(1 + \dfrac{i\gamma^2(j)}{2})}{\pi\tan\xi(j)}\right),$$

$$\frac{\partial \xi_{20}(j)}{\partial \gamma(j)} = -B(j)\frac{\partial \beta(j)}{\partial \gamma(j)}(7\ln 2 - 2\mathrm{Re}\psi(1 + i\beta^2(j))) + \frac{2|p|(j)\left(\dfrac{\partial |p|(j)}{\partial \gamma(j)}\sin 2\xi(j) + |p|(j)\dfrac{\partial \xi(j)}{\partial \gamma(j)}(\cos 2\xi(j) + |p|^2(j))\right)}{1 + |p|^2(j)(2\cos 2\xi(j) + |p|^2(j))}$$

$$\frac{\partial \beta(j)}{\partial \xi_{10}(j)} = \frac{1}{2\pi\beta(j)\tan\xi(j)},$$

$$\frac{\partial \xi_{20}(j)}{\partial \xi_{10}(j)} = \frac{-\dfrac{7}{2}\ln 2 + \mathrm{Re}\psi(1 + i\beta^2(j))}{\pi\tan\xi(j)} - \frac{2|p|^2(j)(\cos 2\xi(j) + |p|^2(j))}{1 + |p|^2(j)(2\cos 2\xi(j) + |p|^2(j))},$$

$$\begin{pmatrix} \dfrac{\partial \beta_1(j)}{\partial \gamma(j)} & \dfrac{\partial \beta_1(j)}{\partial \xi_{10}(j)} \\ \dfrac{\partial \xi_{21}(j)}{\partial \gamma(j)} & \dfrac{\partial \xi_{21}(j)}{\partial \xi_{10}(j)} \end{pmatrix} \tag{A7}$$

with

$$\frac{\partial\beta_1(j)}{\partial\gamma(j)} = \exp(-\frac{k\pi}{2})\frac{\partial\beta(j)}{\partial\gamma(j)} ,$$

$$\frac{\partial\xi_{21}(j)}{\partial\gamma(j)} = -3\beta(j)g_2(k)\frac{\partial\beta(j)}{\partial\gamma(j)} - \frac{\partial\xi_{20}(j)}{\partial\gamma(j)} ,$$

$$\frac{\partial\beta_1(j)}{\partial\xi_{10}(j)} = \frac{\exp(-\frac{k\pi}{2})}{2\pi r\tan\xi(j)} ,$$

$$\frac{\partial\xi_{21}(j)}{\partial\xi_{10}(j)} = -3\beta(j)g_2(k)\frac{\partial\beta(j)}{\partial\xi_{10}(j)} - \frac{\partial\xi_{20}(j)}{\partial\xi_{10}(j)} ,$$

$$\begin{pmatrix} \dfrac{\partial|p_1|^2(j)}{\partial\beta_1(j)} \\ \\ \dfrac{\partial|p_1|^2(j)}{\partial\xi_{21}(j)} \end{pmatrix} = \begin{pmatrix} \beta_1(j)e^{\pi\beta_1^2(j)}(8\pi e^{\pi\beta_1^2(j)} - 4\pi(e^{2\pi\beta_1^2(j)} - 1)^{1/2}\cos(d) - 4\pi e^{2\pi\beta_1^2(j)}(e^{2\pi\beta_1^2(j)} - 1)^{-1/2}\cos(d)) \\ + 2(e^{2\pi\beta_1^2(j)} - 1)^{1/2}\sin(d)(7\ln 2 - 2\mathrm{Re}\psi(1+i\beta_1^2(j))) \\ \\ 2e^{\pi\beta_1^2(j)}(e^{2\pi\beta_1^2(j)} - 1)^{1/2}\sin(d) \end{pmatrix} , \tag{A8}$$

with

$$d = -\frac{3\pi}{4} - \frac{7}{2}\beta_1^2(j)\ln 2 + \arg\Gamma(i\beta_1^2(j)) + \xi_{21}(j) , \tag{A9}$$

$$\begin{pmatrix} \dfrac{\partial\xi_1(j)}{\partial\gamma(j)} \\ \\ \dfrac{\partial\xi_1(j)}{\partial\xi_{10}(j)} \end{pmatrix} = \begin{pmatrix} \dfrac{s\left(\dfrac{|p_1|^2(j) - 1}{|p_1|(j)}\dfrac{\partial|p_1|^2(j)}{\partial\gamma(j)} - 4(1 + |p_1|^2(j))|p_1|(j)\pi\beta_1(j)\dfrac{\partial\beta_1(j)}{\partial\gamma(j)}\right)}{2|p_1|(j)(4|p_1|^2(j)\exp(2\pi\beta_1^2(j)) - (1 + |p_1|^2(j))^2)^{1/2}} \\ \\ \dfrac{s\left(\dfrac{|p_1|^2(j) - 1}{|p_1|(j)}\dfrac{\partial|p_1|^2(j)}{\partial\xi_{10}(j)} - 4(1 + |p_1|^2(j))|p_1|(j)\pi\beta_1(j)\dfrac{\partial\beta_1(j)}{\partial\xi_{10}(j)}\right)}{2|p_1|(j)(4|p_1|^2(j)\exp(2\pi\beta_1^2(j)) - (1 + |p_1|^2(j))^2)^{1/2}} \end{pmatrix} , \tag{A10}$$

$$\begin{pmatrix} \dfrac{\partial\gamma_1(j)}{\partial\gamma(j)} & \dfrac{\partial\gamma_1(j)}{\partial\xi_{10}(j)} \\ \dfrac{\partial\xi_{11}(j)}{\partial\gamma(j)} & \dfrac{\partial\xi_{11}(j)}{\partial\xi_{10}(j)} \end{pmatrix} \tag{A11}$$

with

$$\frac{\partial\gamma_1(j)}{\partial\gamma(j)} = \frac{\partial|p_1|^2(j)/\partial\gamma(j)}{2\pi\gamma_1(j)(1 + |p_1|^2(j))} ,$$

$$\frac{\partial\xi_{11}(j)}{\partial\gamma(j)} = \frac{\partial\gamma_1(j)}{\partial\gamma(j)}\left(3\gamma_1(j)\ln 2 - \gamma_1(j)\mathrm{Re}\psi(1 + \frac{i\gamma_1^2(j)}{2})\right) - \frac{\partial\xi_1(j)}{\partial\gamma(j)} ,$$

$$\frac{\partial\gamma_1(j)}{\partial\xi_{10}(j)} = \frac{\dfrac{\partial|p_1|^2(j)}{\partial\xi_{10}(j)}}{2\pi\gamma_1(j)(1 + |p_1|^2(j))} ,$$

$$\frac{\partial\xi_{11}(j)}{\partial\xi_{10}(j)} = \frac{\partial\gamma_1(j)}{\partial\xi_{10}(j)}\left(3\gamma_1(j)\ln 2 - \gamma_1(j)\mathrm{Re}\psi(1 + \frac{i\gamma_1^2(j)}{2})\right) - \frac{\partial\xi_1(j)}{\partial\xi_{10}(j)} .$$

Using (A5) - (A11) we finally arrive at the following Jacobian of the Poincaré map:

$$\begin{pmatrix} \dfrac{\partial\gamma(j + 1)}{\partial\gamma(j)} & \dfrac{\partial\gamma(j + 1)}{\partial\xi_{10}(j)} \\ \dfrac{\partial\xi_{10}(j + 1)}{\partial\gamma(j)} & \dfrac{\partial\xi_{10}(j + 1)}{\partial\xi_{10}(j)} \end{pmatrix} \tag{A12}$$

with

$$\frac{\partial\gamma(j + 1)}{\partial\gamma(j)} = \exp(-\frac{k\pi}{2})\frac{\partial\gamma_1(j)}{\partial\gamma(j)} ,$$

$$\frac{\partial\xi_{10}(j+1)}{\partial\gamma(j)} = \frac{3}{2}\gamma_2(j)g_1(k)\frac{\partial\gamma_1(j)}{\partial\gamma(j)} - \frac{\partial\xi_{11}(j)}{\partial\gamma(j)} ,$$

$$\frac{\partial\gamma(j+1)}{\partial\xi_{10}(j)} = \exp(-\frac{k\pi}{2})\frac{\partial\gamma_1(j)}{\partial\xi_{10}(j)} ,$$

$$\frac{\partial\xi_{10}(j+1)}{\partial\xi_{10}(j)} = \frac{3}{2}\gamma_1(j)g_2(k)\frac{\partial\gamma_1(j)}{\partial\xi_{10}(j)} - \frac{\partial\xi_{11}(j)}{\partial\xi_{10}(j)} .$$

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Chapter 5

Slow passage through a transcritical bifurcation[4]

Abstract

This paper deals with a class of second order differential equations with two equilibrium solutions. As a slowly varying parameter crosses a critical value a transcritical bifurcation occurs. The local transition behaviour is described by a non-integrable nonlinear differential equation with the Airy equation as linearization. The aim of this study is to predict whether or not an explosion will take place after passage of the bifurcation point given the initial state. For that purpose use is made of averaging methods and matched asymptotic expansions.

1. Introduction

In this study we analyze a transcritical bifurcation problem described by a second order nonlinear differential equation depending on two parameters k and λ:

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} = H(x,\lambda) \ . \tag{1.1}$$

The damping is chosen to be fixed and the parameter λ slowly varies in time. Both parameters depend on a small parameter ε: $k = \kappa\varepsilon$ and $\lambda = \lambda(\varepsilon t)$, $0 < \varepsilon << 1$. The initial value $\lambda(0)$ is chosen smaller than a certain critical value λ_c for which the reduced system ($\varepsilon = 0$) exhibits a transcritical bifurcation. For the reduced system (λ is fixed and no damping) with $\lambda \neq \lambda_c$ the stability of an equilibrium solution $x_E(\lambda)$ is determined by the linearization of (1.1). The critical value λ_c of λ occurs if

$$\frac{\partial H}{\partial x}(x_E(\lambda_c),\lambda_c) = 0 \ . \tag{1.2}$$

[4] by G.J.M. Marée; in revised form to be submitted

For this value of λ a stable and an unstable equilibrium solution of the reduced system intersect and there is an exchange of stabilities. Since $H(x_E,\lambda) = 0$ and therefore $dx_E/d\lambda = -H_\lambda/H_x$, we obtain in our case

$$\frac{\partial H}{\partial \lambda}(x_E(\lambda_c),\lambda_c) = 0 \; . \tag{1.3}$$

In the neighbourhood of $x = x_E(\lambda_c)$, $\lambda = \lambda_c$ we assume H to have the following form:

$$\begin{aligned} H(x,\lambda) &= \alpha_{20}((x - x_E(\lambda_c)) - a_1(\lambda - \lambda_c) - b_1(\lambda - \lambda_c)^2 - \ldots) \\ &\qquad ((x - x_E(\lambda_c)) - a_2(\lambda - \lambda_c) - b_2(\lambda-\lambda_c)^2 - \ldots) \\ &= \alpha_{20}(x - x_E(\lambda_c))^2 + \alpha_{11}(\lambda - \lambda_c)(x - x_E(\lambda_c)) + \alpha_{02}(\lambda - \lambda_c)^2 + \ldots \; , \end{aligned} \tag{1.4}$$

$$\text{where} \quad \alpha_{nm} = \frac{1}{n!m!}\left(\frac{\partial}{\partial x}\right)^n \left(\frac{\partial}{\partial \lambda}\right)^m H(x_E(\lambda_c),\lambda_c) \quad .$$

The transcritical bifurcation is characterized by the inequality:

$$\alpha^2_{11} - 4\alpha_{02}\alpha_{20} > 0. \tag{1.5a}$$

Furthermore, we demand

$$a_2 > a_1 \text{ and } \alpha_{20} < 0 \; . \tag{1.5b,c}$$

Condition (1.5a) implies that two equilibria exist in the neighbourhood of $\lambda = \lambda_c$ and condition (1.5c) indicates that $H(x,\lambda) < 0$ for sufficiently large x. The bifurcation time t_c satisfies:

$$\lambda(\varepsilon t_c) = \lambda_c \; . \tag{1.6}$$

An example of a mechanical system that exhibits transcritical dynamical behaviour is the motion of a current-carrying conductor restrained by strings and subjected to a magnetic field, sketched in figure 1 (see also Stoker (1950)).

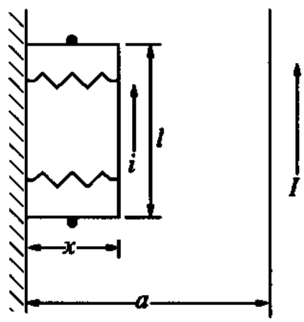


Figure 1 ***Elastically restrained current-carrying wire subjected to a force from a magnetic field due to an other infinitely long fixed parallel wire.***

The differential equation that governs the motion of the conductor is

$$m\frac{d^2x}{dt^2} + c(x - \frac{\rho}{a - x}) = 0 . \tag{1.7}$$

The parameter ρ is given by $\rho = 2Iil/c$, and c is the spring constant. The term $\rho c/(a-x)$ describes the force of attraction between the wires through the magnetic field. We assume that the current in one of the wires is harmonic in time, so that we define

$$\rho = \frac{1}{2}(\frac{a^2}{4} + b) + \frac{1}{2}(\frac{a^2}{4} - b)\sin(\varepsilon t) \ , \ 0 < b < \frac{a^2}{4} . \tag{1.8}$$

This assumption ensures that the force due to the magnetic field is never larger than the spring force. The singularities of (1.7) are the points $(x_1,0)$, $(x_2,0)$ with x_1 and x_2 the roots of $x^2 - ax + \rho = 0$. We define

$$x_1 = \frac{a}{2} + \sqrt{\frac{a^2}{4} - \rho} \ , \quad x_2 = \frac{a}{2} - \sqrt{\frac{a^2}{4} - \rho} \ . \tag{1.9a,b}$$

In figure 2 these roots are sketched as a function in time.

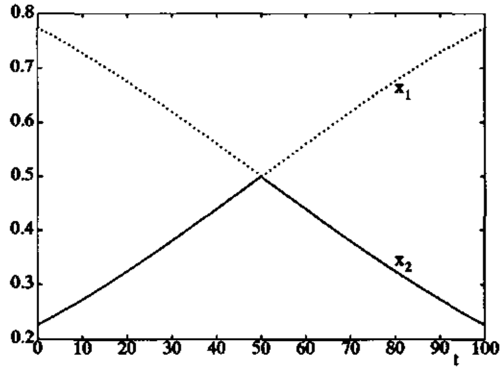


Figure 2 *The roots $x_{1,2}(t)$ of $x^2 - x + (0.175 + 0.075\sin(\varepsilon\pi t)) = 0$ with $\varepsilon = 0.01$.*

If $b \leq \rho < a^2/4$ both roots $x_{1,2}$ are real and positive with $x_1 > x_2$. For $\rho = a^2/4$ both roots have the value $a/2$. The potential energy $F(x)$ of the system is given by

$$F(x) = c\frac{x^2}{2} + c\rho\log(a - x) \ . \tag{1.10}$$

For $\rho < a^2/4$ the solution curves are closed curves surrounding the point $(x_2,0)$ until the energy constant is so large that the solution curve contains the saddle singularity at $(x_1,0)$. The motion of the wire is periodic if the displacement of the wire and the initial velocity are not too large. When $\rho = a^2/4$ the solution curve through $(a/2,0)$ has a cusp at this point and there are no closed solution curves. The critical value of ρ occurs at a time t_c for which $\sin(\varepsilon t_c) = 1$. We aim to predict whether or not the mass will tend to move towards the fixed wire for $t > t_c$.

For λ fixed, smaller than λ_c, the system (1.1) has one stable and one unstable equilibrium. On a large time-scale the system shows a damped oscillation until λ has reached λ_c. The parameter λ slowly passes λ_c corresponding to a transcritical bifurcation, indicating that the two equilibria exchange stability. Because λ is slowly varying, the equilibria are slowly varying in time. The solution in the neighbourhood of the stable slowly varying equilibrium solution is approximated by a harmonic oscillation. For λ outside a certain ε-neighbourhood of λ_c asymptotic approximations are obtained with the aid of averaging methods. The bifurcation is described by a local approximation, being a transition layer. Local scaling analysis yields as approximating differential equation a non-integrable nonlinear Airy equation. Depending on the matching conditions transition layer solutions either increase algebraically or explode through a singularity.

Haberman (1979) also studied this type of nonlinear second order differential equations. We apply averaging and study systems with damping, while Haberman uses the method of eliminating secular terms in problems without damping. The principal difference is that in our study attention is focused on the problem of predicting the behaviour of solutions after passage of the bifurcation point depending on the initial conditions. Moreover, the validity of our approximations is proved. Lebovitz and Schaar (1975) studied the transcritical exchange of stabilities in slowly varying first order systems. Other results on slow passage through some other types of bifurcations are obtained by Baer, Erneux and Rinzel (1989), Holden and Erneux(1993), Marée (1993,1995) and Neishtadt (1987,1988).

In section 2 the transcritical bifurcation is illustrated. We consider the different equilibrium states of the undamped system when the parameter λ is fixed. In section 3 slowly varying equilibrium solutions are computed. Moreover, the solution in the neighbourhood of the slowly varying stable equilibrium is approximated by averaging for an interval in which the solution is bounded away from the bifurcation point. In section 4 we analyze the transition layer equation, and obtain matching conditions for the local asymptotic inner solution. In section 5 and 6 we predict the behaviour of solutions after passage of the bifurcation point at the basis of the initial conditions and the values of the parameters. Moreover, some interesting properties of the transition are observed. In section 7 we make some concluding remarks and discuss some open problems.

2. The qualitative behaviour of the system

Substitution of $\varepsilon = 0$ in (1.1) yields the reduced system

$$\frac{d^2x}{dt^2} = H(x,\lambda) \ , \tag{2.1}$$

in which λ is independent of t. We assume that for λ not equal to a critical value λ_c there are two equilibrium solutions which satisfy the algebraic equation

$$H(x,\lambda) = 0 \ . \tag{2.2}$$

The linear stability of an equilibrium solution $x_E(\lambda)$ is determined by the sign of $H_x(x_E,\lambda)$ (x_E is stable if $H_x(x_E,\lambda) < 0$). Linearization at the equilibrium points yields an equilibrium being a centre point and an unstable point of saddle point type for $\lambda < \lambda_c$. For $\lambda > \lambda_c$ the saddle point has changed in a centre point and the centre point in a saddle point. For $\lambda = \lambda_c$ the equilibria coalesce and the solution curve through the unique equilibrium point has a cusp at this point.

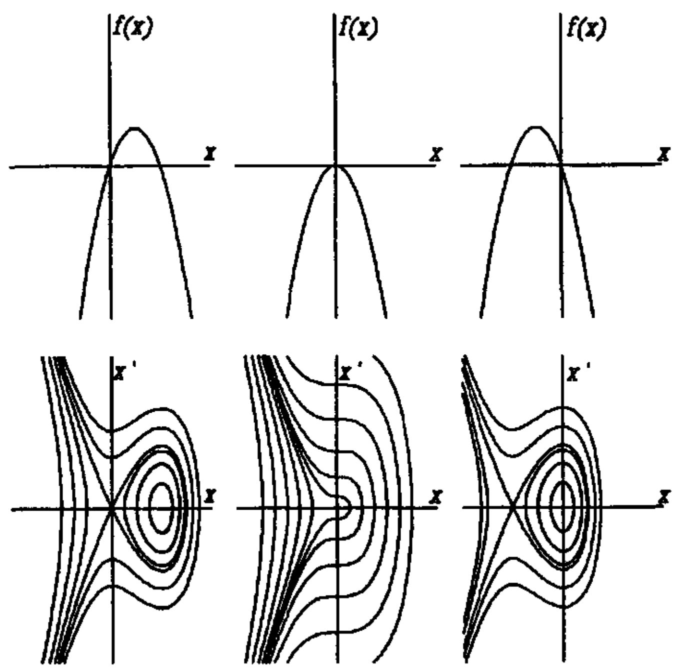
In literature this phenomenon has commonly been called a transcritical bifurcation. (Haberman (1979) calls the transition from a stable line to an other stable line, as λ crosses λ_c, a "straight-straight bifurcation" to indicate its geometric property). In figures 3 and 4 the transcritical bifurcation is illustrated for the case that $H(x,\lambda) = -x^2 + x\lambda$. For $\varepsilon = 0$ the energy integral of this system equals

$$E = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{3}x^3 - \frac{1}{2}x^2\lambda\,. \tag{2.3}$$

The homoclinic orbits for $\lambda < 0$ and $\lambda > 0$ are characterized respectively by

$$E(x,\, dx/dt) = 2\lambda^3/3 \text{ and } E(x,\, dx/dt) = 0. \tag{2.4}$$

The damping turns a centre point into a stable spiral point.



(a) $\lambda > \lambda_c$ (b) $\lambda = \lambda_c$ (c) $\lambda < \lambda_c$

Figure 3 *The graph of the function $f(x) = x(\lambda - x)$ in three cases of fixed λ and the corresponding phase portrait of $d^2x/dt^2 = f(x)$ for different values of λ.*

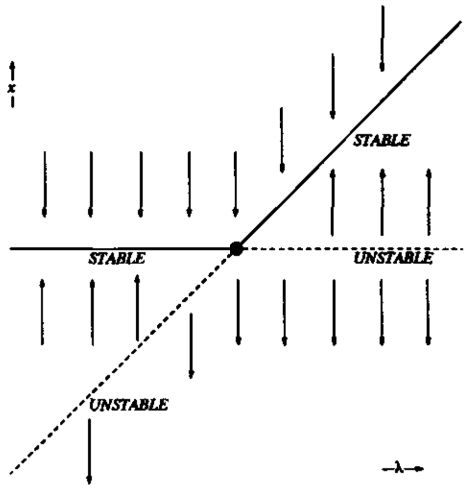


Figure 4 *The branches of the limit solution of $d^2x/dt^2 = x(\lambda - x)$ as a function of λ.*

Depending on the matching conditions (which depend on the initial values and on the values of the parameters) the solution of a second order transcritical bifurcation problem either approaches the new stable equilibrium, exponentially decays, or explodes after passage of the bifurcation point. In figure 5 we give some numerical solutions that pass a transcritical bifurcation. It shows that the dynamic behaviour of the system after the slow evolution through the critical conditions may strongly change with the initial values.

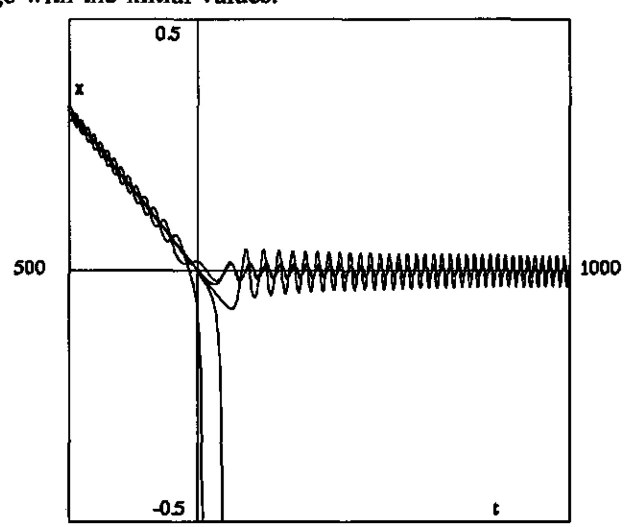


Figure 5 *Numerical solutions of $d^2x/dt^2 = -x^2 - \sin(\varepsilon t - \pi/2)x$ for various values of the initial values when $\varepsilon = 0.0025$.*

3. Local asymptotic expansions

The solution $x_E(\lambda)$ of the reduced problem $H(x,\lambda) = 0$ is the first term of an asymptotic expansion, which forms the slowly varying equilibrium solution. We rewrite (1.1) in order to analyze a slowly varying equilibrium $x_{sve}(\varepsilon t)$:

$$\varepsilon^2\left(\frac{d^2x}{d(\varepsilon t)^2} + \kappa\frac{dx}{d(\varepsilon t)}\right) = H(x,\lambda) \; . \tag{3.1}$$

Assuming that the derivatives in (3.1) are small we obtain a slowly varying equilibrium solution by perturbing the dependent variable at $x_E(\lambda(\varepsilon t))$. Because $H(x_E,\lambda) = 0$ we can use the Taylor series of $H(x,\lambda)$ at $x = x_E(\lambda)$. In this way we obtain an asymptotic expansion for $x_{sve}(\varepsilon t)$:

$$x_{sve}(\varepsilon t) = x_E(\lambda(\varepsilon t)) + \varepsilon^2 x_1(\varepsilon t) + \varepsilon^4 x_2(\varepsilon t) + \varepsilon^6 x_3(\varepsilon t) + \dots \, , \tag{3.2}$$

with

$$x_1(\varepsilon t) = \left(\frac{d^2x_E}{d(\varepsilon t)^2} + \frac{\kappa dx_E}{d(\varepsilon t)}\right)/\frac{\partial H}{\partial x}\Big|_{x_E} \, , \tag{3.3a}$$

$$x_2(\varepsilon t) = \left(\frac{d^2x_1}{d(\varepsilon t)^2} + \frac{\kappa dx_1}{d(\varepsilon t)} - \frac{x_1^2}{2}\frac{\partial^2 H}{\partial x^2}\Big|_{x_E}\right)/\frac{\partial H}{\partial x}\Big|_{x_E} \, , \tag{3.3b}$$

$$x_3(\varepsilon t) = \left(\frac{d^2x_2}{d(\varepsilon t)^2} + \frac{\kappa dx_2}{d(\varepsilon t)} - x_1x_2\frac{\partial^2 F}{\partial u^2}\Big|_{x_E} - \frac{x_1^3}{6}\frac{\partial^3 H}{\partial x^3}\Big|_{x_E}\right)/\frac{\partial H}{\partial x}\Big|_{x_E} \, . \tag{3.3c}$$

For a solution that holds in a ε^μ-neighbourhood of the slowly varying stable solution we write:

$$x(t) = x_{sve}(\varepsilon t) + \varepsilon^\mu u(t) \; , \; \mu > 0 \; . \tag{3.4}$$

Furthermore, we take as initial values

$$x(0) = x_{sve}(0) + \varepsilon^\mu x_0 \, , \tag{3.5a}$$

$$\frac{dx}{dt}(0) = \frac{dx_{sve}}{dt}(0) + \varepsilon^\mu v_0 \, , \tag{3.5b}$$

$$\lambda(0) = \lambda_0 < \lambda_c \; . \tag{3.5c}$$

Substitution in (1.1) yields (using Taylor expansions) a slowly varying oscillating system

$$\frac{d^2u}{dt^2} + \omega^2(\varepsilon t)u = -\kappa\varepsilon\frac{du}{dt} + \frac{\varepsilon^{\mu}}{2}\frac{\partial^2 H}{\partial x^2}\Big|_{x_{sve}} u^2 + \frac{\varepsilon^{2\mu}}{6}\frac{\partial^3 H}{\partial x^3}\Big|_{x_{sve}} u^3 + \ \ldots \tag{3.6}$$

with

$$\omega^2(\varepsilon t) = -\frac{\partial H}{\partial x}(x_{sve}(\varepsilon t), \lambda(\varepsilon t)) > 0 \ . \tag{3.7}$$

From (3.6) and (3.3) we observe that a critical point is approached as $\omega(\varepsilon t)$ tends to zero, which we already expected from the linear stability analysis (see (1.2)).

We now carry out some transformations in order to obtain a system to which the averaging method applies.

- Damping transformation: $u = e^{-\frac{1}{2}\kappa\varepsilon t} w := r_1 w$. (3.8)

- Adiabatic transformation: $w = \omega^{-\frac{1}{2}}(\varepsilon t) q$. (3.9)

- Averaging transformations; introduction of polar coordinates:

$$\frac{dq}{dt} = \omega(\varepsilon t) v \ , \tag{3.10a}$$

$$\frac{dv}{dt} = \frac{1}{\omega(\varepsilon t)}\frac{d^2q}{dt^2} - \frac{\varepsilon}{\omega(\varepsilon t)}\frac{d\omega}{d(\varepsilon t)} v \ , \tag{3.10b}$$

$$q = r\cos\varphi \ , \ v = -r\sin\varphi \tag{3.11a}$$

with initial values

$$q(0) = r_0\cos\varphi_0 \ , \ v(0) = -r_0\sin\varphi_0 \ . \tag{3.11b}$$

- Fast time: $\eta = \int_0^{\varepsilon t}\frac{\omega(\tau)d\tau}{\varepsilon}$. (3.12)

- Angle transformation: $\psi = \varphi - \eta$. (3.13)

We finally arrive at the following initial value problem

$$\frac{dr_1}{d\eta} = -\frac{\kappa\varepsilon}{2\omega(\varepsilon t)}r_1\ ,\ r_1(0) = 1\ , \tag{3.14a}$$

$$\begin{aligned}\frac{dr}{d\eta} = &-\frac{1}{4}\varepsilon^2\kappa^2\omega^{-2}(\varepsilon t)r\cos(\psi+\eta)\sin(\psi+\eta)\\ &-\frac{1}{2}\varepsilon^{\mu}\omega^{-5/2}(\varepsilon t)r_1r^2\cos^2(\psi+\eta)\sin(\psi+\eta)\frac{\partial^2H}{\partial x^2}|_{x_{sve}}\\ &-\frac{1}{6}\varepsilon^{2\mu}\omega^{-3}(\varepsilon t)r_1^2r^3\cos^3(\psi+\eta)\sin(\psi+\eta)\frac{\partial^3H}{\partial x^3}|_{x_{sve}} + \dots\ ,\ r(0) = r_0\ ,\end{aligned} \tag{3.14b}$$

$$\begin{aligned}\frac{d\psi}{d\eta} = &-\frac{1}{4}\varepsilon^2\kappa^2\omega^{-2}(\varepsilon t)\cos^2(\psi+\eta)\\ &-\frac{1}{2}\varepsilon^{\mu}\omega^{-5/2}(\varepsilon t)r_1r\cos^3(\psi+\eta)\frac{\partial^2H}{\partial x^2}|_{x_{sve}}\\ &-\frac{1}{6}\varepsilon^{2\mu}\omega^{-3}(\varepsilon t)r_1^2r^2\cos^4(\psi+\eta)\frac{\partial^3H}{\partial x^3}|_{x_{sve}} + \dots\ ,\ \psi(0) = \varphi_0\ ,\end{aligned} \tag{3.14c}$$

where the dots stand for higher order ε-terms.

We will consider

$$\lambda_c > \lambda_c - \delta(\varepsilon) \geq \lambda \geq \lambda_0 = O(1) \tag{3.15a}$$

with

$$\varepsilon^{4\mu/5} = o(\delta(\varepsilon))\ . \tag{3.15b}$$

Then the following approximations are valid:

$$\varepsilon^{\mu}\omega^{-5/2}(\varepsilon t) = o(1)\ , \tag{3.16a}$$

$$0 \leq \eta = \frac{1}{\varepsilon}\int_0^{\varepsilon t}\omega(\tau)d\tau \leq \frac{1}{\varepsilon}\int_0^{\varepsilon t_c}\omega(\tau)d\tau - \delta_1^{-1}(\varepsilon) := \eta_c - \delta_1^{-1}(\varepsilon) \tag{3.16b}$$

with t_c such that

$$\lambda(\varepsilon t_c) = \lambda_c \tag{3.16c}$$

and

$$\delta_1(\varepsilon) = o(1) \text{ as well as } \varepsilon = O(\delta_1(\varepsilon))\ . \tag{3.16d}$$

With the aid of the lemma of Gronwall a "second order" approximation theorem can be proven for $\eta \in [0, \eta_c - \delta_1^{-1}(\varepsilon)]$, which yields an estimate of $O(\varepsilon^{2\mu}\omega^{-5}(\varepsilon t))$. This theorem is a generalization of the theorems 3.2 and 4.1 of Marée (1995). The proof goes along the same lines. We state this approximation theorem as follows.

Theorem 3.1 Consider the initial value problems

$$\frac{dx}{d\eta} = \frac{\varepsilon^{\mu}}{\omega^{5/2}(\eta,\varepsilon)}f(x,\eta)f_1(r_1)+\frac{\varepsilon^{2\mu}}{\omega^{3}(\eta,\varepsilon)}g(x,\eta)g_1(r_1)+\frac{\varepsilon^{2}}{\omega^{2}(\eta,\varepsilon)}h(x,\eta)h_1(r_1)+o(\varepsilon^{\mu}\omega^{-5/2}(\eta,\varepsilon)) \ , \ x(0) = x_0 \tag{3.17}$$

and

$$\frac{dx_a}{d\eta} = \frac{\varepsilon^{2\mu}}{\omega^{5}(\eta,\varepsilon)}f^{10}(x_a,\eta)f_1(r_1) + \frac{\varepsilon^{2\mu}}{\omega^{3}(\eta,\varepsilon)}g^{0}(x_a,\eta)g_1(r_1) + \frac{\varepsilon^{2}}{\omega^{2}(\eta,\varepsilon)}h^{0}(x_a,\eta)h_1(r_1) \ , \ x_a(0) = x_0 \tag{3.18}$$

with f, g, h: $\mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}^2$, $\eta \in [0, \eta_c - \delta_1^{-1}(\varepsilon)]$ with $\delta_1(\varepsilon) = o(1)$ and $\delta_1(\varepsilon) = O(\varepsilon)$, $\varepsilon^{\mu}\omega^{-5/2}(\eta,\varepsilon) = o(1)$, $\varepsilon \in (0,\varepsilon_0]$, and $r_1 = r_1(\eta)$ bounded.
Furthermore,

$$f^1(x,\eta) = \nabla f(x,\eta)u^1(x,\eta)f_1(r_1) \tag{3.19}$$

and

$$u^1(x,\eta) = \int_0^{\eta} f(x,s)ds - \frac{1}{T}\int_0^{T}\int_0^{\eta} f(x,s)ds d\eta \ . \tag{3.20}$$

Suppose
a) f has a Lipschitz-continuous first derivative in x, the vector-functions f_1, g_1, h_1, g, h, ∇g, ∇h, ∇f and $\nabla^2 f$ are continuous in the variables and bounded by a constant M, independent of ε, for $\eta \in [0, \eta_c - \delta_1^{-1}(\varepsilon)]$;
b) f, g and h are T-periodic in η, averages f^0, g^0 and h^0 (f^1 has average f^{10}). Moreover $f^0 = 0$.
Then $x(\eta) = x_a(\eta) + o(1)$ for $\eta \in [0, \eta_c - \delta_1^{-1}(\varepsilon)]$.

Remark Since $f^0 = 0$, we have to apply second order averaging (see also Sanders and Verhulst (1987)).

Corollary 3.2 For $\lambda_c > \lambda_c - \delta(\varepsilon) \geq \lambda \geq \lambda_0$ - so for $\eta \in [0, \eta_c - \delta_1^{-1}(\varepsilon)]$ (see (3.16b)) - the solution of (1.1) has the following expansion

$$x(t) = x_{sve}(\varepsilon t) + \varepsilon^{\mu}r_a\exp(\frac{-\kappa\varepsilon t}{2})\omega^{-1/2}(\varepsilon t)\cos(\eta + \psi_a) + o(\varepsilon^{\mu}\omega^{-1/2}(\varepsilon t)) \tag{3.21}$$

with $(r_a\,,\,\psi_a)$ the solution of the system

$$\frac{dr_a}{d\eta} = 0\ ,\ r_a(0) = r_0\ , \tag{3.22a}$$

$$\frac{d\psi_a}{d\eta} = -\frac{5}{48}\varepsilon^{2\mu}\omega^{-5}(\varepsilon t)e^{-\kappa\varepsilon t}r_a^2\Big(\frac{\partial^2 H}{\partial x^2}\Big|_{x_{sve}}\Big)^2 - \frac{1}{16}\varepsilon^{2\mu}\omega^{-3}(\varepsilon t)e^{-\kappa\varepsilon t}r_a^2\frac{\partial^3 H}{\partial x^3}\Big|_{x_{sve}} - \frac{1}{8}\varepsilon^2\kappa^2\omega^{-2}(\varepsilon t)\ ,\ \psi_a(0) = \varphi_0\ . \tag{3.22b}$$

Remark The first term on the right hand side of (3.22b) stems from second order averaging, the last two terms from first order averaging.

The proof of this corollary immediately follows, because (3.14) is a special case of the initial value problem stated in theorem 3.1.

4. The transition layer equation and matching conditions

In order to obtain matching conditions for the local asymptotic solution describing the transcritical bifurcation we determine the asymptotic development of x when λ is in the neighbourhood of λ_c. We assume that we can expand λ near the bifurcation time t_c (see (3.16c)) in the following way

$$\lambda = \lambda_c + \lambda_c'(\varepsilon(t - t_c)) + \lambda_c''(\varepsilon(t - t_c))^2/2 + \dots\ ,\ \lambda_c^{(m)} = (d^m/d(\varepsilon t)^m)\lambda(\varepsilon t)\ ,\ \lambda_c' > 0\ . \tag{4.1}$$

The expansion for the slowly varying equilibrium solution in the vicinity of the critical time becomes

$$\begin{aligned} x_{sve} &= x_E(\lambda_c) + a_1(\lambda - \lambda_c) + b_1(\lambda - \lambda_c)^2 + \dots + \varepsilon^2\Big(\frac{a_1\lambda_c''+2b_1(\lambda_c')^2+\kappa a_1\lambda_c'}{\alpha_{20}\lambda_c'(a_1-a_2)(\varepsilon(t-t_c))} + \dots\Big) \\ &+ \varepsilon^4\Big(\frac{2(a_1\lambda_c''+2b_1(\lambda_c')^2+\kappa a_1\lambda_c')}{(\alpha_{20}\lambda_c'(a_1-a_2))^2(\varepsilon(t-t_c))^4} + \dots\Big) + \varepsilon^6\Big(\frac{40(a_1\lambda_c''+2b_1(\lambda_c')^2+\kappa a_1\lambda_c')}{(\alpha_{20}\lambda_c'(a_1-a_2))^3(\varepsilon(t-t_c))^7} + \dots\Big) + \dots\ , \end{aligned} \tag{4.2}$$

where we use the notation of (1.4) and the fact that near $x = x_E(\lambda_c)$, $\lambda = \lambda_c$:

$$\frac{\partial H}{\partial x}\Big|_{x_E} = \alpha_{20}(a_1 - a_2)(\lambda - \lambda_c)\ , \tag{4.3}$$

$$\frac{\partial^2 H}{\partial x^2}\Big|_{x_E} = 2\alpha_{20}\ . \tag{4.4}$$

Consequently, the asymptotic expansion of the slowly varying equilibrium solution does not hold anymore when $\lambda - \lambda_c = O(\varepsilon^{2/3})$, so on a time-scale $t - t_c = O(\varepsilon^{-1/3})$. From (3.7), (4.3) and (3.16a) it follows that the asymptotic averaged expansion breaks down when $\varepsilon^{\mu} = O((\lambda - \lambda_c)^{5/4})$. Thus, if we choose $\mu = 5/6$ the two types of expansions for a transcritical bifurcation break down simultaneously.

It follows that the outer asymptotic expansion ceases to be valid when

$$\lambda = \lambda_c + \varepsilon^{2/3}\lambda_c' s \text{ or } t = t_c + \varepsilon^{-1/3}s \tag{4.5}$$

and

$$x = x_E(\lambda_c) + \varepsilon^{2/3}y_0 + \varepsilon^{4/3}y_1 + \dots \ . \tag{4.6}$$

If we apply these transition layer scalings to (1.1), we obtain the following leading order transition equation:

$$\frac{d^2y_0}{ds^2} = \alpha_{20}(y_0 - a_1\lambda_c' s)(y_0 - a_2\lambda_c' s) \ . \tag{4.7}$$

The $O(\varepsilon^{2/3})$ equation has the following form:

$$\begin{aligned}\frac{d^2y_1}{ds^2} + \kappa\frac{dy_0}{ds} = \alpha_{20}(2y_0y_1 - (a_1 + a_2)(\lambda_c' s y_1 + \frac{\lambda_c''}{2}s^2y_0) + a_1a_2\lambda_c'\lambda_c'' s^3 \\ - (b_1 + b_2)(\lambda_c' s)^2 y_0 + (b_1a_2 + b_2a_1)(\lambda_c' s)^3) \ .\end{aligned} \tag{4.8}$$

Eq. (4.7) could also have been obtained by applying the analysis of significant degenerations of the differential equation (see Marée (1993,1995)).

Setting $t = t_c + \varepsilon^{-1/3}s$ and $\mu = 5/6$ we obtain the following approximations for the terms in the asymptotic expansion (3.21):

$$\begin{aligned}x_{sve} &= x_E(\lambda_c) + a_1\lambda_c'\varepsilon^{2/3}s + (a_1\frac{\lambda_c''}{2} + b_1(\lambda_c')^2)\varepsilon^{4/3}s^2 + \dots \\ &+\varepsilon^{4/3}(\frac{a_1\lambda_c''+2b_1(\lambda_c')^2+\kappa a_1\lambda_c'}{\alpha_{20}\lambda_c'(a_1-a_2)s}(1 + \frac{1}{\alpha_{20}\lambda_c'(a_1-a_2)s^3}\sum_{m=0}^{\infty}\frac{(3m + 2)!}{m!(3\alpha_{20}\lambda_c'(a_1-a_2)s^3)^m})) + \dots ,\end{aligned} \tag{4.9a}$$

$$\omega^2(\varepsilon t) = -\alpha_{20}(a_1 - a_2)\lambda_c' s\varepsilon^{2/3} , \tag{4.9b}$$

$$r_1 = \exp(\frac{-\kappa t_0}{2}) \text{ (with } t_c = \frac{t_0}{\varepsilon}\text{ , } t_0 > 0) \text{ ,} \tag{4.9c}$$

$$r_a = r_0 \text{ ,} \tag{4.9d}$$

$$\eta = \frac{\int_0^{t_0} \omega(\tau)d\tau}{\varepsilon} - \frac{2(\alpha_{20}(a_1 - a_2)\lambda_c')^{1/2}}{3}(-s)^{3/2} + \ \dots \text{ ,} \tag{4.9e}$$

$$\begin{aligned} \psi_a &= \int_0^{t_0 + \varepsilon^{2/3}s} - \frac{5}{48}\frac{\varepsilon^{2/3}}{\omega^4(\tau)}\exp(-\kappa\tau)r_0^2(\frac{\partial^2 H}{\partial x^2}|_{x_{sve}})^2 d\tau + \ \dots \\ &= -\frac{5}{48}(\alpha_{20}(a_1 - a_2)\lambda_c')^{-2}(-s)^{-1}r_0^2\exp(-\kappa t_0)(\frac{\partial^2 H}{\partial x^2}|_{x_{sve}})^2 + \ \dots \ . \end{aligned} \tag{4.9f}$$

Consequently, near the bifurcation point the outer solution behaves asymptotically as

$$x(t) \sim x_E(\lambda_c) + \varepsilon^{2/3}(a_1\lambda_c's + r_0 e^{-\frac{\kappa t_0}{2}}(-\alpha_{20}(a_1 - a_2)\lambda_c's)^{-1/4}\cos(-\frac{2}{3}(\alpha_{20}(a_1 - a_2)\lambda_c')^{1/2}(-s)^{3/2} + \theta)) + \ \dots \quad \text{if } s \rightarrow -\infty \text{ ,} \tag{4.10}$$

where r_0 and θ are constants determined by the initial conditions. At the time that a transcritical bifurcation is expected eq. (4.7) holds. The matching condition for this transition equation is that of a linear turning point problem:

$$y_0 \sim a_1\lambda_c's + r_0 e^{-\frac{\kappa t_0}{2}}\gamma^{-1/4}(-s)^{-1/4}\cos(-\frac{2}{3}\gamma^{1/2}(-s)^{3/2} + \theta) \text{ as } s \rightarrow -\infty \text{ ,} \tag{4.11}$$

where $\gamma = \alpha_{20}(a_1 - a_2)\lambda_c'$. (4.12)

We want to stress that the integration constants are independent of ε, which is in contrast with the jump phenomenon (Marée (1993)) and the slow pitchfork bifurcation (Marée (1995)).

5. The behaviour of solutions after passage of the bifurcation point

Applying the transformation

$$s = \gamma^{-1/3} z \ , \ y_0 = a_1 \lambda_c' s - \frac{\gamma^{2/3}}{\alpha_{20}} y \tag{5.1}$$

we obtain the following normalized version of the transition equation (4.7):

$$\frac{d^2 y}{dz^2} = - y(y - z) \tag{5.2}$$

with the following transformed matching condition as $z \to -\infty$

$$y \sim -r_0 e^{-\kappa t_0/2} \alpha_{20} \gamma^{-5/6} (-z)^{-1/4} \cos(- \frac{2}{3}(-z)^{3/2} + \theta_1) := \alpha_1 (-z)^{-1/4} \cos(- \frac{2}{3}(-z)^{3/2} + \theta_1) \ . \tag{5.3}$$

It can be shown (see e.g. Ince (1956)) that eq. (5.2) does not have the Painlevé property; the solution is not free from movable critical points. The differential equation is non-integrable in the sense that it can not be written as the compatibility condition of an associated system of linear partial differential equations that can be solved through data describing solutions of the linear system. For integrable systems, like the Painlevé transcendents (see e.g. Its and Novokshenov (1986)), these are soliton equations.

Remark A necessary condition for the differential equation to have the Painlevé property is that on can substitute the series

$$y(z) = \sum_{n=0}^{\infty} \beta_n (z - z_0)^{n+r} \ , \ z_0 \text{ arbitrary} \ . \tag{5.4}$$

If we investigate whether (5.2) has the Painlevé property we substitute (5.4) and find that $r = -2$, $\beta_0 = -6$, $\beta_1 = \beta_2 = \beta_5 = 0$, $\beta_3 = 1/2$, $\beta_4 = z_0/12$ and a contradiction appears for $n = 6$.

From the matching condition (5.3) it follows that

$$y \sim \alpha_1\cos(\theta_1 - \frac{\pi}{4})(-z)^{-1/4}\cos(-\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}) - \alpha_1\sin(\theta_1 - \frac{\pi}{4})(-z)^{-1/4}\sin(-\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4})$$
$$\sim \alpha_1\cos(\theta_1 - \frac{\pi}{4})\pi^{1/2}Ai(z) - \alpha_1\sin(\theta_1 - \frac{\pi}{4})\pi^{1/2}Bi(z) \tag{5.5}$$

with $Ai(z)$ and $Bi(z)$ the Airy functions (see Abramowitz and Stegun (1964)). The question arises what type of behaviour the solutions of eq. (5.2) have for $z \to \infty$. This depends on the integration constants α_1 and θ_1. It turns out that there are three possibilities, of which two are illustrated in figure 6.

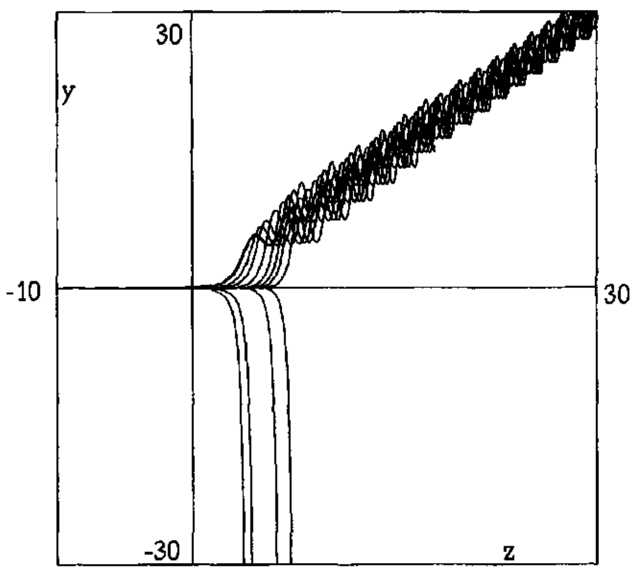


Figure 6 *If $\theta_1 \in (-2.245, 0.687)$ and $\alpha_1=0.1$ solutions of eq. (5.2) approach the stable branch $y = z$ after bifurcation. Other solutions explode with chance 1 and tend to $-\infty$.*

First, an explosion to $-\infty$ can occur in a finite time. Furthermore, a transition to the new stable branch $y = z$ is possible. Solutions show an oscillatory behaviour and from the matching conditions it follows that they must behave asymptotically as

$$y \sim z + \alpha_2(z)^{-1/4}\cos(-\frac{2}{3}(z)^{3/2} + \theta_2) \text{ if } z \to \infty . \tag{5.6}$$

Finally, for a one-parameter family of starting values, the solution approaches the unstable equilibrium beyond the bifurcation point for $z \to \infty$. The chance that this occurs is of measure zero. These solutions behave asymptotically as

$$y \sim z + \alpha_3(z)^{-1/4}\exp(-2z^{3/2}/3)(1 + o(1)) \text{ if } z \to \infty\,. \tag{5.7}$$

This last case separates the other two cases.

In figure 7 two branches of the one parameter family of initial values corresponding with a separating solution have been sketched in the α_1, θ_1 -plane. For α_1 small the phase quantity θ_1 tends to $-0.75\pi + n\pi$, $n \in \mathbf{N}$. It is seen that for an uniformly distributed initial phase θ_1 the chance of a smooth transition to the stable branch after passage of the bifurcation point becomes smaller when the amplitude of the original oscillation is larger. This is illustrated in figure 8.

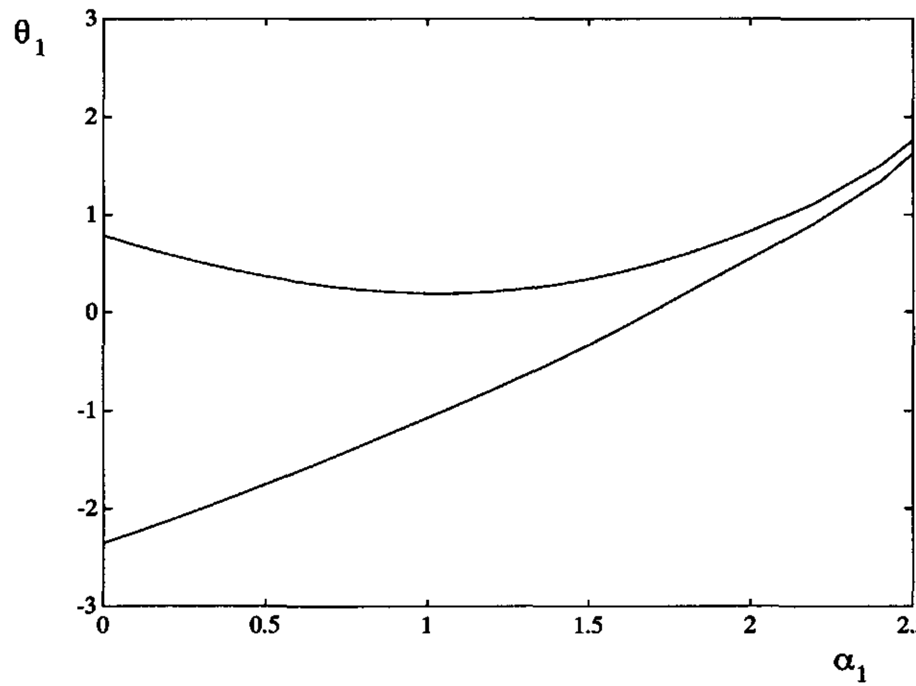


Figure 7 *For values of α_1 and θ_1 in the region between the solid lines the stable branch is approached after passage of the bifurcation point. In the separating case that is represented by the solid lines, solutions will approach the unstable branch. In the remaining part of the parameter plane solutions will explode.*

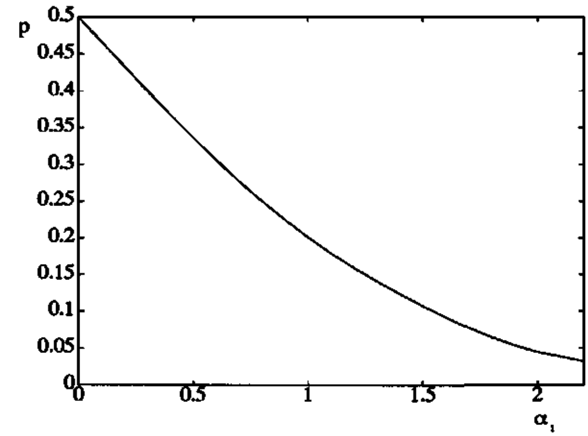


Figure 8 *The chance p as a function of α_1 that a solution of (5.2) approaches the stable branch after passage of the bifurcation point.*

Furthermore, in figure 9 we have depicted in the phase plane of the initial state of an original slowly varying transcritical system when a transition to the new stable branch may occur after the first passage of the bifurcation point.

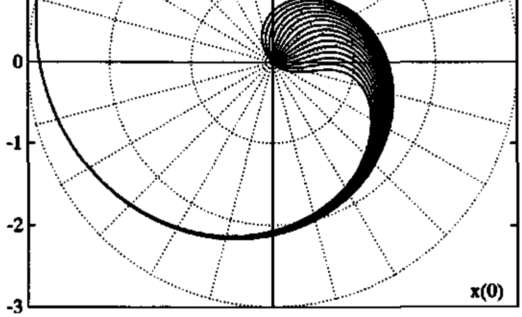


Figure 9 *The behaviour of solutions of $d^2x/dt^2 = -x^2 + \sin(\varepsilon t-\pi/2)x$ for $\varepsilon = 1/(1500\pi)$ sketched in the x(0), x'(0)-plane. For initial values within the shaded region the stable branch will be followed after bifurcation. The bounds of this region reflect approximations of the initial values for which the unstable branch will be approached after bifurcation.*

If α_1 becomes larger it is more likely that solutions of (5.2) will intersect either the line $y = z$ for a negative value of z or the line $y = 0$ for a positive value of z. If the derivative of the solution of (5.2) in such an intersection point is negative, then an explosion will occur in a finite time. The second derivative, formulated in (5.2), is zero in the intersection point and will be negative after passage of this point for $dy/dz < 0$. Therefore, dy/dz will remain negative. This is illustrated in figure 10. Moreover, it is shown in this figure that the amplitude of the oscillation around the new stable branch increases with the delay of the transition.

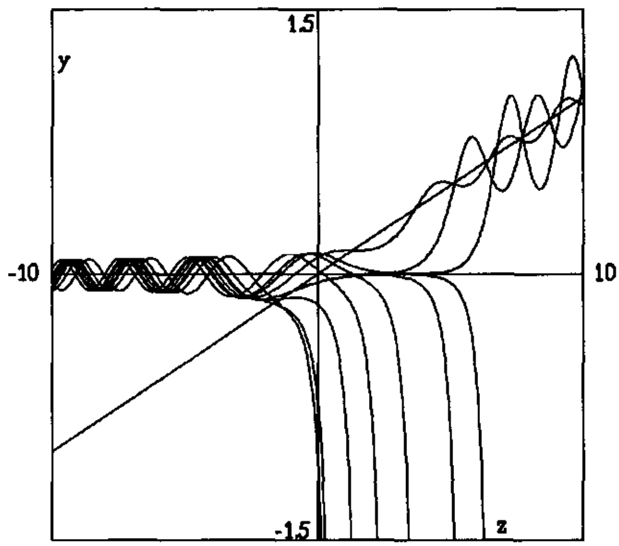


Figure 10 *Solutions of (5.2) for $\alpha_1 = 1.5$ and different values of θ_1. If $\exists z^*$ such that for a solution $y(z)$ of (5.2) $y(z^*) \leq min(z^*,0)$ and $dy/dz(z^*) < 0$, then an explosion will occur in a finite time.*

The interpretation of these results for the physical problem of section 1 is that if the position of the mass is near the fixed wire or if the velocity of the mass towards the fixed wire is high, the mass will further move towards this fixed wire.

We have shown that near the time at which a transcritical bifurcation may be expected, eq. (5.2) holds -after normalization- with matching condition (as $z \to -\infty$) given by (5.3). In case of an explosion the type of singularity for the transition equation is (see (5.4))

$$y \sim \frac{-6}{(z - z_0)^2} \,. \tag{5.8}$$

6. Moderate and small amplitude oscillations

When the damping parameter is large or the oscillation around the slowly varying stable equilibrium is small, the integration constant α_1 is small. Since the chance of an explosion is approximately 1/2 when $\alpha_1 = o(1)$, this case seems the most interesting one for practical problems. Therefore, we will discuss some aspects of small amplitude oscillations in this section. For y small eq. (5.2) can be approximated by the Airy equation

$$\frac{d^2y}{dz^2} - zy = 0 . \tag{6.1}$$

It is necessary to consider separately two cases: α_1 larger than $\varepsilon^{2/3}$ in order of magnitude and α_1 smaller than or equal to $\varepsilon^{2/3}$ in order of magnitude.

Moderate amplitude oscillations

For α_1 small, $\alpha_1 = o(1)$ and $\varepsilon^{2/3} = o(\alpha_1)$, the separating solution that approaches the unstable branch after bifurcation satisfies the following condition:

$$\theta_1 = \frac{\pi}{4} + n\pi + o(1) \ , \ n \in \mathbf{N} . \tag{6.2}$$

The stable branch is followed for $\theta_1 \in (-0.75\pi + 2n\pi, 0.25\pi + 2n\pi)$, $n \in \mathbf{N}$. In this case the constant that multiplies the (exponentially) exploding Airy function *Bi(z)* in (5.5) is positive; there will exist a new inner transition layer and solutions will exhibit an oscillatory behaviour around the new stable branch $y = z$ after bifurcation. The separating condition (6.2) for small α_1 is the same as the one for the second Painlevé transcendent when the initial amplitude is small (see e.g. Marée (1995)). In order to have a more accurate approximation of the moment the solution leaves the stable branch that turned unstable, we transform

$$w = y - z , \tag{6.3}$$

so that equations (5.2) and its linearization (6.1) become respectively

$$\frac{d^2w}{dz^2} + w(w + z) = 0 \tag{6.4}$$

and

$$\frac{d^2w}{dz^2} - z(w + z) = 0 . \tag{6.5}$$

In the figures 11 and 12 the solutions of (6.4) and (6.5) are compared with each other with the same matching condition for $z \to -\infty$:

$$w \sim -z + \alpha_2 \pi^{1/2} Bi(z) := -z + \beta Bi(z) \ . \tag{6.6}$$

If we assume that β is of order $O(\varepsilon^{\rho})$, $0 < \rho < 2/3$, we can substitute in (6.4) the expansion

$$w = -z + \varepsilon^{\rho} w_1 + \varepsilon^{2\rho} w_2 + \varepsilon^{4\rho} w_3 + \ldots \ . \tag{6.7}$$

We then obtain the following transition equations:

$$\frac{d^2 w_1}{dz^2} = w_1 z \ , \tag{6.8a}$$

$$\frac{d^2 w_2}{dz^2} = -w_1^2 + w_2 z \ , \tag{6.8b}$$

and for $n > 2$

$$\frac{d^2 w_n}{dz^2} = -2 w_1 w_{n-1} + w_n z \qquad , n \text{ odd} \ , \tag{6.8c}$$

$$\frac{d^2 w_n}{dz^2} = -w_{n/2}^2 - 2 w_1 w_{n-1} + w_n z \qquad , n \text{ even} \ . \tag{6.8d}$$

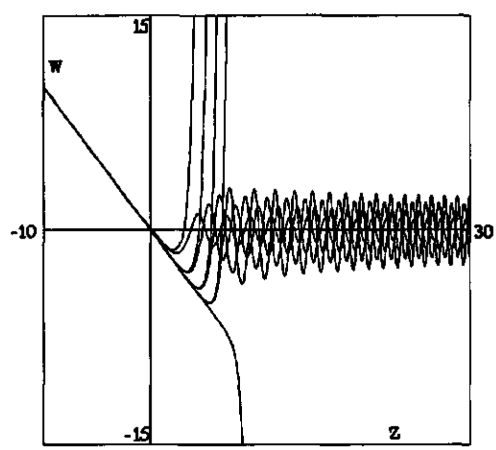


Figure 11 *Solutions of (6.4) and of (6.5) with matching condition $w(z) \sim -z + \alpha_2 \pi^{1/2} Bi(z)$ as $z \to -\infty$ for $\alpha_2 = 10^{-4}$, 10^{-3}, 10^{-2}, and 10^{-1}. Solutions of (6.5) explode to ∞, while solutions of (6.4) show an oscillatory behaviour after passage of $z = 0$. Moreover, it is shown that when $w(z) \sim -z - 10^{-4}\pi^{1/2} Bi(z)$ as $z \to -\infty$ solutions of both equations explode to $-\infty$ in the same way. When α_2 becomes smaller, the moment of the explosion is delayed.*

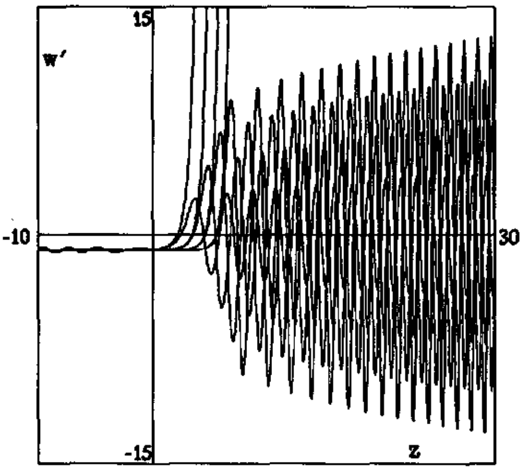


Figure 12 *The derivatives of the solutions of (6.4) and (6.5) given in figure 11 that do not explode to -∞.*

We will see that if $\beta < 0$, solutions of (6.4) indeed will explode to $-\infty$ with chance 1. Otherwise, they will intersect the new stable branch $w = 0$ and solutions of (6.4) will follow this branch in an oscillatory way. The behaviour of $Ai(z)$ and $Bi(z)$ for $z \to \infty$ is (see e.g. Olver (1974))

$$Ai(z) \sim \frac{\exp(-2z^{3/2}/3)}{2\pi^{1/2}z^{1/4}} \quad , \quad Ai'(z) \sim -\frac{z^{1/4}\exp(-2z^{3/2}/3)}{2\pi^{1/2}} \, , \tag{6.9}$$

$$Bi(z) \sim \frac{\exp(2z^{3/2}/3)}{\pi^{1/2}z^{1/4}} \quad , \quad Bi'(z) \sim \frac{z^{1/4}\exp(2z^{3/2}/3)}{\pi^{1/2}} \, . \tag{6.10}$$

The exact solution of (6.8a) is

$$w_1 = \beta_1 Ai(z) + \beta_2 Bi(z) \, . \tag{6.11}$$

It follows that for $z \to \infty$

$$w_1 \sim \beta_2 Bi(z) \sim \beta_2 \pi^{-1/2} z^{-1/4} \exp(2z^{3/2}/3) \, . \tag{6.12}$$

We will show that the expansion (6.7) becomes disordered as $z \to +\infty$. To see this, we calculate the higher order terms asymptotically as $z \to +\infty$:

$$w_2 \sim -\frac{1}{3}\beta_2^2\pi^{-1}z^{-3/2}\exp(4z^{3/2}/3) \ , \tag{6.13a}$$

$$w_n \sim d_n\beta_2^n\pi^{-n/2}z^{(-1-5n)/4}\exp(2nz^{3/2}/3) \ , \tag{6.13b}$$

where the coefficients d_n satisfy a recurrent relation

$$d_1 = 1, \ d_2 = -1/3, \tag{6.14a}$$

and for $n > 2$

$$(n^2 - 1)d_n = -2d_1d_{n-1} \qquad , \ n \text{ odd} \ , \tag{6.14b}$$

$$(n^2 - 1)d_n = -d_{n/2}^2 - 2d_1d_{n-1} \qquad , \ n \text{ even} \ . \tag{6.14c}$$

Since $d_n < 0$ if n is even and $d_n > 0$ if n is odd, it follows that an explosion to $-\infty$ will occur when $\beta_2 < 0$. Each succeeding term in the series (6.7) is exponentially larger than the preceding one diminished by the factor $\pi^{1/2}z^{5/4}/\beta_2$ as $z \to +\infty$. Therefore, we introduce the nonlinear scale transformation

$$\beta_2 e^{\rho}\pi^{-1/2}z^{-5/4}\exp(2z^{3/2}/3) = \Phi \ , \tag{6.15}$$

where Φ is an $O(1)$ variable. In the region where $\Phi = O(1)$ is $z << \varepsilon^{-2/3}$. The first transition layer was characterized by $z = O(1)$ and hence $\varepsilon(t - t_c) = O(\varepsilon^{2/3})$. The second nonlinear transition layer is characterized by $\Phi = O(1)$ and we obtain from (6.15) and (4.5):

$$\frac{2}{3}(\varepsilon(t - t_c))^{3/2} - \frac{5}{4}\varepsilon\ln(\varepsilon(t - t_c)) = -(\rho + \frac{5}{6})\varepsilon\ln\varepsilon + \varepsilon\ln(\Phi\pi^{1/2}/\beta_2) \ . \tag{6.16}$$

It follows from (6.16) that $\varepsilon(t - t_c) = O((-\varepsilon\ln\varepsilon)^{2/3})$. Some algebraic manipulations show that the nonlinear transformation (6.16) is asymptotically equivalent to the following linear scale change

$$\varepsilon^{2/3}z = \varepsilon(t - t_c) = \xi^{2/3}(\varepsilon) + \varepsilon\xi^{-1/3}(\varepsilon)\ln(\Phi\pi^{1/2}/\beta) \tag{6.17}$$

with

$$\xi(\varepsilon) = -\frac{3}{2}\rho\varepsilon\ln\varepsilon + \frac{5}{4}\varepsilon\ln(-\frac{3}{2}\rho\ln\varepsilon) \ . \tag{6.18}$$

It can be concluded that $\varepsilon(t - t_c)$ is shifted by a small amount, $\xi^{2/3}(\varepsilon)$, which is much larger than $O(\varepsilon^{2/3})$. Since $\varepsilon\xi^{-1/3}(\varepsilon) = o(1)$, however, $\varepsilon(t - t_c)$ still varies only by a small amount in this interior transition layer. Consequently, a better approximation of the bifurcation time is obtained:

$$t = t_c + \varepsilon^{-1/3}((-\frac{3}{2}\rho\ln\varepsilon + \frac{5}{4}\ln(-\frac{3}{2}\rho\ln\varepsilon))^{2/3} + (-\frac{3}{2}\rho\ln\varepsilon + \frac{5}{4}\ln(-\frac{3}{2}\rho\ln\varepsilon))^{-1/3}\ln\Phi_c)\,, \quad (6.19)$$

where $\Phi_c = O(1)$ can be found by computing the intersection point of (6.7) for $z \to +\infty$ with the new stable branch $w = 0$ of (6.4).

Oscillations with small amplitude

In the analysis of small amplitude oscillations the slowly varying equilibrium solution plays an important role. Considering the behaviour of the stable slowly varying equilibrium solution we obtain from (4.9) the following matching condition for the local transition expansion if $s \to -\infty$

$$\begin{aligned} x_{sve} = {} & x_E(\lambda_c) + \varepsilon^{2/3}a_1\lambda_c{}'s + \varepsilon^{4/3}(a_1\frac{\lambda_c{}''}{2}s^2 + b_1(\lambda_c{}')^2s^2\,) \\ & -\varepsilon^{4/3}(\,\pi(a_1\lambda_c{}'' + 2b_1(\lambda_c{}')^2 + \kappa a_1\lambda_c{}')\gamma^{-2/3}Hi(\gamma^{1/3}s)\,) + \ldots \end{aligned} \quad (6.20)$$

with γ as defined in (4.12) and the Scorer's function $Hi(s)$ (see e.g. Olver (1974)) the solution of an inhomogeneous Airy equation:

$$\frac{d^2Hi(s)}{ds^2} - Hi(s)s = \frac{1}{\pi}\,. \quad (6.21)$$

The matching condition (6.20) is automatically fulfilled by (4.7) and (4.8); if we take $y_0 = a_1\gamma_c{}'s$ and set y_1 equal to the $O(\varepsilon^{2/3})$-term in (6.20) we obtain exact solutions of (4.7) and (4.8). If $s > 0$ $Hi(s)$ behaves numerically like $Bi(s)$ for sufficiently large s. Therefore, the local asymptotic expansion (6.20) will become disordered when $Hi(\gamma^{1/3}s)$ becomes large. It can be concluded that depending on the sign of

$$c_1 := \pi(a_1\lambda_c{}'' + 2b_1(\lambda_c{}')^2 + \kappa a_1\lambda_c{}')\gamma^{-2/3} \quad (6.22)$$

the stable slowly varying equilibrium solution either explodes to $-\infty$ ($c_1 > 0$) or approaches the new stable branch ($c_1 < 0$) after passage of the bifurcation point.

This is valid because an explosion can not be cancelled out by higher order terms. If $c_1 = 0$ we have to take into consideration the higher order terms in order to determine the separation condition.

If we consider in the original system a solution that holds in a $\varepsilon^{3/2}$-neighbourhood of the slowly varying stable equilibrium solution (so $\mu = 3/2$ in (3.4)), the approximation theorem 3.1 is still valid near the bifurcation time t_c. We now obtain that near the bifurcation point the outer solution behaves asymptotically as

$$
\begin{aligned}
x \sim x_E(\lambda_c) &+ \varepsilon^{2/3} a_1 \lambda_c' s + \varepsilon^{4/3}\Big(a_1 \frac{\lambda_c''}{2} s^2 + b_1 (\lambda_c')^2 s^2\Big) \\
&- \varepsilon^{4/3}\big(\alpha^* Ai(\gamma^{1/3} s) - \beta^* Bi(\gamma^{1/3} s) + c_1 Hi(\gamma^{1/3} s)\big) + \dots \text{ for } s \to -\infty
\end{aligned}
\tag{6.23}
$$

with c_1 as defined in (6.22) and

$$
\alpha^* = -r_0 \gamma^{-1/6} \pi^{1/2} e^{-\kappa t_c/2} \cos\Big(\theta_1 - \frac{\pi}{4}\Big)\,,\ \beta^* = -r_0 \gamma^{-1/6} \pi^{1/2} e^{-\kappa t_c/2} \sin\Big(\theta_1 - \frac{\pi}{4}\Big)\,, \tag{6.24a,b}
$$

which follows from (5.3) and (5.5). Again, the $O(\varepsilon^{2/3})$-term and the $O(\varepsilon^{4/3})$-term are exact solutions of respectively the inner equation (4.7) and the $O(\varepsilon^{2/3})$-equation (4.8). This is true since

$$
w(x) = \beta_1 Ai(x) + \beta_2 Bi(x) + Hi(x)\,, \tag{6.25}
$$

with β_1 and β_2 arbitrary constants, is a general solution of the normalized $O(\varepsilon^{2/3})$-equation

$$
\frac{d^2 w}{dx^2} - x w(x) = \frac{1}{\pi}\,. \tag{6.26}
$$

We can now conclude that solutions in an $O(\varepsilon^{2/3})$-neighbourhood of the slowly varying equilibrium solution of (1.1) explode to $-\infty$ when

$$
-\beta^* + c_1 > 0 \tag{6.27}
$$

with β^* and c_1 as defined in (6.22) and (6.24b). If $-\beta^* + c_1 < 0$ the above mentioned solutions will approach the new stable branch after passage of the bifurcation point. When $-\beta^* + c_1 = 0$ the asymptotic behaviour of the solution y_1 of (4.8) is

$$
y_1 \sim \alpha^* Ai(\gamma^{1/3} s) - c_1 Gi(\gamma^{1/3} s) + a_1 \frac{\lambda_c''}{2} s^2 + b_1 (\lambda_c')^2 s^2 \text{ for } s \to -\infty \tag{6.28}
$$

with the Scorer's function *Gi(s)* the solution of

$$\frac{d^2Gi(s)}{ds^2} - Gi(s)s = -\frac{1}{\pi}\ . \tag{6.29}$$

The Scorer's functions *Gi(x)* and *Hi(x)* satisfy the equality

$$Gi(s) + Hi(s) = Bi(s)\ . \tag{6.30}$$

The separation condition $-\beta^* + c_1 = 0$ depends on the value of the damping parameter κ, on the initial values, and on the behaviour of the solutions of (1.1) near $x = x_E(\lambda_c)$ and $\lambda = \lambda_c$. If we take $y_0 = a_1\gamma_c's$ and y_1 equal to the r.h.s. of (6.28) we obtain exact solutions of (4.7) and (4.8). Moreover, the matching condition for the local transition expansion of the unstable slowly varying equilibrium solution for $s \to \infty$ is automatically fulfilled. Higher order terms have to be taken into account in order to conclude if this "separating solution" will approach the unstable branch after bifurcation (and so no transition has taken place). This approach will be exponentially since the Airy function *Ai(s)* that appears in the matching condition (6.28) exponentially decays if $s \to \infty$ and *Gi(s)* matches for $s \to \infty$ exactly to the slowly varying equilibrium terms that follow from (5.3). In fact,

$$Gi(s) \sim \frac{1}{\pi s}\left(1 + \frac{1}{s^3}\sum_{m=0}^{\infty}\frac{(3m+2)!}{m!(3s^3)^m}\right) \quad \text{for } s \to +\infty\ . \tag{6.31}$$

After introducing the inner expansion

$$y = y_0 + \varepsilon^{2/3}y_1 + \varepsilon^{4/3}y_2 + \ldots \tag{6.32}$$

we obtain if $\mu = 3/2$ the following asymptotic behaviour of y_0 and y_1 as $s \to +\infty$ when the new stable branch is approached:

$$y_0 \sim a_1\lambda_c's\ \ ,\ \ y_1 \sim (\beta^* - c_1)Hi(\gamma^{1/3}s) \sim (\beta^* - c_1)(\gamma^{1/3}s)^{-1/4}\exp(\frac{2}{3}(\gamma^{1/3}s)^{3/2})\ . \tag{6.33}$$

In a similar manner we obtain that y_2 is proportional to $c_2(\gamma^{1/3}s)^{-3/2}\exp(4(\gamma^{1/3}s)^{3/2}/3)$ as $s \to +\infty$. Each succeeding term is exponentially larger than the preceding one diminished by the factor $(\gamma^{1/3}s)^{5/4}$. Therefore, we now introduce the nonlinear scale transformation

$$\varepsilon^{2/3}(\gamma^{1/3}s)^{-5/4}\exp(2(\gamma^{1/3}s)^{3/2}/3) = \Phi\ , \tag{6.34}$$

where Φ is an $O(1)$ variable. With the similar methods as we used in the case of moderate amplitude oscillations we obtain that the nonlinear transformation (6.34) is asymptotically equivalent to the following linear scale change

$$\varepsilon^{2/3}s = \varepsilon(t - t_c) = \gamma^{-1/3}(\xi^{2/3}(\varepsilon) + \varepsilon\xi^{-1/3}(\varepsilon)\ln\Phi) \tag{6.35}$$

with

$$\xi(\varepsilon) = -\varepsilon\ln\varepsilon + \frac{5}{4}\varepsilon\ln(-\ln\varepsilon) \,. \tag{6.36}$$

Consequently, in the case of small amplitude oscillations a better approximation of the bifurcation time is

$$t = t_c + \varepsilon^{-1/3}\gamma^{-1/3}((-\ln\varepsilon + \frac{5}{4}\ln(-\ln\varepsilon))^{2/3} + (-\ln\varepsilon + \frac{5}{4}\ln(-\ln\varepsilon))^{-1/3}\ln\Phi_c) \,, \tag{6.37}$$

where $\Phi_c = O(1)$.

7. Some remarks

In this study we have analyzed a nonlinear second order differential equation with a slowly varying parameter. Solutions in the neighbourhood of a parameter-dependent equilibrium are studied as the parameter crosses a critical value corresponding to a transcritical bifurcation. The aim of this study was to predict whether or not a solution will explode when it slowly evolves through such a bifurcation point. The leading order transition equation for this problem is a nonlinear non-integrable second order differential equation with the Airy equation as linearization. Solutions of this transition equation either explode, exponentially decay or algebraically grow, corresponding to a transition from the equilibrium that turned unstable to the new stable branch. It turned out to be possible to derive a separation condition which yields a prediction of the behaviour of solutions beyond the bifurcation point when we consider small amplitude oscillations around the slowly varying originally stable equilibrium solution ($\mu > 5/6$ in (3.4)). This separation condition depends on the damping parameter, on the initial conditions, and on the behaviour of the function $H(x,\lambda)$ of (1.1) near the bifurcation point.

With the aid of both averaging and boundary layer methods the solution of the problem is approximated asymptotically. With perturbation techniques we obtain an asymptotic expansion for the slowly varying equilibrium solution and an averaged asymptotic expansion for the oscillation around this solution. For a certain order of the amplitude of the oscillation ($\mu = 5/6$ in (3.4)) both expansions break down simultaneously. Matching conditions for the local transition equation that describes the transcritical bifurcation are obtained from the behaviour of the outer asymptotic expansions near the bifurcation point. The transition occurs on a relatively short time-scale of order $\varepsilon t = O(\varepsilon^{2/3})$. For small amplitude oscillations we have found a solution that automatically satisfies the matching conditions. This special solution is a linear combination of a polynomial, the Airy functions *Ai* and *Bi*, and the Scorer's functions *Gi* and *Hi*.

Although a separation condition has been found for small amplitude oscillations, some open problems still remain. For solutions for which the averaged asymptotic expansion for the oscillation and the asymptotic expansion for the slowly varying equilibrium break down simultaneously, we have noticed that the chance of an explosion is larger when the amplitude of the oscillation is increased. We obtained a numerical approximation for the separation condition. The question remains whether this condition can be obtained analytically. The answer lies in the transitional equation which hopefully has analytical properties comparable with those of a Painlevé transcendent. Such an approach may reveal a dependence of the possible singularity of the nonlinear transition equation on the integration constants. Furthermore, we want to investigate the matching of an exponentially increasing Airy-like solution to a solution with a slowly decreasing oscillation in the neighbourhood of a new unconnected stable branch. If more equilibria exist the higher order transition equations have a stronger singularity than the significant degeneration as the transformed time variable z approaches this singularity z_0 (see (5.8)). We want to study in what way we then can connect the slowly varying oscillating solutions before and after passage of the bifurcation point, see figure 13. For that purpose we have to analyze a secondary transition layer in the neighbourhood of the singularity. In this case, the explosion transforms the state variable x from being in an $O(\varepsilon^{2/3})$-neighbourhood of its critical bifurcation value to differing from it by $O(1)$. We expect a jump transition to a slowly varying periodic nonlinear wave around the new unconnected stable equilibrium (see Marée (1993)).

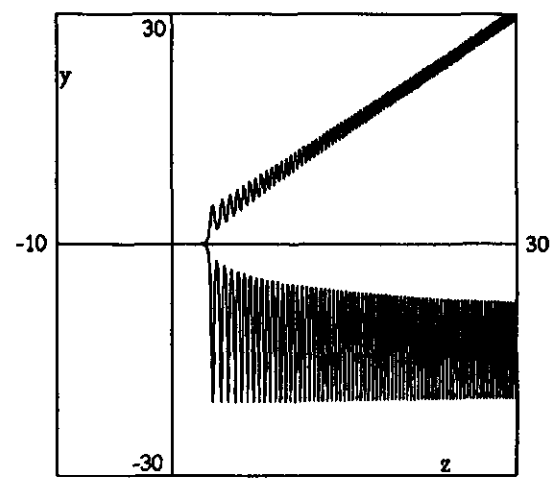


Figure 13 *If two stable equilibria exist after the slow passage through the transcritical bifurcation point, solutions either approach a new equilibrium with a slow oscillatory decay after a smooth transition or oscillate around the other equilibrium after a jump transition. This is illustrated for the system* $d^2x/dt^2 = -x(x - \varepsilon t + 1)(x + 1)$ *for* $\varepsilon = 0.01$ *with* $(x(0), dx/dt(0)) = (-0.02, 0)$ *(jump), and* $(x(0), dx/dt(0)) = (0.02, 0)$ *(transcritical bifurcation).*

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Summary

Dynamical systems modelling physical processes often evolve on several time-scales with different orders of magnitude. In modelling oscillating systems some simplifying assumptions have to be made. When the short-term behaviour of a natural system is considered the parameters that appear in mathematical models of such systems can be assumed constant. In the long term, however, these parameters will vary slowly because of gradual changes in the nature of the system. Moreover, system parameters can be varied deliberately by the experimenter. This slow change of the system parameter can produce an enormous effect on the state of the system at a certain moment, which can lead to undesirable responses; "small causes produce large effects". In this thesis we study sudden changes in systems that can be modeled by second order nonlinear differential equations. The model parameter slowly changes in a dynamical way and is a function of the (slow) time.

Outside a certain transition region the system exhibits on a large time-scale a damped oscillation around a stable, slowly varying, parameter-dependent equilibrium solution. The dynamics of the regarded problems on a large time-scale are described with the aid of averaging methods. When the bifurcation parameter approaches a critical value, however, this asymptotic averaged approximation is not valid anymore. A sudden, rapid transition takes place, since the stability of an equilibrium changes or an equilibrium vanishes. In order to offer a quantitative analysis it is important to understand in what way the solutions of the differential equations behave in the vicinity of the bifurcation point. In order to describe a bifurcation or jump phenomenon a local approximation has to be made. Local analysis yields that Painlevé equations can play an important role in the bifurcation process. In specific cases the solutions of the nonlinear transition equations can exhibit algebraic growth or they can explode (via a singularity). In this study the validity of the local approximations has been proven for a large class of systems. With the aid of matching techniques and a thorough analysis of the transition equation an accurate prediction has been made of the behaviour of solutions after passage of the bifurcation point. Examples of nonlinear systems that are of the same type as the problems which are considered in this thesis can be found in mechanics, climatology, biology, astronomy, and space craft technology. The power of the mathematical analysis, that has been performed in this thesis, is that it can be applied to a large class of dynamical systems. The analysis in this thesis has been carried out with the use of singular perturbation techniques. The problems that we consider are related to physical systems. We distinguish between variables, which

are related to the dynamics of the system, and parameters that exhibit a slow change on a large time-scale. The problem is treated dynamically, since we take into consideration a slowly varying parameter.

In chapter 2 an elementary bistable system is considered that corresponds qualitatively to the Euler arc from mechanics. For this system a sudden moment of snap-through occurs, since the parameter that describes the "stiffness" of the problem slowly varies in time and causes a dynamical bifurcation. Solutions of the system, that originally oscillate around a slowly decreasing equilibrium solution, exhibit a sudden jump behaviour and a transition to an other equilibrium state occurs. From the point of view of the qualitative analysis it is important to make a good prediction of the moment in time at which this jump will take place. The significant degeneration, which describes the jump phenomenon, is a nonlinear differential equation that can not be solved in terms of known functions or combinations of known functions; the local transition behaviour is described by the first Painlevé transcendent. Although it is not possible to deduce an asymptotic expression for the exact moment of snap-through, we are able to obtain an expression for the upper and lower limit of the expected moment of snap-through. In order to achieve this goal we apply numerical methods. A specific solution of the first Painlevé equation can be derived that matches the parameter-dependent equilibrium solution which had existed before the jump took place. The zero of this specific solution can be considered as a natural constant for the first Painlevé equation. The limit expansion of this solution has an asymptotic series. The zero of the solution of the first Painlevé equation with certain matching conditions yields a better approximation for the moment that the system snaps through. The matching condition (and therefore also the approximation of the jump moment) depend on the amplitude and the phase of the original oscillation, and on the velocity at which the parameter slowly changes.

In chapters 3 and 4 systems are analyzed which are modelled by second order nonlinear differential equations that pass a pitchfork bifurcation. In the vicinity of a certain critical value of the state parameter a transition occurs from a stable "straight" equilibrium to a parabolic equilibrium curve. The leading order transition equation, which describes the pitchfork bifurcation, is the second Painlevé transcendent. The transcendental solutions of this equation either algebraically grow, which corresponds with a transition from the linear equilibrium to one of the two stable branches of the parabolic equilibrium curve, or exponentially decay, which corresponds with a transition to the unstable, slowly varying equilibrium solution after bifurcation.

In chapter 3 we analyze the validity of the different asymptotic expansions. In order to obtain a global picture of the system we apply matching techniques and approximation theorems, that are obtained by extending existing first and second order averaging methods, and we extend the local solutions. It is proven that the different local solutions overlap. The matching conditions depend on the initial conditions and on the values of the parameters. Analytical analysis of the transition equation provides information about the required matching procedures. Moreover, it is seen that the asymptotic approximation remains valid before, during, and after the pitchfork bifurcation. There is a connection between the slowly oscillating solutions before and after passage of the bifurcation point. It is possible to predict accurately which stable branch of the parabolic equilibrium curve will be followed after bifurcation dependent on the state of the system "far away" from the bifurcation point. In this thesis an interesting connection has been discovered between the recent theory of Painlevé equations and the applications of singular perturbation techniques.

In chapter 4 we are concerned with the dynamics of a slowly varying Hamiltonian system for which the phase portrait for a fixed value of the forcing function qualitatively changes with time. This phase portrait periodically changes and a figure-eight separatrix periodically disappears and reappears. As the system parameter changes a double homclinic loop is born which grows to a maximum, shrinks back into the origin, lies dormant, and then is born again; the bifurcation parameter periodically crosses a critical value corresponding to a supercritical pitchfork bifurcation. Dependent on the initial state of the system and the values of the parameters the system can exhibit chaotic or (quasi) periodic behaviour. The sequence of stable upper and lower branches which a given trajectory follows after passage of the pitchfork bifurcation can be irregular and so the system can exhibit sensitive dependence on initial conditions. The attraction properties can be analyzed with the aid of a Poincaré map. Chaotic dynamics almost always occur in a system without friction. For a dissipative system it is more likely that it will exhibit a periodic behaviour from a certain moment in time. The validity of the approximating Poincaré map and of the matched asymptotic approximations can be proven on a large time-scale. The proof of this validity has been carried out with the aid of an approximation theorem that concerns the averaging of oscillating functions with a slowly varying frequency, an extension theorem, matching techniques, and connection formulas for the solutions of the second Painlevé equation. Numerical simulations confirm the results that are obtained with analytical methods. A mechanical example is the motion of a simple pendulum that is connected to a rotating, rigid body.

Finally, in chapter 5 we study the general class of nonlinear second order problems with a slowly varying parameter that passes a critical value corresponding to a transcritical bifurcation. The jump phenomenon and the pitchfork bifurcation can be generalized in the same way as the transcritical bifurcation problem has been generalized in chapter 5. In this case, in the vicinity of a certain critical value of the parameter a transition occurs from a stable "straight" equilibrium to an other "straight" equilibrium, whereas the originally stable equilibrium becomes unstable. Again, the local solution that is obtained with averaging methods is valid outside a certain nonlinear transition layer and yields matching conditions for the second order differential equation that is generic for this type of bifurcation. This significant degeneration, however, does not possess the Painlevé property. Solutions of the transition equation decrease exponentially, explode or exhibit algebraic growth, which corresponds to a transition from the one stable equilibrium to another. The chance of an explosion becomes larger when the amplitude of the original oscillation is larger. With the aid of local asymptotic approximations and an analysis of the transition equation it can be investigated whether or not the system will explode on a certain moment. The "explosion condition" depends on the initial conditions and the values of the parameters.

Samenvatting

Natuurlijke processen ontwikkelen zich vaak op tijdschalen waarvan de lengtes nogal sterk kunnen verschillen. Bij het modelleren van oscillerende systemen moeten vereenvoudigde veronderstellingen gemaakt worden. Op een korte termijn kunnen de parameters in deze mathematische modellen constant verondersteld worden. Op een lange termijn zullen deze parameters echter geleidelijk variëren wegens veranderingen in de aard van het systeem (bijvoorbeeld metaalmoeheid) of doordat deze parameters met opzet gevarieerd worden door iemand die een experiment uitvoert. De langzame verandering van de systeemparameter kan op een gegeven moment leiden tot een grote toestandsverandering; "kleine oorzaken hebben grote gevolgen". In dit proefschrift worden plotselinge veranderingen bestudeerd in systemen die kunnen worden beschreven door niet-lineaire differentiaalvergelijkingen van de tweede orde met een parameter die op een dynamische wijze langzaam verandert als functie van de tijd.

Buiten een zeker overgangsgebied vertoont het systeem op een lange tijdschaal een gedempte oscillatie rond een stabiele, langzaam veranderende evenwichtsoplossing. Met uitgebreide middelingsmethoden wordt de dynamica op de lange tijdschaal beschreven. Wanneer de bifurcatieparameter echter een kritische waarde nadert, verliest de oorspronkelijke asymptotische benadering zijn geldigheid. Er vindt een plotselinge overgang plaats, doordat een evenwichtsoplossing van aard verandert of verdwijnt. Het is belangrijk om precies te weten hoe de oplossing zich rond het bifurcatiepunt gedraagt. Om de bifurcatie of sprong te beschrijven moet een lokale benadering gemaakt worden. Deze lokale analyse laat zien dat Painlevé vergelijkingen een belangrijke rol kunnen spelen in het bifurcatieproces. In specifieke gevallen kunnen de oplossingen van de niet-lineaire overgangsvergelijkingen een algebraïsche groei vertonen of ze kunnen exploderen (via een singulariteit). In deze studie wordt voor een grote klasse van problemen de geldigheid van de lokale asymptotische benaderingen bewezen. Met behulp van aansluitingstechnieken en een grondige analyse van de overgangsvergelijking wordt een accurate voorspelling gemaakt van het gedrag van het systeem nadat het bifurcatiepunt gepasseerd is. Voorbeelden van de in dit proefschrift bestudeerde niet-lineaire systemen kunnen worden gevonden in de mechanica, de klimatologie, de biologie, de sterrenkunde en de ruimtevaarttechniek. De kracht van de wiskundige analyse, zoals die hier uitgevoerd wordt, is dat een grote klasse van systemen onder één noemer gebracht kan worden. De analyse in dit proefschrift is uitgevoerd met methoden uit de singuliere storingsrekening. Er worden problemen beschouwd die verwant zijn aan fysische systemen waarbij een onderscheid gemaakt wordt tussen de variabelen, die

verband houden met de dynamiek van het dynamische systeem, en een parameter die een trend vertoont op een grotere tijdschaal. Er ontstaat een dynamisch probleem, omdat de trend van de parameter in aanmerking wordt genomen.

In hoofdstuk 2 wordt een elementair bistabiel systeem beschouwd dat kwalitatief overeenkomt met de Eulerboog uit de mechanica. Bij dit systeem treedt op een gegeven moment een doorslag op, omdat wegens een langzame afname van een "stijfheidsparameter" een bifurcatie optreedt waarbij het systeem, dat oorspronkelijk oscilleerde rond een bepaalde evenwichtsstand, plotseling spronggedrag blijkt te vertonen en in de omgeving van een andere evenwichtsstand gaat verblijven. Vanuit het oogpunt van de kwantitatieve analyse is het belangrijk om het moment te voorspellen waarop de sprong plaatsvindt. De significante degeneratie, die dit sprongverschijnsel beschrijft, is een niet-lineaire differentiaalvergelijking die niet op te lossen is in termen van bekende functies of combinaties van bekende functies; het lokale gedrag wordt beschreven door de eerste Painlevé transcendent. Hoewel het niet mogelijk is een asymptotische uitdrukking af te leiden voor het exacte moment van doorslag, kan met behulp van numerieke methoden wel een uitdrukking afgeleid worden voor een boven- en ondergrens van het verwachte doorslagtijdstip. Een specifieke oplossing van de eerste Painlevé vergelijking kan worden gevonden die aansluit bij de oorspronkelijke parameter-afhankelijke evenwichtsoplossing. Deze specifieke oplossing heeft een nulpunt dat beschouwd kan worden als een natuurlijke constante voor de Painlevé vergelijking. De limietexpansie wordt beschreven door een asymptotische reeks. Het nulpunt van de oplossing van de eerste Painlevé vergelijking met bepaalde aansluitingsvoorwaarden levert een betere benadering op voor het moment van doorslag. Deze benadering hangt af van de amplitudo en fase van de oorspronkelijke oscillatie en van de snelheid waarmee de toestandsparameter een verandering ondergaat.

De systemen, die in de hoofdstukken 3 en 4 bestudeerd worden, betreffen tweede orde niet-lineaire problemen die een stemvorkbifurcatie ondergaan. In de omgeving van een zekere (kritieke) waarde van de toestandsparameter treedt een overgang op van een stabiel "recht" evenwicht naar een parabolische evenwichtscurve. De significante degeneratie, die de stemvorkbifurcatie beschrijft, is de tweede Painlevé vergelijking. De transcendente oplossingen van deze vergelijking vertonen of een algebraïsch groeigedrag, hetgeen correspondeert met een overgang van het lineaire evenwicht naar één van de beide stabiele takken van de parabolische curve, ofwel een exponentieel verval, corresponderend met een overgang naar de onstabiele, langzaam variërende evenwichtsoplossing.

In hoofdstuk 3 wordt met behulp van approximaties -verkregen met uitgebreide eerste en tweede orde middelingsmethoden-, aansluitingstechnieken en uitbreiding van lokale oplossingen onderzoek gedaan naar de geldigheid van de asymptotische benaderingen. Er wordt bewezen dat er een overlap plaatsvindt tussen de verschillende lokale oplossingen. De aansluitingsvoorwaarden hangen af van de beginvoorwaarden en de waarden van de parameters. Analytisch onderzoek van de overgangsvergelijking verschaft informatie over de vereiste aansluitingsprocedures en leert dat de asymptotische benadering "door de grenslaag heen" uitgevoerd kan worden waarbij een verband afgeleid wordt tussen de langzaam oscillerende oplossingen voor en na de bifurcatie. Het blijkt mogelijk om, afhankelijk van het gedrag van een oplossing "ver weg" van het bifurcatietijdstip, te voorspellen welke stabiele tak van de parabolische evenwichtscurve na de bifurcatie benaderd zal worden. Er blijkt een interessante relatie te bestaan tussen de recente theorie over het gedrag van Painlevé transcendenten en de toepassingen van singuliere storingstechnieken.

In hoofdstuk 4 wordt de dynamica beschouwd van een langzaam veranderend Hamilton systeem waarbij het faseportret van het gereduceerde systeem kwalitatief verandert in de tijd. Periodiek ondergaat dit faseportret een verandering waarbij een dubbele homocliene baan ontstaat die groeit tot een maximum, weer krimpt, verdwijnt en weer opnieuw geboren wordt; het systeem ondergaat periodiek een (superkritische) stemvorkbifurcatie. Afhankelijk van de begintoestand en de waarden van de parameters blijkt het systeem chaotisch of (quasi)periodiek gedrag te kunnen vertonen, omdat de opeenvolging van stabiele (boven- en onder)takken, die een oplossing volgt na het ondergaan van de stemvorkbifurcatie, irregulier kan zijn en een gevoelige afhankelijkheid van de beginvoorwaarden kan vertonen. Met behulp van de terugkeer- of Painlevé-afbeelding kunnen de attractie eigenschappen van het systeem onderzocht worden. Bij een systeem zonder demping treedt bijna altijd chaos op. Een dissipatief systeem kan op een gegeven moment met een grotere kans periodiek worden. De geldigheid van de benaderende Poincaré-afbeelding en van de aangesloten asymptotische benaderingen kan bewezen worden op een grote tijdschaal. Hierbij wordt gebruik gemaakt van een benaderingsstelling met betrekking tot het middelen van oscillerende systemen met een langzaam variërende frequentie, een uitbreidingsstelling, aansluitingstechnieken en connectieformules voor oplossingen van de tweede Painlevé vergelijking. Numerieke simulaties maatstaven de met analytische methoden verkregen resultaten. Een mechanisch voorbeeld is de beweging van een slinger die vastzit aan een roterend star lichaam.

In hoofdstuk 5 tenslotte bestuderen we een algemene klasse van problemen met een langzaam variërende parameter die een transkritische bifurcatie ondergaan. Op dezelfde wijze als in dit hoofdstuk gedaan wordt voor de transkritische bifurcatie kunnen het sprongverschijnsel en de stemvorkbifurcatie veralgemeniseerd worden. In de omgeving van een bepaalde kritieke parameterwaarde treedt nu een overgang op van een stabiel "recht" evenwicht naar een nieuw stabiel "recht" evenwicht, waarbij het oorspronkelijk stabiele evenwicht instabiel wordt. Opnieuw levert de met middeling verkregen lokale oplossing, die geldig is buiten een zeker overgangsgebied waar de transkritische bifurcatie optreedt, aansluitingsvoorwaarden op voor de niet-lineaire tweede orde differentiaalvergelijking die deze bifurcatie beschrijft. Deze significante degeneratie bezit echter niet de Painlevé eigenschap. Oplossingen van de overgangsvergelijking kunnen algebraïsche groei vertonen, corresponderend met een overgang van het ene stabiele evenwicht naar het andere, exponentieel afnemen, of exploderen. De kans op een explosie wordt groter wanneer de amplitudo van de oorspronkelijke oscillatie vergroot wordt. Met behulp van de lokale asymptotische benaderingen en een analyse van de overgangsvergelijking kan geanalyseerd worden of afhankelijk van de beginvoorwaarden en de waarden van de parameters het systeem op een gegeven moment wel of niet zal exploderen.

Curriculum Vitae

Gregorius Johannes Maria Marée werd op 6 november 1966 in Oosterbeek (gemeente Renkum) geboren. Vanaf augustus 1979 was hij leerling aan het Katholiek Gelders Lyceum te Arnhem, waar hij in mei 1985 het VWO-einddiploma behaalde. Daarna begon hij aan de studie wiskunde aan de Rijksuniversiteit Utrecht. Het propaedeutisch examen wiskunde werd behaald in juni 1986 en het propaedeutisch examen informatica in april 1990. In augustus 1991 behaalde hij zijn doctoraal examen wiskunde met afstudeerwerk in de richting van de Dynamische Systemen met als keuzevakken econometrie, bedrijfseconomie/ commerciële economie en marketing en beleid en management. Gedurende zijn studietijd was hij actief bij de studievereniging A-Eskwadraat en de studentenverenigingen Unitas SR, SBU en AEGEE. Een hoogtepunt van zijn studieperiode vormde de organisatie van een uitwisselingsprogramma tussen Utrechtse studenten en wetenschappers uit Moskou en Leningrad. Van september 1987 tot juli 1991 is hij werkzaam geweest als studentassistent aan de Rijksuniversiteit Utrecht.
Van 1 september 1991 tot 1 september 1995 was hij als onderzoeker in opleiding werkzaam bij de vakgroep wiskunde van de Landbouwuniversiteit Wageningen. Zijn voornaamste werkzaamheden aldaar bestonden uit het verrichten van wetenschappelijk onderzoek (in de richting van de Toegepaste Analyse) en de verzorging van onderwijs. In het kader van een door de Stichting Mathematisch Centrum ondersteund project deed hij onderzoek naar plotselinge veranderingen in mechanische systemen die mathematisch gemodelleerd kunnen worden als niet-lineaire differentiaalvergelijkingen met een parameter die langzaam verandert in de tijd. Het onderzoek, waarvan de belangrijkste resultaten in dit proefschrift staan beschreven, werd verricht onder begeleiding van Prof.dr.ir. J. Grasman en Prof.dr. F. Verhulst.

Let's find a Way,
Today
That can take us to tomorrow,
Follow that Way
A way like flowing water

Let's leave
Behind
the things that do not matter
And turn
Our lives
To a more important chapter.

Let's take the time
Let's try to find
What real life has to offer
And maybe then
We'll find again
What we had long forgotten.
Like a friend,
True 'til the end,
It will help us unward.

The sun is high,
The road is wide,
And it starts where we are standing.
No one knows
How far it goes,
For the road is never ending.

It goes
Away,
Beyond what we have thought o.f
It flows
Away,
Away like flowing water.

Uit *The Te of Piglet*
Benjamin Hopf

DEMONEN

Kindlief, nu wil ik àlles voor je doen;
maar doe het mij dan -alsjeblieft- niet aan,
dat ik je als een ancilla -goor met groen
gevlekt- naar dat examen op zie gaan.

Om zo'n examen-demon te verslaan,
is kwestie van een diep intern fatsoen.
Jouw angst is tegenpool van eigenwaan.
Schud niet je hoofd: mijzelf paste jouw schoen.

Ik was een angsthaas van de ergste soort.
Bevend vloog ik door de tentamens voort,
tot, om een ander, ik mijzelf vergat:

Op mijn promotie, in een volle zaal.
Toen tartte ik de toga's, allemaal,
omdat daarginds mijn grijze vader zat.

Uit: *Sonnetten van een leraar*, Ida Gerhardt