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Analysis of the Exit Problem for Randomly Perturbed Dynamical Systems in Applications

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STELLINGEN

behorende bij het proefschrift

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1. Het is in de literatuur gebruikelijk om bij de constructie van contouren in de toestandsruimte waarlangs de eikonaalfunctie een constante waarde aanneemt uit te gaan van een beginwaarde-probleem. Deze methode is problematisch vanwege de sterke afhankelijkheid van beginwaarden. Deze moeilijkheid wordt vermeden door het probleem te herformuleren als randwaarde-probleem. Zie hoofdstuk 4 van dit proefschrift.

2. De uitdrukking (3.28) in [1], zijnde de kans dat op een bepaald tijdstip de stochastisch belaste slinger een kritieke energie heeft overschreden, is incorrect omdat niet aan de rechter-randvoorwaarde is voldaan. Rekening houdend met verschillen in notatie wordt de juiste uitdrukking gegeven door formule (5.15) in hoofdstuk 2 van dit proefschrift.
   [1] A. Katz and Z. Schuss (1985), *Reliability of elastic structures driven by random loads.* SIAM J. Appl. Math. Vol. 45, No. 3, 383-402.

3. In de praktijk worden verdelingen voor de levensduur van mechanische systemen dikwijls verkregen door meetresultaten te benaderen met een bekende statistische verdeling (zoals de gammaverdeling, de Weibull-verdeling, etc.). M.b.v. het uittreemodel is het soms mogelijk om deze verdelingen (en/of hun momenten) af te leiden uit de dynamica van het systeem [1],[2].
   [1] J. Grasman en H. Roozen (1989), *Reliability of stochastically forced systems.* Proc. of the second workshop on road-vehicle-systems and related mathematics, june 20-25, 1987, Torino, 219-235.
   [2] Dit proefschrift hoofdstuk 2.

4. Bij toepassing van de stralenmethode kan een cusp-singulariteit optreden. In de standaardvorm is deze singulariteit gegeven door $P(\xi, a, b) = \frac{1}{4}\xi^4 - a\xi^2 - b\xi$. De lijnen $P_\xi = 0$ in het (a, b)-parametervlak komen overeen met stralen. Dit zijn precies de lijnen die in [1] gebruikt worden om de cusp-singulariteit uit te beelden.

   [1] T. Poston and I. Stewart (1978), *Catastrophe theory and its applications.*

5. Dynamische systemen vormen een uitgebreid studie- en onderzoeksgebied voor wiskundigen, maar bieden in een aantal gevallen ook niet-wiskundigen de mogelijkheid om met elementaire middelen (zoals een microcomputer) op relevante wijze modelbouw te verrichten in hun eigen discipline.

6. De oplossing van een tweede-graads algebraïsche vergelijking is voor velen standaardkennis. Weinigen echter kunnen een derdegraadsvergelijking oplossen. Zie hiervoor [1].

   [1] R.M. Miura (1980), *Explicit roots of the cubic polynomial and applications.* CMS Applied Math. Notes, 5, 22-40.

7. Een goede bibliotheek is voor wiskundigen van primair belang. Bezuinigen op het boekenbudget moet dan ook opgevat worden als een zeer ernstige maatregel.

8. Een voorwaarde voor succesvolle practische toepassing van het inverse model van de electrocardiografie is dat men van elk te onderzoeken persoon de individuele geometrie van hart- en torso-oppervlak in rekening brengt.

9. Wie in boekwinkel of bibliotheek zoekt naar boeken over lineair programmeren, zijnde een onderdeel van de operationele analyse, treft deze gewoonlijk aan tussen de computerboeken.

10. Wie zich op een autoweg bevindt waarvan bekend is dat een van de twee rijstroken (die dezelfde verkeersrichting hebben) verderop geblokkeerd is, schiet het snelst op door op de geblokkeerde rijstrook te gaan/blijven rijden en pas vlak voor de blokkade op de andere rijstrook in te voegen.

CONTENTS

OVERVIEW

In the preface of his book entitled 'Theory and applications of stochastic differential equations', Z. Schuss (1980) noticed a gap between the theory of stochastic differential equations and its applications. In addition to the work of Schuss and many others in the field, the present work aims at narrowing this gap.

This thesis deals with randomly perturbed dynamical systems. Such systems frequently arise in the modelling of phenomena in biology, mechanics, chemistry, and physics. In some cases random perturbations form a minor aspect of the problem under study. Then a deterministic description can be used. In the present work the behaviour of the dynamical systems depends essentially on the random perturbations. We encounter systems with so-called 'diffusion across the flow' (Chapters 1,2) and systems with 'diffusion against the flow' (Chapters 1,3–5). The stability of equilibria of these systems (and thus, the lifetime, reliability of these systems) is affected by random perturbations.

In the study of so-called 'exit problems' we consider a domain in the state space of the dynamical system and try to compute statistical quantities related to escape from this domain, such as the probability density function of the exit-time, the probability density function of exit points on the boundary of the domain (or, less ambitiously, the first few statistical moments of these densities: mean, variance, etc.). The expectation value of the exit-time can be used to express the stochastic stability of the system.

We speak of randomly *perturbed* dynamical systems, so we assume that the stochastic fluctuations are small. This is often a realistic assumption. To

derive expressions for the statistical quantities mentioned above, we employ asymptotics where the small parameter is related to the intensity of the random perturbations. The asymptotic method used in Chapter 2 is well-established. The asymptotics in Chapters 3–6 are of a formal character. The asymptotic analysis is performed to the lowest order necessary to incorporate the essential effects. In view of the complexity of this simplest approach, we did not carry out higher order calculations.

The first chapter forms an introduction to some important topics in the theory of exit problems. We discuss the relevant (initial-) boundary-value problems, the classification of boundaries of domains of stochastic dynamical systems, and we give elementary examples of systems with 'diffusion across the flow' and 'diffusion against the flow' and their asymptotic solution. This chapter facilitates access to literature on exit problems and to the remaining chapters of this thesis. A more detailed treatment of the topics touched upon in this chapter is found in the cited literature.

Chapter 2 is concerned with the dynamics of a loaded stiff rod. The load consists of a deterministic part and a small stochastic part. An accumulation of stochastic load fluctuations may drive the energy of the rod across some critical level. The expectation value of the time to reach this critical energy level is a measure for the reliability of the system that contains the rod. According to the directions in which the loads act, various cases are distinguished. We derive expressions for the expectation value of the exit-time and (for some of the cases) of a number of other statistical quantities, as the exit-time density, its moments and cumulants and the probability density function of the square root of the energy (as a function of time). We use an asymptotic method known as the averaging technique. As a matter of fact, the model is a randomly loaded slightly damped oscillator. Since many practical systems near equilibrium behave essentially like a slightly damped oscillator, the results obtained may be expected to have a wide range of application.

In Chapter 3 we study the exit problem for a stochastic dynamical system of interacting biological populations. Exit from the domain (the positive orthant) corresponds with extinction of a population. We start with a birth and death process, having a discrete state space, and subsequently formulate an 'approximate' Fokker-Planck (or forward Kolmogorov) equation in a continuous state space. It is assumed that the deterministic system associated with the stochastic dynamical system has a point attractor in the positive orthant. The

biological system will remain for some (probably long) time in a neighbourhood of the attracting point, but after a (rare) succession of random fluctuations, one of the populations will get extinct. Determining the expected time of exit (of whichever of the populations), and of which population will probably get extinct first, requires the numerical solution of a system of so-called 'ray equations' (obtained from the Fokker-Planck equation by the WKB-method). In literature these differential equations are provided with initial conditions, which entails difficulties in the numerical construction of contours in the state space on which the eikonal function attains a constant value (confidence contours). We define boundary conditions instead of initial conditions and thereby resolve these difficulties. The ideas are illustrated by a two-dimensional generalized Lotka-Volterra model. This model allows a nice demonstration of the concepts of deterministic stability and stochastic stability. Numerically constructed confidence contours are shown for predator-prey, mutualism and competition variants of the model. We carry out numerical simulations of birth-death processes to check the results.

A discussion of various ways of numerical solution of the system of ray equations is found in Chapter 4. In particular we explain the boundary-value method referred to above. Moreover we give some details on the numerical construction of rays and confidence contours. At the end we present an example with intersecting rays. This phenomenon is investigated analytically in Chapter 6.

In Chapter 5 we are concerned again with a stochastic version of the two-dimensional generalized Lotka-Volterra model. The approach differs from that in Chapter 3 in that now we pay attention to what happens near the boundaries of the domain (the positive coordinate axes). The main difficulty is caused by the fact that the normal components of both the drift and the diffusion coefficients vanish near the boundaries, as linear functions of the distance to the boundaries. To obtain expressions for the statistical quantities of interest, we generalize a method of other authors in the study of a similar one-dimensional problem. The asymptotic expressions contain some unknown constants, that can be obtained numerically. Explicit calculations are carried out for a predator-prey system as an example.

Applying the WKB-method to the forward Kolmogorov equation, we obtain the ray equations. In the solution of the ray equations one sometimes observes intersecting rays forming caustic surfaces. This phenomenon is stud-

ied in Chapter 6. Near locations of intersecting rays, the WKB-approximation does not hold. We derive a uniform asymptotic expansion in terms of new canonical integrals whose validity extends over regions containing caustics. We start with the simple case of a cusp arising in a diffusion problem for which explicit results can be obtained. Subsequently, we generalize it to a formal approach to singularities arising in the forward Kolmogorov equation.

The text of each of the chapters has appeared as a report or a publication in a scientific journal or is going to:

[1] H. Roozen (1989), *A short introduction to exit problems.* CWI Quarterly, Vol. 2, No. 1, 45-65.

[2] H. Roozen (1989), *Stochastic stability of the loaded stiff rod.* Journal of Engineering Mathematics 23, 357-376.

[3] H. Roozen (1987), *Equilibrium and extinction in stochastic population dynamics.* Bull. Math. Biol., Vol. 49, No. 6, 671-696.

[4] H. Roozen (1986), *Numerical construction of rays and confidence contours in stochastic population dynamics.* Technical Note, Centre for Mathematics and Computer Science, Amsterdam.

[5] H. Roozen (1989), *An asymptotic solution to a two-dimensional exit problem arising in population dynamics.* SIAM J. Appl. Math. Vol. 49, No. 6, 1793-1810.

[6] H. Roozen (1989), *Singularities arising in the asymptotic solution of the forward Kolmogorov equation.* Report (submitted for publication).

A SHORT INTRODUCTION TO EXIT PROBLEMS

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Many phenomena that occur in nature and technology exhibit a stochastic behaviour. When the stochastic element is relevant, it has to be included in the modeling of such phenomena. We discuss models with a deterministic component and a small stochastic component. The short term behaviour of these models is determined mainly by the deterministic component, whereas the long term behaviour is influenced considerably by the stochastic component. For the description of the long term behaviour, deterministic stability concepts (stable, neutral equilibrium) are inadequate and have to be replaced by stochastic stability concepts (the expected exit time from a region containing such a deterministic equilibrium). In the study of so-called exit problems we consider a domain in the state space of a stochastic dynamical system and try to determine statistical quantities (such as the mean exit time, the distribution of exit points over the boundary of the domain, etc.) related to leaving this domain. We treat the exit problems from an asymptotic (in the limit for small noise) point of view.

1. Introduction

In this contribution we will study some aspects of stochastic dynamical systems that have a deterministic part (defining the associated 'deterministic system') and a small stochastic part consisting of Gaussian white noise (referred to as 'stochastic fluctuations').

In some of these systems, the dynamical characteristics of interest are dominated by the deterministic system, while the stochastic fluctuations are only

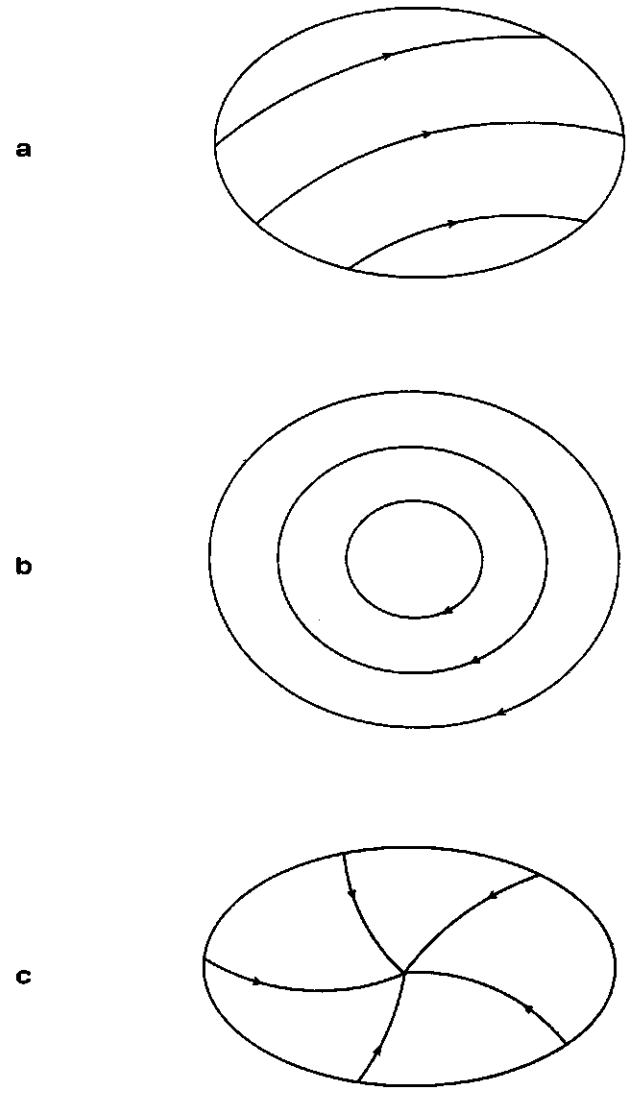

Figure 1. Illustration of diffusion (a) with, (b) across and (c) against the flow.

of secondary importance, in the sense that omission of the stochastic fluctuations does not essentially alter these characteristics. This if demonstrated, for example, by a 'diffusion *with* the flow', see Figure 1*a*. Starting at a point in a bounded domain D, the trajectories of the stochastic dynamical system leave the domain D with probability close to one in the same time as the deterministic trajectory through that point. The probability density function defined on the boundary ∂D, describing the point of exit from D of the stochastic dynamical system, is concentrated near the deterministic exit point. Stochastic systems of this type will not be considered here.

In other stochastic dynamical systems, the stochastic fluctuations, though small, are of great importance to the dynamical characteristics of interest. Without stochastic fluctuations these characteristics are essentially changed. One such example is a 'diffusion *across* the flow', as depicted in Figure 1*b*. The deterministic system consists of a center point, surrounded by closed trajectories. Consider a domain D, enclosed by one of these trajectories. In the deterministic system no exit from D can occur, since we follow ceaselessly the closed trajectory through the starting point. In contrast with this fact, in the stochastic dynamical system, i.e. in the deterministic system perturbed by stochastic fluctuations, exit will occur in finite time with probability one. Another such example is a 'diffusion *against* the flow', depicted in Figure 1*c*. A bounded domain D is entered at its boundary ∂D by deterministic trajectories that converge to an asymptotically stable limit point contained in D. In this deterministic system, if we start at some point in D, we approach the limit point along the trajectory through the starting point. Again, the deterministic system does not allow exit from D, but when this system is perturbed by stochastic fluctuations, exit will happen in finite time with probability one. Although more complicated systems exist that exhibit a similar behaviour, such as attracting limit cycles or strange attractors, etc., notably in higher dimensional domains, we will confine ourselves to systems of the two simple types described here, in particular to systems of the last type.

We will concentrate on a few statistical characteristics related to the problem of exit from a domain, like the expectation value of the time of first exit (which provides a measure for the stability of the stochastic system) and the distribution of exit points over the boundary of the domain.

2. The equations

A stochastic system is frequently described either in terms of a stochastic differential equation (that, as an extension to an ordinary differential equation, contains stochastic terms) [1], or in terms of a Kolmogorov equation. In the former case, an equivalent description in terms of a Kolmogorov equation may be possible. In this section we formulate the forward and backward Kolmogorov equations [19,55], which form the starting point of our analysis.

We consider a stochastic dynamical system that has been defined on the n-dimensional domain D in the state space. Let $v(x,t)dx$ denote the probability that the system is in the infinitesimal subregion $(x, x+dx) \in D$ at time t. This function satisfies the forward (Kolmogorov) equation (also called the Fokker-Planck equation)

$$\frac{\partial v}{\partial t} = M_\epsilon v, \quad x \in D, \tag{2.1}$$

where the differential operator M_ϵ is defined by

$$M_\epsilon v \equiv -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(x)v) + \frac{\epsilon}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}(x)v). \tag{2.2}$$

Equation (2.1) has to be supplemented with the relevant initial and boundary conditions. The first term on the right side of (2.2) represents the deterministic part of the dynamical system, b is called the deterministic or drift vector. The second term on the right side represents the stochastic fluctuations. The matrix ϵa is known as the diffusion matrix and is symmetric and positive (semi-) definite. The parameter ϵ, $0 < \epsilon \ll 1$, indicates that the stochastic fluctuations are small relative to the deterministic part. When $\epsilon = 0$ the stochastic fluctuations are absent and equation (2.1) reduces to the Liouville equation. Then, if the initial position is deterministic, the initial probability density is a delta function, say $\delta(x - x_0)$, and the solution of the Liouville equation corresponds to the solution of the system of ordinary differential equations

$$\frac{dx_i}{dt} = b_i(x), \quad i = 1, 2, \ldots, n \tag{2.3a}$$

with initial conditions

$$x(0) = x_0. \tag{2.3b}$$

This system is defined as the deterministic system corresponding to the stochastic dynamical system.

In order to determine the distribution of exit points over ∂D, as well as the expected time of first exit from D, we use backward equations. Let $p(x, y)\ dS_y$ be the probability of exit at $dS_y \in \partial D$, given that we started at $x \in D$, i.e. p is the exit density. Note that p is a probability density function with respect to y. We define the function $u_s(x)$ as follows:

$$u_s(x) = \int_{\partial D} f(y)p(x, y)\ dS_y, \tag{2.4}$$

where f is a function on ∂D that can be chosen arbitrarily. With f defined as the indicator function

$$f = \begin{cases} 1 & \text{on } \partial_1 D, \text{ where } \partial_1 D \subseteq \partial D, \\ 0 & \text{on } \partial_0 D = \partial D \setminus \partial_1 D, \end{cases} \tag{2.5}$$

$u_s(x)$ is the probability of exit at $\partial_1 D$, given that we started at $x \in D$. The function u_s is the solution of the stationary backward equation

$$L_\epsilon u_s = 0, \quad x \in D, \tag{2.6a}$$

subject to the boundary condition

$$u_s = f(x), \quad x \in \partial D, \tag{2.6b}$$

where the differential operator L_ϵ is defined by

$$L_\epsilon u \equiv \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + \frac{\epsilon}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}, \tag{2.7}$$

and a and b are the same functions as above.

We consider the time-dependent backward equation

$$\frac{\partial u}{\partial t} = L_\epsilon u \tag{2.8a}$$

as well. With the boundary condition

$$u = f(x), \quad x \in \partial D, \tag{2.8b}$$

where f is the indicator function (2.5), and the initial condition

$$u(x, 0) = 0, \quad x \in D, \tag{2.8c}$$

$u(x,t)$ is the probability that exit occurs at $\partial_1 D$ on the time interval $(0,t]$, given that we started at $x \in D$ on time $t = 0$.

The time of (first) exit from D is defined as

$$\tau(x) = \inf \{t | x(t) \in \partial D,\ x(0) = x \in D\}. \tag{2.9}$$

Its expectated value $T(x) = \mathrm{E}\tau(x)$ is the solution of the boundary value problem:

$$L_\epsilon T = -1, \quad x \in D, \tag{2.10a}$$

$$T = 0, \quad x \in \partial D. \tag{2.10b}$$

Equation (2.10a) is known as the Dynkin equation.

The reader interested in the details of the equations and the corresponding conditions that we have given here, and in related material, is referred to the literature [19,55]. In later sections we shall be concerned with the asymptotic solution of (2.1),(2.6),(2.8) and (2.10) for small ϵ.

The backward and forward differential operators L_ϵ and M_ϵ defined above are formal adjoints, which means that the following relation holds [48]:

$$\int\int_D (vL_\epsilon u - uM_\epsilon v)\, dx = \int_{\partial D} P.\xi\, dS_x, \tag{2.11}$$

where P is the vector with components

$$P_i = \sum_{j=1}^{n} \frac{\epsilon}{2} \left[a_{ij} v \frac{\partial u}{\partial x_j} - u \frac{\partial (a_{ij} v)}{\partial x_j} \right] + b_i uv, \quad i = 1, 2, \ldots, n \tag{2.12}$$

dS_x is an infinitesimal surface element containing x, and ξ denotes the outward normal on ∂D.

3. The boundary

In the study of exit problems, the behaviour of the stochastic system at and near the boundary of the domain deserves special attention, since the domain is left via the boundary. For a given stochastic system we must verify whether the boundary can actually be reached from the interior domain.

In many practical situations the type of the boundary is determined by the drift vector and the diffusion matrix. For one-dimensional stochastic systems there is a classification of such boundaries originating from Feller [16]. In

integrable				type of boundary	boundary is attainable	interior is attainable
I_1	I_2	I_3	I_4			
yes	yes			regular	yes	yes
yes	no	yes		exit	yes	no
yes	no	no		natural attracting	no	no
no	yes		yes	entrance	no	yes
no	yes		no	natural repelling	no	no
no	no					

Table 1. Boundary classification for one-dimensional stochastic systems. Five boundary types are defined according to the integrability of some of the functions I_1 to I_4. The last two columns indicate whether the boundary is attainable from the interior domain and whether the interior domain is attainable from the boundary.

a semi-group approach to adjoint forward and backward equations he distinguished the regular, exit, entrance and natural boundaries. In table 1 we have repeated schematically the boundary classification as it has been described in [54]. The type of boundary depends on the integrability at the boundary point of some of the following functions:

$$\begin{aligned} I_1(x) &= \exp\left[-\frac{4}{\epsilon}\int^x b(t)/a(t)\ dt\right], \\ I_2(x) &= \frac{2}{\epsilon a(x)}\exp\left[\frac{4}{\epsilon}\int^x b(t)/a(t)\ dt\right], \\ I_3(x) &= I_1(x)\int^x I_2(t)\ dt, \\ I_4(x) &= I_2(x)\int^x I_1(t)\ dt. \end{aligned} \tag{3.1}$$

If there are sample paths that hit the boundary in finite time, the boundary is attainable, otherwise it is unattainable. Table 1 indicates that it makes sense to talk of an exit problem only if at least one of the boundaries of the domain (an interval) is a regular or exit boundary (which are the only cases that permit the boundary to be reached in finite time from the interior domain). When we have a regular boundary we should speak of the problem of *first* exit from a domain, since in that case the exited domain can be re-entered and subsequently re-exited. For higher dimensional stochastic systems a similar classification has never been published.

In other situations we dispose of a stochastic system, defined by a drift vector and a diffusion matrix on a domain D, and we want to erect a boundary of a desired type at any place in D, thereby restricting the domain to a subdomain D' of D. Examples of such boundaries are absorbing and reflecting boundaries [19]. On reaching an absorbing boundary from the interior domain D', the system is taken apart (is absorbed) so that this domain cannot be entered again, comparable with an exit boundary. At a reflecting boundary no probability can pass, so that exit at this boundary is impossible. With respect to the solution of the forward equation an absorbing boundary implies the boundary condition $v(x,t) = 0$, where $x \in \partial D'$, and a reflecting boundary implies the condition $\xi.J(x,t) = 0$, where $x \in \partial D'$, ξ is the outward normal on $\partial D'$ and J is the probability current, i.e. the vector with components

$$J_i(x,t) = b_i(x)v - \frac{\epsilon}{2}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(a_{ij}(x)v), \quad i = 1,2,\ldots,n. \tag{3.2}$$

Absorbing and reflecting boundaries can be set up in domains of any dimension.

4. An example of diffusion across the flow

In this section we treat a simple example of diffusion across the flow. Consider an oscillator with a small damping, that is subjected to a stochastic forcing. The damping effect is introduced here since it leads to a realistic model, without essential complications for the analysis that follows. As a consequence of the stochastic effects, the energy of the oscillator can reach some critical level after some time. This critical level can for example be chosen as the energy at which the system, the oscillator is part of, breaks down. We will derive an expression for the expected time needed to reach this critical energy level, which is a measure for the stochastic stability of the oscillator. In nondimensional form, the differential equation for this problem is formally [53]

$$\ddot{x} + \epsilon\alpha\dot{x} + x = \sqrt{\epsilon}q(x)\,\xi, \tag{4.1}$$

where x is the deviation from the equilibrium position, the dot denotes differentiation with respect to the time t, ϵ is a small positive parameter, and $\epsilon\alpha$ is a nonnegative $O(\epsilon)$ damping constant. The right side of (4.1) represents

Gaussian white noise with intensity $\epsilon q^2(x)$. The function q is approximated by the first two terms of its Taylor expansion around $x = 0$

$$q(x) \approx \beta_0 + \beta_1 x. \tag{4.2}$$

It is assumed that not both β_0 and β_1 are zero. The second order formal differential equation (4.1) is replaced by the system of first order stochastic differential equations [1]

$$\begin{aligned} dx &= \dot{x}dt, \\ d\dot{x} &= -(\epsilon\alpha\dot{x} + x)dt + \sqrt{\epsilon}(\beta_0 + \beta_1 x)\, dW, \end{aligned} \tag{4.3}$$

where W represents Brownian motion. The undisturbed ($\epsilon = 0$) system (4.1) is an undamped oscillator, whose dynamics are described by closed trajectories around the origin in the $(x, \dot{x})$-phase space. Each trajectory corresponds to an energy level. The energy is larger for orbits farther away from the origin. The effect of nonzero ϵ is that the trajectories tend to spiral slightly inwards to approach the origin as a consequence of damping if $\alpha \neq 0$ and contain stochastic fluctuations in the $\dot{x}$-direction. The backward equation corresponding to (4.3) reads [19]

$$\frac{\partial u}{\partial t} = \dot{x}\frac{\partial u}{\partial x} - (\epsilon\alpha\dot{x} + x)\frac{\partial u}{\partial \dot{x}} + \frac{\epsilon}{2}(\beta_0 + \beta_1 x)^2 \frac{\partial^2 u}{\partial \dot{x}^2}, \tag{4.4}$$

with u defined as in Section 2. This equation is studied asymptotically for small ϵ and on the time scale of $O(\epsilon^{-1})$. With

$$t = \tilde{t}/\epsilon, \quad u = u^0 + \epsilon u^1 + \cdots, \tag{4.5}$$

and the transformations $(x, \dot{x}) \to (r, \theta)$ defined by

$$x = \sqrt{2}\, r\cos\theta, \quad \dot{x} = \sqrt{2}\, r\sin\theta, \tag{4.6}$$

we obtain to leading order in ϵ

$$\frac{\partial u^0}{\partial \theta} = 0, \tag{4.7}$$

implying that u^0 is a function of r and $\tilde{t}$ only. The variable r is the square root of the dimensionless energy of the undisturbed ($\epsilon = 0$) system. The nondimensionalization process can be carried out such that the critical energy corresponds to $r^2 = 1$, thus $r \in [0, 1]$. To the next order in ϵ we obtain an

equation in terms of u^0 and u^1. Terms with u^1 vanish by integration of this equation with respect to θ from 0 to 2π and the additional assumption that u^1 is periodic in θ with period 2π. The resulting equation for u^0 reads

$$\frac{\partial u^0}{\partial \tilde{t}} = \left(\frac{a_0}{r} + a_2 r\right) \frac{\partial u^0}{\partial r} + \left(a_0 + a_1 r^2\right) \frac{\partial^2 u^0}{\partial r^2}, \tag{4.8a}$$

with

$$a_0 = \beta_0^2/8, \quad a_1 = \beta_1^2/16, \quad a_2 = 3\beta_1^2/16 - \alpha/2. \tag{4.8b}$$

The description to this order in ϵ includes the effects of both damping and stochastic fluctuations. If, as a consequence of the latter effect the critical energy $r^2 = 1$ is reached in finite time with probability one, starting from $r \in [0,1]$, the oscillator is said to be stochastically unstable. In that case, the stability of the oscillator is measured by the expected time of exit from the unit interval at 1.

In the present discussion we only consider the case $|\beta_0|, |\beta_1| \gg O(\epsilon^{\frac{1}{2}})$, so that a_0 and a_1 do not vanish in the asymptotics leading to equation (4.8a) and thus appear in this equation indeed. The boundary $r = 0$ is then an entrance boundary and at $r = 1$ we adopt an absorbing boundary in order to model the breakdown of the oscillator at the critical energy. Thus exit from the unit interval can take place only at $r = 1$. Let $u_s(r)$ be the probability of exit at $r = 1$, given that we started at r on time $t = 0$. The leading order term $u_s^0(r)$ in the expansion of $u_s(r)$ in powers of ϵ is obtained by solving the stationary equation (4.8a) with boundary condition $u_s^0(1) = 1$. The only relevant solution (i.e. yielding values $u_s^0(r) \in [0,1]$) is $u_s^0(r) \equiv 1$. There is no freedom to specify an arbitrary boundary condition at $r = 0$. We conclude that if we start somewhere on the interval $[0,1]$, exit at $r = 1$ will occur with probability one, so that the oscillator is stochastically unstable. Next we consider the expected exit time $T(r)$, starting from a point r. Similar to the time scaling in (4.5) we put $T = \tilde{T}/\epsilon$ and similar to the expansion of u in (4.5) we put $\tilde{T} = \tilde{T}^0 + \epsilon \tilde{T}^1 + \cdots$, so that $T = \tilde{T}^0/\epsilon + \tilde{T}^1 + \cdots$. An approximation for T is found by solving the Dynkin equation

$$-1 = \left(\frac{a_0}{r} + a_2 r\right) \frac{\partial \tilde{T}^0}{\partial r} + \left(a_0 + a_1 r^2\right) \frac{\partial^2 \tilde{T}^0}{\partial r^2}, \tag{4.9a}$$

with the boundary conditions

$$\tilde{T}^0(0) \text{ is finite}, \tag{4.9b}$$

$$\tilde{T}^0(1) = 0. \tag{4.9c}$$

For $a_2 \neq a_1$ we find

$$T(r) \sim \frac{1}{\epsilon(a_1 - a_2)} \int_r^1 \Big[\Big(\frac{a_1}{a_0}s^2 + 1\Big)^{-\frac{a_2}{2a_1} + \frac{1}{2}} - 1\Big]\frac{1}{s}\, ds, \quad (a_2 \neq a_1). \quad (4.10)$$

If $a_2 = a_1$, this is substituted into equation (4.9a). Solving the corresponding boundary value problem we find

$$T(r) \sim \frac{1}{2\epsilon a_1} \int_r^1 \frac{1}{s} \log\Big(\frac{a_1}{a_0}s^2 + 1\Big)\, ds. \quad (4.11)$$

The reader is asked to take notice of the order of magnitude of the results (4.10) and (4.11) in order to compare this with results to be derived later for diffusion against the flow systems. The cases that either β_0 or β_1 are of order $O(\epsilon^{\frac{1}{2}})$ must be treated separately. In the case $\beta_0 = O(\epsilon^{\frac{1}{2}})$ it can be shown that if the damping is larger than a certain value the oscillator is stochastically stable on the time scale under consideration. This means that on this time scale the probability of exit is less than one, in contrast with the result above.

A more detailed description of the exit problem for oscillators as described here can be found in [53]. The stochastic stability of oscillators with a different type of damping (as cubic damping) or noise (red, dichotomic, etc.) and with a forcing described by a potential function has been treated in [14]. The asymptotics that we have used in this example to arrive at equation (4.8) are well established and are known under the names of averaging technique [3,31,35,50,56] and adiabatic elimination of fast variables [19].

5. Diffusion against the flow

In this section we discuss the exit problem for systems that are of diffusion against the flow type. First we treat a one-dimensional system, then a multidimensional potential system that can be treated with essentially the same means, and we will conclude with more general multidimensional systems.

5.1 A one-dimensional system

Consider the stochastic system defined on $[\alpha, \beta]$, where $\alpha < 0$ and $\beta > 0$, with drift coefficient $b(x)$ satisfying

$$b(x) \begin{cases} > 0, & x \in [\alpha, 0), \\ = 0, & x = 0, \\ < 0, & x \in (0, \beta], \end{cases} \quad (5.1.1a)$$

so that $x = 0$ is an attractor, and diffusion coefficient $\epsilon a(x)$, $0 < \epsilon \ll 1$, with

$$a(x) > 0, \quad x \in [\alpha, \beta]. \tag{5.1.1b}$$

For this system we will find the functions u_s and T defined in Section 2, asymptotically for small ϵ. The boundary value problem for u_s reads

$$L_\epsilon u_s \equiv \frac{\epsilon}{2} a(x) \frac{d^2 u_s}{dx^2} + b(x) \frac{du_s}{dx} = 0, \tag{5.1.2a}$$

$$u_s(\alpha) = c_\alpha, \quad u_s(\beta) = c_\beta, \tag{5.1.2b}$$

where c_α and c_β are given constants. The choice of boundary conditions (5.1.2b) incorporates the following cases

1. $u_s(\alpha) = 1, u_s(\beta) = 0$: $u_s(x)$ is the probability of exit (in finite time) at the boundary α, given we start in x.
2. $u_s(\alpha) = 0, u_s(\beta) = 1$: $u_s(x)$ is the probability of exit (in finite time) at the boundary β, given we start in x.
3. $u_s(\alpha) = 1, u_s(\beta) = 1$: $u_s(x)$ is the probability of exit (in finite time and independent at which of the two boundaries), given we start in x.

Remark that $u_s(x) \equiv 1$ solves (5.1.2a) and satisfies choice 3 of the boundary conditions. Thus, with probability one exit occurs (in finite time). This result is independent of ϵ and independent of the starting point x.

The reduced equation (5.1.2a), i.e. equation (5.1.2a) with $\epsilon = 0$, is solved by any constant c_0. This solution is valid away from α and β but not near these points since the boundary conditions (5.1.2b) cannot be satisfied. We assume that the functions a and b have the Taylor series expansions

$$\begin{aligned} a(x) &= a(\alpha) + a'(\alpha)(x-\alpha) + \cdots \quad \text{near } x = \alpha, \\ b(x) &= b(\alpha) + b'(\alpha)(x-\alpha) + \cdots \quad \text{near } x = \alpha, \\ a(x) &= a(\beta) + a'(\beta)(x-\beta) + \cdots \quad \text{near } x = \beta, \\ b(x) &= b(\beta) + b'(\beta)(x-\beta) + \cdots \quad \text{near } x = \beta. \end{aligned} \tag{5.1.3}$$

As an abbreviation we use the notation $\tilde{b}(x) = 2b(x)/a(x)$. It follows from (5.1.1) that $\tilde{b}(\alpha) > 0$ and $\tilde{b}(\beta) < 0$. A boundary layer analysis near $x = \alpha$ and $x = \beta$ shows the presence of $O(\epsilon)$ boundary layers near these points. An asymptotic expression for u_s to leading order in ϵ that is uniformly valid on $[\alpha, \beta]$ is given by

$$u_s(x) \sim c_0 + (c_\alpha - c_0) \exp[-\tilde{b}(\alpha)(x-\alpha)/\epsilon] + (c_\beta - c_0) \exp[-\tilde{b}(\beta)(x-\beta)/\epsilon]. \tag{5.1.4}$$

Note that the constant c_0 is left undetermined by the given asymptotics. To find c_0 we utilize a variational formulation of the boundary value problem (5.1.2), following [23], see also [61]. After multiplication by the factor

$$g(x) = \exp\left[\int_0^x \frac{2b(s) - \epsilon a'(s)}{\epsilon a(s)}\, ds\right], \tag{5.1.5}$$

equation (5.1.2a) can be written as the Euler equation

$$\frac{dF_{u_s'}}{dx} - F_{u_s} = 0, \tag{5.1.6}$$

with $F = \frac{\epsilon}{4}(u_s')^2 ag$. Consequently, the solution of (5.1.2) corresponds to an extremal of the functional

$$J[u_s] = \int_\alpha^\beta \frac{\epsilon}{4}(u_s')^2 ag\, dx, \tag{5.1.7}$$

with respect to functions u_s satisfying the boundary conditions (5.1.2b), see [5,9,46,47]. The expression (5.1.4) for u_s is substituted into the integral in (5.1.7), and this integral is evaluated asymptotically for small ϵ by the method of Laplace [2,4]. The constant c_0 is determined by the requirement that the corresponding function u_s is an extremal of the functional J thus obtained, that is, by

$$\frac{dJ}{dc_0} = 0. \tag{5.1.8}$$

In addition to (5.1.1) we shall henceforth assume that

$$\tilde{b}'(x) < 0, \quad x \in [\alpha, \beta]. \tag{5.1.9}$$

Carrying out the above procedure we then find that the largest contributions to the integral in (5.1.7) are from the neighbourhoods of α and β, and c_0 is given by

$$c_0 = \frac{c_\alpha \tilde{b}(\alpha)\exp[-I(\alpha)/\epsilon] - c_\beta \tilde{b}(\beta)\exp[-I(\beta)/\epsilon]}{\tilde{b}(\alpha)\exp[-I(\alpha)/\epsilon] - \tilde{b}(\beta)\exp[-I(\beta)/\epsilon]}, \tag{5.1.10}$$

where $I(x)$ is defined as

$$I(x) = -\int_0^x \tilde{b}(s)\, ds, \quad (> 0 \text{ for } x \neq 0). \tag{5.1.11}$$

The result (5.1.10) simplifies to

$$c_0 = \begin{cases} c_\alpha, & \text{if } I(\alpha) < I(\beta), \\ c_\beta, & \text{if } I(\beta) < I(\alpha), \\ \dfrac{c_\alpha \tilde{b}(\alpha) - c_\beta \tilde{b}(\beta)}{\tilde{b}(\alpha) - \tilde{b}(\beta)}, & \text{if } I(\alpha) = I(\beta). \end{cases} \tag{5.1.12}$$

In the limit $\epsilon \to 0$ we thus have the following result. The position of the starting point is of importance only if we start in $O(\epsilon)$-neighbourhoods of the boundaries α and β. All other starting points exit with the same probability at a certain boundary; this is with probability one at the boundary with the lowest value of I. If $I(\alpha) = I(\beta)$ and if we start outside $O(\epsilon)$-neighbourhoods of the boundaries, the probability of exit at α and the probability of exit at β are constants with values between zero and one (adding to one), depending on $\tilde{b}(\alpha)$ and $\tilde{b}(\beta)$.

The asymptotic result that we have derived above is found alternatively by evaluation for small ϵ of the exact solution of the boundary value problem (5.1.2). For the higher dimensional problems in the following subsections no exact solution is possible in general, and can we only use asymptotic methods.

Next we derive an expression for the expected time T of exit from the interval $[\alpha, \beta]$. The function T satisfies the inhomogeneous equation

$$\frac{\epsilon}{2} a(x) \frac{d^2T}{dx^2} + b(x) \frac{dT}{dx} = -1, \tag{5.1.13a}$$

with the conditions

$$T(\alpha) = 0, \quad T(\beta) = 0. \tag{5.1.13b}$$

The approach to this boundary value problem is largely the same as above, the only additional difficulty is the appearance of the inhomogeneous term in (5.1.13a). We anticipate that T is of the form

$$T(x) = c_0(\epsilon) \tau(x), \tag{5.1.14a}$$

where c_0 is a constant with respect to x that depends on ϵ in the following way

$$1/c_0(\epsilon) = o(\epsilon), \tag{5.1.14b}$$

asymptotically for small ϵ. Expression (5.1.14a) is substituted into (5.1.13) and the corresponding boundary value problem is asymptotically solved to obtain τ. For T we find

$$T(x) \sim c_0(\epsilon) \left\{ 1 - \exp[-\tilde{b}(\alpha)(x - \alpha)/\epsilon] - \exp[-\tilde{b}(\beta)(x - \beta)/\epsilon] \right\}, \tag{5.1.15}$$

to leading order in ϵ uniformly on $[\alpha, \beta]$. The unknown constant c_0 is determined again from a variational principle. Equation (5.1.13a) is multiplied by the factor g defined in (5.1.5). The solution of the boundary value problem (5.1.13) then corresponds to an extremal of the functional

$$J[T] = \int_\alpha^\beta \left[\frac{\epsilon}{4}(T')^2 a - T\right] g \, dx, \tag{5.1.16}$$

with respect to functions $T(x)$ satisfying the boundary conditions (5.1.13b). This functional is evaluated by substitution of (5.1.15) into (5.1.16) and application of the method of Laplace. The major contributions to the integral in (5.1.16) are from neighbourhoods of α and β and from a neighbourhood of $x = 0$. Putting (5.1.8) it is found that

$$c_0 = \frac{\frac{4}{a(0)}\sqrt{\frac{2\pi\epsilon}{-b'(0)}}}{\tilde{b}(\alpha)\exp[-I(\alpha)/\epsilon] - \tilde{b}(\beta)\exp[-I(\beta)/\epsilon]}. \tag{5.1.17}$$

This result simplifies to

$$c_0 = 4\sqrt{\frac{\pi\epsilon}{-b'(0)a(0)}} \cdot \begin{cases} \frac{1}{\tilde{b}(\alpha)}\exp[I(\alpha)/\epsilon], & \text{if } I(\alpha) < I(\beta), \\ \frac{1}{-\tilde{b}(\beta)}\exp[I(\beta)/\epsilon], & \text{if } I(\beta) < I(\alpha), \\ \frac{1}{\tilde{b}(\alpha)-\tilde{b}(\beta)}\exp[I(\beta)/\epsilon], & \text{if } I(\alpha) = I(\beta). \end{cases} \tag{5.1.18}$$

Thus, in the limit $\epsilon \to 0$, if we start outside an $O(\epsilon)$-neighbourhood of the boundaries α and β, the expected exit time is independent of the starting point x and equals one of the constants given in (5.1.18), depending on the magnitude of $I(\alpha)$ and $I(\beta)$. Note that the expected value of the exit time is an exponential function of the reciprocal of the small parameter ϵ, which is very large.

Other asymptotic approaches to the type of problem we encountered in this subsection can be found in de Groen [24], who used an eigenfunction expansion method, in Jiang Furu [26], who used the two-scale method, and in Matkowsky and Schuss [41], whose method will be explained further on. A biologically relevant model in which at one of the boundaries of the domain both the drift and diffusion coefficients vanish, linearly with the distance to this boundary, has been treated in [25,52,59,60].

5.2 Potential systems

The method to determine c_0 in the previous section is based on the fact that with the factor g defined in (5.1.5) the nonself-adjoint backward differential operator L_ϵ turns into a self-adjoint operator, so that consequently variational formulations of the boundary value problems of exit become feasible. In this subsection we shall see that for multidimensional stochastic systems a similar factor g exists only for a class of so-called potential systems. Results for these systems will be derived.

We consider an n-dimensional stochastic system with a domain D that contains a deterministic point attractor and that has a boundary ∂D at which the deterministic trajectories *enter* D. We assume that the diffusion matrix is positive definite. First we study the asymptotic solution of the boundary value problem (2.6). Equation (2.6a) with $\epsilon = 0$ is solved by a constant c_0. It can be shown [19,55] that this solution is valid outside an $O(\epsilon)$-neighbourhood of ∂D (this is related to the fact that deterministic trajectories enter D). We assume that ∂D is smooth. For points $x \in D$ near ∂D, we introduce $n-1$ new coordinates along ∂D, and the new coordinate $\rho = |x - x'|$, where x' is the projection of x on ∂D. Using the stretching transformation

$$z = \epsilon\rho \tag{5.2.1}$$

we then obtain from (2.6a) the boundary layer equation

$$\frac{1}{2}\bar{a}(x')\frac{\partial^2 u_s}{\partial z^2} + \bar{b}(x')\frac{\partial u_s}{\partial z} = 0, \tag{5.2.2a}$$

with $\bar{a}$ and $\bar{b}$ defined as

$$\bar{a}(x') = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x')\xi_i\xi_j, \quad \bar{b}(x') = -\sum_{i=1}^{n} b_i(x')\xi_i, \tag{5.2.2b}$$

where ξ denotes the outward normal on ∂D. Equation (5.2.2a) is solved with the conditions (2.6b) and $\lim_{z\to\infty} u_s = c_0$. In the original variable x we find

$$u_s(x) \sim c_0 + (f(x') - c_0)\exp[-\tilde{b}(x')|x - x'|/\epsilon], \tag{5.2.3}$$

uniformly on D, where $\tilde{b}(x') = 2\bar{b}(x')/\bar{a}(x')$. We intent to determine the unknown constant c_0 from a variational principle again. In general the backward

operator L_ϵ defined in (2.7) is nonself-adjoint. A factor $g(x)$ is sought such that gL_ϵ is self-adjoint. This requirement leads to the following expression

$$\epsilon \frac{\partial \log g}{\partial x_i} = \sum_{j=1}^{n} a_{ij}^{-1} \left[2b_j - \epsilon \sum_{k=1}^{n} \frac{\partial a_{jk}}{\partial x_k} \right] \equiv V_i, \quad i = 1, 2, \ldots, n \tag{5.2.4}$$

where a_{ij}^{-1} denotes the inverse of a_{ij} (we assume this matrix is invertible). A function g satisfying (5.2.4) exists only if the vector field V is irrotational, that is, can be described by a potential function ϕ as follows

$$V_i = -\frac{\partial \phi}{\partial x_i}. \tag{5.2.5}$$

Stochastic systems for which (5.2.5) holds are called potential systems. The remaining analysis in this subsection will be restricted to such systems. In order for the vector field V to be irrotational independent of the value of ϵ, we assume in addition that

$$\phi = \phi_0 + \epsilon \phi_1. \tag{5.2.6}$$

From (5.2.4),(5.2.5) and (5.2.6) it follows for g that

$$g(x) = \exp[-\phi_0(x)/\epsilon + \phi_1(x)], \tag{5.2.7a}$$

with

$$\begin{aligned} \phi_0(x) &= -\int_{x_0}^{x} \sum_{i=1}^{n} \sum_{j=1}^{n} 2a_{ij}^{-1} b_j \, dx_i, \\ \phi_1(x) &= -\int_{x_0}^{x} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{-1} \sum_{k=1}^{n} \frac{\partial a_{jk}}{\partial x_k} \, dx_i. \end{aligned} \tag{5.2.7b}$$

The integrals in (5.2.7b) are functions of x that are independent of the path of integration. The integrals equal zero at the point x_0, which is chosen to coincide with the position of the deterministic attractor. Using the relationship (5.2.4) with the matrix a brought to the left side, we find that equation (2.6a) multiplied by g can be written as the Euler equation

$$-F_u + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_{u_{x_i}} = 0, \tag{5.2.8a}$$

with F equal to

$$F = \frac{\epsilon}{4} g \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} u_{x_i} u_{x_j}. \tag{5.2.8b}$$

In these expressions we suppressed the subscript s of u for the reason of clarity. Thus, the solution of the boundary value problem (2.6) corresponds to an extremal of the functional

$$J[u_s] = \int_D F \, dx, \tag{5.2.9}$$

taken over functions u_s satisfying the boundary condition (2.6*b*). Expression (5.2.3) for u_s is substituted into the integral in (5.2.9), which subsequently is evaluated for small ϵ by the method of Laplace. To be definite we assume that the drift vector and the diffusion matrix are such that the major contributions to this integral come from the boundary ∂D. From (5.1.8) we then find

$$c_0 = \frac{\int_{\partial D} f(y)\, b(y).\xi(y)\ \exp[-\phi_0(y)/\epsilon + \phi_1(y)]\ dS_y}{\int_{\partial D} b(y).\xi(y)\ \exp[-\phi_0(y)/\epsilon + \phi_1(y)]\ dS_y}. \tag{5.2.10}$$

Using the definition (2.4) of u_s we write

$$\lim_{\epsilon \to 0} \int_{\partial D} f(y) \left[p(x,y) - \frac{b(y).\xi(y)\ \exp[-\phi_0(y)/\epsilon + \phi_1(y)]}{\int_{\partial D} b(y).\xi(y)\ \exp[-\phi_0(y)/\epsilon + \phi_1(y)]\ dS_y} \right] dS_y = 0. \tag{5.2.11}$$

This result indicates that for small ϵ the exit density p is independent of the starting point x, given $|x - x'| \gg O(\epsilon)$, and that this density is sharply peaked near the boundary point(s) with minimal potential ϕ_0. In typical situations, there is a unique y^* such that

$$\phi_0(y) > \phi_0(y^*), \quad \text{for } y \neq y^*, \quad y, y^* \in \partial D, \tag{5.2.12}$$

see Figure 2. Then (5.2.11) implies that in the limit $\epsilon \to 0$ the exit density becomes:

$$p(x,y) = \delta(y - y^*), \tag{5.2.13}$$

that is, exit occurs with probability one at y^*. For cases that the minimum of ϕ_0 on ∂D is attained on a set larger than one point, the reader is referred to the literature [41].

An asymptotic expression for the expected time of exit from a region, for systems of the potential type considered above, can be derived as in subsection 5.1. This is left as an exercise for the reader.

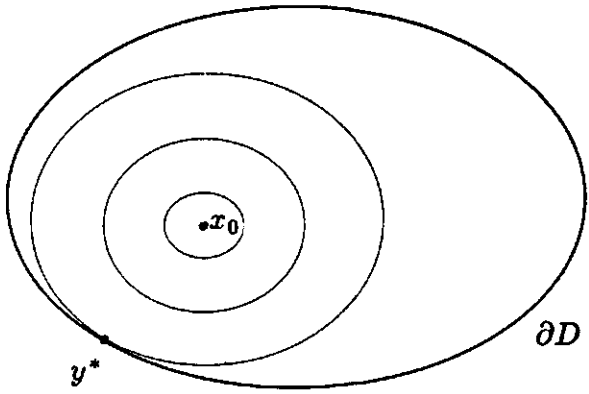

Figure 2. Contours on which ϕ_0 attains a constant value. This value is higher for contours farther away from x_0. The lowest value of ϕ_0 on ∂D is attained at y^*.

5.3 More general multidimensional systems

As we have seen in Section 5.2, the method to determine c_0 described in Section 5.1 for one-dimensional stochastic systems is applicable to multidimensional systems only if they are of a particular potential type. In the present section we give a brief outline of the approach to more general multidimensional systems, due to Matkowsky and Schuss [41].

We take over the discussion of Section 5.2 until the determination of the constant c_0. The idea is to determine this constant by the employment of the relationship (2.11) between the backward operator (2.7) and the adjoint forward operator (2.2). To this aim, we first construct a solution of the stationary forward equation (2.1). This is done by means of the WKB-method [37], which assumes that this solution is of the form

$$v(x) = w(x)\exp[-Q(x)/\epsilon], \tag{5.3.1a}$$

for small ϵ, where

$$Q(x_0) = 0, \quad w(x_0) = 1. \tag{5.3.1b}$$

The condition on w is a normalization. Substitution of (5.3.1a) into (2.1) with the left side set equal to zero yields to leading order in ϵ the eikonal equation

$$\sum_{i=1}^{n} b_i \frac{\partial Q}{\partial x_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{a_{ij}}{2} \frac{\partial Q}{\partial x_i}\frac{\partial Q}{\partial x_j} = 0, \tag{5.3.2}$$

and to the next order in ϵ the transport equation

$$\sum_{i=1}^{n}\left[\sum_{j=1}^{n} a_{ij}\frac{\partial Q}{\partial x_j}+b_i\right]\frac{\partial w}{\partial x_i}+\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\left(\frac{a_{ij}}{2}\frac{\partial^2 Q}{\partial x_i\partial x_j}+\frac{\partial a_{ij}}{\partial x_i}\frac{\partial Q}{\partial x_j}\right)+\frac{\partial b_i}{\partial x_i}\right]w=0. \tag{5.3.3}$$

The functions Q and w are solutions of equations (5.3.2) and (5.3.3). Then the relation (2.11) is evaluated with the function v defined by (5.3.1),(5.3.2),(5.3.3) and the expression (5.2.3) for u_s. In the limit $\epsilon \to 0$ we obtain

$$c_0=\frac{\int_{\partial D} f(y)\; b(y).\xi(y)\; w(y)\exp[-Q(y)/\epsilon]\; dS_y}{\int_{\partial D} b(y).\xi(y)\; w(y)\exp[-Q(y)/\epsilon]\; dS_y}. \tag{5.3.4}$$

Following the argument of the previous subsection, we find that for small ϵ the exit density p is peaked near the boundary point(s) with the lowest value of Q. Thus, the role played by the potential ϕ_0 in Section 5.2 is taken over here by the function Q. The potential ϕ_0 was expressed explicitly in terms of the drift vector and the diffusion matrix by (5.2.7*b*). Except in some special cases, no such explicit expression exists for Q. In practice this function is obtained through numerical integration of the eikonal equation by the ray method [37]. Such an integration scheme may include the transport equation as well in order to determine w.

The method described in the present section is powerful in the sense that it can be applied to a large class of problems in arbitrary dimension. However, the asymptotics to the stationary forward equation (2.1) are not (yet) supported by a solid mathematical background. The asymptotic method described above is related to an asymptotic method used frequently in geometrical optics and diffraction theory. For the latter method a more or less extensive literature exists, see for example the publications of Keller and coworkers [7,30] and Ludwig [36], and the more recent work of Duistermaat [13], Maslov [39], Maslov and Fedoriuk [40], etc.. For the former method, i.e. the asymptotic method to the partial differential equations related to exit problems, the literature is limited, see for example Cohen and Lewis [8], Ludwig [37] and a more recent paper of Brannan [6].

For the expected exit time the following formula has been derived [41]:

$$T(x)\sim\frac{\sqrt{2\pi\epsilon}\;\exp[Q(y^*)/\epsilon]}{H_1^{\frac{1}{2}}(x_0)\left(b.\xi\; w\; H_2^{-\frac{1}{2}}\right)(y^*)}\left\{1-\exp[-\tilde{b}(x')|x-x'|/\epsilon]\right\}, \tag{5.3.5a}$$

in which:

$$H_1(x_0) = \det\left\{\frac{\partial^2 Q}{\partial x_i \partial x_j}(x_0)\right\}_{i,j=1,2,\ldots,n}, \qquad (5.3.5b)$$

$$H_2(y^*) = \det\left\{\frac{\partial^2 Q}{\partial y_i \partial y_j}(y^*)\right\}_{i,j=1,2,\ldots,n-1}, \qquad (5.3.5c)$$

where x_0 is the deterministic equilibrium point and y^* is the unique (by assumption, for other cases see the literature) point on ∂D with the lowest value of Q.

Now that we have obtained expressions for the expected exit time for a diffusion across the flow in Section 4 and for diffusions against the flow in Section 5, it is interesting to compare them in their dependence on the small parameter ϵ. For the former type of diffusion this dependence is algebraic, while for the latter it is exponential. Thus, the persistence of diffusions against the flow is much greater than of diffusions across the flow.

In the stochastic systems under consideration the deterministic flow was directed inward at the boundary of the domain. Other systems, in which the deterministic flow at the boundary coincides with the boundary, have been analysed in [42] and [43]. In the first paper there are no critical points of the deterministic system located on this boundary, whereas in the second paper there are.

Above we studied exit problems using formal asymptotic methods. The same subject has been studied by Ventcel and Freidlin [17,58], Friedman [18] and others from a probabilistic point of view. Rigorous mathematical methods have been used by Day [11,12], Evans and Ishii [15], Kamin [27,28] and others.

The stochastic systems that we considered had a continuous domain. In chemistry, physics, biology and other areas one meets processes with a discontinuous domain, for example birth or birth-death processes. For these processes, asymptotic methods that resemble the method described in this subsection have been presented in [32,33,34,44].

6. Some applications

Exit models have many applications. We mention only a few of them. There are applications in population genetics, see for example Crow and Kimura [10], who describe the change in gene frequency of biological populations by means of a stochastic diffusion model. Exit from a domain here corresponds

to the fixation of a gene. See also Maruyama [38] and Gillespie [20]. Another application in biology is the description of the dynamics of stochastic populations. In such applications, exit corresponds to extinction of a species. Examples can be found in Goel and Richter-Dyn [21], Ludwig [37], May [45], Nisbet and Gurney [49], Roozen [51,52], Roughgarden [54]. Other applications are in mechanics and reliability theory. Many mechanical systems near equilibrium behave essentially like the diffusion across the flow model or the diffusion against the flow model that we studied in this paper. The stochastic domain can be chosen as the domain in which the system is known to function properly. Exit may correspond to break down of the system. The expected exit time is a measure for the reliability of the system. See for example Grasman [22], Katz and Schuss [29], Roozen [53]. For an application of an exit model to the dynamics of the atmospheric circulation, see de Swart and Grasman [57]. The expected exit times predict lifetimes of alternative circulation types. Other applications of exit models can be found in the literature.

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STOCHASTIC STABILITY OF THE LOADED STIFF ROD

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A stiff rod, held in a vertical position by an elastic hinge, is subjected to a load consisting of a deterministic and a small stochastic component, both acting in fixed directions. The rod is slightly damped and carries out small oscillations around an equilibrium position. Above a critical energy level, the mechanical system of which the rod forms a part, may get damaged. At some time, an accumulation of stochastic effects can lead to an excess of this critical energy level. In this paper we derive various statistical expressions related to the time needed to reach the critical energy level. These expressions can be adopted as a measure of the reliability of the mechanical system.

1. Introduction

In their paper [7], Katz and Schuss considered the reliability of elastic structures with random loads. The reliability of such structures was treated as an exit problem in the theory of stochastic dynamical systems. Starting with the simple pendulum, a sequence of models has been considered with increasing complexity, viz. the double pendulum, the n-fold pendulum and the elastic continuous column. It has been shown that the exit behaviour of these more complex pendula is essentially the same as that of the simple pendulum. In this paper we will study the simple pendulum (or stiff rod) into more detail than was done by Katz and Schuss. In contrast to their approach, in which both

the deterministic and stochastic loads to the simple pendulum were applied vertically, we will allow these loads to act independently from each other, in arbitrary fixed directions.

In Section 2 we will give a description of the stiff rod model and derive the stochastic equation in dimensionless form. The deterministic load is of order $O(\epsilon^0)$, the stochastic load has intensity of order $O(\epsilon)$, and the damping is of order $O(\epsilon)$, where $0 < \epsilon \ll 1$ is a small parameter. In Section 3 we will derive the backward equation, valid on the time scale of order $O(\epsilon^{-1})$, which is needed in the study of the problem of exit from an energy interval. This interval, bounded below by zero and above by a critical energy, is scaled to the unit interval. In the Sections 4, 5 and 6 we will distinguish three cases, according to the magnitude of the angles in which the deterministic and the stochastic loads act. The regular case is treated in Section 4, while Sections 5 and 6 treat two special cases. For each of these cases, we will derive expressions for the expected exit time from the unit interval. For some special cases the exit time density, the corresponding cumulants and some other related expressions will be derived as well. In Section 7 we give some of the results in dimensional form. The discussion in Section 8 is directed towards some practical aspects of the results obtained.

2. The model

An unloaded stiff rod of length l, with mass m at a distance l' from the hinge O, and spring constant μ at O, carries out small oscillations around the equilibrium position $\varphi = 0$. Next a deterministic load P_d is applied to the rod, acting under the fixed angle φ_d, as indicated in Figure 1. The potential energy due to P_d is given by

$$-P_d l[1 - \cos(\varphi - \varphi_d)] \approx -P_d l(1 - \cos\varphi_d - \varphi \sin\varphi_d + \frac{\varphi^2}{2}\cos\varphi_d), \qquad (2.1)$$

where we assumed that $|\varphi|$ is small so that

$$\begin{aligned} &\sin\varphi \approx \varphi, \\ &\cos\varphi \approx 1 - \varphi^2/2. \end{aligned} \qquad (2.2)$$

The potential energy due to the spring property of the hinge equals

$$\frac{1}{2}\mu\varphi^2 l^2. \qquad (2.3)$$

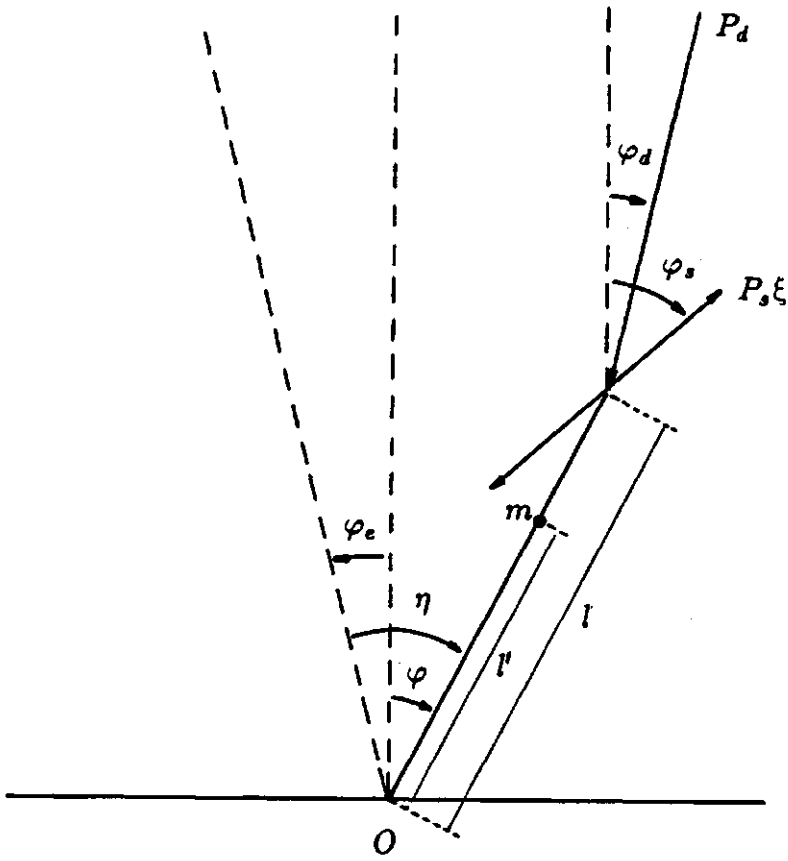

Figure 1. The loaded stiff rod.

Differentiating the total potential energy, which is obtained by adding (2.1) and (2.3), with respect to φ and equating this to zero we obtain the new equilibrium position φ_e

$$\varphi_e = \frac{-P_d \sin \varphi_d}{\mu l - P_d \cos \varphi_d}. \tag{2.4}$$

It is assumed that $\mu l \gg |P_d|$, so that $|\varphi_e|$ is small, in agreement with (2.2). The arbitrary constant that may be added to the potential energy is chosen such that the total potential energy equals zero at φ_e. The kinetic energy of the rod is given by

$$T = \frac{1}{2} m l'^2 \left(\frac{d\varphi}{dt}\right)^2. \tag{2.5}$$

The Lagrange equation of motion reads

$$m \frac{l'^2}{l^2} \frac{d^2 z}{dt^2} + \mu z - \left(\frac{z}{l} \cos \varphi_d - \sin \varphi_d\right) P_d = 0, \quad z = l\varphi. \tag{2.6}$$

To describe a more realistic system, we extend the model with a damping term and a stochastic load component:

$$m\frac{l'^2}{l^2}\frac{d^2z}{dt^2}+\alpha\frac{dz}{dt}+\mu z-\left(\frac{z}{l}\cos\varphi_d-\sin\varphi_d\right)P_d=\left(\frac{z}{l}\cos\varphi_s-\sin\varphi_s\right)\ \xi(\gamma t)P_s. \tag{2.7}$$

Here $\xi(.)$ is a Gaussian white noise process and γ denotes a constant frequency (an appropriate choice is the unit frequency), so that γt is a dimensionless quantity. P_s has the dimension of a load. Note that the stochastic load component is zero for $\varphi=\varphi_s$ (which is possible only if φ_s is small). Here φ_s is the fixed angle under which the stochastic load acts. The damping constant is denoted by α. With the abbreviations

$$\tilde{m}=m\frac{l'^2}{l^2},\quad \tilde{\mu}=\mu-\frac{P_d}{l}\cos\varphi_d, \tag{2.8}$$

and the change of variable from z to η given by

$$z=\eta-\frac{P_d\sin\varphi_d}{\tilde{\mu}}, \tag{2.9}$$

where η is the deviation from the equilibrium position (2.4), we have

$$\tilde{m}\frac{d^2\eta}{dt^2}+\alpha\frac{d\eta}{dt}+\tilde{\mu}\eta=\left(\frac{\cos\varphi_s}{l}\eta-\frac{P_d\sin\varphi_d\cos\varphi_s}{\tilde{\mu}l}-\sin\varphi_s\right)\ \xi(\gamma t)P_s. \tag{2.10}$$

Suppose the rod is part of a mechanical structure, that functions well as long as the energy of the rod is below a critical value R^2 (recall that the energy of the rod takes its minimal value zero, when it is at rest at the equilibrium position φ_e determined by the deterministic load). With the transformations

$$\begin{aligned}\eta^*&=\eta\sqrt{\tilde{\mu}}/R, & t^*&=t\sqrt{\tilde{\mu}/\tilde{m}}, & P_d^*&=P_d/R\sqrt{\tilde{\mu}},\\ P_s^*&=P_s/R\sqrt{\tilde{\mu}}, & \gamma^*&=\gamma\sqrt{\tilde{m}/\tilde{\mu}},\end{aligned} \tag{2.11a}$$

and the white noise scaling property

$$\xi(\gamma t)=\xi(\gamma^*t^*)=\frac{1}{\sqrt{\gamma^*}}\,\xi(t^*), \tag{2.11b}$$

we obtain the equation of motion in dimensionless variables, denoted by *

$$\frac{d^2\eta^*}{dt^{*2}}+\frac{\alpha}{\sqrt{\tilde{m}\tilde{\mu}}}\frac{d\eta^*}{dt^*}+\eta^*=\frac{P_s^*}{\sqrt{\gamma^*}}\left(\frac{R\cos\varphi_s}{l\sqrt{\tilde{\mu}}}(\eta^*-P_d^*\sin\varphi_d)-\sin\varphi_s\right)\ \xi(t^*). \tag{2.12}$$

We make the following assumptions on the magnitude of the various terms

$$\frac{P_s^*}{\sqrt{\gamma^*}} = \sqrt{\epsilon}, \quad \frac{R\cos\varphi_s}{l\sqrt{\mu}} = k_1, \quad k_1 P_d^* \sin\varphi_d + \sin\varphi_s = k_0, \quad \frac{\alpha}{\sqrt{m\mu}} = \epsilon k, \tag{2.13}$$

in which $0 < \epsilon \ll 1$ and k, k_0, k_1 are $O(\epsilon^0)$ constants. Equation (2.12) becomes:

$$\ddot{\eta} + \epsilon k\dot{\eta} + \eta = \sqrt{\epsilon}\,(k_1\eta - k_0)\,\xi, \tag{2.14}$$

where we suppressed the * of η^* and the dot denotes differentiation with respect to t^*. Equation (2.14) describes an $O(\epsilon)$ damped oscillation of the stiff rod from a load consisting of a deterministic part of $O(\epsilon^0)$ and a stochastic part with intensity of $O(\epsilon)$. We note that φ_d and φ_s are arbitrary fixed angles. Equation (2.14) can be written as the system

$$\begin{aligned} \frac{d\eta}{dt^*} &= \dot{\eta}, \\ \frac{d\dot{\eta}}{dt^*} &= -(\epsilon k\dot{\eta} + \eta) + \sqrt{\epsilon}\,(k_1\eta - k_0)\,\xi. \end{aligned} \tag{2.15}$$

3. The backward equation

The undisturbed ($\epsilon = 0$) system (2.15) is an undamped oscillator, whose dynamics are described by closed trajectories around the origin in the $(\eta, \dot{\eta})$-phase plane. Each trajectory corresponds to an energy level. The energy is larger for orbits farther away from the origin. The effect of a nonzero ϵ is that the trajectories tend to spiral inwards to approach the origin (as a consequece of damping) and exhibit stochastic fluctuations in the $\dot{\eta}$-direction. The backward Kolmogorov equation corresponding to (2.15) reads [10, Ch.5] or [4, Ch.4]:

$$\frac{\partial u}{\partial t^*} = \dot{\eta}\frac{\partial u}{\partial \eta} - (\epsilon k\dot{\eta} + \eta)\frac{\partial u}{\partial \dot{\eta}} + \frac{\epsilon}{2}\,(k_1\eta - k_0)^2\frac{\partial^2 u}{\partial \dot{\eta}^2}. \tag{3.1}$$

We will use this equation with the function $u(\eta, t^*)$ and with various other functions in the place of u, which will be defined later on. The remainder of this section applies to all these functions. Equation (3.1) is studied asymptotically for small ϵ on a time scale of order ϵ^{-1}. With

$$t^* = \tilde{t}/\epsilon, \tag{3.2a}$$

$$u = u^0 + \epsilon u^1 + \cdots, \tag{3.2b}$$

and the transformation $(\eta, \dot{\eta}) \rightarrow (r, \theta)$ defined by

$$\eta = \sqrt{2} r \cos\theta, \qquad \dot{\eta} = \sqrt{2} r \sin\theta, \tag{3.3}$$

we obtain to leading order in ϵ

$$\frac{\partial u^0}{\partial \theta} = 0, \tag{3.4}$$

implying $u^0 = u^0(r, \tilde{t})$. Note that r^2 is the dimensionless energy of the undisturbed ($\epsilon = 0$) system, $r \in [0, 1]$ and $r^2 = 1$ corresponds to the critical energy.

To the next order in ϵ an equation is obtained in terms of u^0 and u^1. Terms with u^1 vanish by integration of this equation with respect to θ from 0 to 2π and the additional assumption that u^1 is periodic in θ with period 2π. The resulting equation for u^0 reads

$$\frac{\partial u^0}{\partial \tilde{t}} = \left(\frac{a_0}{r} + a_2 r\right) \frac{\partial u^0}{\partial r} + \left(a_0 + a_1 r^2\right) \frac{\partial^2 u^0}{\partial r^2}, \tag{3.5a}$$

with

$$a_0 = \frac{1}{8} k_0^2, \qquad a_1 = \frac{1}{16} k_1^2, \qquad a_2 = \frac{3}{16} k_1^2 - \frac{1}{2} k. \tag{3.5b}$$

Thus, the description to this order in ϵ includes damping and stochastic effects. If, as a consequence of the latter effect, the critical energy $r^2 = 1$ is reached in finite time with probability one, starting from $r \in [0, 1]$, the rod is *stochastically unstable* [10]. In that case, the mean exit time from the unit interval at 1 is a measure for the stability (reliability) of the rod (and thus an index of reliability). Below we will discuss this problem of exit from the unit interval. Several cases are distinguished, according to whether k_0, k_1 are equal or unequal to zero.

4. The regular case

Let the regular case be defined by $k_0 \neq 0, k_1 \neq 0$. This case occurs, in general, when k_0 and k_1 are chosen arbitrarily. In this case $a_0 > 0, a_1 > 0$. The boundary $r = 0$ of the unit interval is classified as an *entrance* boundary, see Feller [3], meaning that $r = 0$ cannot be reached in finite time from the interior of the interval, and the interior can be reached in finite time from $r = 0$. At $r = 1$ an *absorbing* boundary [4] is adopted. It can be reached in finite time from the interior of the interval. On reaching this boundary, absorption occurs,

so that the interval cannot be entered again. Thus, exit from the unit interval can take place only at $r = 1$. Let $u_s(r)$ be the probability of exit at $r = 1$, starting from a point $r \in [0, 1]$. Its leading order part $u_s^0(r)$ in an expansion in powers of ϵ satisfies the stationary backward equation (3.5a) with the boundary condition $u_s^0(1) = 1$. We find $u_s(r) \sim u_s^0(r) \equiv 1$ as the only relevant (i.e. finite) solution. There is no freedom to specify an arbitrary boundary condition at $r = 0$. We conclude that if we start somewhere on the interval $[0, 1]$, exit at $r = 1$ will occur with probability one, so that the oscillator is stochastically unstable. Next we consider the expected exit time $T(r)$, starting from a point r. Similar to the time scaling (3.2a) we put $T = \tilde{T}/\epsilon$ and similar to the expansion (3.2b) of u we put $\tilde{T} = \tilde{T}^0 + \epsilon\tilde{T}^1 + \cdots$, so that $T = \tilde{T}^0/\epsilon + \tilde{T}^1 + \cdots$. An approximation for T is found by solving the Dynkin equation [10, p.118]

$$-1 = \Big(\frac{a_0}{r} + a_2 r\Big)\frac{\partial \tilde{T}^0}{\partial r} + \Big(a_0 + a_1 r^2\Big)\frac{\partial^2 \tilde{T}^0}{\partial r^2}, \tag{4.1a}$$

with the conditions

$$\tilde{T}^0(0) \text{ is finite}, \tag{4.1b}$$

$$\tilde{T}^0(1) = 0. \tag{4.1c}$$

We find

$$T(r) \sim \frac{1}{\epsilon(a_1 - a_2)} \int_r^1 \Big[\Big(\frac{a_1}{a_0}s^2 + 1\Big)^{-\frac{a_2}{2a_1} + \frac{1}{2}} - 1\Big]\frac{1}{s}\, ds, \quad (a_2 \neq a_1). \tag{4.2}$$

Only if the exponent in the integrand is a simple rational number, the integral reduces to a simple expression. If the exponent is equal to a positive integer, the binomial expansion can be used to evaluate the integral. If $a_2 = a_1$, this is substituted into equation (4.1a). Solving the corresponding boundary value problem we arrive at:

$$T(r) \sim \frac{1}{2\epsilon a_1} \int_r^1 \frac{1}{s} \log\Big(\frac{a_1}{a_0}s^2 + 1\Big)\, ds. \tag{4.3}$$

5. The first special case: angles of deterministic and stochastic load are related in a particular way.

We assume $k_0 = 0$ while $k_1 \neq 0$. In this case $a_0 = 0, a_1 > 0$. The angles φ_d, φ_s are related by

$$0 = k_1 P_d^* \sin\varphi_d + \sin\varphi_s. \tag{5.1}$$

An important case is the vertically loaded pendulum ($\varphi_d = \varphi_s = 0$). Since $a_0 = 0$, the behaviour is qualitatively different from that in Section 4. At $r = 0$ we now have a *natural* boundary, meaning that $r = 0$ cannot be reached from the interior of the unit interval, and the interior of the unit interval cannot be reached from $r = 0$, in finite time. Let $u_s(r)$ now be defined as the probability of exit at $r = 1$, starting from a point r of the half open interval $(0, 1]$. This probability is obtained by solving the stationary backward equation (3.5a) on the interval $\delta \leq r \leq 1$ with $0 < \delta \ll 1$, where the boundary conditions are $u_s^0(\delta) = 0,\ u_s^0(1) = 1$. In the limit $\delta \to 0$ we obtain

$$u_s(r) \sim \begin{cases} 1, & a_2 > a_1, \quad \text{(damping below level } a_2 = a_1\text{)}, \\ r^{1-a_2/a_1}, & a_2 < a_1, \quad \text{(damping beyond level } a_2 = a_1\text{)}. \end{cases} \tag{5.2}$$

In the first case of (5.2), exit at $r = 1$ occurs with probability one, as in Section 4 the rod is stochastically unstable. In the second case of (5.2), the probability of exit at $r = 1$ can be made arbitrarily small by starting close enough to $r = 0$, the rod is *stochastically stable*. We will continue with the stochastically unstable case here; the treatment of the stochastically stable case is postponed until subsection 5.4. The expected exit time is obtained by solving equation (4.1a) with $a_0 = 0$ under the conditions $\tilde{T}^0(1) = 0,\ \tilde{T}^0(\delta) = 0,\ 0 < \delta \ll 1$. In the limit $\delta \to 0$

$$T(r) \sim \frac{1}{\epsilon(a_2 - a_1)} \log\frac{1}{r}, \qquad (a_2 > a_1). \tag{5.3}$$

Because $a_0 = 0$, the differential equations describing the exit problem have a simple form, so that a variety of interesting expressions can be derived.

5.1 The probability of exit before a certain time ($a_2 > a_1$)

Henceforth, the dimensionless time t^* will be denoted by t, for convenience. Let $u(r, t)$ be the probability that exit through $r = 1$ takes place on the time

interval $[0,t]$, starting in $r \in (0,1]$ on $t = 0$. With $a_0 = 0$ the backward equation (3.5a) reads

$$\frac{\partial u^0}{\partial \tilde{t}} = a_2 r \frac{\partial u^0}{\partial r} + a_1 r^2 \frac{\partial^2 u^0}{\partial r^2}. \tag{5.4a}$$

This equation has to be solved with the conditions

$$u^0(r,0) = 0, \tag{5.4b}$$

$$u^0(\delta,\tilde{t}) = 0, \qquad \delta \to 0, \tag{5.4c}$$

$$u^0(1,\tilde{t}) = 1. \tag{5.4d}$$

Taking the Laplace transform with respect to $\tilde{t}$

$$\bar{u}^0(r,s) = \int_0^\infty e^{-s\tilde{t}}\, u^0(r,\tilde{t})\, d\tilde{t}, \tag{5.5}$$

the following boundary value problem is obtained

$$a_1 r^2 \frac{\partial^2 \bar{u}^0}{\partial r^2} + a_2 r \frac{\partial \bar{u}^0}{\partial r} = s\bar{u}^0 - u^0(r,0) = s\bar{u}^0, \tag{5.6a}$$

$$\bar{u}^0(\delta,s) = 0, \qquad \delta \to 0, \tag{5.6b}$$

$$\bar{u}^0(1,s) = \frac{1}{s}. \tag{5.6c}$$

The equation (5.6a) is an equidimensional or Euler equation [1], that by the change of variable

$$r = e^\rho, \tag{5.7}$$

transforms into

$$a_1 \frac{\partial^2 \bar{u}^0}{\partial \rho^2} + (a_2 - a_1)\frac{\partial \bar{u}^0}{\partial \rho} - s\bar{u}^0 = 0, \tag{5.8}$$

where the coefficients of the derivatives have become constants. The characteristic equation

$$a_1 k^2 + (a_2 - a_1)k - s = 0 \tag{5.9}$$

corresponding to this equation is solved by

$$k_1(s) = \frac{-(a_2 - a_1) + \sqrt{(a_2 - a_1)^2 + 4a_1 s}}{2a_1}, \tag{5.10a}$$

$$k_2(s) = \frac{-(a_2 - a_1) - \sqrt{(a_2 - a_1)^2 + 4a_1 s}}{2a_1}. \tag{5.10b}$$

The solution of (5.8) is

$$\bar{u}^0 = C_1(s)\ e^{k_1(s)\rho} + C_2(s)\ e^{k_2(s)\rho}, \tag{5.11}$$

and that of (5.6*a*)

$$\bar{u}^0 = C_1(s)\ r^{k_1(s)} + C_2(s)\ r^{k_2(s)}. \tag{5.12}$$

The unknown functions C_1 and C_2 follow from the boundary conditions (5.6*bc*)

$$\lim_{\delta\to 0} C_2(s) = \lim_{\delta\to 0} \frac{-1}{s(\delta^{k_2-k_1}-1)} = 0, \tag{5.13a}$$

in which we used the positivity of a_1, and, in the same limit

$$C_1(s) = 1/s. \tag{5.13b}$$

The solution of (5.6) is

$$\bar{u}^0(r,s) = r^{k_1(s)}/s. \tag{5.14}$$

Using some elementary properties of Laplace transforms and the inverse transform formula 5.129 [8, p.264], we obtain as solution of problem (5.4)

$$u^0(r,\frac{\tilde{t}}{a_1}) = \frac{1}{2}\mathrm{erfc}\left[\frac{\log\frac{1}{r}}{2\sqrt{\tilde{t}}} - c\sqrt{\tilde{t}}\right] + \frac{1}{2}r^{-2c}\mathrm{erfc}\left[\frac{\log\frac{1}{r}}{2\sqrt{\tilde{t}}} + c\sqrt{\tilde{t}}\right], \tag{5.15a}$$

where

$$c = \frac{a_2 - a_1}{2a_1}, \qquad (0 < c \le 1, \quad c = 1\text{: no damping}). \tag{5.15b}$$

With $u(r,t) \sim u^0(r,\tilde{t})$ we obtain an approximation of $u(r,t)$. It is easily verified that this result is in agreement with

$$\lim_{t\to\infty} u(r,t) = u_s(r), \tag{5.16}$$

with $u_s(r) = 1$, as in the first case of (5.2), where $u_s(r)$ was the probability that exit at $r = 1$ will occur, starting at $r \in (0,1]$. The function u has been depicted in Figure 2*a*. The result derived in this subsection differs from formula (3.28) in Katz and Schuss [7], since we took into account the right hand boundary condition.

As an application, we consider the following problem. What is the probability of failure of the mechanical system during an operating time $\hat{t}$? Let $v_0(r)$ be the probability density function of the initial r (recall that r is the square

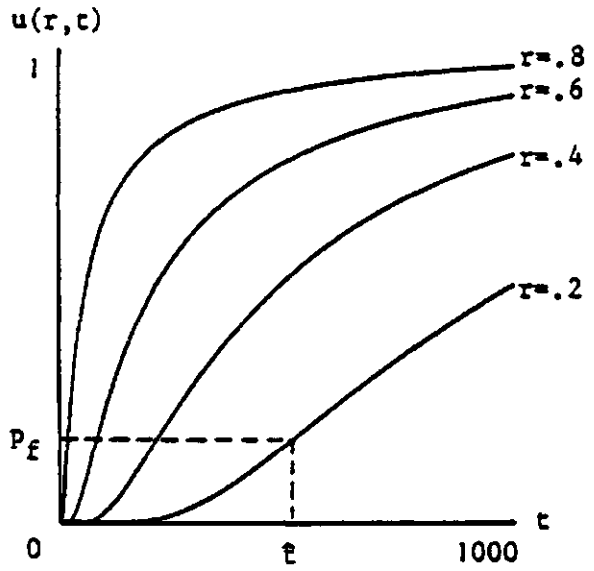

Figure 2a. The probability $u(r,t)$ for the undamped rod with $\epsilon = 0.01$ and $a_1 = 1/16$. For $r = 0.2$ the probability of failure exit p_f corresponding to an operating time $\hat{t}$ has been indicated.

root of the dimensionless energy). The probability of failure p_f is determined by

$$p_f = \int_0^1 v_0(r)u(r,\hat{t})\ dr. \tag{5.17}$$

In Figure 2a, the probability of failure for $v_0(r) = \delta(r-0.2)$ has been indicated.

5.2 The exit time density, its moments and cumulants ($a_2 > a_1$)

We define $\tau(r,t)dt$ as the probability that the time of exit through $r = 1$ is in the time interval $(t, t+dt)$, given that we started at $r \in (0,1]$ on $t = 0$. The function $\tau(r,t)$ is a probability density with respect to t. In this subsection, we will derive expressions for $\tau(r,t)$ and its n-th moment $\mu_n(r)$ and cumulant $\kappa_n(r)$. The function

$$1 - \int_0^t \tau(r,t')\ dt', \tag{5.18}$$

is known to satisfy the backward equation [4]. By differentiation with respect to t, it follows that τ satisfies the backward equation as well, so that for its

leading order term τ^0 we have

$$\frac{\partial \tau^0}{\partial \tilde{t}} = a_1 r^2 \frac{\partial^2 \tau^0}{\partial r^2} + a_2 r \frac{\partial \tau^0}{\partial r}. \tag{5.19}$$

Taking the Laplace transform of τ^0 with respect to $\tilde{t}$,

$$\bar{\tau}^0(r,s) = \int_0^\infty e^{-s\tilde{t}}\ \tau^0(r,\tilde{t})\ d\tilde{t}, \tag{5.20}$$

equation (5.19) becomes

$$a_1 r^2 \frac{\partial^2 \bar{\tau}^0}{\partial r^2} + a_2 r \frac{\partial \bar{\tau}^0}{\partial r} = s\bar{\tau}^0 - \tau^0(r,0) = s\bar{\tau}^0, \tag{5.21}$$

in which the initial condition

$$\tau^0(r,0) = 0 \tag{5.22}$$

has been used, valid for $\delta < r < 1$, where again $0 < \delta \ll 1$. For $r = \delta$ and $r = 1$ the probability density function $\tau(r,t)$ equals the delta function $\delta(t-0)$, or

$$\mu_0^0(r) = 1, \tag{5.23a}$$

$$\mu_m^0(r) = 0, \quad m > 0, \tag{5.23b}$$

where $\mu_m^0(r)$ is the m-th moment about the origin $\tilde{t} = 0$ of $\epsilon^{-1}\tau^0(r,\tilde{t})$. Expression (5.23$a$) results from the normalisation of the density $\tau(r,t)$ and the expansion

$$\tau(r,t) \sim \tau^0(r,\tilde{t}). \tag{5.24}$$

By the relation [6, p.6]

$$\epsilon^{-1}\bar{\tau}^0(r,s) = G^0(r,is) = \sum_{m=0}^{\infty} \frac{(-s)^m}{m!}\ \mu_m^0(r), \tag{5.25}$$

in which G^0 is the characteristic or moment generating function of $\epsilon^{-1}\tau^0$, the conditions (5.23) translate into boundary conditions for (5.21)

$$\bar{\tau}^0(1,s) = \bar{\tau}^0(\delta,s) = \epsilon, \quad (\delta \to 0). \tag{5.26}$$

The boundary value problem (5.21, 5.26) is solved in a way similar to that in the previous section. We find

$$\lim_{\delta\to 0} C_2(s) = \lim_{\delta\to 0} \epsilon\ \frac{\delta^{-k_1} - 1}{\delta^{k_2-k_1} - 1} = 0, \tag{5.27a}$$

because $a_1 > 0$, and, in this limit,

$$C_1(s) = \epsilon, \tag{5.27b}$$

so that

$$\bar{\tau}^0(r,s) = \epsilon\, r^{k_1(s)}. \tag{5.28}$$

Using the inverse Laplace transform formula 5.85 [8, p.258], the following result is obtained

$$\frac{1}{\epsilon a_1}\tau^0(r,\frac{\tilde{t}}{a_1}) = \frac{r^{-c}}{2\sqrt{\pi}}\Big(\log\frac{1}{r}\Big)\tilde{t}^{-\frac{3}{2}}\, e^{-c^2\tilde{t} - \frac{(\log r)^2}{4\tilde{t}}}. \tag{5.29}$$

With (5.24), an approximation to $\tau(r,t)$ follows, see Figure 2*b*.

Next we will derive expressions for the moments and cumulants of the density $\tau(r,t)$. The moments $\mu_n(r)$ about $t = 0$ of $\tau(r,t)$ are found starting from the definition

$$\mu_n(r) = \int_0^\infty t^n \tau(r,t)\, dt. \tag{5.30}$$

Using the obtained approximation of $\tau(r,t)$, and formulas 3.471:9 [5, p.340] and 8.468 [5, p.967], it is found after some calculation that

$$\mu_n(r) \sim (4\epsilon a_1 c^2)^{-n} \sum_{j=1}^{n} \frac{(2n-j-1)!}{(j-1)!(n-j)!}\,(2c\log r^{-1})^j, \quad n \geq 1. \tag{5.31}$$

With (5.28) the moment generating function G of $\tau(r,t)$ yields

$$G(r,is) = \bar{\tau}(r,s) \sim \frac{1}{\epsilon}\,\bar{\tau}^0(r,s/\epsilon) = r^{k_1(s/\epsilon)}. \tag{5.32}$$

Taylor expanding the logarithm of $G(r,is)$ around $s = 0$ we find

$$\log G(r,is) \sim \sum_{n=1}^{\infty} \frac{(-s)^n}{n!}\,\kappa_n(r), \tag{5.33}$$

with the cumulants $\kappa_n(r)$ given by

$$\kappa_n(r) \sim \frac{(2n-3)!!}{(2a_1\epsilon)^n c^{2n-1}}\,\log\frac{1}{r}, \qquad n \geq 1, \tag{5.34}$$

the double exclamation mark being defined as in [5, p.xliii].

In this subsection we have obtained expressions for the lifetime density of a mechanical system, as well as its moments and cumulants, as functions

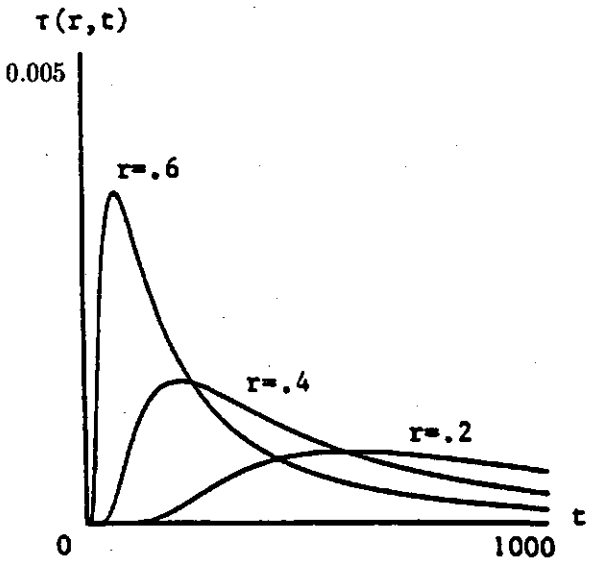

Figure 2b. The probability density function of the exit time, depending on the starting point r, for the undamped rod with $\epsilon = 0.01$ *and* $a_1 = 1/16$.

of the initial r. When the initial r is not known, but its density $v_0(r)$ is, these expressions should be integrated with respect to r from 0 to 1 with the weight function $v_0(r)$, as at the end of the previous subsection. This remark applies also to other results in this paper, in particular to the expressions of the expectation of the exit time.

5.3 Solution of the forward equation ($a_2 > a_1$)

Let us define $v(r,t)dr$ as the probability of being in the interval $(r, r+dr)$ at time t, given the probability $v_0(r')dr'$ of being in the interval $(r', r'+dr')$ at time $t = 0$, where $r, r' \in (0,1]$. In this subsection, an expression for $v(r,t)$ will be derived. The Fokker-Planck or forward Kolmogorov equation associated with (5.4a) reads

$$\frac{\partial v^0}{\partial \tilde{t}} = -a_2 \frac{\partial}{\partial r}(r v^0) + a_1 \frac{\partial^2}{\partial r^2}(r^2 v^0), \tag{5.35}$$

which has to be solved with the conditions

$$v^0(r,0) = v_0(r), \tag{5.36a}$$

$$v^0(\delta, \tilde{t}) = 0, \quad \delta \to 0, \tag{5.36b}$$

$$v^0(1, \tilde{t}) = 0. \tag{5.36c}$$

The last two conditions result from erecting absorbing boundaries at $r = \delta$ and $r = 1$, and v_0 in (5.36a) is the initial density. Equation (5.35) is rewritten as

$$\frac{\partial v^0}{\partial \tilde{t}} = a_1 r^2 \frac{\partial^2 v^0}{\partial r^2} + (4a_1 - a_2) r \frac{\partial v^0}{\partial r} + (2a_1 - a_2) v^0. \tag{5.37}$$

Taking the Laplace transform of v with respect to $\tilde{t}$

$$\bar{v}^0(r, s) = \int_0^\infty e^{-s\tilde{t}}\, v^0(r, \tilde{t})\, d\tilde{t}, \tag{5.38}$$

(5.36, 5.37) changes into the boundary value problem

$$a_1 r^2 \frac{\partial^2 \bar{v}^0}{\partial r^2} + (4a_1 - a_2) r \frac{\partial \bar{v}^0}{\partial r} + (2a_1 - a_2 - s)\bar{v}^0 = -v_0(r), \tag{5.39a}$$

$$\bar{v}^0(\delta, s) = 0, \quad \delta \to 0, \tag{5.39b}$$

$$\bar{v}^0(1, s) = 0. \tag{5.39c}$$

The change of variable (5.7) turns equation (5.39a) into the equation

$$a_1 \frac{\partial^2 \bar{v}^0}{\partial \rho^2} + (3a_1 - a_2) \frac{\partial \bar{v}^0}{\partial \rho} + (2a_1 - a_2 - s)\bar{v}^0 = -v_0(e^\rho). \tag{5.40}$$

By solving its characteristic equation, the homogeneous equation associated with (5.40) is found to have the independent solutions

$$e^{(-1-k_1(s))\rho}, \; e^{(-1-k_2(s))\rho}, \tag{5.41}$$

with k_1 and k_2 defined in (5.10), and Wronskian

$$W(\rho) = (k_1 - k_2)\, e^{(-k_1-k_2-2)\rho}. \tag{5.42}$$

The inhomogeneous equation (5.40) is solved by the method of variation of parameters [1]. In the original variable r its solution reads

$$\begin{aligned} \bar{v}^0(r, s) = {} & \frac{r^{-1-k_1}}{a_1(k_1 - k_2)} \left[C_1(s) + \int_\delta^r v_0(r')\, (r')^{k_1}\, dr' \right] \\ & + \frac{r^{-1-k_2}}{a_1(k_1 - k_2)} \left[C_2(s) + \int_r^1 v_0(r')\, (r')^{k_2}\, dr' \right], \end{aligned} \tag{5.43}$$

in which it is assumed that the initial density v_0 is sufficiently regular so that the integrands remain finite in the integration domain (note that $k_1 > 0,\ k_2 < 0$). The functions C_1 and C_2 are determined by the boundary conditions (5.39*bc*). For C_2 we find

$$C_2(s) = \frac{\delta^{-k_1} \int_\delta^1 v_0(r')\ (r')^{k_1}\ dr' - \delta^{-k_2} \int_\delta^1 v_0(r')\ (r')^{k_2}\ dr'}{\delta^{-k_2} - \delta^{-k_1}}. \tag{5.44}$$

The second term in the nominator is bounded by $v_0(\delta)$. This bound is finite if v_0 is bounded. Then, consequently, in the limit for $\delta \to 0$

$$C_2(s) = -\int_0^1 v_0(r')\ (r')^{k_1}\ dr', \tag{5.45a}$$

$$C_1(s) = 0. \tag{5.45b}$$

The solution of (5.39) can be written in the following form

$$\begin{aligned} \bar{v}^0(r,s) = & \frac{1}{a_1(k_1-k_2)} \int_0^1 \frac{v_0(r')}{r} \left(\frac{r'}{r}\right)^{\frac{k_1+k_2}{2}} \\ & \left[e^{-\left|\log \frac{r}{r'}\right| \frac{k_1-k_2}{2}} - e^{-\left(\log \frac{1}{r'r}\right) \frac{k_1-k_2}{2}} \right] dr'. \end{aligned} \tag{5.46}$$

The solution of problem (5.35, 5.36) is obtained using the inverse Laplace transform 5.87 [8, p.258]

$$v^0(r, \frac{\tilde{t}}{a_1}) = \int_0^1 v_0(r') \frac{1}{r} \left(\frac{r}{r'}\right)^c \frac{e^{-c^2\tilde{t}}}{2\sqrt{\pi\tilde{t}}} \left[e^{-\frac{(\log \frac{r}{r'})^2}{4\tilde{t}}} - e^{-\frac{(\log r'r)^2}{4\tilde{t}}} \right] dr', \tag{5.47}$$

and with $v(r,t) \sim v^0(r,\tilde{t})$ we have an approximation of $v(r,t)$, see Figure 2*c*. The results (5.15, 5.29, 5.47) are related by

$$\frac{\partial}{\partial \tilde{t}} u^0(r, \frac{\tilde{t}}{a_1}) = \frac{1}{\epsilon a_1} \tau^0(r, \frac{\tilde{t}}{a_1}), \tag{5.48a}$$

$$1 - \int_0^1 v^0(r, \frac{\tilde{t}}{a_1})\ dr = u^0(r^*, \frac{\tilde{t}}{a_1}), \quad \text{for } v_0(r) = \delta(r - r^*). \tag{5.48b}$$

The example in Figure 2*c* shows the decay of v with time due to exit at $r = 1$. For small r we see an increase of v with time because of systems that are driven to $r = 0$, but cannot exit there since $r = 0$ is a natural boundary.

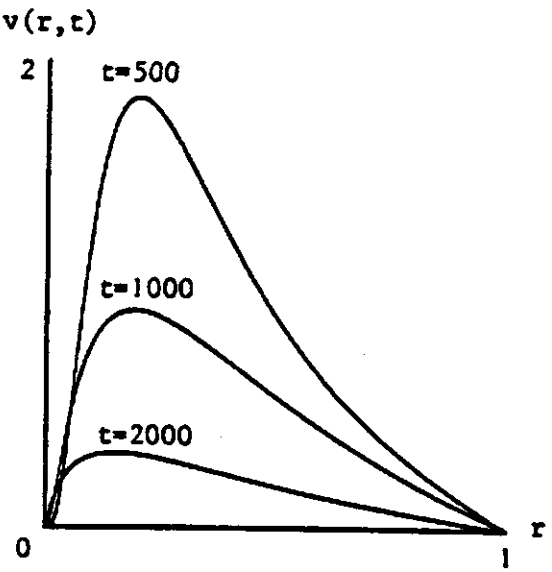

Figure 2c. The function $v(r,t)$ corresponding to the initial density $v_0(r) = \delta(r-0.2)$, for the undamped rod with $\epsilon = 0.01$ and $a_1 = 1/16$.

As an application of the result derived here, we discuss the construction of confidence intervals. Suppose we are interested in the confidence interval of probability p at time $\hat{t}$, for systems not yet exited (the probability that the system has not yet exited is given by the integral of $v(r,\hat{t})$ with respect to r from 0 to 1). Let r_1 and r_2 be the lower and upper boundary of the confidence interval, respectively, so that $0 < r_1 < r_2 < 1$. The values of r_1 and r_2 are determined by the conditions

$$v(r_1,\hat{t}) = v(r_2,\hat{t}), \tag{5.49a}$$

$$\int_{r_1}^{r_2} v(r,\hat{t})\, dr \Big/ \int_0^1 v(r,\hat{t})\, dr = p, \tag{5.49b}$$

from which they can be computed numerically.

5.4 The stochastically stable case ($a_2 < a_1$)

As we have seen at the beginning of Section 5, for stochastically stable systems the probability of exit at $r = 1$ is given by

$$u_s^0(r) = r^{1-a_2/a_1}. \tag{5.50}$$

Let $T(r)$ be defined here as the expected time of exit through $r = 1$, where it is given that exit through $r = 1$ will occur. An expression for T is found by solving the boundary value problem [4, p.142]

$$-u_s^0 = a_2 r \frac{\partial}{\partial r}(u_s^0 \tilde{T}^0) + a_1 r^2 \frac{\partial^2}{\partial r^2}(u_s^0 \tilde{T}^0), \tag{5.51a}$$

$$u_s^0(\delta)\tilde{T}^0(\delta) = 0, \quad (\delta \to 0), \tag{5.51b}$$

$$u_s^0(1)\tilde{T}^0(1) = 0. \tag{5.51c}$$

We find

$$T(r) \sim \frac{1}{\epsilon(a_1 - a_2)} \log \frac{1}{r}. \tag{5.52}$$

Starting at a given position r, with increasing damping (decreasing a_2) the probability of exit at $r = 1$ decreases according to (5.50), and the mean time of exit at $r = 1$, given that exit occurs, decreases as well, according to (5.52). Thus, in systems with damping far beyond the level $a_2 = a_1$ (i.e. $a_2 \ll a_1$), starting away from $r = 1$, exit is not likely to occur, and if it occurs, it probably is relatively shortly after starting. In such systems, failure mechanisms of a type, different from that discussed in this paper, are of importance. See the remark in Section 8.

6. The second special case: horizontal stochastic load.

We assume $k_1 = 0$. It follows that $a_1 = 0$. This case corresponds to a horizontal stochastic loading. Using (2.13) we find that $k_0^2 = 1$. The backward equation similar to (3.5) reads

$$\frac{\partial u^0}{\partial \tilde{t}} = \left(\frac{a_0}{r} + a_2 r\right) \frac{\partial u^0}{\partial r} + a_0 \frac{\partial^2 u^0}{\partial r^2}, \tag{6.1a}$$

with

$$a_0 = \frac{1}{8}, \qquad a_2 = -\frac{1}{2}k. \tag{6.1b}$$

This is the backward equation of a Rayleigh process [4,12]. As in Section 4, $r = 0$ is an entrance boundary, $r = 1$ is an absorbing boundary, and it can be verified that the probability of exit at $r = 1$, starting at any point of the closed unit interval, equals one. For the mean exit time we find

$$T(r) \sim \frac{1}{\epsilon a_2} \int_r^1 \left[1 - e^{-\frac{a_2}{2a_0}s^2}\right] \frac{1}{s}\, ds, \quad (a_2 \neq 0). \tag{6.2}$$

6.1 The subcase of no damping

In the special case that there is no damping, i.e. $k = 0$ and thus $a_2 = 0$, equation (6.1a) becomes

$$\frac{\partial u^0}{\partial \tilde{t}} = a_0\Big(\frac{1}{r}\frac{\partial u^0}{\partial r} + \frac{\partial^2 u^0}{\partial r^2}\Big). \tag{6.3a}$$

For the leading order term u^0 in the expansion of $u(r,t)$, defined as in Section 5, this equation is supplemented with the conditions

$$u^0(r,0) = 0, \tag{6.3b}$$

$$u^0(0,\tilde{t}) \text{ is finite}, \tag{6.3c}$$

$$u^0(1,\tilde{t}) = 1. \tag{6.3d}$$

The initial-boundary value problem (6.3) is the same as for axisymmetric heat conduction in a cylinder, which has the solution [11, p.175]

$$u^0(r,\tilde{t}) = 1 - 2\sum_i \frac{J_0(\xi_i r)\; e^{-a_0\xi_i^2\tilde{t}}}{\xi_i J_1(\xi_i)}, \tag{6.4}$$

where the summation extends over the positive roots of

$$J_0(\xi_i) = 0, \tag{6.5}$$

J_i being the Bessel function of the first kind of order i. The exit time density yields

$$\epsilon^{-1}\tau^0(r,\tilde{t}) = \frac{\partial}{\partial \tilde{t}}u^0(r,\tilde{t}) = 2a_0\sum_i \frac{\xi_i J_0(\xi_i r)\; e^{-a_0\xi_i^2\tilde{t}}}{J_1(\xi_i)}. \tag{6.6}$$

Next we compute the characteristic function and the first few cumulants corresponding to this density. The differential equation for τ^0 yields

$$\frac{\partial \tau^0}{\partial \tilde{t}} = a_0\Big(\frac{1}{r}\frac{\partial \tau^0}{\partial r} + \frac{\partial^2 \tau^0}{\partial r^2}\Big). \tag{6.7}$$

Taking the Laplace transform of τ^0 with respect to $\tilde{t}$ we obtain

$$\frac{\partial^2 \bar{\tau}^0}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\tau}^0}{\partial r} - \frac{s}{a_0}\bar{\tau}^0 = 0, \tag{6.8}$$

where we used the initial condition

$$\tau^0(r,0) = 0. \tag{6.9}$$

The general solution of (6.8) is

$$\bar{\tau}^0(r,s) = c_1 I_0(\sqrt{s/a_0}\, r) + c_2 K_0(\sqrt{s/a_0}\, r), \tag{6.10}$$

where I_0, K_0 are modified Bessel functions of the first and second kind, respectively. Using the boundary conditions

$$\bar{\tau}^0(1,s) = \epsilon, \tag{6.11a}$$

$$\bar{\tau}^0(0,s) \text{ is finite}, \tag{6.11b}$$

we find

$$\bar{\tau}^0(r,s) = \epsilon\ I_0(\sqrt{s/a_0}\, r)/I_0(\sqrt{s/a_0}). \tag{6.12}$$

The characteristic function of $\tau(r,t)$ is given by

$$G(r,is) \sim \epsilon^{-1}\bar{\tau}^0(r, s/\epsilon). \tag{6.13}$$

The cumulants $\kappa_n(r)$ of $\tau(r,t)$ are generated by

$$\log G(r,is) \sim \sum_{n=1}^{\infty} \frac{(-s)^n}{n!}\ \kappa_n(r). \tag{6.14}$$

Using the Taylor expansion [5, p.961]

$$I_0(s) = \sum_{k=0}^{\infty} \frac{\left(\frac{s}{2}\right)^{2k}}{(k!)^2}, \tag{6.15}$$

the cumulants are obtained as

$$\kappa_n(r) \sim \frac{\beta_n}{(\epsilon a_0)^n}\ (1 - r^{2n}), \tag{6.16}$$

where the β_n are found from the recurrent relations

$$\sum_{i=1}^{n} \beta_i\ (-4)^i\ (i!)\ i\ \binom{n}{i}^2 = -n, \quad (n = 1,2,3,\ldots). \tag{6.17}$$

The first few β's are given by

$$\beta_1 = \frac{1}{4}, \quad \beta_2 = \frac{1}{32}, \quad \beta_3 = \frac{1}{96}. \tag{6.18}$$

7. Examples in dimensional variables

Below we give some of the results in dimensional form. Let exclusively for the present section T denote the dimensional exit time and r, R the dimensional energy and dimensional critical energy, respectively. Using (2.11),(2.13) we find in the case of the undamped rod ($a_2 = 3a_1$), according to formulas (4.2),(5.3) and (6.16) respectively, that for the regular case

$$T(r) \sim \frac{8\gamma\tilde{m}\tilde{\mu}l^2}{P_s^2} \log \sqrt{\frac{\tilde{\mu}R^2\cos^2\varphi_s + 2(P_d \sin\varphi_d \cos\varphi_s + l\tilde{\mu}\sin\varphi_s)^2}{\tilde{\mu}r^2\cos^2\varphi_s + 2(P_d \sin\varphi_d \cos\varphi_s + l\tilde{\mu}\sin\varphi_s)^2}}, \quad (7.1)$$

for the vertically loaded rod

$$T(r) \sim \frac{8\gamma\tilde{m}\tilde{\mu}l^2}{P_s^2} \log \frac{R}{r}, \quad (7.2)$$

and for a horizontal stochastic loading

$$T(r) \sim \frac{(R^2 - r^2)2\gamma\tilde{m}}{P_s^2}, \quad (7.3)$$

with $\tilde{m}, \tilde{\mu}$ defined in (2.8). Recall that in order for these formulas to be applicable, the dimensionless parameter combinations in (2.13) must have the required order of magnitude. This is especially important for the last parameter group in (2.13). If the damping constant is large, so that this group is larger than indicated, for example of order 1, then the exit time is exponentially large, see [10, Ch.7.4] or [4, p.362], and the model presently described clearly doesn't apply.

8. Discussion

In this paper we have studied the exit problem on the time scale of order $O(\epsilon^{-1})$. We have distinguished three cases, depending on the values of k_0 and k_1. The case treated in Section 4 has been called regular, in the sense that an arbitrary choice of k_0 and k_1 (or φ_d and φ_s) will almost always lead to this case. In Sections 5 and 6 the special cases $k_0 = 0$ and $k_1 = 0$ have been treated, respectively. Confining ourselves to the first special case (a similar remark applies to the second special case) it can be remarked that in practice k_0 will

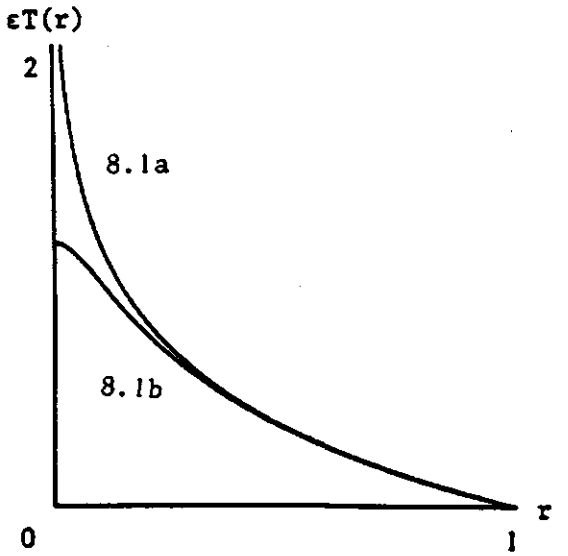

Figure 3. The expected exit time for the nearly vertically loaded undamped rod according to the formulas (8.1*a*) *and* (8.1*b*).

never be equal to zero exactly. For example, due to practical imperfections, an exactly vertically loaded rod does not exist. In view of the asymptotics that we have used, the left side of (5.1) may be replaced by $O(\epsilon^{\frac{1}{2}})$ in order for the first special case to remain applicable. As an example, consider a nearly vertically loaded rod with $a_2 > a_1$. If relation (5.1) is satisfied within $O(\epsilon^{\frac{1}{2}})$, the mean exit time is given by (5.3), otherwise it is given by (4.2) with small a_0 measuring the deviation from exactly vertical. For the undamped rod ($a_2 = 3a_1$) these exit times yield respectively

$$T(r) \sim \frac{1}{2\epsilon a_1} \log \frac{1}{r}, \tag{8.1a}$$

$$T(r) \sim \frac{1}{2\epsilon a_1} \log \sqrt{\frac{a_1 + a_0}{a_1 r^2 + a_0}}, \tag{8.1b}$$

which have been depicted in Figure 3 for $a_0 = 0.01$ and $a_1 = 1$, r being the square root of the dimensionless energy. Moreover, we see that (8.1*a*) is a good approximation to (8.1*b*) away from $r = 0$.

The expressions (4.2), (4.3), (5.3), (6.2), and (6.16) with $n = 1$, give the expected time $T(r)$ needed in the various cases considered to reach a critical energy level (which, in these cases, will happen with probability one). This

critical energy was chosen as the upper bound of the normal operating energy range of a system. Given a probability density function $v_0(r)$ of the initial r, we can adopt the constant

$$T_f = \int_0^1 v_0(r)T(r)\ dr \tag{8.2}$$

as the expected failure time for that system.

The formulas (5.34), (6.16) with $n = 2$ (higher cumulants are often of less practical interest) can be used to indicate the variance in failure time. It was not possible to obtain similar simple expressions for all cases considered. In such cases the variance is easily obtained numerically by solving a recurrent system of elliptic boundary value problems [4, p.138 and p.171].

Due to the simplicity of the equations in some special cases, we were able to find expressions for the probability density of the exit time, see (5.29) and (6.6). In practice the failure density is obtained by fitting experimental data with some statistical density (exponential, gamma, Weibull or lognormal), see for example [9]. In the present paper we showed how to derive such a density by employing the dynamics of the system.

A stock or investment policy can be based on the expression for u obtained in Section 5.1. Given an operating time $\hat{t}$ and a probability density of initial energies, the probability of failure is determined by (5.17). The expression for v in Section 5.3 can be used for the construction of confidence intervals.

The stochastic stability of oscillators with a different type of damping (as cubic damping) or noise (red, dichotomic, etc.), and with a forcing that can be described by a potential function, has been treated by Dygas, Matkowsky and Schuss [2].

Finally, it may be remarked that in practice failure mechanisms, different from the type considered here, may be present. One such a mechanism, to mention, is wear out, which becomes important, especially in systems with damping beyond the level $a_2 = a_1$. An approach to wear out could be as follows. Assume that wear out depends on the state $r \in [0, 1]$ and the time $t \in (0, \infty)$, expressed for some application by a penalty function $P(r, t)$. Let $S(r, t)dr$ be the expected time spent in $(r, r + dr)$ during the time interval $(0, t)$

$$S(r, t) = \int_0^t v(r, t')\ dt', \tag{8.3}$$

where $v(r,t)dr$ is the probability of being in $(r, r+dr)$ on time t (i.e. the solution of the forward equation). A wear out function can be defined by

$$W(t) = \int_0^1 P(r,t)S(r,t)\ dr. \tag{8.4}$$

The failure time t_f follows from

$$W(t_f) = W_c, \tag{8.5}$$

where W_c is a critical value, so that excess of this value leads to failure of the system. In general, the computations involved become very complicated, so that in practice this scheme has to be simplified somewhere.

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EQUILIBRIUM AND EXTINCTION IN STOCHASTIC POPULATION DYNAMICS

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Stochastic models of interacting biological populations, with birth and death rates depending on the population size are studied in the quasi-stationary state. Confidence regions in the state space are constructed by a new method for the numerical solution of the ray equations. The concept of extinction time, which is closely related to the concept of stability for stochastic systems, is discussed. Results of numerical calculations for two-dimensional stochastic population models are presented.

1. Introduction. This paper describes properties of stochastic population systems defined by birth and death rates depending on the population size. The state variables are the non-negative population numbers, so that the state space is formed by the positive orthant and its boundary. The deterministic dynamical system associated with the stochastic system is assumed to have a stable equilibrium point lying in the interior of the state space. For large population sizes the statistically quasi-stationary state of the stochastic system is described by a probability density function (p.d.f.) defined on the state space, with a maximum at the equilibrium point. Close to the equilibrium point this p.d.f. is approximately multivariate Gaussian. At a larger distance, deviations from the Gaussian shape become pronounced. Part of this paper deals with the construction of the contours of the associated confidence regions in state space. A method has been developed for the numerical solution of the ray equations, needed in the computation of the contours. The method does not have the disadvantages of the numerical methods that are normally used to solve the ray equations, such as the initial value approach or shooting methods.

An important concept in stochastic population dynamics is the concept of stability. Once a stochastic system of populations is caught within the attraction domain of a stable equilibrium point of the corresponding deterministic system, it will remain there for a long time, attracted by the equilibrium point. Stochastic fluctuations give rise to deviations from the equilibrium point. Large departures may occur, which can lead to escape from the domain of attraction of the equilibrium. With probability one this will happen within a finite time. On hitting the boundary of the state space by stochastic fluctuations, the domain of attraction is left: one population becomes extinct. The birth and death rates for that population are zero, the boundary is absorbing. The ecological stability of a system is expressed in terms

of the mean extinction time, for which expressions are given. A related problem that will be touched upon is which of the populations of a stochastic system is expected to become extinct first.

In the second part of this paper the foregoing theory is applied to a stochastic system consisting of two interacting populations. The associated deterministic system is a generalized Lotka–Volterra system with two components. A systematic treatment of this system is given. The stability condition for the interior equilibrium point is derived. Close to the equilibrium the quasi-stationary state of the stochastic system is approximately described by a two-dimensional Gaussian p.d.f. The parameters of this Gaussian p.d.f. are expressed in the parameters of the stochastic model. As a consequence, some conclusions about the behaviour of the stochastic model can be drawn. For a number of particular cases, contours of confidence regions in the state space are depicted. There is a good agreement with the results of numerical simulations of the original stochastic birth and death processes. Finally, the concept of extinction is treated. Results obtained by the ray method and by the approximate method in which the p.d.f. is assumed to be Gaussian, are compared with the data, obtained by the numerical simulations.

2. Derivation of the Fokker–Planck Equation. In this section the Fokker–Planck or forward Kolmogorov equation is derived. This is a transport equation for the joint probability density function for the number of individuals in each of the populations of a population system at time t. First a system with a single population is considered. Then the straightforward generalization to n populations is made.

One population. Let the number of individuals in a single population at some time t be given by the non-negative integer N. Assume that the population has an infinitesimal birth rate $\bar{B}(N)$ and an infinitesimal death rate $\bar{D}(N)$. This means that the probability of a single birth or death in the small time interval $(t, t+\Delta t)$ is given by $\bar{B}(N)\Delta t$ and $\bar{D}(N)\Delta t$. The probability of multiple births and deaths in the time interval is proportional to $(\Delta t)^2$ and may be neglected for small Δt. The probability $\bar{P}(N, t)$ of having N individuals at time t, satisfies the equation:

$$\bar{P}(N, t+\Delta t) = \bar{P}(N-1, t)\bar{B}(N-1)\,\Delta t + \bar{P}(N+1, t)\bar{D}(N+1)\,\Delta t + \bar{P}(N, t)[1-\{\bar{B}(N)+\bar{D}(N)\}\,\Delta t]. \tag{1}$$

Thus the number of individuals at some time is obtained by either one of three mutually exclusive events in the foregoing small time interval of length Δt, i.e. a birth, a death, neither birth nor death. In the limit $\Delta t \to 0$ the master equation is obtained:

$$\frac{\partial \bar{P}(N,t)}{\partial t} = \bar{P}(N-1,t)\bar{B}(N-1) + \bar{P}(N+1,t)\bar{D}(N+1) - \bar{P}(N,t)\{\bar{B}(N)+\bar{D}(N)\}. \tag{2}$$

In this way a discrete state space Markov process has been defined. An approximating process in a continuous state space is constructed in the following way. A new population variable $x = N/K$ is introduced by scaling N with K and the corresponding p.d.f. $P(x, t)$ is defined by

$$\bar{P}(N,t) = Pr\left\{N - \frac{1}{2} \le N(t) \le N + \frac{1}{2}\right\} = Pr\left\{x - \frac{1}{2K} \le x(t) \le x + \frac{1}{2K}\right\}$$
$$= P(x,t)\frac{1}{K}. \tag{3}$$

The parameter K is typical for the population size. A natural choice is the size of the equilibrium of the deterministic system, which by assumption is large. Then x may approximately be treated as a continuous variable and functions of x as continuous functions. With

$$\bar{B}(N) = \bar{B}(xK) = B(x),$$
$$\bar{D}(N) = \bar{D}(xK) = D(x), \tag{4}$$

we obtain from the master equation:

$$\frac{\partial P(x,t)}{\partial t} = P\left(x - \frac{1}{K}, t\right)B\left(x - \frac{1}{K}\right) + P\left(x + \frac{1}{K}, t\right)D\left(x + \frac{1}{K}\right) - P(x,t)\{B(x) + D(x)\}. \tag{5}$$

The birth and death rates B and D are assumed to be expressible in van Kampen's (1981) canonical form, that is, in a power series in K^{-1} of the following kind:

$$B(x) = f(K)\left[{}^0B\left(\frac{N}{K}\right) + {}^1B\left(\frac{N}{K}\right)K^{-1} + {}^2B\left(\frac{N}{K}\right)K^{-2} + \cdots\right],$$
$$D(x) = f(K)\left[{}^0D\left(\frac{N}{K}\right) + {}^1D\left(\frac{N}{K}\right)K^{-1} + {}^2D\left(\frac{N}{K}\right)K^{-2} + \cdots\right]. \tag{6}$$

It is assumed that B and D are smooth functions. By a Taylor series expansion (Kramers–Moyal expansion) of the functions on the right side of equation (5) up to second order in K^{-1}, the following Fokker–Planck equation is obtained:

$$\frac{1}{f(K)}\frac{\partial P(x,t)}{\partial t} = -\frac{1}{K}\frac{\partial}{\partial x}\left[\left\{({}^0B(x) - {}^0D(x)) + \frac{1}{K}({}^1B(x) - {}^1D(x))\right.\right.$$

$$+\frac{1}{K^2}\left({}^2B(x)-{}^2D(x)\right)+\cdots\Big\}P(x,t)\Big]$$

$$+\frac{2}{K^2}\frac{\partial^2}{\partial x^2}\Big[\Big\{\left({}^0B(x)+{}^0D(x)\right)$$

$$+\frac{1}{K}\left({}^1B(x)+{}^1D(x)\right)$$

$$+\frac{1}{K^2}\left({}^2B(x)+{}^2D(x)\right)+\cdots\Big\}P(x,t)\Big]. \tag{7}$$

In the simple case that

$$f(K)=K,$$
$${}^iB(x)\equiv{}^iD(x)\equiv 0, \text{ for } i>0, \tag{8}$$

so that B and D are given by

$$B(x)=K.{}^0B\left(\frac{N}{K}\right),$$
$$D(x)=K.{}^0D\left(\frac{N}{K}\right), \tag{9}$$

the Fokker–Planck equation takes the form

$$\frac{\partial P(x,t)}{\partial t}=-\frac{\partial}{\partial x}\left[\left\{{}^0B(x)-{}^0D(x)\right\}P(x,t)\right]$$
$$+\frac{1}{2K}\frac{\partial^2}{\partial x^2}\left[\left\{{}^0B(x)+{}^0D(x)\right\}P(x,t)\right]. \tag{10}$$

The first term on the right side expresses the drift of the system, the second one the diffusion. For large values of K the diffusion is small compared to the drift.

The approximating continuous process and the original discrete process have the same first and second jump moments. However, as a consequence of truncating the Taylor expansion after two terms, higher moments do not agree. The higher moments of the continuous process are all zero, while for the discrete process the odd moments are equal to the first moment and the even moments are equal to the second moment.

An approximation to birth–death processes incorporating higher jump moments can be found in Knessl *et al.* (1984, 1985). The methods described in this paper are also applicable to that approximation, as it gives rise to a Hamilton–Jacobi system similar to (28).

n populations. For every population i from a system of n populations, the birth rate is $\bar{B}_i(N_1, N_2, \ldots, N_n)$ and the death rate is $\bar{D}_i(N_1, N_2, \ldots, N_n)$. Let the $B_i(x_1, x_2, \ldots, x_n)$ and the $D_i(x_1, x_2, \ldots, x_n)$ be defined by

$$\bar{B}_i(N_1, N_2, \ldots, N_n) = \bar{B}_i(x_1K_1, x_2K_2, \ldots, x_nK_n) = B_i(x_1, x_2, \ldots, x_n),$$
$$\bar{D}_i(N_1, N_2, \ldots, N_n) = \bar{D}_i(x_1K_1, x_2K_2, \ldots, x_nK_n) = D_i(x_1, x_2, \ldots, x_n). \quad (11)$$

New variables $x_i = N_i/K_i$ were introduced by scaling the old ones by the sizes of the equilibrium populations. On assuming the canonical forms

$$B_i(x_1, x_2, \ldots, x_n) = K_i \cdot {}^0B_i\left(\frac{N_1}{K_1}, \frac{N_2}{K_2}, \ldots, \frac{N_n}{K_n}\right),$$
$$D_i(x_1, x_2, \ldots, x_n) = K_i \cdot {}^0D_i\left(\frac{N_1}{K_1}, \frac{N_2}{K_2}, \ldots, \frac{N_n}{K_n}\right), \quad (12)$$

the following Fokker–Planck equation is obtained:

$$\frac{\partial P(x_1, x_2, \ldots, x_n, t)}{\partial t}$$
$$= \sum_{i=1}^{n} \left\{ \frac{-\partial}{\partial x_i} [\{{}^0B_i(x_1, x_2, \ldots, x_n) - {}^0D_i(x_1, x_2, \ldots, x_n)\} P(x_1, x_2, \ldots, x_n, t)] + \frac{1}{2K_i} \frac{\partial^2}{\partial x_i^2} [\{{}^0B_i(x_1, x_2, \ldots, x_n) + {}^0D_i(x_1, x_2, \ldots, x_n)\} P(x_1, x_2, \ldots, x_n, t)] \right\}. \quad (13)$$

The population sizes K_i are assumed to be large and of equal order:

$$K_i = O\left(\frac{1}{\varepsilon}\right), \ (\varepsilon \text{ small}) \quad (14)$$

so that

$$\frac{1}{K_i} = \kappa_i \varepsilon, \ \kappa_i = O(1). \quad (15)$$

Substitution in the Fokker–Planck equation gives:

$$\frac{\partial P(x, t)}{\partial t} = \sum_{i=1}^{n} \left[-\frac{\partial}{\partial x_i} \{b_i(x) P(x, t)\} + \frac{\varepsilon}{2} \frac{\partial^2}{\partial x_i^2} \{a_i(x) P(x, t)\} \right], \quad (16)$$

in which

$$b_i(x) = {}^0B_i(x) - {}^0D_i(x), \tag{17a}$$

$$a_i(x) = \kappa_i({}^0B_i(x) + {}^0D_i(x)). \tag{17b}$$

This form of the Fokker–Planck equation, valid for a system of n populations having large equilibrium values of equal order, is the starting point of our analysis.

For very large populations the diffusion term may be neglected. The system is then described by the Liouville equation:

$$\frac{\partial P(x, t)}{\partial t} = \sum_{i=1}^{n} \left[-\frac{\partial}{\partial x_i} \{b_i(x) P(x, t)\} \right]. \tag{18}$$

It can be shown (Gardiner, 1983) that this equation with initial condition

$$P(x, t_0 | x_0, t_0) = \delta(x - x_0) \tag{19}$$

describes a deterministic motion which can also be found by solving the system of differential equations:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = b_i(x), \; i = 1, 2, \ldots, n \tag{20}$$

with initial conditions

$$x(t_0) = x_0. \tag{21}$$

This system of differential equations defines the deterministic system associated with the Fokker–Planck equation. The equilibrium points of the deterministic system are found by putting

$$b_i(x) = 0, \; i = 1, 2, \ldots, n \tag{22}$$

that is, by equating birth and death rates.

Boundary classification. For one-dimensional stochastic systems a complete classification of boundaries exists (Gardiner, 1983; Feller, 1952; Roughgarden, 1979). In order for the one-dimensional population system to have an exit boundary at $x = 0$, the following conditions must be satisfied:

$$J(0, t) < 0, \tag{23a}$$

$$b(0) = 0, \; a(0) = 0, \tag{23b}$$

in which J is the probability current defined by

$$J(x, t) \equiv b(x)P(x, t) - \frac{\varepsilon}{2}\frac{\partial}{\partial x}\{a(x)P(x, t)\}. \tag{24}$$

By (23), $x=0$ can be reached from the interior of the state space, while the interior cannot be reached from $x=0$. Using (23b), the requirement (23a) results in a degeneration condition for a:

$$\frac{\partial a}{\partial x}(0) > 0. \tag{25}$$

For higher dimensions no complete classification of boundaries exists. An examination of the probability current and drift and diffusion coefficients at the boundaries $x_i = 0$ can reveal whether these boundaries can be reached from the interior of the state space and vice versa.

3. Asymptotic solution of the Fokker–Planck equation. In this section, the asymptotic analysis of Ludwig (1975) for small noise strength ε is carried out, leading to the ray equations. A local analysis in the neighbourhood of the equilibrium point is given. Moreover, the numerical solution of the ray equations is discussed.

The analysis will be restricted to a study of the quasi-stationary state. A non-trivial stationary state does not exist for a stochastic process with absorbing boundaries. For a discussion of this point in relation to population dynamics, see Nisbet and Gurney (1982). Apart from possibly a short initial period of time, the quasi-stationary state is supposed to give a close description of the stochastic process during a long period of time. The quasi-stationary state is obtained by putting the l.h.s. of equation (16) equal to zero:

$$0 = \sum_{i=1}^{n}\left[-\frac{\partial}{\partial x_i}\{b_i(x)P_s(x)\} + \frac{\varepsilon}{2}\frac{\partial^2}{\partial x_i^2}\{a_i(x)P_s(x)\}\right]. \tag{26}$$

The function $P_s(x)$ is the p.d.f. corresponding to the quasi-stationary state. On the assumption that the deterministic system (20) has a stable equilibrium point lying in the interior of the state space, we apply the asymptotic analysis of Ludwig for small ε. A simple WKB-Ansatz to the solution of equation (26) is:

$$P_s(x) = C\exp\left(-\frac{Q(x)}{\varepsilon}\right), \tag{27}$$

in which C is a normalization constant. Substitution of this expression in equation (26) and rearranging terms in equal powers of ε yields to leading order in ε:

$$\sum_{i=1}^{n}\left[b_i(x)\frac{\partial Q}{\partial x_i}+\frac{1}{2}a_i(x)\left(\frac{\partial Q}{\partial x_i}\right)^2\right]=0. \tag{28}$$

We assume that the stochastic system is described sufficiently accurately by (27), (28). Otherwise, higher order terms in ε must be included in the WKB-approximation, see Ludwig (1975). Equation (28) is called the eikonal equation. It is a Hamilton–Jacobi equation and can be written as:

$$H(x, p)=0, \tag{29}$$

in which H is the Hamiltonian

$$H(x, p)=\sum_{i=1}^{n}\left(b_i p_i+\frac{1}{2}a_i p_i^2\right) \tag{30}$$

with

$$p_i=\frac{\partial Q}{\partial x_i}. \tag{31}$$

The corresponding system of ordinary differential equations is:

$$\frac{\mathrm{d}x_i}{\mathrm{d}s}=\frac{\partial H}{\partial p_i}=b_i+a_i p_i, \tag{32a}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}s}=-\frac{\partial H}{\partial x_i}=-\sum_{j=1}^{n}\left[\frac{\partial b_j}{\partial x_i}p_j+\frac{1}{2}\frac{\partial a_j}{\partial x_i}p_j^2\right], \tag{32b}$$

in which s is a parameter along the characteristics. The rate of change of Q with s is given by

$$\frac{\mathrm{d}Q}{\mathrm{d}s}=-H(x, p)+\sum_{i=1}^{n}\frac{\mathrm{d}x_i}{\mathrm{d}s}p_i=\sum_{i=1}^{n}\frac{1}{2}a_i p_i^2\geq 0. \tag{32c}$$

The dynamical system defined by the equations (32) is assumed to have an equilibrium point given by

$$\left.\begin{aligned}x_i&=1\\p_i&=0\end{aligned}\right\}\quad i=1, 2, \ldots, n.$$
$$Q=0 \tag{33}$$

It is seen that the projection of this equilibrium point on the x-space coincides with the equilibrium point of the deterministic system. The equations (32) are called the ray equations. The projections of solutions of (32) on the x-space are

called rays. All rays emanate from the equilibrium point. Rays may be interpreted as paths of maximum likelihood joining (points in the neighbourhood of) the equilibrium point with points in x-space. See Ludwig (1975) and the references given there.

Local analysis near the equilibrium. In the neighbourhood of the equilibrium point of the ray equations, Q is approximated by a quadratic form:

$$Q(x) \approx \tilde{Q}(x) = \sum_{i,j} \frac{1}{2} P_{ij}(x_i - 1)(x_j - 1), \tag{34}$$

in which P_{ij} is a symmetric matrix:

$$P_{ij} = P_{ji}. \tag{35}$$

Differentiation of expression (34) gives an approximation for the p_i:

$$p_i = \frac{\partial Q}{\partial x_i} \approx \sum_j P_{ij}(x_j - 1),\ i = 1, 2, \ldots, n. \tag{36}$$

The deterministic vector field b_i near the equilibrium point is approximated by

$$b_i \approx \sum_j \frac{\partial b_i}{\partial x_j}(x_j - 1),\ i = 1, 2, \ldots, n. \tag{37}$$

Substitution of the approximations (36) and (37) in the eikonal equation gives:

$$\sum_i \left[\sum_j \frac{\partial b_i}{\partial x_j}(x_j - 1) \sum_k P_{ik}(x_k - 1) + \frac{1}{2} a_i \left\{ \sum_j P_{ij}(x_j - 1) \right\}^2 \right] = 0. \tag{38}$$

Making use of the symmetry of P_{ij}, this can be rewritten in the matrix form:

$$PAP + PB + B^t P = 0, \tag{39}$$

in which $B = (\partial b_i / \partial x_j)$, t denotes the transpose and A is a diagonal matrix containing the values of a_i at the equilibrium point. Left and right multiplication with $S = P^{-1}$ gives:

$$A + BS + SB^t = 0. \tag{40}$$

If the matrices S and A are written columnwise as vectors, a linear system with n^2 equations is obtained, which can be solved for S. The matrix P is obtained by inversion of S. All eigenvalues of B are negative, because of the assumed stability of the equilibrium point of the deterministic system. Consequently, the last two operations can be carried out. The elements of the matrix P can be

substituted in expression (34), resulting in an approximation of Q in the neighbourhood of the equilibrium point.

Confidence regions. From the Ansatz (27) it is clear that contours of constant Q (hypersurfaces) in the state space, are contours of constant probability. Let Q_z be the value of Q corresponding to the contour, for which the probability of being in the region R enclosed by this contour, is equal to z:

$$\int_R P_s(x')\,dx' = z,\ 0 \leq z \leq 1. \tag{41}$$

In order to construct the contour enclosing the confidence region of probability z, the corresponding value Q_z of Q has to be determined. The following heuristic method is used. According to the local analysis, in a first approximation P_s has an n-variate normal distribution around the equilibrium point, given by

$$P_s(x) = C \exp\left(-\frac{\tilde{Q}(x)}{\varepsilon}\right). \tag{42}$$

Then by a standard result in probability theory (Hogg and Craig, 1970), $2\tilde{Q}(x)/\varepsilon$ has a chi-square distribution with n degrees of freedom. The value $2Q_z/\varepsilon$ which will not be exceeded by $2\tilde{Q}(x)/\varepsilon$ with probability z, can be found in a table of the chi-square distribution with n degrees of freedom. In the case $n=2$, used in the examples in Section 5, the chi-square distribution has a simple form from which it can be derived that

$$Q_z = -\varepsilon \ln(1-z). \tag{43}$$

Numerical solution of the ray equations. The local analysis near the equilibrium point may not be a sufficiently accurate approximation away from this point. Then the ray equations have to be solved numerically.

(*i*) *The initial value approach*

For the system (32) a starting point $x(0)$ at $s=0$ is chosen close to the equilibrium point. The formulas (34) and (36) give the initial values for Q and p_i ($i=1, 2, \ldots, n$). The ray equations are solved numerically by using a routine for solving a system of ordinary differential equations written in first-order form with conditions in the form of initial values. For this purpose the NAG-library contains Runge-Kutta Merson routines or variable order, variable step Adam routines. On applying such a routine, the solution $Q(s)$, $x_i(s)$, $p_i(s)$ ($i=1, 2, \ldots n$) is obtained along the ray defined by the initial point $x(0)$. Once

the initial point has been chosen, there is no control over the way the ray develops through space. Generally there is a very strong dependence on the initial point. Especially when the eigenvalues of the deterministic system in the equilibrium point do not have ratios close to one, it is impracticable to choose the initial points in such a way that a bundle of rays is obtained, which uniformly covers the state space around the equilibrium point. Thus, the method is not well suited for the construction of the contours of the confidence regions.

Even shooting methods, in which the initial point is manipulated systematically as to obtain the desired rays, did not solve the difficulty (Ludwig, 1975).

(ii) The boundary value approach

Instead of specifying the $2n+1$ conditions at a starting point close to the equilibrium, $n+1$ conditions are imposed at the starting point coinciding with the equilibrium and n conditions are imposed at an endpoint which can be chosen freely:

$$s \to -\infty: \quad Q=0,\ x_i=1, \quad i=1,2,\ldots,n \tag{44a}$$

$$s=0: \quad x_i=e_i, \quad i=1,2,\ldots,n \tag{44b}$$

where the e_i are the coordinates of the endpoint. At $s \to -\infty$ the condition for x has been retained and that for p has been dropped. This choice is understood as follows. Trajectories of (32ab) exist, that for decreasing s leave a neighbourhood of the equilibrium point and in the limit $s \to -\infty$ approach the subspace with negative eigenvalues. Condition (44a) forbids such solutions.

The limit $s \to -\infty$ (in numerical computations replaced by $s=-s^*$, s^* a sufficiently large number) will cause the characteristic to start at (close to) $x=1, p=0$. The characteristic lies in the unstable manifold through $x=1, p=0$. It can be shown (Roozen, 1986) that the plane tangent to this manifold at this point satisfies (39), so that the boundary value solution corresponds to an initial value solution.

The two point boundary value problem thus described is solved by using an appropriate routine, for example the NAG-routine D02RAF. This routine uses a deferred correction technique and Newton iteration. On a grid of s-values, a first approximation to the solution has to be given, from which the routine iteratively tries to find the solution. Experience has shown that the method works well provided that a good first approximation is given and a sufficient number of grid points are used.

Endpoints may exist, for example in the neighbourhood of a caustic, which can be connected with the equilibrium by more than one characteristic. In that case, the characteristic that is constructed by the boundary value approach depends on the first approximation to the solution.

Because the endpoints can be chosen at will, contours of confidence regions can be constructed by this method quite efficiently. The contours and rays of the stochastic two population models shown in Section 5 of this paper, have been obtained by this method. Details of the numerical construction of rays and contours can be found in Roozen (1986).

4. Extinction. In this section, exit from a region R with boundary S is treated. Questions of interest are the following. What is the expected first exit time and which population is expected to exit first? In the birth–death models treated in this paper, exit at the boundary $x_i=0$ means extinction of population i.

The expected time of first exit $T(x)$, starting in a point x in a region R with boundary S, satisfies the Dynkin equation, which can be derived from the backward Kolmogorov equation (Gardiner, 1983; Schuss, 1980) and is given by:

$$\sum_{i=1}^{n} \left\{ b_i(x) \frac{\partial T(x)}{\partial x_i} + \frac{\varepsilon}{2} a_i(x) \frac{\partial^2 T(x)}{\partial x_i^2} \right\} = -1. \tag{45}$$

It has to be solved with the boundary condition

$$T(x)=0 \text{ for } x \in S. \tag{46}$$

The problem is rewritten as

$$\begin{aligned} L_\varepsilon T &= -1 \text{ in } R \\ T &= 0 \text{ on } S. \end{aligned} \tag{47}$$

The operator L_ε is the formal adjoint of the operator on the right side of equation (16) working on the p.d.f.

The probability $P(x, x')$ of exit at $x' \in S$, starting from $x \in R$ is related to the solution $u(x)$ of the boundary value problem:

$$\begin{aligned} L_\varepsilon u &= 0 \text{ in } R \\ u &= f(x) \text{ on } S \end{aligned} \tag{48}$$

by the relation

$$u(x)=\int_S f(x')P(x, x')\, \mathrm{d}S_{x'}. \tag{49}$$

After choosing the function f, we can solve the problem (48) and obtain the corresponding function $u(x)$. If for example $f(x')=\delta(x'-a)$, then by solving (48), $u(x)=P(x, a)$ is obtained.

In the asymptotic analysis for small ε, see Ludwig (1975), Matkowsky and Schuss (1977), it follows for the expected exit time that

$$T \sim e^{Q(x^*)/\varepsilon}, \tag{50}$$

in which $x^* \in S$ is the boundary point at which Q takes its minimal value at the boundary. The most likely point of exit is the boundary point x^*. The asymptotic results have been derived for the case that at the boundary S the trajectories of the deterministic system enter the region R.

In the study of extinction the boundaries $x_i = 0$ $(i = 1, 2, \ldots, n)$ are of interest. If $x_i = 0$ is the boundary which contains x^*, then i is the population which most likely will get extinct first. However, in the application to the birth and death models, some complications arise. At $x_i = 0$ $(i = 1, 2, \ldots, n)$ the trajectories of the deterministic system remain in the boundary $x_i = 0$. The functions $a_i(x)$ and $b_i(x)$ tend to zero at the boundary $x_i = 0$, which requires a new type of local asymptotic analysis. As a consequence of the fact that $a_i(x)$ and $b_i(x)$ vanish near the boundaries, the rays there deflect and large gradients in the ray variables are found.

The problems are avoided by studying the mean time needed for a system of populations to reach one of the small positive levels $x_i = l_i$ $(i = 1, 2, \ldots, n)$, assuming the WKB-Ansatz to be valid for $x_i \geq l_i$ $(i = 1, 2, \ldots, n)$. The computation of x^* and $Q(x^*)$ can be carried out efficiently by using a variant of the boundary value approach. With respect to the boundary $x_i = l_i$ the boundary conditions are:

$$\begin{aligned} s \to -\infty: &\quad Q = 0,\ x_j = 1, \quad j = 1, 2, \ldots, n \\ s = 0: &\quad x_i = l_i,\ p_j = 0, \quad j = 1, 2, \ldots, n \text{ and } j \neq i. \end{aligned} \tag{51}$$

The variant of the boundary value approach with conditions (51) has to be solved for each of the boundaries $x_i = l_i$ $(i = 1, 2, \ldots, n)$. The boundary at which the smallest value for Q is found is the expected exit boundary and the point on this boundary where this value is taken is the expected exit point. For the details of the numerical solution, see Roozen (1986).

5. *A stochastic two population model.* In this section the theory is applied to a stochastic two population model. The model under consideration is described by birth and death rates of the form:

$$\begin{aligned} B_1(N_1, N_2) &= N_1(\lambda_{10} + \lambda_{11}N_1 + \lambda_{12}N_2) \\ B_2(N_1, N_2) &= N_2(\lambda_{20} + \lambda_{21}N_1 + \lambda_{22}N_2) \\ D_1(N_1, N_2) &= N_1(\mu_{10} + \mu_{11}N_1 + \mu_{12}N_2) \\ D_2(N_1, N_2) &= N_2(\mu_{20} + \mu_{21}N_1 + \mu_{22}N_2), \end{aligned} \tag{52}$$

in which the λ_{ij} and μ_{ij} are constants, such that the birth and death rates are positive for all admitted values of N_1 and N_2.

The deterministic system. In the original variables N_1 and N_2, the deterministic system is given by the generalized Lotka–Volterra system:

$$\frac{dN_1}{dt} = N_1(b_{10} + b_{11}N_1 + b_{12}N_2)$$

$$\frac{dN_2}{dt} = N_2(b_{20} + b_{21}N_1 + b_{22}N_2), \tag{53}$$

in which $b_{ij} = 1, \lambda_{ij} - \mu_{ij}$ for $i = 1, 2$ and $j = 0, 1, 2$. The equilibrium populations K_1 and K_2 are given by

$$K_1 = \frac{b_{22}b_{10} - b_{12}b_{20}}{b_{21}b_{12} - b_{11}b_{22}}, \; K_2 = \frac{b_{11}b_{20} - b_{21}b_{10}}{b_{21}b_{12} - b_{11}b_{22}}. \tag{54}$$

Introduction of new variables

$$x_i = \frac{N_i}{K_i}, \; i = 1, 2 \tag{55}$$

gives:

$$\frac{dx_1}{dt} = k_1 x_1(1 + \alpha - x_1 - \alpha x_2)$$

$$\frac{dx_2}{dt} = k_2 x_2(1 + \beta - \beta x_1 - x_2), \tag{56}$$

in which:

$$k_1 = -b_{11}K_1, \quad k_2 = -b_{22}K_2, \quad \alpha = \frac{b_{12}K_2}{b_{11}K_1}, \quad \beta = \frac{b_{21}K_1}{b_{22}K_2}. \tag{57}$$

Assuming that both populations have a self-limiting growth (i.e. that b_{11} and b_{22} are negative) the factors k_1 and k_2 are positive. They can be interpreted as the reciprocals of time scales for the respective populations. The type of interaction between the populations is determined by the parameters α and β as shown in Fig. 1.

For the subsequent analysis it is important to know the condition under which the equilibrium of the deterministic system at (1, 1) is stable. Linearization of the system $b_1(x_1, x_2)$, $b_2(x_1, x_2)$ which is given by equations (56), in the neighbourhood of the equilibrium point gives:

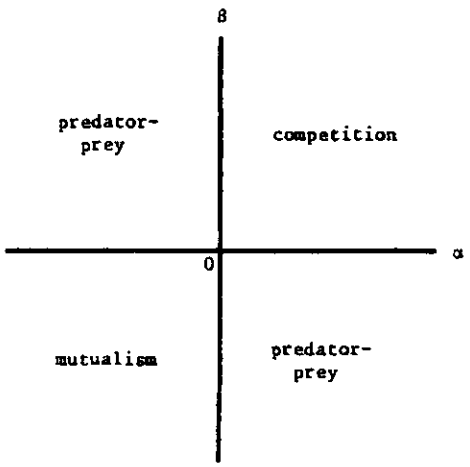

Figure 1. The type of interaction between the two populations depending on the parameters α and β.

$$\frac{d\tilde{x}}{dt} = B\tilde{x}, \tag{58}$$

in which $\tilde{x} = (x_1 - 1, x_2 - 1)^t$ and the matrix B is given by

$$B = \begin{bmatrix} -k_1 & -\alpha k_1 \\ -\beta k_2 & -k_2 \end{bmatrix}. \tag{59}$$

The eigenvalues of B are:

$$\lambda_{1,2} = \frac{-(k_1 + k_2) \pm \sqrt{(k_1 - k_2)^2 + 4k_1 k_2 \alpha\beta}}{2}. \tag{60}$$

The condition for stability is that the real parts of the eigenvalues are negative, resulting in

$$\alpha\beta < 1. \tag{61}$$

Figure 2 shows the region of stability in the α,β-parameter plane. From Figs 1 and 2 it can immediately be concluded that the equilibrium (1, 1) is stable for all predator–prey models, while for mutualism and competition models this equilibrium is stable for parameter values of α and β in only a small region of the α,β-parameter plane.

The same kind of stability analysis as given above can be carried out for the equilibrium points at (0, 0), $(1 + \alpha, 0)$ and $(0, 1 + \beta)$, which are of interest only if both coordinates are non-negative. It can be shown that also for these equilibrium points the type of stability depends on α and β only.

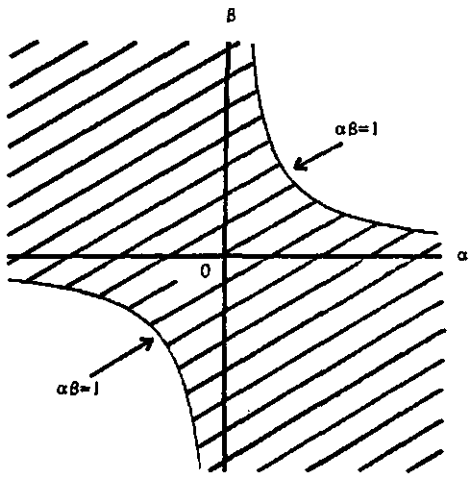

Figure 2. Deterministic stability. The region of stability (shaded) of the deterministic system at the equilibrium point (1, 1) depending on the parameters α and β.

Local analysis of the stochastic system near the equilibrium. On the assumption (61) the theory of the preceding sections may be applied. Here the local analysis is made, which is valid in the neighbourhood of the equilibrium point. Generally the system (40) has to be solved numerically. In this case however, it is possible to find explicit expressions. In scaled variables x_i the functions $a_i(x)$ defined by equation (17b) are given by

$$a_1(x_1, x_2)=\kappa_1 x_1(a_{10}+a_{11}K_1x_1+a_{12}K_2x_2)$$
$$a_2(x_1, x_2)=\kappa_2 x_2(a_{20}+a_{21}K_1x_1+a_{22}K_2x_2), \tag{62}$$

in which

$$a_{ij}=\lambda_{ij}+\mu_{ij} \text{ for } i=1, 2 \text{ and } j=0, 1, 2$$

and

$$\varepsilon^{-1}=\frac{K_1+K_2}{2}, \kappa_i=\frac{K_1+K_2}{2K_i}, i=1, 2. \tag{63}$$

The matrix A is diagonal with elements a_1 and a_2 equal to:

$$a_1=a_1(1, 1)=\kappa_1(a_{10}+a_{11}K_1+a_{12}K_2)$$
$$a_2=a_2(1, 1)=\kappa_2(a_{20}+a_{21}K_1+a_{22}K_2). \tag{64}$$

The matrix B is defined by (59). Equation (40) leads to the following linear system:

$$\begin{bmatrix} -2k_1 & -\alpha k_1 & -\alpha k_1 & 0 \\ -\beta k_2 & -k_1-k_2 & 0 & -\alpha k_1 \\ -\beta k_2 & 0 & -k_1-k_2 & -\alpha k_1 \\ 0 & -\beta k_2 & -\beta k_2 & -2k_2 \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{21} \\ s_{12} \\ s_{22} \end{bmatrix} = \begin{bmatrix} -a_1 \\ 0 \\ 0 \\ -a_2 \end{bmatrix}. \quad (65)$$

The solution of this system is given by

$$S = \frac{1}{2k_1k_2(k_1+k_2)(1-\alpha\beta)} \times \begin{bmatrix} a_1k_2[k_2+k_1(1-\alpha\beta)]+a_2\alpha^2k_1^2 & -a_1\beta k_2^2-a_2\alpha k_1^2 \\ -a_1\beta k_2^2-a_2\alpha k_1^2 & a_1\beta^2k_2^2+a_2k_1[k_1+k_2(1-\alpha\beta)] \end{bmatrix}. \quad (66)$$

The matrix P in expression (34) is found by inversion of the matrix S:

$$P = S^{-1} = \frac{2(k_1+k_2)}{(a_1\beta k_2-a_2\alpha k_1)^2+a_1a_2(k_1+k_2)^2} \times \begin{bmatrix} a_1\beta^2k_2^2+a_2k_1[k_1+k_2(1-\alpha\beta)] & a_1\beta k_2^2+a_2\alpha k_1^2 \\ a_1\beta k_2^2+a_2\alpha k_1^2 & a_1k_2[k_2+k_1(1-\alpha\beta)]+a_2\alpha^2k_1^2 \end{bmatrix}. \quad (67)$$

By using (42) the bivariate normal p.d.f. is determined. The corresponding confidence contours in the state space are ellipses. In some special cases the expression for P can be simplified. For example, the competition model, treated by May [1974, p. 123–129], is a particular case of our more general model. The p.d.f. derived by May is easily found from the formulas given above.

The expressions show the complex dependence of the ellipses on the parameters k_1, k_2, α, β of the deterministic system and the noise components a_1 and a_2 at the equilibrium point. It may be noted that multiplication of k_1, k_2, a_1, a_2 with the same constant leaves the resulting p.d.f. invariant. The effect of an increase (decrease) of velocity by which the system returns to the deterministic equilibrium cancels the effect of an increase (decrease) of the stochastic fluctuations.

The covariance matrix corresponding to a bivariate normal distribution is given by Batschelet (1981):

$$\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \quad (68)$$

in which σ_1^2 and σ_2^2 are the variances and ρ is the correlation coefficient. On equating this matrix to the actual covariance matrix εS, the following expressions for the variances and the correlation coefficient are obtained:

$$\sigma_1^2 = \varepsilon \frac{a_1 k_2 [k_2 + k_1(1-\alpha\beta)] + a_2 \alpha^2 k_1^2}{2k_1 k_2 (k_1 + k_2)(1-\alpha\beta)}, \tag{69}$$

$$\sigma_2^2 = \varepsilon \frac{a_2 k_1 [k_1 + k_2(1-\alpha\beta)] + a_1 \beta^2 k_2^2}{2k_1 k_2 (k_1 + k_2)(1-\alpha\beta)}, \tag{70}$$

$$\rho = \frac{-(a_1 \beta k_2^2 + a_2 \alpha k_1^2)}{\sqrt{\{a_1 k_2 [k_2 + k_1(1-\alpha\beta)] + a_2 \alpha^2 k_1^2\}\{a_2 k_1 [k_1 + k_2(1-\alpha\beta)] + a_1 \beta^2 k_2^2\}}}. \tag{71}$$

Note the dependence of the first two expressions on ε, the reciprocal of the mean of the equilibrium populations. The parameters $k_1, k_2, a_1, a_2, \varepsilon$ are all positive. The condition for a stable equilibrium is that $1-\alpha\beta$ is positive. Then, it is easily seen that the sign of the correlation coefficient ρ equals the sign of the numerator on the right side of equation (71), from which it follows that for competition ($\alpha>0, \beta>0$) the correlation coefficient is negative and for mutualism ($\alpha<0, \beta<0$) the correlation coefficient is positive. For predator–prey systems both positive and negative values are possible.

For the cases that the local analysis is also valid far from the equilibrium, an expression for the expected exit time can be derived. Let C_i be the minimal value of Q in equation (34) for which the ellipse touches the axis $x_i = 0$ and define T_i by

$$T_i = e^{C_i/\varepsilon}, \; i = 1, 2. \tag{72}$$

It can easily be shown that

$$T_1 = \exp\left\{\frac{1}{\varepsilon} \frac{(k_1 + k_2) k_1 k_2 (1-\alpha\beta)}{a_2 \alpha^2 k_1^2 + a_1 k_2 [k_2 + k_1(1-\alpha\beta)]}\right\} = e^{1/2\sigma_1^2}, \tag{73}$$

$$T_2 = \exp\left\{\frac{1}{\varepsilon} \frac{(k_1 + k_2) k_1 k_2 (1-\alpha\beta)}{a_1 \beta^2 k_2^2 + a_2 k_2 [k_1 + k_2(1-\alpha\beta)]}\right\} = e^{1/2\sigma_2^2}. \tag{74}$$

The last equalities in (73) and (74) follow from (69) and (70). For the expected extinction time T it follows that

$$T \sim \min(T_1, T_2). \tag{75}$$

The ecological stability index ξ, defined in Nisbet and Gurney (1982, p. 10) by

$$\xi \equiv \ln T \tag{76}$$

then satisfies:

$$\xi \sim \tilde{\xi} \equiv \min\left(\frac{1}{2\sigma_1^2}, \frac{1}{2\sigma_2^2}\right). \tag{77}$$

Recalling that scaled population variables are used, this result can be seen as a two-dimensional generalization of the one-dimensional result in Nisbet and Gurney (1982, p. 202).

Given the values of a_1, a_2, k_1, k_2 it may be wondered which combination of α and β leads to the largest ecological stability. Figure 3 shows the curves of equal $\bar{\xi}$ in the α, β-parameter plane. Typically, the largest ecological stability is found in predator–prey systems. As an illustration of the different meaning of stability in deterministic and stochastic systems, Fig. 3 should be compared with Fig. 2.

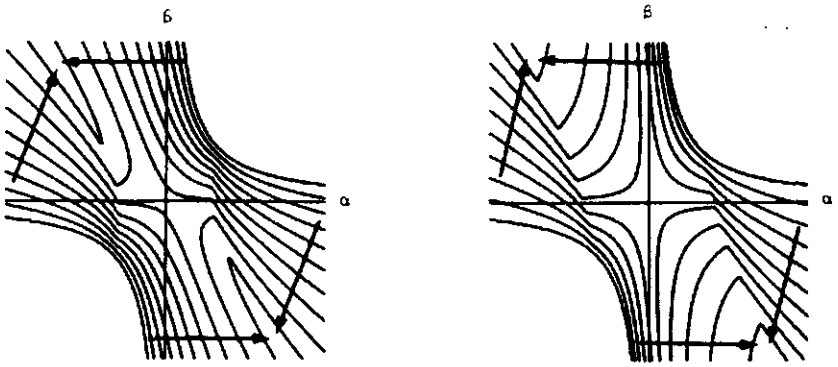

Fig. 3. Stochastic stability. Curves of equal $\bar{\xi}$, derived from the local analysis, in the α,β-parameter plane. The arrows indicate increasing values of $\bar{\xi}$. The difference in $\bar{\xi}$-values between successive curves in each of the figures is constant. In the atypical case $a_1/k_1 = a_2/k_2$ the maximum of $\bar{\xi}$ is adopted on $\beta = -a_1\alpha/a_2$, which is a line through the origin with a negative slope. Fig. 3a shows a case $a_1/k_1 \approx a_2/k_2$ and Fig. 3b shows a case $a_1/k_1 \not\approx a_2/k_2$. The actual values of a_1, a_2, k_1, k_2 are 1.0, 1.3, 1.1, 0.8 for case a and 1.0, 8.0, 1.1, 0.7 for case b.

Apart from the time to extinction it may be convenient to have an expression for the expected time for a system of populations to reach one of the levels $x_i = l_i$ (expressed in units of the equilibrium populations) with l_i not necessarily equal to zero. It is found that with

$$T_i^{l_i} = e^{(l_i - 1)^2/2\sigma_i^2},\ i = 1, 2 \tag{78}$$

the expected time T satisfies:

$$T \sim \min(T_1^{l_1}, T_2^{l_2}). \tag{79}$$

The points of contact of the ellipses with the axes $x_1 = l_1$ and $x_2 = l_2$ are given by

$$x_1 = l_1,\ x_2 = 1 + \frac{P_{12}}{P_{22}}(1 - l_1) \tag{80}$$

TABLE I

(A) The Birth and Death Rates for Three, Two-population Models.
(B) Parameters of the Corresponding Deterministic System.

		1 Predator–prey			2 Mutualism			3 Competition		
A	(λ_{ij})	1.	.004	0.	.6	.004	.0056	1.	.004	0.
		.6	.0056	0.	.8	.0016	.0016	.8	0.	.0016
	(μ_{ij})	.280	.012	.0064	.480	.012	0.	.64	.008	.0032
		.480	0.	.008	.480	0.	.0096	.56	.0008	.0056
B	α	.8			−.7			.8		
	β	−.7			−.2			.2		
	k_1	.4			.4			.2		
	k_2	.4			.4			.2		
	K_1	50			50			50		
	K_2	50			50			50		

(A) The coefficients appear in the same order as in (52). (B) The values of the parameters can be calculated from the birth and death rates.

TABLE II

Results of Numerical Birth–Death Experiments

$\frac{1}{\varepsilon}$	Number of experiments	$\log T$	Exit at $x_1=0.1$ at $x_2=$	% of exp	Exit at $x_2=0.1$ at $x_1=$	% of exp
10	200	2.175	1.14	57	1.21	43
20	200	3.067	1.11	59.5	1.30	40.5
40	150	4.554	1.21	61.3	1.48	38.7
60	50	5.530	1.14	82	1.61	18
80	50	6.879	1.26	94	1.65	6

and

$$x_1 = 1 + \frac{P_{12}}{P_{11}}(1-l_2),\ x_2 = l_2, \tag{81}$$

respectively.

In cases that the local analysis is valid only in a small neighbourhood of the equilibrium, the expressions given here cannot be used. Instead, the full system of ray equations (32) has to be solved numerically.

Construction of contours. For the cases, defined by the birth and death rates in Table I, the ray equations (32) have been solved numerically by the method,

presented in this paper as the boundary value approach. Figure 4 shows the confidence contours and rays for the various cases. The cases 1 and 3 are similar to the examples 1 and 2 of Ludwig (1975). The difficulties reported by Ludwig in the construction of rays in his second example, the competition model, were also experienced by the author, when the initial value approach was used, in which the initial values were chosen equidistantly on a small circle around the equilibrium point. As a consequence of the fact that the eigenvalues of the

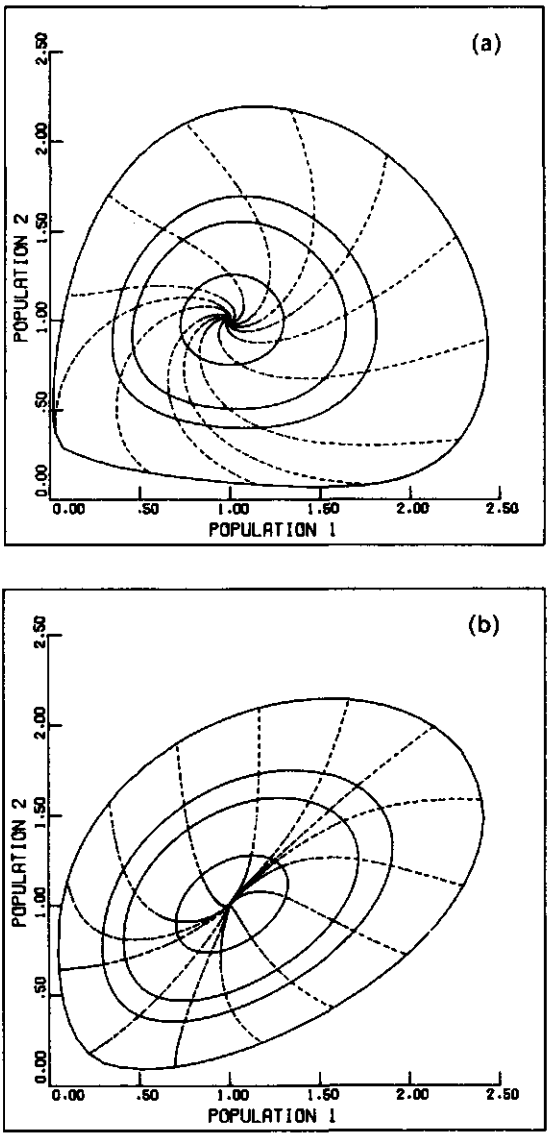

Figure 4(a) and (b).

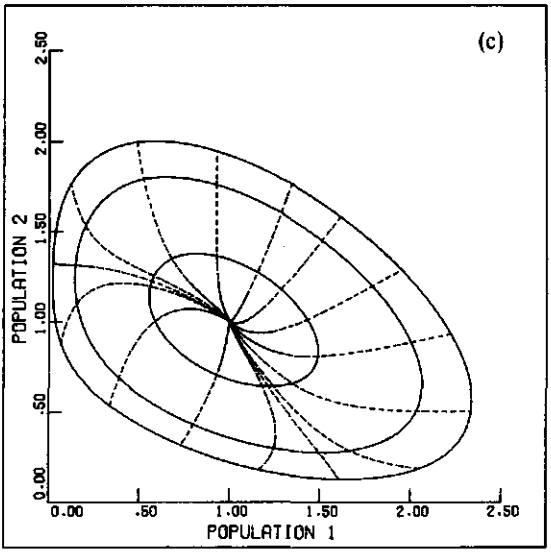

Figure 4. Rays and confidence contours for (a) the predator–prey model, (b) the mutualism model, (c) the competition model, defined in Table I, as obtained by solving the ray equations by the boundary value approach. From the inside outwards, the three (inner) contours correspond to the 50%, 95% and 99% confidence regions, respectively.

linearized deterministic system at the equilibrium point do not have a ratio close to one [ratio 5.7 in Ludwig (1982)], almost all rays follow very closely and nearly indistinguishable from each other one of two paths, as shown in Fig. 5. The boundary value approach introduced in this paper overcomes these problems, see Fig. 4c. From the figures it is apparent that close to the equilibrium the contours have the elliptic shape, while further away deviations from the elliptic shape tend to come in.

Figure 6 shows the result of a numerical simulation (see the Appendix) of the predator–prey system. Each of about 35,000 dots represents a visit. Because the population sizes can take on only integer values, the dots should lie on a two-dimensional grid. However, the dots have been plotted slightly away from their grid positions in a random way, in order to get a good idea of the corresponding p.d.f. The agreement with the constructed contours is quite good.

Exit boundary and exit time. As an illustration of the theory dealing with the expected exit point and the expected exit time, a number of numerical simulations (see the Appendix) have been carried out for the competition model given in Table I, with different values of the noise parameter ε. The results are shown in Table II. The first column shows the value of ε^{-1}. The second column shows the number of experiments carried out for the corresponding case. Column 3 shows the values found for $\log T$, in which T is

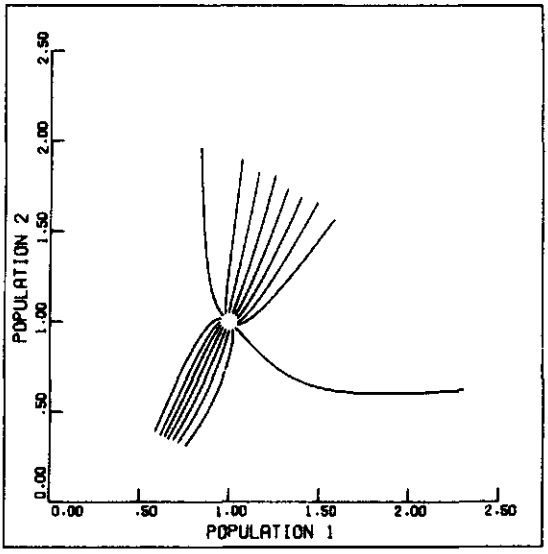

Figure 5. Rays obtained by the initial value approach. The initial points were chosen equidistantly on a circle around the equilibrium point. Instead of covering the state space uniformly in the neighbourhood of the equilibrium, there is a tendency for the rays to follow one of two main directions. For this illustration an extremely large radius (= 0.05) of the circle was chosen. For a small radius (for example = 0.001) the effect is stronger and the rays leave the equilibrium almost indistinguishable from each other in one of two directions.

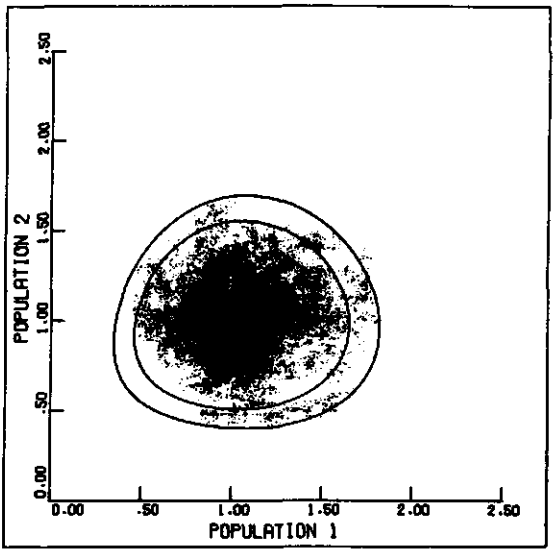

Figure 6. Result of a numerical simulation of the predator–prey birth–death process, defined in Table I, together with the 95% and 99% confidence contours as obtained by the boundary value approach. The 50% contour is almost invisible.

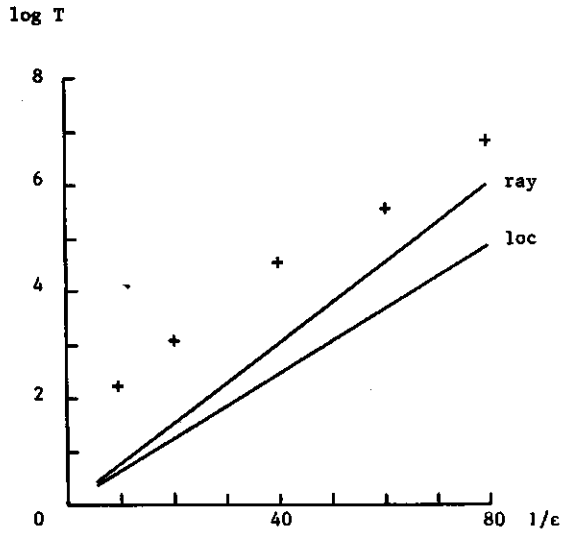

Figure 7. Relationship between log T and $1/\varepsilon$. The crosses are values obtained by the experiments. The lines correspond to the expressions (82a) and (82b) with $C_{loc}=C_{ray}=1$.

the mean time needed for one of the populations of the system to reach 10% of its equilibrium value. The resulting columns 4 and 5 show the mean point of exit at the boundary $x_1=0.1$, $x_2=0.1$ (in units of the equilibrium values) respectively, with the percentage of exits at that boundary.

The approach based on the local analysis, as described in this paper, indicates that (0.1, 1.31) is the most probable exit point, so that population 1 is expected to exit first. The expected exit time is $T_{loc}\sim e^{0.06062/\varepsilon}$.

Solution of the ray equations, in the way described at the end of section 4, indicates that (0.1, 1.25) is the most probable exit point, so that population 1 is expected to exit first. See Fig. 4c, which indeed suggests that the boundary $x_1=0.1$ is tangent to a contour line in the neighbourhood of (0.1, 1.25). The corresponding value of Q is lower than is the case for the contour, to which the boundary $x_2=0.1$ is tangent. The expected exit time is $T_{ray}\sim e^{0.07508/\varepsilon}$.

As is seen, the mean exit point obtained from the experiments agrees well with the value from the asymptotic theory for small ε. Notice the increasing percentage of exits in the neighbourhood of the predicted place for decreasing values of ε in the experiments. The approach based on the local analysis leads to results which agree reasonably well with the results of the experiments.

We write the relationships between T and ε^{-1} as

$$T_{loc}=C_{loc}\,e^{0.06062/\varepsilon}, \tag{82a}$$

$$T_{ray}=C_{ray}\,e^{0.07508/\varepsilon}, \tag{82b}$$

in which C_{loc} and C_{ray} are constants. In a graph of $\ln T$ against ε^{-1} these relationships are straight lines which, asymptotically for small ε, should be parallelled by a line fitted through the data points of the experiments, see Fig. 7. At this point the experimental data agree reasonably well with the theory.

It must be noted that in the numerical simulations above, the (largest) values of ε are not small compared with the minimal value 0.07508 of Q, so that the validity of the first order WKB-approximation may be questioned here. However, there is a practical reason (a limited computing time) that makes it almost impossible to obtain simulation results for smaller values of ε.

The author thanks J. Grasman, J. B. T. M. Roerdink and H. E. de Swart for remarks on the manuscript and/or discussions. The idea to formulate the conditions of the ray equations as boundary conditions originated from R. M. M. Mattheij.

APPENDIX

Numerical simulation of stochastic birth–death processes. Numerical simulations have been carried out in order to check the results obtained by the theory. The simulations are discussed here for a two population model, the generalization to higher dimensions being straightforward.

Let a system of two populations be in the state (N_1, N_2) at time t. Then in the small time interval of length Δt succeeding t, one of the following five mutually exclusive events occurs:

(1) a birth in population 1 with probability $B_1(N_1, N_2)\Delta t$

(2) a death in population 1 with probability $D_1(N_1, N_2)\Delta t$

(3) a birth in population 2 with probability $B_2(N_1, N_2)\Delta t$

(4) a death in population 2 with probability $D_2(N_1, N_2)\Delta t$

(5) neither a birth nor a death in one of the populations with probability
$1-[B_1(N_1, N_2)+D_1(N_1, N_2)+B_2(N_1, N_2)+D_2(N_1, N_2)]\Delta t$. (A1)

In a more convenient form for numerical simulation, the process is described as follows. When the process has arrived in a state (N_1, N_2) there is a waiting time T_{N_1,N_2} in that state, followed by a jump away from that state, which then is with probability one to one of the states (N_1+1, N_2), (N_1-1, N_2), (N_1, N_2+1), (N_1, N_2-1). The waiting time T_{N_1,N_2} is distributed exponentially:

$$P(T_{N_1,N_2} \geq t) = e^{-[B_1(N_1,N_2)+D_1(N_1,N_2)+B_2(N_1,N_2)+D_2(N_1,N_2)]t} \tag{A2}$$

see for example Karlin and Taylor (1975/1981). Using the inverse method [Abramowitz and Stegun (1972, p. 950, 953)] we obtain as random deviates from this exponential distribution:

$$t = -\frac{1}{B_1+D_1+B_2+D_2}\ln U, \tag{A3}$$

in which U is a random number on the interval (0, 1). Note that t depends on N_1 and N_2.

After the generation of the waiting time, a jump has to be made to one of the four neighbouring states indicated above, with total probability equal to one. These jump probabilities:

$$\frac{B_1}{S}, \frac{D_1}{S}, \frac{B_2}{S}, \frac{D_2}{S} \tag{A4}$$

are obtained from the old probabilities by scaling with S:

$$S = B_1 + D_1 + B_2 + D_2. \tag{A5}$$

The jump that is actually carried out is determined by a random number generator. To this end the interval (0, 1) is divided into four disjunct subintervals, each of which correponds to one of the jumps, the length of the interval being equal to the probability of the jump. A number is randomly chosen from the interval (0, 1) and the jump corresponding to the interval in which the random number lies is carried out.

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NUMERICAL CONSTRUCTION OF RAYS AND CONFIDENCE CONTOURS IN STOCHASTIC POPULATION DYNAMICS

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We consider a stochastic system with small stochastic fluctuations, where the associated deterministic system has a stable equilibrium point. The stationary Fokker-Planck (or forward Kolmogorov) equation can be solved by the WKB-method, leading to a system of ray equations. This technical note deals with the numerical solution of the ray equations. The methods which are described here have been applied to stochastic birth-death models [1]. This technical note is a supplement to paper [1].

1. Introduction

We consider the n-dimensional stochastic system which is described by the stationary Fokker-Planck equation:

$$0 = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}[b_i(x)P_s(x)] + \frac{\epsilon}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i x_j}[a_{ij}(x)P_s(x)], \qquad (1.1)$$

in which b is the deterministic vector field, a the symmetric positive definite diffusion matrix, and ϵ a small parameter. It is assumed that in a bounded region R of the state space x, the deterministic system

$$\frac{dx}{dt} = b(x) \qquad (1.2)$$

has a single stable equilibrium point. Without loss of generality this equilibrium is positioned at the origin. We assume that at the boundary S of the region R the deterministic vector field is directed inwards to R. The asymptotic theory for small ϵ yields expressions for the most probable exit point $x^* \in S$ from the region R, and the expected time of exit from R [2].

To the solution of (1.1) a simple WKB-Ansatz [3] is made

$$P_s(x) = C \exp[-Q(x)/\epsilon]. \tag{1.3}$$

Substitution in (1.1) leads to leading order in ϵ to the eikonal equation:

$$0 = \sum_i b_i(x) \frac{\partial Q}{\partial x_i} + \sum_{i,j} \frac{1}{2} a_{ij}(x) \frac{\partial Q}{\partial x_i} \frac{\partial Q}{\partial x_j}. \tag{1.4}$$

This is a Hamilton-Jacobi equation and can be written as $H(x,p) = 0$, in which H is the Hamiltonian

$$H(x,p) = \sum_i b_i p_i + \sum_{i,j} \tfrac{1}{2} a_{ij} p_i p_j, \tag{1.5}$$

with $p_i = \partial Q / \partial x_i$. The corresponding system of ordinary differential equations is:

$$\frac{dx_i}{ds} = \frac{\partial H}{\partial p_i} = b_i + \sum_j a_{ij} p_j, \quad (i = 1, 2, \ldots n) \tag{1.6a}$$

$$\frac{dp_i}{ds} = -\frac{\partial H}{\partial x_i} = -\sum_j \frac{\partial b_j}{\partial x_i} p_j - \sum_{j,k} \tfrac{1}{2} \frac{\partial a_{jk}}{\partial x_i} p_j p_k, \quad (i = 1, 2, \ldots n) \tag{1.6b}$$

$$\frac{dQ}{ds} = -H(x,p) + \sum_i \frac{dx_i}{ds} p_i = \sum_{i,j} \tfrac{1}{2} a_{ij} p_i p_j, \tag{1.6c}$$

in which s is a parameter along the characteristics. The projection of a characteristic on the x-space is called a ray. The system (1.6) is called the system of ray equations. It can be considered as a dynamical system in which Q and the elements of x and p are the state variables. The variable Q is passive in the sense that it depends on the other state variables, while the other state variables do not depend on Q. If desired, Q could be omitted from the system and be computed afterwards from the values of x and p along a characteristic. The dynamical system (1.6) is assumed to have the equilibrium point

$$x = p = 0, \quad Q = 0. \tag{1.7}$$

The projection of this equilibrium point on the x-space coincides with the stable equilibrium point of the deterministic system. The value of Q is zero at the equilibrium point. All characteristics start in a close neighbourhood of the equilibrium point. As is seen from equation (1.6c), the value of Q along a characteristic is nondecreasing. The function Q is useful for the construction of confidence contours. The confidence contour of probability z $(0 \leq z \leq 1)$ encloses the smallest region in the x-space where with probability z the system can be found. By assumption (1.3) the confidence contours are $(n-1)$-dimensional surfaces in the x-space on which Q has a constant value. The function Q is also useful in finding the most probable exit point and the expected time of exit from R. The asymptotic theory in [2] shows that the point x^* on S which has the lowest Q-value is the most probable point of exit, and that the expected exit time is expressed in the value of Q at x^* by the relation $T \sim \exp[Q(x^*)/\epsilon]$. Various ways of numerically solving the system of ray equations are discussed below.

2. The initial value approach

The initial point $x(s = 0)$ of a ray is chosen close to the equilibrium (1.7). A local analysis in the neighbourhood of this equilibrium yields values for $p(s = 0)$ and $Q(s = 0)$. In this local analysis, Q is approximated by a quadratic form

$$Q(x) \approx \tfrac{1}{2} x^t P x, \tag{2.1}$$

in which t denotes the transpose and P is a symmetric matrix. It follows that

$$p = \frac{dQ}{dx} \approx Px. \tag{2.2}$$

The deterministic vector field b is approximated by its linearization at $x = 0$:

$$b \approx Bx, \tag{2.3}$$

in which $B = (\partial b_i/\partial x_j(0))$. Substitution of (2.2) and (2.3) in the eikonal equation gives:

$$PAP + PB + B^t P = 0, \tag{2.4}$$

in which the matrix A is given by $A = (a_{ij}(0))$. Left and right multiplication of (2.4) with $S = P^{-1}$ gives:

$$A + BS + SB^t = 0. \tag{2.5}$$

If the matrices S and A are written columnwise as vectors, a linear system with n^2 equations is obtained, which can be solved for S. The matrix P is obtained by inversion of S. All eigenvalues of B are negative because of the stability of the equilibrium of the deterministic system. Consequenty, the last two operations can be carried out. The elements of the matrix P can be substituted in expressions (2.2) and (2.3), resulting in approximations for $p(s=0)$ and $Q(s=0)$.

The ray equations are solved numerically by a routine for solving a system of ordinary differential equations in first order form, with initial conditions. Such routines can be found for example in the NAG-library [4]. The solution $x(s), p(s), Q(s)$ is obtained along the characteristic determined by the initial point $x(s=0)$. In cases that one is interested in the ray starting from a particular point where the values of p, x and Q are known (or can be approximated, as above), the initial value method performs well, and by using an appropriate numerical routine, the solution can be obtained with a high accuracy. There is no control however, over the way the ray develops through x-space. Generally, there is a strong dependence on the choice of the initial point. Thus, the method is not well suited for the construction of confidence contours. This has been demonstrated for a two-dimensional stochastic model in [1]. In some numerical integration routines there is the possibility of using a termination criterion based upon the value of one of the state variables. If it is desirable to terminate the computation along a characteristic at a specific value of Q, then the treatment of Q as a state variable is advantageous.

3. The boundary value method

In this approch $n+1$ conditions are imposed at the beginning of the ray and n conditions at the end of the ray, respectively:

$$s \to -\infty : \; Q = 0, x = 0, \qquad s = 0 : \; x = e, \tag{3.1}$$

where e is the position of the end point which can be chosen freely. In the numerical computations the limit $s \to -\infty$ is replaced by $s = -s^*$, with s^* a sufficiently large number. The problem is solved using the NAG-routine D02RAF, ment for a system of ordinary first order differential equations with boundary conditions. This routine uses a deferred correction technique and

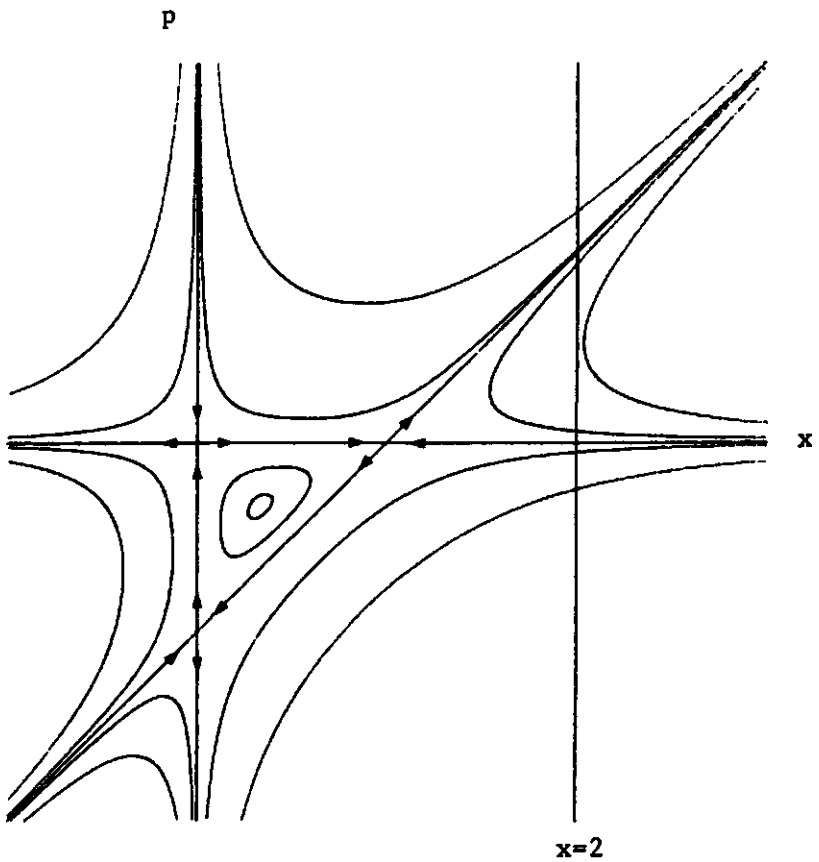

Figure 1. Some trajectories of the system (3.3) in the (x, p)-state space.

Newton iteration. On a grid of s-values, an initial estimate to the solution has to be given, from which the routine iteratively tries to find the solution. The initial estimate for the x-coordinates on the grid follows from a linear interpolation between the coordinates of the equilibrium point and the end point. The initial estimate for the corresponding values of p and Q is based on the local analysis formulas (see section 2). An alternative way to obtain initial estimates is to use known values along a characteristic close to the desired characteristic.

For $s \to -\infty$ the condition $x = 0$ has been imposed and not the condition $p = 0$. The motivation for doing this is explained by means of an example.

Example 1. We consider the one-dimensional stochastic system defined by

$$b(x) = x(1-x), \quad a(x) = 2x. \tag{3.2}$$

The deterministic system has an unstable equilibrium at $x = 0$ and a stable equilibrium at $x = 1$. A solution is sought in the form of the WKB-Ansatz, Q having a minimum at $x = 1$. The system of ray equations yields:

$$\frac{dx}{ds} = x(1-x) + 2xp, \tag{3.3a}$$

$$\frac{dp}{ds} = (2x-1)p - p^2, \tag{3.3b}$$

$$\frac{dQ}{ds} = xp^2. \tag{3.3c}$$

The system (3.3*ab*) has the following equilibrium points:

$$(x,p) = (1,0), (0,0), (0,-1), (\tfrac{1}{3}, -\tfrac{1}{3}). \tag{3.4}$$

Trajectories of the system (3.3*ab*) are depicted in Fig. 1. The local analysis in the neighbourhood of $(x,p) = (1,0)$ shows that the solution of this example is situated along the line $p = x - 1$. Linearization of this system at $(x,p) = (1,0)$ gives:

$$\begin{pmatrix} \frac{dx}{ds} \\ \frac{dp}{ds} \end{pmatrix} \approx \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x-1 \\ p \end{pmatrix}. \tag{3.5}$$

The matrix has the following eigenvalues and corresponding eigenvectors:

$$\lambda_1 = -1, \quad w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = 1, \quad w_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3.6}$$

So in a neighbourhood of the equilibrium $(x,p) = (1,0)$ there is a stable manifold corresponding to the first eigenvalue and eigenvector and an unstable manifold corresponding to the second eigenvalue and eigenvector. Let in the present example the end of the ray be specified by for example:

$$s = 0: \ x = 2. \tag{3.7}$$

From Fig. 1 it is apparent that by the condition

$$s \to -\infty: \ p = 0, \tag{3.8}$$

the solution is not uniquely determined. Apart from the desired solution along the characteristic $p = x - 1$, there is an infinite number of other solutions satisfying condition (3.7) as well as condition (3.8). In contrast, the condition

$$s \rightarrow -\infty : \quad x = 1, \tag{3.9}$$

together with condition (3.7) determine the solution uniquely. From this example we see that the condition for $s \rightarrow -\infty$ must not be such that it determines a surface in the (x,p)-space which coincides with the stable manifold at the equilibrium point.

Remark. Apart from this, Fig.1 exhibits an interesting feature. The region enclosed by the trajectories connecting the equilibrium points $(x,p) = (1,0),(0,-1)$,
$(0,0)$ contains periodic orbits. Along such an orbit s increases and by (3.3c), Q increases as well. After every rotation the value of Q has increased. In this region, Q is a multi-valued function of x. In the region $0 < p < x - 1$, $Q(x)$ is two-valued. The solution we are interested in is situated along $p = x - 1$ (which is a straight line; generally in one-dimensional stochastic systems we have a curve). Suppose we try to find the solution by the initial value method of section 2. Small errors occur in the initial values of the ray variables and are introduced by the numerical integration of the ray equations. As a consequence, the line $p = x - 1$ is not followed exactly. In the limit $s \rightarrow \infty$, instead of approaching the point $(x,p) = (0,-1)$ asymptotically, a trajectory close to $p = x - 1$ is followed, which in the neighbourhood of (0,-1) curves upward or downward along the p-axis. When this happens, the computation can be terminated because the results have lost significance. This demonstrates the difficulty involved in numerically constructing the characteristic that connects two equilibria. Moreover, we see that a picture of the state space can be helpful in understanding the behaviour of numerical methods.

Let us return to the original system (1.6). Linearization of the equations (1.6ab) at the equilibrium $(x,p) = (0,0)$ yields:

$$\begin{pmatrix} \frac{dx}{ds} \\ \frac{dp}{ds} \end{pmatrix} = \begin{pmatrix} B & A \\ 0 & -B^t \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}, \tag{3.10}$$

in which the matrices A and B consist of the values of (a_{ij}) and $(\partial b_i/\partial x_j)$ at the equilibrium $(x,p) = (0,0)$. Let the eigenvalues of the linearized determin-

istic system at $x = 0$ be given by λ_i, $(i = 1, 2, \ldots, n)$ and the corresponding eigenvectors by w_i. By the assumption of stability of the deterministic equilibrium, the real parts of the eigenvalues are negative. The eigenvalues of the matrix in (3.10) are given by $\pm\lambda_i$ $(i = 1, 2, \ldots, n)$. There is a stable manifold at $(x, p) = (0, 0)$ of dimension n formed by the eigenvectores corresponding to the eigenvalues with negative real part, and an unstable manifold of dimension n formed by the eigenvectors corresponding to the eigenvalues with positive real part. The eigenvectors corresponding to the eigenvalues with negative real part are given by $(w_i^t, 0^t)^t$. When the conditions for $s \to -\infty$ are chosen as:

$$s \to -\infty : \; x = 0 \text{ (and of course, } Q = 0), \tag{3.11}$$

then in the (x, p)-space the surface $(0^t, p^t)^t$ is introduced which is perpendicular to the stable manifold at $(x, p) = (0, 0)$. Thus, by this choice of conditions, the kind of non-uniqueness as discussed in example 1 does not occur.

Construction of a single ray. The following discussion is based upon the experience with the two-dimensional population models described in [1]. Other applications may require slight adaptations. The ray to be constructed is determined by the boundary conditions (3.1). The NAG-routine D02RAF has been used without the continuation facility. The grid consisted of 64 equidistant (in the parameter s) points. Experience has shown that the performance of the boundary value method depends critically on a sufficient number of grid points. The magnitude of s^* is less critical. The parameter *tol*, which expresses the maximum absolute deviation of the computed value from the true value in each component of the solution, was set equal to 0.01. The Jacobians which have to be computed were based on the exact analytical expressions. However, at an early stage of programming, it may be convenient to use the facility by which the Jacobians are approximated numerically. The initial estimate for the x-coordinates on the grid was based on a linear interpolation between the coordinates of the equilibrium point and the end point. The initial Q- and p-values were derived by the local analysis.

Only when the end point is close to the equilibrium point the solution is obtained. If not, the initial estimate is not accurate enough and the routine fails. Rays with end points far from the equilibrium point are constructed successfully by one of the following methods

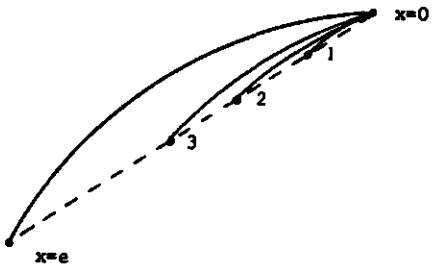

Figure 2. Construction of a ray (growing ray method).

(1) *Growing ray method.* A line is drawn from the equilibrium point to the end point. At point 1 on this line close to the equilibrium, the solution is obtained by applying the routine. The solution for the ray to point 1 is used as initial estimate for the ray to point 2 on the line. The solution for the ray to point 2 is used as initial estimate for the ray to point 3, etc.. This procedure is repeated until the point $x = e$ has been reached. See Fig. 2.

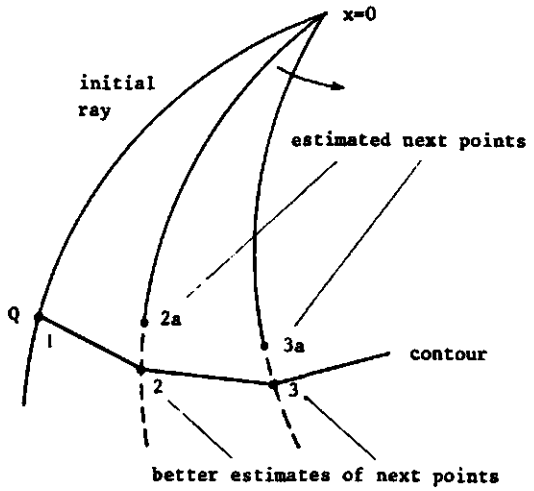

Figure 3. Construction of a confidence contour.

(2) *Neighbouring ray method.* An alternative method is to construct a ray passing close to $x = e$ by the initial value method and using the values along this ray as initial estimate to the desired ray through $x = e$.

Construction of a confidence contour. This is treated most easily for $n = 2$. The generalization to arbitrary n is straightforward. A single ray is constructed, for example by the initial value method. The precise course of the ray is immaterial, but the ray must contain the Q-value corresponding to the desired contour. The point on the ray at which this value is attained is the first point of the contour. Moreover, the values of $p_1 = \partial Q/\partial x_1$ and $p_2 = \partial Q/\partial x_2$ are known at this point. An arbitrary but small distance d is specified. Based on the values of p_1 and p_2 an estimate of the second point $2a$ of the contour can be made, lying a distance d away from the first point of the contour. The ray with end point $2a$ is then computed. The known values along the first ray are used as initial estimate to the solution corresponding to the second ray. From the solution of the second ray a better estimate can be made of the position of the second point of the contour, which will be close to the point $2a$. By connecting this point with the first point we have obtained a small segment of the contour. The procedure is repeated for obtaining the following rays, see Fig.3. A mechanism must be built in to assure that a clockwise (or anti-clockwise) direction is followed. The procedure terminates after a complete rotation. A result of this procedure for a 2-dimensional stochastic population model as treated in [1] is shown in Fig. 4.

The exit problem. The determination of the most probable exit point x^* and the value of Q at this point can be carried out in several ways. In a variant of the boundary value method for confidence contours, rays are constructed with end points a small distance away from each other on the exit boundary. The solution along a ray serves as initial estimate for the solution along the next ray. The point on the boundary where the lowest value of Q is found is an estimate for x^* and the corresponding value of Q is an estimate for $Q(x^*)$.

Another approach has been followed in [1]. For a two-dimensional system in which the confidence contours are convex curves, exit was studied at the boundaries $x_1 = 0.1$ and $x_2 = 0.1$. Exit at $x_2 = 0.1$ is treated as follows. To obtain the point at $x_2 = 0.1$ which has the lowest Q-value, the boundary conditions are formulated as:

$$\begin{aligned} &s \to -\infty : \; Q = 0, \; x_1 = x_2 = 1 \quad \text{(the equilibrium)}, \\ &s = 0 : \; x_2 = 0.1, \; p_1 = 0. \end{aligned} \tag{3.28}$$

The condition $p_1 = \partial Q/\partial x_1 = 0$ indicates that we are looking for an extremum of Q as a function of x_1. The initial estimate to the solution is a ray constructed

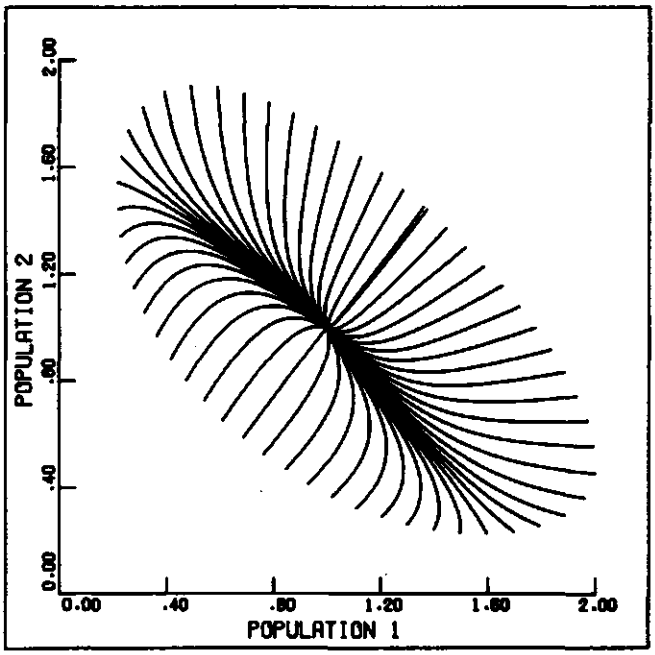

Figure 4. The rays used in the construction of a confidence contour. The confidence contour is obtained by connecting the end points of the rays.

by the initial value method which is believed to be close to the desired ray. Alternatively, the problem is solved first with $x_2 = 0.9$ instead of 0.1. The solution of this problem is used as initial estimate to the solution of the problem with $x_2 = 0.8$, etc. The procedure is repeated until the solution at $x_2 = 0.1$ has been obtained. The boundary $x_1 = 0.1$ is treated in the same way. The boundary which contains the lowest value of Q is the most probable exit boundary, and the point on this boundary where the minimum value is attained is the most probable exit point.

4. The shooting method

Shooting methods, in which the initial point of a characteristic is manipulated systematically in order to obtain the desired characteristic, have been used by Ludwig [3] to find the most probable exit point in a population model and by De Swart and Grasman [5] to find the characteristic connecting equilib-

rium points in a stochastic model in meteorology. The use of shooting methods is rather elaborate because of the sensitivity of the solution to the choice of the initial point of the characteristic. Results for these problems can be obtained easily by means of a boundary value method.

5. Intersecting rays

Rays starting in a neighbourhood of the equilibrium of a system of ray equations, with initial values chosen in accordance with the local analysis, may intersect. This leads to non-uniqueness of $Q(x)$. In this section we confine ourselves to some remarks on this phenomenon and give an example.

In the one-dimensional stochastic system of example 1, the line introduced by the local analysis is tangent to the unstable manifold of the system of ray equations (1.6ab) at the equilibrium point (in this example, both are coincident in the whole (x,p)-space). More general we have: the plane $p = Px$ in the (x,p)-space with P satisfying equation (2.4) is tangent to the unstable manifold of the system of ray equations (1.6ab) at the equilibrium $x = p = 0$.

In the following it is shown that the linearized unstable manifold at $x = p = 0$ satisfies equation (2.4). Let the eigenvalues of B be given by λ_i $(i = 1,2,\ldots,n)$. By the assumption that the deterministic equilibrium is stable, the real parts of the λ_i are negative. The eigenvalues of the system (3.10), which is the linearization of the system (1.6ab) at $x = p = 0$, are given by the λ_i with the corresponding eigenvectors $(w_i^t, 0^t)^t$ and by the $-\lambda_i$ with the corresponding eigenvectors $(v_i^t, z_i^t)^t$, $i = 1,2,\ldots,n$. Let $V = (v_1, v_2, \ldots, v_n)$, $Z = (z_1, z_2, \ldots, z_n)$ and let L be the diagonal matrix with elements $\lambda_1, \lambda_2, \ldots, \lambda_n$. With respect to the eigenvalues $-\lambda_i$ corresponding to the unstable manifold we have by the definition of eigenvalues and eigenvectors:

$$\begin{aligned} BV + AZ &= -VL, \\ -B^t Z &= -ZL. \end{aligned} \tag{5.1}$$

The unstable manifold is given by

$$\begin{pmatrix} x \\ p \end{pmatrix} = \sum_{i=1}^{n} \alpha_i \begin{pmatrix} v_i \\ z_i \end{pmatrix}, \tag{5.2}$$

in which the parameters α_i and the elements of v_i and z_i generally are complex numbers. The last n equations in (5.2) are used to eliminate the α_i:

$$\alpha = Z^{-1} p, \tag{5.3}$$

in which α is the vector $(\alpha_1, \alpha_2, \ldots, \alpha_n)^t$. Substitution in the first n equations in (5.2) gives:

$$x = VZ^{-1}p, \tag{5.4}$$

and, by inversion,

$$p = ZV^{-1}x. \tag{5.5}$$

This is the equation for the unstable manifold of the system (1.6*ab*) at the equilibrium point. With the equations (5.1) it is easily shown that ZV^{-1} satisfies the equation (2.4) for P. So ZV^{-1} equals P. It is assumed that the inverse matrices above exist.

In Example 1 the unstable manifold is $p = x - 1$, which is a single-valued function of x. For an n-dimensional stochastic system the characteristics, with initial values according to the local analysis, lie on an n-dimensional hypersurface through the equilibrium point in the $2n$-dimensional (x, p)-space. On this hypersurface the characteristics do not intersect. Results in [3,6] exhibit non-uniqueness in $Q(x)$. Consequently, we conclude that this hypersurface may be folded so that the projections of the characteristics on the x-space (the rays) do intersect. Regions in the x-space where Q is non-unique are bounded by caustics. Caustic points can be detected numerically at the cost of a large amount of extra computation by keeping up the value of a Jacobian along the rays [3,6]. The occurrence of caustics is also demonstrated in the following example.

Example 2. We consider the system with $n = 2$:

$$\begin{aligned} &b_1(x) = x_1(1 - x_1) + x_2, \quad b_2(x) = -\beta x_2, \quad (\beta > 0, \beta \neq 1) \\ &a_{11} = a_{22} = 1, \quad a_{12} = a_{21} = 0. \end{aligned} \tag{5.6}$$

The deterministic system has the equilibrium points $x_1 = 0, x_2 = 0$ and $x_1 = 1, x_2 = 0$. The latter equilibrium is stable. A WKB-Ansatz leads to the ray equations

$$\frac{dx_1}{ds} = x_1(1 - x_1) + x_2 + p_1, \quad \frac{dx_2}{ds} = -\beta x_2 + p_2, \tag{5.7a}$$

$$\frac{dp_1}{ds} = (2x_1 - 1)p_1, \quad \frac{dp_2}{ds} = -p_1 + \beta p_2, \tag{5.7b}$$

$$\frac{dQ}{ds} = (p_1^2 + p_2^2)/2. \tag{5.7c}$$

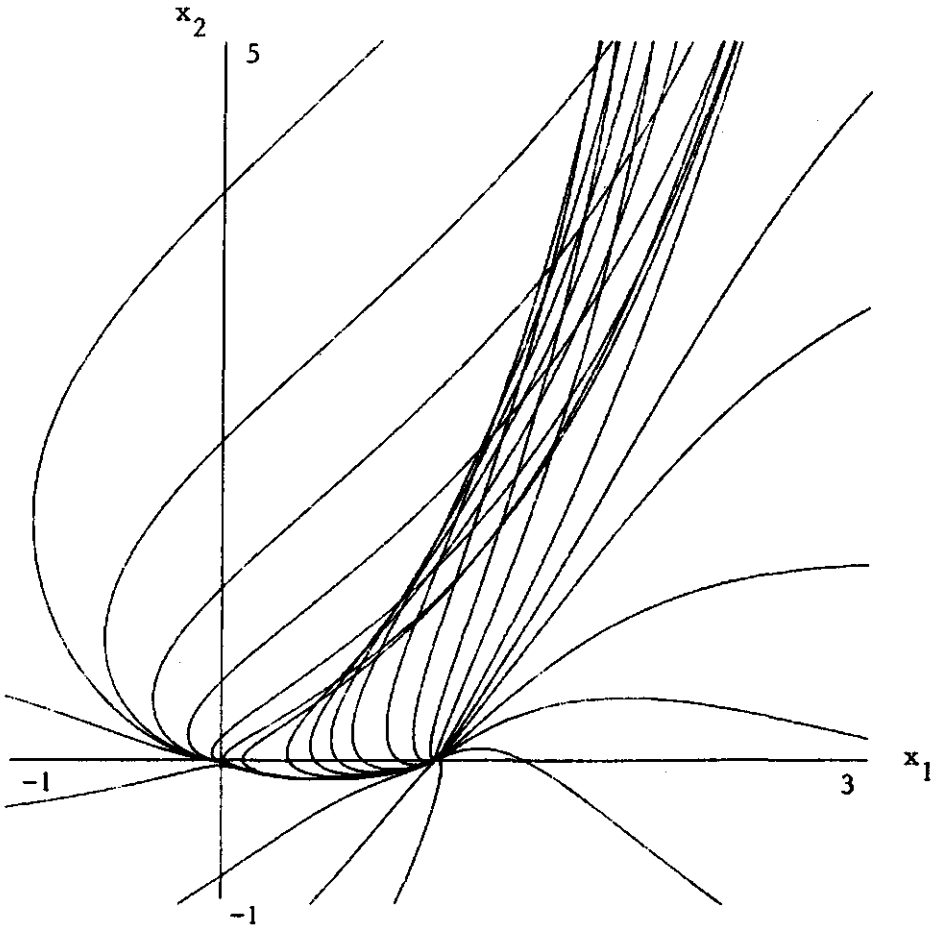

Figure 5. A cusp singularity arising in the solution of system (5.7) with $\beta = 3/2$.

The system (5.7*ab*) has the equilibrium points

$$\begin{aligned} &\left(\frac{1}{2}, -1/4(1+\beta^2), -\beta^2/4(1+\beta^2), -\beta/4(1+\beta^2)\right)^t, \\ &(0,0,0,0)^t, \\ &(1,0,0,0)^t. \end{aligned} \tag{5.8}$$

The matrix P, following from the local analysis is given by

$$P = \frac{2(1+\beta)}{\beta^2+2\beta+2}\begin{pmatrix} 1+\beta & -1 \\ -1 & \beta^2+\beta+1 \end{pmatrix}. \tag{5.9}$$

The system (5.7) has been solved numerically by the initial value method, in which the initial points were chosen on a circle with radius 0.01 around $x_1 = 1, x_2 = 0$ and the corresponding valuse of p_1, p_2 and Q were chosen according to the local analysis. Fig. 5 shows rays for $\beta = 3/2$. A number of rays were chosen very close to the ray connecting $x_1 = 1, x_2 = 0$ with $x_1 = 0, x_2 = 0$. In the neighbourhood of $x_1 = 0, x_2 = 0$ part of those rays turn away to the right and intersect other rays. The figure shows the projection of a cusped manifold. In the region in the x-space between the caustics (the projection of the cusp edges) $Q(x)$ is a 3-valued function.

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AN ASYMPTOTIC SOLUTION TO A TWO-DIMENSIONAL EXIT PROBLEM ARISING IN POPULATION DYNAMICS

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A study is made of a two-dimensional stochastic system with small stochastic fluctuations, that arises in population biology. At the boundary of the state space the diffusion matrix becomes singular. By an asymptotic analysis, expressions are derived that determine the probability of exit at each of the two boundaries and the expectation and variance of the exit time. These expressions contain constants that can be computed numerically.

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1. Introduction

Consider a two-dimensional stochastic system that has a stable deterministic equilibrium and in which the stochastic fluctuations are small. Various systems of this type have been studied in literature, see Matkowsky and Schuss [9,10], Matkowsky, Schuss and Tier [11], Hanson and Tier [7], Wazwaz and Hanson [14,15]. With respect to the behaviour of the deterministic system at the boundary of the region under consideration, different cases can be distinguished: the deterministic vector field enters the region [9], or it is tangent to

the boundary of the region. In the latter case, there may be no deterministic critical points on the boundary [10], or there may be [11]. It is assumed in [9,10,11] that the diffusion tensor is nonsingular. The asymptotic theories for small stochastic fluctuations lead to expressions for the exit distribution and the (lowest) statistical moments of the exit time. An asymptotic analysis of a one-dimensional stochastic system in which the diffusion coefficient becomes singular at the boundary is given in [7,14,15]. In this system, both the drift and the diffusion coefficients vanish, linearly with the distance to the boundary.

The two-dimensional stochastic system treated in this paper arises in population dynamics [8,12]. The diffusion matrix is diagonal and becomes singular at the boundary. There, the normal components of both the drift and the diffusion vanish linearly with the distance to the boundary. This system differs from the system treated in [11] in that the diffusion tensor becomes singular at the boundary, and from the system in [7,14,15] in the dimension. Extending the methods presented in [7,11], asymptotic expressions are derived for the probabilities of exit at the two boundaries as well as the expectation and variance of the exit time.

Section 2 describes the stochastic model and formulates the boundary value problems with respect to exit boundary and exit time. In section 3 we find asymptotic expressions for the probability of exit at each of the boundaries, valid uniformly outside an asymptotic small neighbourhood of the origin. In section 4, a derivation largely analogous to that in section 3 leads to asymptotic expressions for the expectation and variance of the exit time that are uniformly valid. Section 5 is concerned with the numerical determination of constants that appear in the formulas obtained in sections 3 and 4. As an example, section 6 treats a predator-prey system.

2. The stochastic model and the boundary value problems

We consider the two-dimensional stochastic system with small stochastic fluctuations, described by the following forward Fokker-Planck (or forward Kolmogorov) equation

$$\frac{\partial v(x,t)}{\partial t} = M_\epsilon v \equiv \sum_{i=1}^{2} \left[-\frac{\partial}{\partial x_i}[b_i(x)v(x,t)] + \frac{\epsilon}{2}\frac{\partial^2}{\partial x_i^2}[a_i(x)v(x,t)] \right], \qquad (2.1)$$

in which v is the probability density function and $0 < \epsilon \ll 1$ is a small parame-

ter. The variables x_1 and x_2 denote the densities of two biological populations. The state space consists of the region $\bar{R}$:

$$R = \{(x_1, x_2) | x_1, x_2 \text{ real and } > 0\}. \tag{2.2}$$

The diffusion matrix is diagonal with elements

$$\begin{aligned} a_1(x) &= x_1(a_{10} + a_{11}x_1 + a_{12}x_2), \\ a_2(x) &= x_2(a_{20} + a_{21}x_1 + a_{22}x_2), \end{aligned} \tag{2.3}$$

in which the a_{ij} are positive real numbers. This diffusion matrix is singular at $x_1 = 0$ and $x_2 = 0$. The drift vector is of the generalized Lotka-Volterra form

$$\begin{aligned} b_1(x) &= x_1(b_{10} + b_{11}x_1 + b_{12}x_2), \\ b_2(x) &= x_2(b_{20} + b_{21}x_1 + b_{22}x_2), \end{aligned} \tag{2.4}$$

where the b_{ij} are real numbers that are restricted by assumptions made below. Thus, $x_1 = 0$ and $x_2 = 0$ are characteristic boundaries. The deterministic system

$$\begin{aligned} \frac{dx_1}{dt} &= b_1(x), \\ \frac{dx_2}{dt} &= b_2(x), \end{aligned} \tag{2.5}$$

associated with the stochastic system (2.1) has the equilibria

$$(0, 0), \tag{2.6a}$$

$$(0, -b_{20}/b_{22}), \tag{2.6b}$$

$$(-b_{10}/b_{11}, 0), \tag{2.6c}$$

$$x^e = (x_1^e, x_2^e) \equiv \left(\frac{b_{22}b_{10} - b_{12}b_{20}}{b_{21}b_{12} - b_{11}b_{22}}, \frac{b_{11}b_{20} - b_{21}b_{10}}{b_{21}b_{12} - b_{11}b_{22}} \right). \tag{2.6d}$$

By assumption the critical points $(2.6b, c)$ lie on the positive x_2-axis, x_1-axis respectively, with order $O(1)$ distance from the origin

$$-b_{20}/b_{22} > 0, \quad -b_{20}/b_{22} = O(1), \quad -b_{10}/b_{11} > 0, \quad -b_{10}/b_{11} = O(1), \tag{assumption 1}$$

and are attracting along the x_2-axis and x_1-axis respectively:

$$b_{20} > 0, \quad b_{10} > 0. \tag{assumption 2}$$

The deterministic system has an equilibrium in R with coordinates of order $O(1)$:

$$x_1^e > 0, \quad x_1^e = O(1), \quad x_2^e > 0, \quad x_2^e = O(1). \qquad \text{(assumption 3)}$$

The following assumption is made with respect to the stability of the deterministic system at x^e. In the neighbourhood of x^e we have by linearization of the deterministic vector field:

$$b \equiv (b_1(x), b_2(x))^t \approx B(x - x^e), \tag{2.7}$$

where the matrix B is given by

$$B = (B_{ij}) \equiv \left(\frac{\partial b_i}{\partial x_j}(x^e)\right) = (b_{ij} x_i^e). \tag{2.8}$$

The eigenvalues of B are

$$\lambda_{1,2} = \frac{1}{2}\left[b_{11}x_1^e + b_{22}x_2^e \pm \sqrt{(b_{11}x_1^e + b_{22}x_2^e)^2 - 4(b_{11}b_{22} - b_{12}b_{21})x_1^e x_2^e}\right]. \tag{2.9}$$

The condition for stability of the deterministic system at x^e is that the real parts of λ_1 and λ_2 are negative. With the use of the assumptions (1-3) this condition results in

$$b_{11}b_{22} > b_{12}b_{21}. \qquad \text{(assumption 4)}$$

By the assumptions (1-4) the equilibria (2.6b,c) are saddle points. The equilibrium (2.6a) is an unstable node.

At the boundary $x_i = 0$ we have

$$J_i(x,t) \equiv b_i(x)v(x,t) - \frac{\epsilon}{2}\frac{\partial}{\partial x_i}[a_i(x)v(x,t)] < 0, \tag{2.10a}$$

$$b_i(x) = 0, \quad a_i(x) = 0, \tag{2.10b}$$

for $i = 1, 2$. By (2.10a) the probability current J_i at $x_i = 0$ is negative, which indicates that the boundary $x_i = 0$ can be reached from R. Once $x_i = 0$ has been reached, by (2.10b) it cannot be left. Thus, $x_1 = 0$ and $x_2 = 0$ are exit boundaries.

Starting away from $x_1 = 0$ and $x_2 = 0$, the stochastic system described above will likely remain in the neighbourhood of the stable equilibrium x^e of the deterministic system for a long time. With small probabilities large excursions

from x^e occur. In such an excursion the system may exit at $x_1 = 0$ or $x_2 = 0$. This will happen within a finite time with probability one.

The boundary value problems describing exit are commonly defined on a bounded region. However, for the asymptotic analysis held in this paper, the use of the unbounded region R and the boundary ∂R defined by

$$\partial R \equiv \bar{R} \setminus R = \{(x_1, x_2) | x_1 x_2 = 0 \text{ and } x_1 + x_2 \geq 0\} \tag{2.11}$$

will not lead to any difficulty. This is confirmed by results for the analogous one-dimensional exit model, which can be computed explicitly.

In order to determine the probabilities of exit at $x_1 = 0$ and $x_2 = 0$, a study is made of the stationary backward Fokker-Planck (or backward Kolmogorov) equation

$$0 = L_\epsilon u \equiv \sum_{i=1}^{2} \left[b_i(x) \frac{\partial u}{\partial x_i} + \frac{\epsilon}{2} a_i(x) \frac{\partial^2 u}{\partial x_i^2} \right] \quad \text{in } R, \tag{2.12a}$$

with the boundary condition

$$u = f(x) \quad \text{on } \partial R, \tag{2.12b}$$

in which

$$u(x) = \int_{\partial R} f(x') P(x, x') \, dS_{x'}, \tag{2.12c}$$

where $P(x, x')$ is the probability of exit at $x' \in \partial R$, starting from $x \in R$. With the definition

$$f(x) = \begin{cases} 1, & \text{for } x_i = 0, \\ 0, & \text{else}, \end{cases} \tag{2.13}$$

the function $u(x)$ is the probability of exit at the particular boundary $x_i = 0$, starting from $x \in R$. In this paper only boundary conditions of the form

$$f(x) = \begin{cases} C_{b1}, & x_1 = 0, \\ C_{b2}, & x_2 = 0, \end{cases} \tag{2.14}$$

are considered with C_{b1}, C_{b2} constants that are equal to either zero or one.

Another point of interest is the determination of the expectation $\mathrm{E}\,T(x)$ and variance $\mathrm{Var}\,T(x)$ of the exit time $T(x)$, starting from $x \in R$. By

$$\mathrm{E}\,T(x) = T_1, \quad \mathrm{Var}\,T(x) = T_2 - T_1^2, \tag{2.15}$$

the expectation and variance of T are expressed in the moments

$$T_i(x) = < T^i > \tag{2.16}$$

of T, which satisfy the equations

$$L_\epsilon T_i = g_i(x) \quad \text{in } R, \tag{2.17a}$$

and conditions

$$T_i = 0 \quad \text{on } \partial R, \tag{2.17b}$$

for $i = 1, 2$ with

$$g_1(x) \equiv -1, \quad g_2(x) \equiv -2T_1(x). \tag{2.17c}$$

Equation (2.17a) with $i = 1$ is the Dynkin equation. Although higher moments can be determined as well, the analysis of the exit time in this paper is restricted to its expectation and variance. For a derivation of the boundary value problems (2.12, 2.17), the reader is referred to [4,13].

In biological terms, exits means extinction of a species. The expected exit time is a measure for the stochastic persistence of the ecosystem, see Ludwig [8]. The type of interaction between the two populations is mutualism for $b_{12} > 0$, $b_{21} > 0$, competition for $b_{12} < 0$, $b_{21} < 0$, and predation-prey in the other cases.

The motivation for the present study is the following one. In [12] we analyzed the exit problem for the system of populations described above by the method of Ludwig [8]. In that approach, the boundaries $x_1 = 0$ and $x_2 = 0$ of R were replaced by the boundaries $x_i = l_i$, where the l_i were small positive numbers, $i = 1, 2$. Thus the region R was restricted to the smaller region R_l. Whereas the deterministic flow was tangent to the boundaries $x_i = 0$ of R, it was directed inward to R_l at the boundaries $x_i = l_i$ of R_l. Consequently, the asymptotics of Matkowsky and Schuss [9] could be applied to the problem of exit from R_l, as an approximation to the problem of exit from R. In the present paper we adopt a different point of view. The behaviour of the stochastic system near the boundaries $x_1 = 0$ and $x_2 = 0$ will be treated by a variant of [7,11]. In this approach we may take the limits $l_i \to 0$, where this was not allowed in the previous study [12]. Thus we obtain expressions to the problem of exit from R without the need to approximate R by a rather arbitrary smaller region R_l.

In [5] Gillespie treated an exit problem related to a multidimensional singular diffusion arising in genetics. His study differs in various aspects from the present one. In our model the deterministic system is more complicated since the behaviour near the boundaries $x_1 = 0$ and $x_2 = 0$ is dominated by the critical points $(0, -b_{20}/b_{22})$ and $(-b_{10}/b_{11}, 0)$, where no such critical points

are present in [5]. Moreover, for our model no explicit solution of the stationary forward equation is available and no zero probability flux condition holds. These facts give rise to a substantially more elaborate study.

3. The exit boundary

In this section the exit problem (2.12) with f as in (2.14) is solved asymptotically for small ϵ. The solution contains an unknown constant. To obtain an expression for this constant we use an integral formula that results from the divergence theorem. In the integral formula, a formal solution of the forward equation adjoint to (2.12a) is needed. This adjoint equation is solved by the WKB-method, see Ludwig [8]. Near the boundaries $x_1 = 0$ and $x_2 = 0$, the solution of the adjoint equation is peaked at the critical points $(0, -b_{20}/b_{22})$ and $(-b_{10}/b_{11}, 0)$ respectively. Neighbourhoods of these critical points play an important role in the subsequent analysis. This approach is a variant of the method of Matkowsky, Schuss and Tier [11], that was indicated in Hanson and Tier [7].

3.1. The backward equation

An asymptotic analysis of the boundary value problem (2.12) reveals the existence of an outer solution, valid away from $x_1 = 0$ and $x_2 = 0$. Near these boundaries, an examination of different stretchings of the normal coordinate shows the presence of a boundary layer of width $O(\epsilon)$. Inside the boundary layers, the diffusion parallel to the boundary is negligible, except near critical points of the deterministic system. Thus, the following regions are distinguished

$$
\begin{aligned}
&\text{region } A: \quad x_1 = M_1\epsilon, \quad x_2 + b_{20}/b_{22} > M_2\sqrt{\epsilon},\\
&\text{region } B: \quad x_1 = M_3\epsilon, \quad |x_2 + b_{20}/b_{22}| = M_4\sqrt{\epsilon},\\
&\text{region } C: \quad x_1 = M_5\epsilon, \quad M_6\epsilon < x_2 < -b_{20}/b_{22} - M_7\sqrt{\epsilon},\\
&\text{region } D: \quad x_1 = M_8\epsilon, \quad x_2 = M_9\epsilon,\\
&\text{region } A': \quad x_1 + b_{10}/b_{11} > M_{10}\sqrt{\epsilon}, \quad x_2 = M_{11}\epsilon,\\
&\text{region } B': \quad |x_1 + b_{10}/b_{11}| = M_{12}\sqrt{\epsilon}, \quad x_2 = M_{13}\epsilon,\\
&\text{region } C': \quad M_{14}\epsilon < x_1 < -b_{10}/b_{11} - M_{15}\sqrt{\epsilon}, \quad x_2 = M_{16}\epsilon,
\end{aligned}
\tag{3.1}
$$

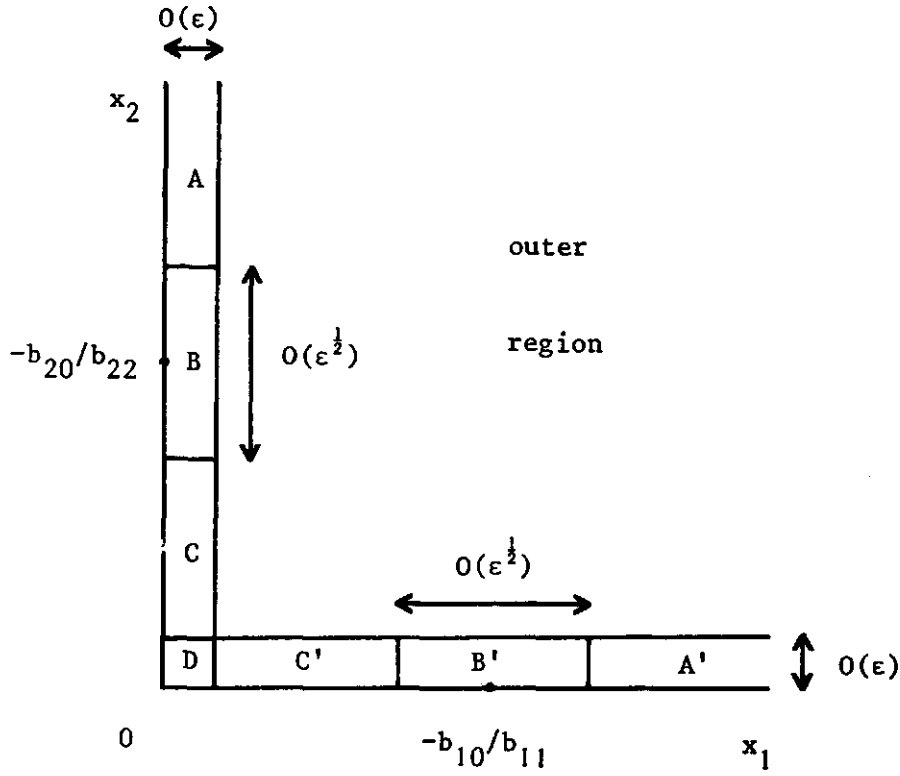

Figure 1. The outer region and the boundary layer regions. In the regions A, A', C, C' the diffusion parallel to the boundary is negligible, while this is not the case in the regions B, B' and D.

where the M_i are arbitrary positive numbers independent of ϵ, see Figure 1.

3.1.1. The outer solution

The reduced equation corresponding to (2.12a) reads

$$\sum_{i=1}^{2} b_i(x) \frac{\partial u}{\partial x_i} = 0, \tag{3.2}$$

which has the solution

$$u = C_b, \tag{3.3}$$

with C_b a constant with respect to x, which is yet undetermined. An expression for C_b will be found in subsection 3.3. The solution (3.3) is valid in R except near $x_1 = 0$ and $x_2 = 0$ because the boundary condition (2.12b, 2.14) cannot be satisfied.

3.1.2. The boundary layer solution in the regions B and B'

Near the critical point $(0, -b_{20}/b_{22})$ of the deterministic system, we introduce the stretched coordinates

$$\bar{x}_1 = x_1/\epsilon, \quad \bar{x}_2 = (x_2 + b_{20}/b_{22})/\sqrt{\epsilon}, \tag{3.4}$$

and the boundary layer function

$$U(\bar{x}_1, \bar{x}_2) = u(\epsilon\bar{x}_1, -b_{20}/b_{22} + \sqrt{\epsilon}\bar{x}_2). \tag{3.5}$$

Substitution into (2.12a) leads to the boundary layer equation

$$k_1\bar{x}_1\frac{\partial U}{\partial \bar{x}_1} + k_2\bar{x}_1\frac{\partial^2 U}{\partial \bar{x}_1^2} - k_3\bar{x}_2\frac{\partial U}{\partial \bar{x}_2} + k_4\frac{\partial^2 U}{\partial \bar{x}_2^2} = 0, \tag{3.6}$$

in which

$$\begin{aligned} k_1 &= b_{10} - b_{12}b_{20}/b_{22}, \\ k_2 &= \frac{1}{2}(a_{10} - a_{12}b_{20}/b_{22}), \\ k_3 &= b_{20}, \\ k_4 &= -\frac{1}{2}(a_{20} - a_{22}b_{20}/b_{22})b_{20}/b_{22}. \end{aligned} \tag{3.7}$$

From the assumptions (1-4) and the positivity of the a_{ij} it follows that the constants k_i are positive. By the separation of variables

$$U(\bar{x}_1, \bar{x}_2) = w(\bar{x}_1)z(\bar{x}_2), \tag{3.8}$$

equation (3.6) leads to the ordinary differential equations

$$k_2\bar{x}_1\frac{d^2 w}{d\bar{x}_1^2} + k_1\bar{x}_1\frac{dw}{d\bar{x}_1} - \lambda w = 0, \tag{3.9a}$$

$$k_4\frac{d^2 z}{d\bar{x}_2^2} - k_3\bar{x}_2\frac{dz}{d\bar{x}_2} + \lambda z = 0, \tag{3.9b}$$

in which λ is a separation constant. The general solution of (3.9a) is

$$w(\bar{x}_1) = \{c_1 W_{\lambda_1,\frac{1}{2}}(-\tilde{x}_1) + c_2 W_{-\lambda_1,\frac{1}{2}}(\tilde{x}_1)\} \exp[-\tilde{x}_1/2], \tag{3.10}$$

in which

$$\tilde{x}_1 = k_1 \bar{x}_1/k_2, \quad \lambda_1 = \lambda/k_1, \tag{3.11}$$

$W_{\lambda_1,\frac{1}{2}}$ and $W_{-\lambda_1,\frac{1}{2}}$ are Whittaker functions [6] and c_1, c_2 are arbitrary constants. The general solution of (3.9b) is

$$z(\bar{x}_2) = \{c_3 D_{\lambda_2}(\tilde{x}_2) + c_4 D_{-\lambda_2-1}(i\tilde{x}_2)\} \exp[\tilde{x}_2^2/4], \tag{3.12}$$

in which

$$\tilde{x}_2 = \sqrt{k_3/k_4}\, \bar{x}_2, \quad \lambda_2 = \lambda/k_3, \tag{3.13}$$

D_{λ_2} and $D_{-\lambda_2-1}$ are parabolic cylinder functions [3] and c_3, c_4 are arbitrary constants. At $\bar{x}_1 = 0$ we have the boundary condition

$$U(0, \bar{x}_2) = C_{b1}. \tag{3.14}$$

This condition can be satisfied only if $\lambda = 0$. The matching condition with the outer solution (3.3) is stated as

$$\lim_{x_1 \to \infty} U(\bar{x}_1, \bar{x}_2) = C_b. \tag{3.15}$$

The boundary layer solution satisfying both conditions (3.14) and (3.15) is given by

$$U(\bar{x}_1, \bar{x}_2) = C_b + (C_{b1} - C_b) \exp[-\bar{x}_1] \tag{3.16a}$$

or, in the original notation:

$$u(x_1, x_2) = C_b + (C_{b1} - C_b) \exp[-k_1 x_1/k_2 \epsilon]. \tag{3.16b}$$

The boundary layer region B' around the critical point $(-b_{10}/b_{11}, 0)$ yields a similar result:

$$u(x_1, x_2) = C_b + (C_{b2} - C_b) \exp[-k_1' x_2/k_2' \epsilon], \tag{3.17}$$

in which

$$k_1' = b_{20} - b_{21} b_{10}/b_{11}, \quad k_2' = \frac{1}{2}(a_{20} - a_{21} b_{10}/b_{11}). \tag{3.18}$$

3.1.3. The boundary layer solution in the regions A,A' and C,C'

Introduction of the stretched coordinate $\bar{x}_1$ in (3.4) and the boundary layer function

$$U(\bar{x}_1, x_2) = u(\epsilon \bar{x}_1, x_2) \tag{3.19}$$

into equation (2.12a) leads to the boundary layer equation

$$\bar{x}_1(b_{10} + b_{12}x_2)\frac{\partial U}{\partial \bar{x}_1} + x_2(b_{20} + b_{22}x_2)\frac{\partial U}{\partial x_2} + \bar{x}_1 \frac{1}{2}(a_{10} + a_{12}x_2)\frac{\partial^2 U}{\partial \bar{x}_1^2} = 0. \tag{3.20}$$

In order to make this equation separable, the variable $\bar{x}_1$ is replaced by the new variable

$$y = \bar{x}_1 \gamma(x_2), \tag{3.21}$$

with the function γ still to be determined. Equation (3.20) becomes

$$y\frac{\partial^2 W}{\partial y^2} + \Gamma(x_2) y \frac{\partial W}{\partial y} + \frac{x_2(b_{20} + b_{22}x_2)}{\frac{1}{2}(a_{10} + a_{12}x_2)\gamma}\frac{\partial W}{\partial x_2} = 0, \tag{3.22}$$

where

$$W(y, x_2) = U(\bar{x}_1, x_2), \tag{3.23}$$

and

$$\Gamma(x_2) \equiv \frac{1}{\frac{1}{2}(a_{10} + a_{12}x_2)\gamma}\left[b_{10} + b_{12}x_2 + x_2(b_{20} + b_{22}x_2)\gamma'/\gamma\right]. \tag{3.24}$$

The function γ is chosen such that

$$\Gamma(x_2) \equiv 1. \tag{3.25}$$

Then (3.24) is a Bernoulli equation. In terms of the reciprocal γ^{-1} it is a linear equation that can be solved by the method of variation of constants [1]. Since (3.24,3.25) is a first order differential equation there is one integration constant. This constant follows from a matching condition, see below. The partial differential equation (3.22) with (3.25) can be solved by separation of variables:

$$W(y, x_2) = w(y)z(x_2), \tag{3.26}$$

which leads to the ordinary differential equations

$$y\frac{d^2 w}{dy^2} + y\frac{dw}{dy} - \lambda w = 0, \tag{3.27a}$$

$$\frac{x_2(b_{20} + b_{22}x_2)}{\frac{1}{2}(a_{10} + a_{12}x_2)\gamma}\frac{dz}{dx_2} + \lambda z = 0, \tag{3.27b}$$

in which λ is a separation constant. To satisfy the matching conditions

$$U(0, x_2) = C_{b1}, \quad \lim_{\bar{x}_1 \to \infty} U(\bar{x}_1, x_2) = C_b, \tag{3.28}$$

λ must equal zero and the solution of (3.20),(3.28) is obtained as

$$U(\bar{x}_1, x_2) = C_b + (C_{b1} - C_b) \exp[-\gamma(x_2)\bar{x}_1], \tag{3.29a}$$

or, in the original notation:

$$u(x_1, x_2) = C_b + (C_{b1} - C_b) \exp[-\gamma(x_2)x_1/\epsilon]. \tag{3.29b}$$

The integration constant in the problem (3.24),(3.25) for γ is chosen such that (3.29) matches the solution (3.16), that is, by the condition

$$\lim_{x_2 \to -b_{20}/b_{22}} \gamma(x_2) = k_1/k_2. \tag{3.30}$$

For future purpuses we remark that at $x_2 = -b_{20}/b_{22}$ and $x_2 = 0$ the function γ^{-1} has the Taylor series expansions

$$\gamma^{-1}(x_2) = \frac{k_2}{k_1} + \frac{a_{12}b_{10} - a_{10}b_{12}}{2k_1(b_{20} + k_1)} \left(x_2 + \frac{b_{20}}{b_{22}}\right) + \cdots, \tag{3.31a}$$

$$\gamma^{-1}(x_2) = \frac{a_{10}}{2b_{10}} - \frac{a_{12}b_{10} - a_{10}b_{12}}{2b_{10}(b_{20} - b_{10})} x_2 + \cdots, \tag{3.31b}$$

respectively.

The boundary layer regions A' and C' along the x_1-axis are treated similarly. There the solution is given by

$$u(x_1, x_2) = C_b + (C_{b2} - C_b) \exp[-\tilde{\gamma}(x_1)x_2/\epsilon]. \tag{3.32}$$

in which $\tilde{\gamma}(x_1)$ solves a Bernoulli problem analogous to (3.24),(3.25),(3.30). The treatment above in the direction along the boundary leads to a correct result only for constant boundary conditions. Readers interested in boundary conditions (2.12b) with nonconstant f are referred to the approach in [11].

3.1.4. Summary

A boundary layer analysis in the region D leads to a complicated expression for u. This is due in part to the conditions to be satisfied by u. Besides matching

conditions with the solutions in C (see (3.29b) and (3.31b) with $x_2 = 0$) and C' these are the conditions (2.12b, 2.14) along $x_1 = 0$ and $x_2 = 0$. In order to retain simplicity and since it will turn out in the remainder of section 3 that for small ϵ the asymptotic small region D is not important to the dynamics elsewhere, we discard the solution in this region. It is easily verified that the results of subsection 3.1 can then be summarized as follows. The uniform asymptotic expansion for small ϵ in $\bar{R} \setminus \bar{D}$, D an $O(\epsilon)$-neighbourhood of the origin, of the boundary value problem (2.12),(2.14) is given by

$$u(x) = \frac{1}{C_b}\{C_b + (C_{b1} - C_b)\exp[-\gamma(x_2)x_1/\epsilon]\}\{C_b + (C_{b2} - C_b)\exp[-\tilde{\gamma}(x_1)x_2/\epsilon]\}, \tag{3.33}$$

in which $\gamma, \tilde{\gamma}$ solve Bernoulli problems as dicussed in section 3.1.3. Expression (3.33) is the uniform asymptotic expansion in $\bar{R}$ of the boundary value problem (2.12) with the boundary conditions

$$\begin{aligned} u(0, x_2) &= C_{b1}\{1 + (C_{b2}/C_b - 1)\exp[-2b_{20}x_2/a_{20}\epsilon]\}, \\ u(x_1, 0) &= C_{b2}\{1 + (C_{b1}/C_b - 1)\exp[-2b_{10}x_1/a_{10}\epsilon]\}, \end{aligned} \tag{3.34}$$

which are different from the boundary conditions (2.14) in the region D. The remainder of section 3 concerns the determination of C_b, which is yet unknown.

3.2. The adjoint equation

The forward equation adjoint to (2.12a) is given by

$$M_\epsilon v = 0, \tag{3.35}$$

with the operator M_ϵ defined in (2.1). The function $v(x)$ describes the probability density corresponding to the (quasi-) stationary state of the system (2.1). The solution of equation (3.35) is needed in section 3.3.

3.2.1. The WKB-approximation

A solution of (3.35) is sought in the form of the WKB-Ansatz [8]

$$v(x_1, x_2) = w(x_1, x_2)\exp[-Q(x_1, x_2)/\epsilon], \quad \epsilon \to 0, \tag{3.36a}$$

where

$$Q(x_1^e, x_2^e) = 0, \tag{3.36b}$$

$$w(x_1^e, x_2^e) = 1, \quad \text{(normalization)}. \tag{3.36c}$$

Substitution of this form into (3.35) leads to leading order $O(\epsilon^{-1})$ to the eikonal equation

$$\sum_{i=1}^{2} \left[b_i \frac{\partial Q}{\partial x_i} + \frac{1}{2} a_i \left(\frac{\partial Q}{\partial x_i} \right)^2 \right] = 0, \tag{3.37}$$

and to order $O(\epsilon^0)$ to the transport equation

$$\sum_{i=1}^{2} \left[\frac{\partial}{\partial x_i}(b_i w) + \frac{\partial Q}{\partial x_i} \frac{\partial}{\partial x_i}(a_i w) + \frac{1}{2} a_i w \frac{\partial^2 Q}{\partial x_i^2} \right] = 0. \tag{3.38}$$

The numerical computation of the functions Q and w subject to the conditions (3.36b,c) is treated in section 5.

3.2.2. Behaviour near the boundary

To investigate the asymptotic behaviour of Q in the x_2-direction for small x_1, the expansion

$$Q(x_1, x_2) = \bar{Q}_0(x_2) + \bar{Q}_1(x_2) x_1 + \frac{1}{2} \bar{Q}_2(x_2) x_1^2 + \cdots \tag{3.39}$$

is substituted into (3.37). Terms of order $O(x_1^0)$ are collected, which results in

$$\frac{d\bar{Q}_0}{dx_2} = -\frac{b_{20} + b_{22} x_2}{\frac{1}{2}(a_{20} + a_{22} x_2)}. \tag{3.40}$$

This expression indicates that inside the interval $x_2 \in [0, \infty)$ the only extremum of $\bar{Q}_0$ is a minimum, situated at the critical point $x_2 = -b_{20}/b_{22}$. By (3.36a) the probability density function v is sharply peaked at this critical point. Therefore, the probability of meeting the stochastic system in the boundary layer $x_1 = O(\epsilon)$, asymptotically equals the probability of meeting the system in the boundary layer region B.

To study the WKB-solution in the region B the new variable

$$\hat{x}_2 = x_2 + b_{20}/b_{22} \tag{3.41}$$

is introduced and Q is approximated by the Taylor series expansion

$$Q(x_1, x_2) = Q_0 + Q_2\hat{x}_2 + Q_1 x_1 + \frac{1}{2}Q_3\hat{x}_2^2 + \cdots \tag{3.42}$$

Note that x_1 is of the order $O(\hat{x}_2^2)$ in the region B. Substitution of (3.42) into the eikonal equation (3.37) determines the constants

$$Q_2 = 0, \quad Q_1 = -k_1/k_2, \quad Q_3 = k_3/k_4, \tag{3.43}$$

and leaves the constant Q_0 undetermined. The value of Q_0 is obtained by solving the problem (3.35, 3.36) numerically.

A boundary layer analysis is carried out to reveal the behaviour of the transport function w in the region B. The WKB-solution of the stationary forward equation of a one-dimensional variant of our model can be calculated explicitly and indicates a singular behaviour. The stretched coordinates (3.4) and the boundary layer function

$$V(\bar{x}_1, \bar{x}_2) = v(\epsilon\bar{x}_1, -b_{20}/b_{22} + \sqrt{\epsilon}\bar{x}_2) \tag{3.44}$$

are introduced. Substitution into (3.35) leads to the boundary layer equation

$$-k_1\frac{\partial}{\partial\bar{x}_1}(\bar{x}_1 V) + k_2\frac{\partial^2}{\partial\bar{x}_1^2}(\bar{x}_1 V) + k_3\frac{\partial}{\partial\bar{x}_2}(\bar{x}_2 V) + k_4\frac{\partial^2 V}{\partial\bar{x}_2^2} = 0, \tag{3.45}$$

with the k_i defined in (3.7). By the separation assumption

$$V(\bar{x}_1, \bar{x}_2) = r(\bar{x}_1)s(\bar{x}_2), \tag{3.46}$$

(3.45) leads to the ordinary differential equations

$$k_2\frac{d^2}{d\bar{x}_1^2}(\bar{x}_1 r) - k_1\frac{d}{d\bar{x}_1}(\bar{x}_1 r) - \mu r = 0, \tag{3.47a}$$

$$k_4\frac{d^2 s}{d\bar{x}_2^2} + k_3\frac{d}{d\bar{x}_2}(\bar{x}_2 s) + \mu s = 0, \tag{3.47b}$$

in which μ is a separation constant. The general solution of (3.47a) is

$$r(\bar{x}_1) = \{c_1 W_{\mu_1,\frac{1}{2}}(-\tilde{x}_1) + c_2 W_{-\mu_1,\frac{1}{2}}(\tilde{x}_1)\}\ \tilde{x}_1^{-1}\exp[\tilde{x}_1/2], \tag{3.48}$$

with

$$\mu_1 = \mu/k_1, \tag{3.49}$$

c_1, c_2 arbitrary constants and $\tilde{x}_1$ defined in (3.11). The general solution of (3.47*b*) is

$$s(\bar{x}_2) = \{c_3 D_{\mu_2}(\tilde{x}_2) + c_4 D_{-\mu_2-1}(i\tilde{x}_2)\} \exp[-\tilde{x}_2^2/4], \tag{3.50}$$

with

$$\mu_2 = \mu/k_3, \tag{3.51}$$

c_3, c_4 arbitrary constants and $\tilde{x}_2$ defined in (3.13). Putting

$$\mu = 0, \quad c_2 = 0, \quad c_4 = 0, \tag{3.52}$$

the boundary layer solution

$$V(\bar{x}_1, \bar{x}_2) = \text{const. } \tilde{x}_1^{-1} \exp[\tilde{x}_1 - \tilde{x}_2^2/2] \tag{3.53}$$

is obtained. The leading order part of the WKB-solution (3.36*a*) with Q given by (3.42,3.43) agrees with the exponential function in the boundary layer solution (3.53). The solution (3.53) indicates that the transport equation w behaves according to

$$w \sim x_1^{-1} \tag{3.54}$$

in the region B. Substitution of the expansion

$$w(x_1, x_2) = x_1^{-1} \left(w_0 + w_2 \hat{x}_2 + w_1 x_1 + \frac{1}{2} w_3 \hat{x}_2^2 + \cdots\right) \tag{3.55}$$

into the transport equation (3.38) and using (3.42,3.43) leaves the constant w_0 undetermined. Its value is obtained by solving the problem (3.35,3.36) numerically.

As a conclusion, in the boundary layer region B the WKB-solution (3.36) behaves as

$$v(x_1, x_2) = C_1(\epsilon)\, x_1^{-1} \exp[(k_1 x_1/k_2 - k_3 \hat{x}_2^2/2k_4)/\epsilon], \tag{3.56}$$

in which

$$C_1(\epsilon) = w_0 \exp[-Q_0/\epsilon], \tag{3.57}$$

where w_0, Q_0 as in (3.55),(3.42) respectively, have to be determined numerically. A similar result can be derived in the boundary layer region B'. There, the WKB-solution is

$$v(x_1, x_2) = C_2(\epsilon)\, x_2^{-1} \exp[(k_1' x_2/k_2' - k_3' \hat{x}_1^2/2k_4')/\epsilon], \tag{3.58}$$

where the constants in C_2, which is the analogue of C_1, have to be determined numerically. The constants k_3', k_4' are given by

$$k_3' = b_{10}, \quad k_4' = -\frac{1}{2}(a_{10} - a_{11}b_{10}/b_{11})b_{10}/b_{11}, \tag{3.59}$$

and $\hat{x}_1$ by

$$\hat{x}_1 = x_1 + b_{10}/b_{11}. \tag{3.60}$$

3.3. Application of the divergence theorem

Using the divergence theorem the following integral relation can be derived:

$$\begin{aligned} &\int_{R'} (vL_\epsilon u - uM_\epsilon v)\, dR' = \\ &= \int_{\partial R'} \sum_{i=1}^{2} \nu_i \left[\frac{\epsilon}{2} a_i \left(v\frac{\partial u}{\partial x_i} - u\frac{\partial v}{\partial x_i} \right) + \left(b_i - \frac{\epsilon}{2}\frac{\partial a_i}{\partial x_i} \right) uv \right] dS, \end{aligned} \tag{3.61}$$

where R' is a region with boundary $\partial R'$ on which the operators L_ϵ, M_ϵ are defined and ν denotes the outward normal on $\partial R'$. In the right side of (3.61) v and its conormal derivative must be evaluated at the boundary. By (3.56),(3.58) these functions become singular at $x_1 = 0$ and $x_2 = 0$. To avoid singular functions, R' is chosen as a slight modification of the region R:

$$R' = \{(x_1, x_2) | x_1, x_2 > \delta\}, \tag{3.62}$$

and

$$\partial R' \equiv \bar{R}' \setminus R' = \{(x_1, x_2) | (x_1 - \delta)(x_2 - \delta) = 0 \text{ and } x_1 + x_2 \geq 2\delta\}, \tag{3.63}$$

with

$$0 < \delta \ll \epsilon. \tag{3.64}$$

By (2.12a) and (3.35) the left side of (3.61) equals zero. First the boundary $x_1 = \delta$ of R' is considered. There, the right side of (3.61) is written as

$$\int_\delta^\infty \left[\frac{\epsilon}{2} a_1 \left(-v\frac{\partial u}{\partial x_1} + u\frac{\partial v}{\partial x_1} \right) - \left(b_1 - \frac{\epsilon}{2}\frac{\partial a_1}{\partial x_1} \right) uv \right]_{x_1 = \delta} dx_2. \tag{3.65}$$

The only significant contribution to this integral comes from the boundary layer region B. Using the expression (3.16b) for u and (3.56) for v, the integrand in

(3.65) is evaluated. Subsequently the limit $\delta \to 0$ is taken and asymptotically for small ϵ the following result is obtained:

$$\int_0^\infty \Big[(a_{10} - b_{10}k_2/k_1)C_{b1} - \frac{1}{2}a_{10}C_b + \{(a_{12} - b_{12}k_2/k_1)C_{b1} + \\ -\frac{1}{2}a_{12}C_b\}x_2\Big](k_1/k_2)C_1(\epsilon)\exp[-k_3\tilde{x}_2^2/2k_4\epsilon]\, dx_2. \tag{3.66}$$

This integral is evaluated by the method of Laplace [1]. The boundary $x_2 = \delta$ is treated similarly. Both results are used in the divergence formula (3.61) to obtain the following expression for C_b:

$$C_b = \frac{C_{b1}C_1(\epsilon)K_1 + C_{b2}C_2(\epsilon)K_2}{C_1(\epsilon)K_1 + C_2(\epsilon)K_2}, \tag{3.67}$$

with the abbreviations

$$K_1 = k_1\sqrt{k_4/k_3}, \quad K_2 = k_1'\sqrt{k_4'/k_3'}. \tag{3.68}$$

Expression (3.67) completes the analysis of section 3. With (3.33) the following result is obtained. Denoting the probability of exit at the boundary $x_i = 0$, starting at x, by $u_i(x)$, we have

$$u_1(x) = \frac{C_1(\epsilon)K_1 + C_2(\epsilon)K_2\exp[-\gamma(x_2)x_1/\epsilon]}{C_1(\epsilon)K_1 + C_2(\epsilon)K_2} \cdot \\ \cdot\Big[1 - \exp[-\tilde{\gamma}(x_1)x_2/\epsilon]\Big], \tag{3.69a}$$

$$u_2(x) = \frac{C_1(\epsilon)K_1\exp[-\tilde{\gamma}(x_1)x_2/\epsilon] + C_2(\epsilon)K_2}{C_1(\epsilon)K_1 + C_2(\epsilon)K_2} \cdot \\ \cdot\Big[1 - \exp[-\gamma(x_2)x_1/\epsilon]\Big], \tag{3.69b}$$

asymptotically for small ϵ in $\bar{R} \setminus \bar{D}$, D an $O(\epsilon)$-neighbourhood of the origin. It is easily verified that in the region $\bar{R} \setminus \bar{D}$, the expressions (3.69$a$) and (3.69$b$) add up to one. Rewriting (3.57) and the analogous expression for C_2 as

$$C_i(\epsilon) = w_{0i}\exp[-Q_{0i}/\epsilon], \quad i = 1, 2, \tag{3.70}$$

and using the fact that w_{0i}, K_i are order $O(1)$ constants, (3.69) can be simplified. In the case $Q_{01} < Q_{02}$ we find

$$u_1(x) \sim 1 - \exp[-\tilde{\gamma}(x_1)x_2/\epsilon], \quad u_2(x) \sim \exp[-\tilde{\gamma}(x_1)x_2/\epsilon], \tag{3.71a}$$

and in the case $Q_{01} > Q_{02}$:

$$u_1(x) \sim \exp[-\gamma(x_2)x_1/\epsilon], \quad u_2(x) \sim 1 - \exp[-\gamma(x_2)x_1/\epsilon]. \tag{3.71b}$$

4. The expectation and variance of the exit time

In this section the boundary value problems (2.17) are solved asymptotically for small ϵ. Assume that $T_i(x)$ is of the form

$$T_i(x) = C_{T,i}(\epsilon)\tau_i(x), \tag{4.1}$$

in which

$$C_{T,i}^{-1}(\epsilon)g_i(x) = o(\epsilon), \quad \epsilon \to 0. \tag{4.2}$$

Substitution of (4.1) into (2.17a) yields to leading order the reduced equation

$$\sum_{j=1}^{2} b_j(x)\frac{\partial \tau_i}{\partial x_j} = 0, \tag{4.3}$$

which is solved by a constant that is taken 1 without loss of generality (any other value can be incorporated in $C_{T,i}$):

$$\tau_i(x) = 1. \tag{4.4}$$

This is the outer solution, valid away from $O(\epsilon)$ boundary layers along $x_1 = 0$ and $x_2 = 0$. A boundary layer analysis can be held as in section 3, with u replaced by τ_i. The only difference is in the boundary condition, for this case stated by (2.17b). The following uniform asymptotic expansion for τ_i is obtained:

$$\tau_i(x_1, x_2) = \Big[1 - \exp[-\gamma(x_2)x_1/\epsilon]\Big]\Big[1 - \exp[-\tilde{\gamma}(x_1)x_2/\epsilon]\Big], \tag{4.5}$$

valid in R (the region D included, due to the bounday conditions, which are simpler here than in section 3). The unknown $C_{T,i}(\epsilon)$ are determined using the integral relation (3.61) with u replaced by T_i. After some calculation, this integral relation reduces to

$$-\frac{2}{\epsilon}\int_\delta^\infty \int_\delta^\infty v g_i \, dx_1 dx_2 = \int_\delta^\infty \left[a_1 v \frac{\partial T_i}{\partial x_1}\right]_{x_1=\delta} dx_2 + \int_\delta^\infty \left[a_2 v \frac{\partial T_i}{\partial x_2}\right]_{x_2=\delta} dx_1. \tag{4.6}$$

On the right side, the largest contributions to the integrals are from the boundary layer regions B and B'. These integrals are evaluated by the method of Laplace, using expressions (4.1),(4.5) with

$$\gamma(x_2) = k_1/k_2, \quad \tilde{\gamma}(x_1) = k_1'/k_2', \tag{4.7}$$

for T and expressions (3.56),(3.58) for v. The left side of (4.6) is evaluated using the WKB-expression (3.36a) for v and the method of Laplace for double integrals. Letting $\delta \to 0$, the following expressions are found for $C_{T,i}(\epsilon)$:

$$C_{T,1}(\epsilon) = \frac{\sqrt{2\pi\epsilon/He(x^e)}}{C_1(\epsilon)K_1 + C_2(\epsilon)K_2}, \quad C_{T,2}(\epsilon) = 2C_{T,1}^2(\epsilon), \tag{4.8}$$

in which $He(x^e)$ is the determinant of the Hessian matrix of Q at x^e. With $Q^* = \min(Q_{01}, Q_{02})$, see (3.70), and using the fact that $w_{0i}, K_i, He(x^e)$ are of order $O(1)$, $C_{T,1}(\epsilon)$ is of the order

$$C_{T,1}(\epsilon) \sim \sqrt{\epsilon}\exp[Q^*/\epsilon]. \tag{4.9}$$

In the evaluation of the left side of (4.6) we let $\epsilon \to 0$, while the WKB-expression (3.36a) for v fails to be integrable in this limit. This procedure was proposed by Ludwig [8]. Its correctness has not been proven. See also the remarks in [7] at this point. By (2.15) the resulting uniform asymptotic expansions in $\bar{R}$ of the expectation and variance of the exit time are given as

$$\mathrm{E}\,T(x) = C_{T,1}(\epsilon)\left[1 - e^{-\gamma(x_2)x_1/\epsilon}\right]\left[1 - e^{-\tilde{\gamma}(x_1)x_2/\epsilon}\right], \tag{4.10a}$$

$$\mathrm{Var}\,T(x) = C_{T,1}^2(\epsilon)\Big[1 - \Big\{e^{-\gamma(x_2)x_1/\epsilon} + e^{-\tilde{\gamma}(x_1)x_2/\epsilon} + -e^{-\gamma(x_2)x_1/\epsilon - \tilde{\gamma}(x_1)x_2/\epsilon}\Big\}^2\Big], \tag{4.10b}$$

respectively.

5. Numerical determination of the WKB-solution

To obtain the constants $Q_{01}, w_{01}, Q_{02}, w_{02}$ in (3.70), The WKB-solution (3.36) of the adjoint equation (3.35) is determined numerically. By the Hamilton-Jacobi theory [2], the eikonal equation (3.37) is written in terms of the Hamiltonian H:

$$H(x,p) \equiv \sum_{i=1}^{2}[b_i p_i + \frac{1}{2}a_i p_i^2] = 0, \tag{5.1a}$$

where

$$p_i = \frac{\partial Q}{\partial x_i}. \tag{5.1b}$$

The corresponding system of bicharacteristics reads

$$\frac{dx_i}{ds} = \frac{\partial H}{\partial p_i} = b_i + a_i p_i, \quad (i = 1, 2) \tag{5.2a}$$

$$\frac{dp_i}{ds} = -\frac{\partial H}{\partial x_i} = -\sum_{j=1}^{2} \left[\frac{\partial b_j}{\partial x_i} p_j + \frac{1}{2} \frac{\partial a_j}{\partial x_i} p_j^2 \right], \quad (i = 1, 2) \tag{5.2b}$$

The rate of change of Q with $s$is given by

$$\frac{dQ}{ds} = -H + \sum_{i=1}^{2} \frac{dx_i}{ds} p_i = \sum_{i=1}^{2} \frac{1}{2} a_i p_i^2, \quad \text{(which is } \geq 0). \tag{5.2c}$$

Here s is a parameter along the characteristics. At $s = 0$ all characteristics start in a neighbourhood of the equilibrium

$$x = x^e, \quad p = 0, \quad Q = 0, \tag{5.3}$$

of the system (5.2). The initial position of a characteristic is specified on a circle around x^e with radius $r \ll 1$, by the variable θ:

$$x_1 = x_1^e + r\cos\theta, \quad x_2 = x_2^e + r\sin\theta. \tag{5.4}$$

The corresponding initial values of p_1, p_2, Q are obtained by the following local analysis. In the neighbourhood of (5.3), Q is approximated by the quadratic form

$$Q \approx \frac{1}{2}(x - x^e)^t P (x - x^e), \tag{5.5}$$

in which P is a symmetric matrix and t denotes the transpose. It follows that

$$p = \frac{dQ}{dx} \approx P(x - x^e). \tag{5.6}$$

Substitution of the approximations (5.6),(2.7) and the approximation

$$\begin{pmatrix} a_1(x^e) & 0 \\ 0 & a_2(x^e) \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \equiv A \tag{5.7}$$

of the diffusion matrix into the eikonal equation (3.37) leads to the matrix equation

$$PAP + PB + B^t P = 0, \tag{5.8}$$

which is solved to give

$$P = \frac{-2(B_{11}+B_{22})}{(B_{21}A_1 - B_{12}A_2)^2 + A_1A_2(B_{11}+B_{22})^2}\, G, \tag{5.9a}$$

where G is the matrix with elements

$$\begin{aligned} G_{11} &= B_{21}^2 A_1 + [B_{11}B_{22} - B_{12}B_{21} + B_{11}^2]A_2, \\ G_{12} &= G_{21} = B_{22}B_{21}A_1 + B_{11}B_{12}A_2, \\ G_{22} &= B_{12}^2 A_2 + [B_{11}B_{22} - B_{12}B_{21} + B_{22}^2]A_1. \end{aligned} \tag{5.9b}$$

The initial values of p_1, p_2, Q are determined by (5.6),(5.5),(5.9). Notice that (5.9) also determines the determinant

$$He(x^e) = P_{11}P_{22} - P_{12}P_{21} \tag{5.10}$$

in (4.8). Next we consider the transport equation (3.38). With (5.2a) and

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\frac{dx_i}{ds}\right) = \frac{d}{ds}\ln|J|, \tag{5.11a}$$

see [8], in which J is the Jacobian

$$J = \begin{vmatrix} \dfrac{dx_1}{ds} & \dfrac{\partial x_1}{\partial \theta} \\ \dfrac{dx_2}{ds} & \dfrac{\partial x_2}{\partial \theta} \end{vmatrix}, \tag{5.11b}$$

equation (3.38) is rewritten as

$$\frac{d}{ds}(\ln w^2|J|) = -\sum_{i=1}^{2}\left[\frac{\partial b_i}{\partial x_i} + p_i\frac{\partial a_i}{\partial x_i}\right]. \tag{5.12}$$

Differentiation of $(5.2a,b)$ with respect to θ leads to the equations

$$\frac{d}{ds}\left(\frac{\partial x_i}{\partial \theta}\right) = \frac{\partial b_i}{\partial \theta} + \frac{\partial a_i}{\partial \theta}p_i + a_i\frac{\partial p_i}{\partial \theta}, \tag{5.13a}$$

$$\begin{aligned} \frac{d}{ds}\left(\frac{\partial p_i}{\partial \theta}\right) = -\sum_{j=1}^{2}\Big[&\frac{\partial^2 b_j}{\partial x_i \partial \theta}p_j + \frac{\partial b_j}{\partial x_i}\frac{\partial p_j}{\partial \theta} + \frac{1}{2}\frac{\partial^2 a_j}{\partial x_i \partial \theta}p_j^2 + \\ &+ \frac{\partial a_j}{\partial x_i}\frac{\partial a_j}{\partial \theta}p_j\Big], \end{aligned} \tag{5.13b}$$

$i = 1, 2$, which describe the rate of change with s of $\partial x_i/\partial\theta$ and, using (5.2a), of J. The initial value at $s = 0$ of w is chosen according to (3.36c). The

initial values of $\partial x_i/\partial\theta$ and $\partial p_i/\partial\theta$ are obtained by differentiation of the initial expressions (5.4) for x_i and (5.6) for p_i with respect to θ.

To obtain Q_{01}, w_{01}, the system of 10 ordinary differential equations (5.2), (5.12),(5.13) is integrated. By trial and error the angle θ of the initial point is manipulated in order to obtain a characteristic containing points close to $(0, -b_{20}/b_{22})$. Once a (Δ^2, Δ)-neighbourhood of $(0, -b_{20}/b_{22})$ is reached, $0 < \Delta \ll 1$, the integration is terminated. Using the values of Q, w obtained numerically at the end point(s) of the characteristic and the formulas (3.42),(3.43) for Q and (3.55) for w, valid in the case $x_1 = O(\hat{x}_2^2)$, $\hat{x}_2$ small, we approximate the values of Q_{01}, w_{01}.

The solutions of Q and w obtained by the numerical method described above are not always unique functions of x. By assumption, the solution is unique along the characteristic which starts at the initial point with $r = 0$ and ends at $(0, -b_{20}/b_{22})$. In numerical computations, this characteristic can not be followed exactly. Near $(0, -b_{20}/b_{22})$ the characteristics curve upward or downward along $x_1 = 0$ and get into caustic surfaces, as indicated by a change of sign in the determinant (5.11*b*). There, the solution is not a unique function of x. The numerical integration has been terminated before the determinant vanishes. The boundary $x_1 = 0$ cannot be approached too close. Consequently, Δ cannot be taken arbitrary small, which limits the accuracy of the computed values of Q_{01}, w_{01}. In the subsequent example, the numerical computation was stopped at $x_1 \approx 0.02$.

6. An example

Consider the predator-prey system defined by the diffusion

$$\begin{aligned} a_1(x) &= x_1(1.28 + 0.80x_1 + 0.32x_2), \\ a_2(x) &= x_2(1.08 + 0.28x_1 + 0.40x_2), \end{aligned} \tag{6.1a}$$

and the drift

$$\begin{aligned} b_1(x) &= x_1(0.72 - 0.40x_1 - 0.32x_2), \\ b_2(x) &= x_2(0.12 + 0.28x_1 - 0.40x_2), \end{aligned} \tag{6.1b}$$

in which x_1 and x_2 denote the prey and predator density, respectively. The stochastic system defined by (6.1) has previously been studied in [12]. Some trajectories of the deterministic system are depicted in Figure 2. The numerical

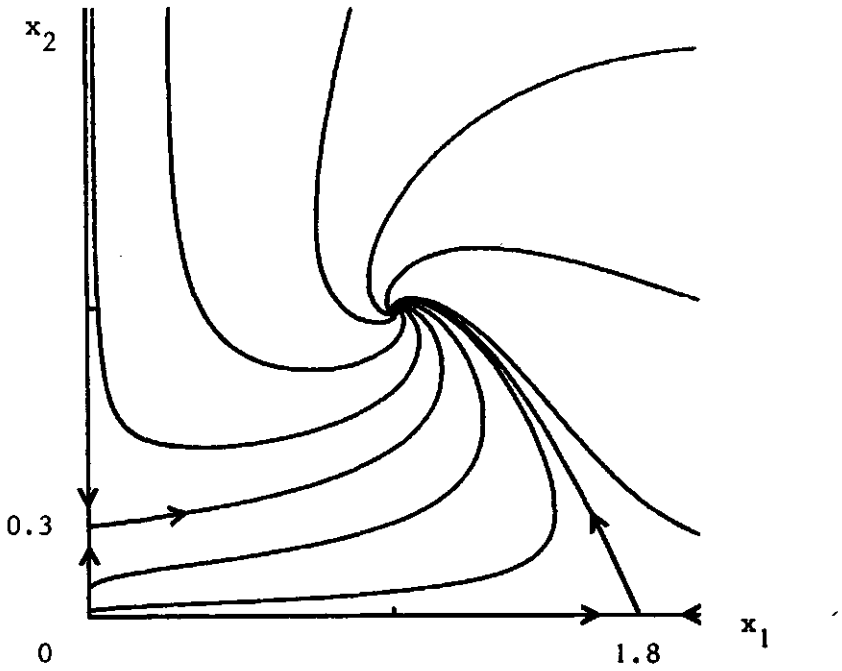

Figure 2. Trajectories of the deterministic system associated with the stochastic system (6.1). The critical points are (0,0),(0,0.9),(1.8,0) and (1,1).

computation described in section 5 produces the values

$$Q_{01} = 0.26, \quad w_{01} = 1.1, \quad Q_{02} = 0.30, \quad w_{02} = 1.5. \tag{6.2}$$

The projection on the x-plane of the characteristics (called rays), used in this computation, are depicted in Figure 3. Outside the region D, the probability of exit at the boundary $x_i = 0$ is given by u_i:

$$u_1(x) = \frac{0.69e^{-0.26/\epsilon} + 1.03e^{-(0.30+x_1\gamma(x_2))/\epsilon}}{0.69e^{-0.26/\epsilon} + 1.03e^{-0.30/\epsilon}} \left[1 - e^{-\tilde{\gamma}(x_1)x_2/\epsilon}\right]$$
$$\sim 1 - e^{-\tilde{\gamma}(x_1)x_2/\epsilon}, \tag{6.3a}$$

$$u_2(x) = \frac{0.69e^{-(0.26+x_2\tilde{\gamma}(x_1))/\epsilon} + 1.03e^{-0.30/\epsilon}}{0.69e^{-0.26/\epsilon} + 1.03e^{-0.30/\epsilon}} \left[1 - e^{-\gamma(x_2)x_1/\epsilon}\right],$$
$$\sim e^{-\tilde{\gamma}(x_1)x_2/\epsilon}, \tag{6.3b}$$

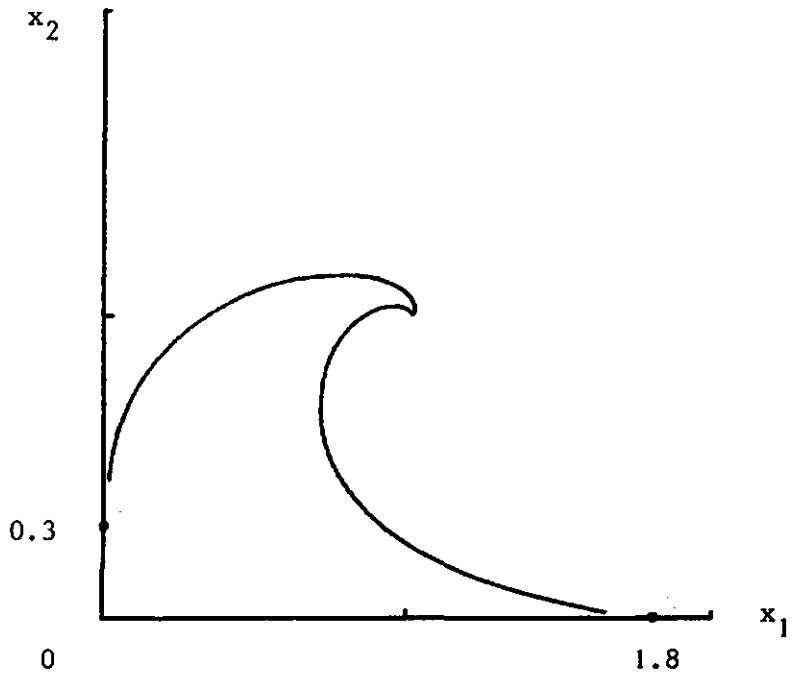

Figure 3. The rays used in the numerical computation of Q_{01}, w_{01}, Q_{02} *and* w_{02}.

according to (3.69),(3.71). We conclude that if the starting point of the sytem (6.1) is outside an $O(\epsilon)$-neighbourhood of the x_1-axis, i.e. if the initial predator density is not very small, then the prey will get extinct before the predator, with probability one in the limit for $\epsilon \to 0$. The expectation and variance of the exit time satisfy equations (4.10), uniformly in R, with

$$C_{T,1}(\epsilon) = \frac{\sqrt{\epsilon}}{0.11e^{-0.26/\epsilon} + 0.16e^{-0.30/\epsilon}} \sim 9.1\sqrt{\epsilon}e^{0.26/\epsilon}. \tag{6.4}$$

which indicates that if the starting point of the system (6.1) is outside $O(\epsilon)$-neighbourhoods of the coordinate axes, i.e. if the initial prey and predator densities are not very small, the expected time of extinction of one of the populations is exponentially large. The functions γ and $\tilde{\gamma}$ are computed numerically. The expression (3.31a) supplies the starting values of γ at $-b_{20}/b_{22} \pm \eta$, where η is a small number, and a forward (backward) finite difference scheme, based on the Bernoulli-equation (3.24,3.25) is used to obtain $\gamma(x_2)$ for $x_2 > -b_{20}/b_{22} + \eta$ ($x_2 < -b_{20}/b_{22} - \eta$). The graphs of $\gamma, \tilde{\gamma}$ are shown in figure 4. In the boundary

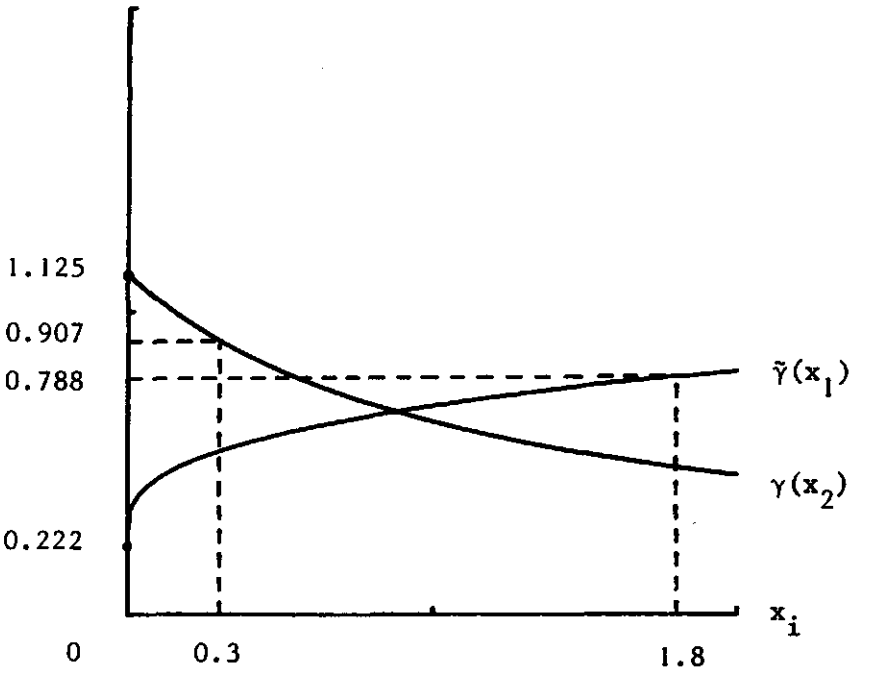

Figure 4. The functions $\gamma, \tilde{\gamma}$. The critical points, at which the initial condition for the Bernoulli differential equation is specified, are indicated. The values denoted along the vertical axis follow from (3.30), $\lim_{x_2 \to 0} \gamma(x_2) = 2b_{10}/a_{10}$, *see* (3.31*b*), *and similar formulas for the other boundary.*

layers along $x_1 = 0$ and $x_2 = 0$ a small (large) value of $\gamma, \tilde{\gamma}$ respectively, may be interpreted as a relatively weak (strong) stochastic stability. From figure 4 we conclude that for low prey density the stochastic stability of the system (6.1) decreases with increasing predator density; for low predator density the stochastic stability increases with increasing prey density.

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SINGULARITIES ARISING IN THE ASYMPTOTIC SOLUTION OF THE FORWARD KOLMOGOROV EQUATION

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In the study of stochastic dynamical systems one is interested in the solution of the forward Kolmogorov equation. In case of small stochastic fluctuations, a WKB-Ansatz can be made to this solution. This approach leads to a system of ray equations. When solving these equations, it is observed that rays may intersect in caustic surfaces. Near such locations a WKB-approximation does not hold. In this paper we obtain a uniform asymptotic expansion that remains valid near caustics. This expansion is expressed in terms of new canonical integrals.

1. Introduction

In the study of stochastic dynamical systems we are interested in the solution of the forward Kolmogorov equation for various reasons. It describes the probability density function of the position of the system in the state space. Moreover, it is related to statistical quantities describing the problem of exit from a region in the state space. Such quantities are the probability density function or expectation value of the exit time, the probability density function

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of the exit point, the most probable exit point, etc. [19,24]. It is assumed that the stochastic fluctuations are small, that is, of intensity of order k^{-1}, where k is a large parameter. For simplicity we assume that the spatial domain extends to infinity. We consider initial data involving the large parameter. We look at the forward Kolmogorov equation either in time dependent form (a parabolic differential equation) or in stationary form (an elliptic differential equation). A formal asymptotic solution to these equations can be found by application of the WKB-method, as described in [17]. This method leads to a system of ordinary differential equations along characteristics. The projection of a characteristic on the (temporal-) spatial domain is called a ray. It may occur that at certain locations (the caustic surfaces) rays intersect. Near such locations the WKB-method does not hold. We derive an asymptotic expansion that is valid uniformly and thus near a caustic. We study in detail one of the simplest cases: the cusp caustic arising in a diffusion problem with one spatial variable and with a constant diffusion. We conclude with the formulation of a formal approach to singularities arising in the forward Kolmogorov equation.

Section 2 starts with the formulation of a diffusion problem. Application of the WKB-method leads to an explicit expression for the asymptotic solution of the diffusion problem in terms of the ray variables. For some initial conditions the ray pattern shows intersecting rays that form a cusp.

In Section 3 we give the exact solution of the diffusion problem in the form of an integral. This integral is expanded asymptotically by the method of Laplace. This approach gives us an interpretation of rays, from which we learn how to handle regions in the (time-) space domain where more than one ray go through one point. The asymptotic solution is not valid near certain singular surfaces. The approach in this section is the same as in [7]. There it served to motivate the application of the WKB-method to linear partial differential equations.

In Section 4 we continue with the integral of Section 3. The singularity in the integral is assumed to be of the typical form. With the asymptotic theory of Chester, Friedman and Ursell [6] we obtain a uniform asymptotic expansion of the integral. This expansion is also valid near caustics. It contains canonical integrals U_0, U_1 and U_2 that are not found in earlier literature. In this section reference is made to the appendices. Appendix A treats the standard cusp and the solution of a third degree algebraic equation. Appendix B lists some properties of U_0. The method in this section is similar to that

of the uniform asymptotic expansion of oscillatory integrals [14,18] arising in hyperbolic problems, such as the reduced wave equation [16].

Section 5 is concerned with the evaluation of the uniform asymptotic expansion near caustics. It turns out that near the fold lines and away from the cusp point, the asymptotic solution is obtained with the ray method, by taking into account the contribution of only one (nonsingular) ray. Near the cusp point we need the uniform expansion involving canonical integrals. The determination of the functions arising in this expansion is discussed.

Section 6 generalizes the approach of the cusp in a diffusion problem to singularities arising in the asymptotic solution of the forward Kolmogorov equation.

2. The diffusion problem and its solution by the WKB-method

For a description of the WKB-method applied to linear partial differential equations of parabolic and elliptic type we refer to Cohen and Lewis [7], Ludwig [17] and Brannan [5]. The last two papers are addressed directly to stochastic dynamical systems. In the present paper we consider the problem [7]:

$$kv_t = v_{xx}, \qquad t > 0, \quad x \in (-\infty, +\infty), \tag{2.1a}$$

$$v(0,x) = z(x)\exp(-ks(x)), \qquad z \geq 0, \tag{2.1b}$$

$$\int_{-\infty}^{+\infty} v(t,x)\,dx < \infty, \tag{2.1c}$$

asymptotically for $k \to \infty$, i.e. for a small diffusion. Note that the initial condition (2.1b) is in the WKB-form. From physical considerations we assumed z nonnegative and adopted condition (2.1c). Further assumptions on z and s will be made implicitly below.

The WKB-method presupposes a solution of (2.1) in the form

$$v(t,x) = \exp[-k\{Q(t,x) - k^{-1}\log w(t,x) + \cdots\}], \quad \text{as } k \to \infty, \tag{2.2}$$

where we indicated only the two highest order terms in k. Substitution of this form in (2.1a) gives to leading order in k the eikonal equation and to the next order in k the transport equation, respectively

$$Q_t + Q_x^2 = 0, \tag{2.3a}$$

$$w_t + 2Q_x w_x + Q_{xx} w = 0. \tag{2.3b}$$

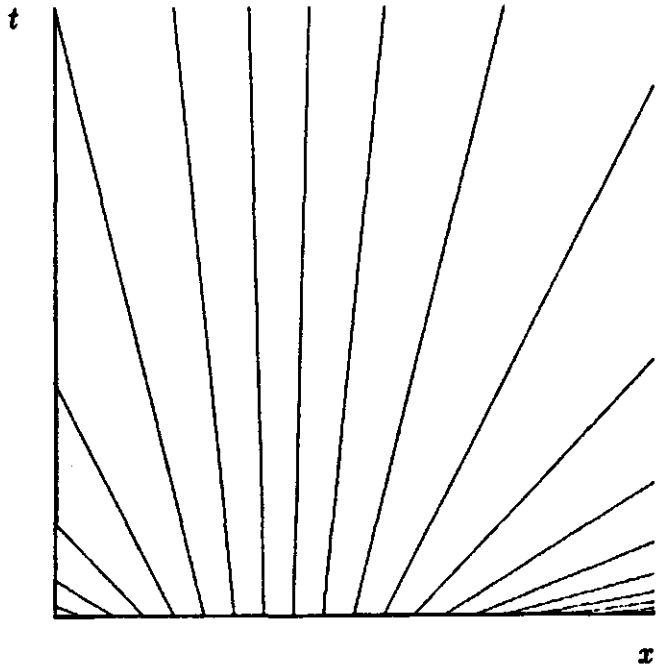

Fig. 1a. For $s''(\gamma) \geq 0$ the (t, x)-space is simply covered by rays.

The partial differential equations (2.3) can be written as the system of ordinary differential equations

$$\frac{dx}{dt} = 2p, \quad \frac{dp}{dt} = 0, \quad \frac{dQ}{dt} = p^2, \tag{2.4a}$$

$$\frac{d}{dt}(\log w + \tfrac{1}{2}\log J) = 0. \tag{2.4b}$$

Equations (2.4a) are the so-called ray equations. These follow from (2.3*a*) by the theory of Hamilton-Jacobi [13]. Here p denotes Q_x. Solutions $x(t, \gamma)$ of the first equation of (2.4*a*) form rays in the (t, x)-space. As variable along a ray we have chosen t and as a variable that distinguishes rays we take γ, which is the position x of a ray at $t = 0$. Equation (2.4*b*) follows from (2.3*b*), and J denotes the Jacobian $\partial x/\partial\gamma$ corresponding to the transformation from γ to x. The explicit solution of system (2.4) with initial condition (2.1*b*) reads in terms of the ray variables

$$x(t, \gamma) = \gamma + 2s'(\gamma)t, \tag{2.5a}$$

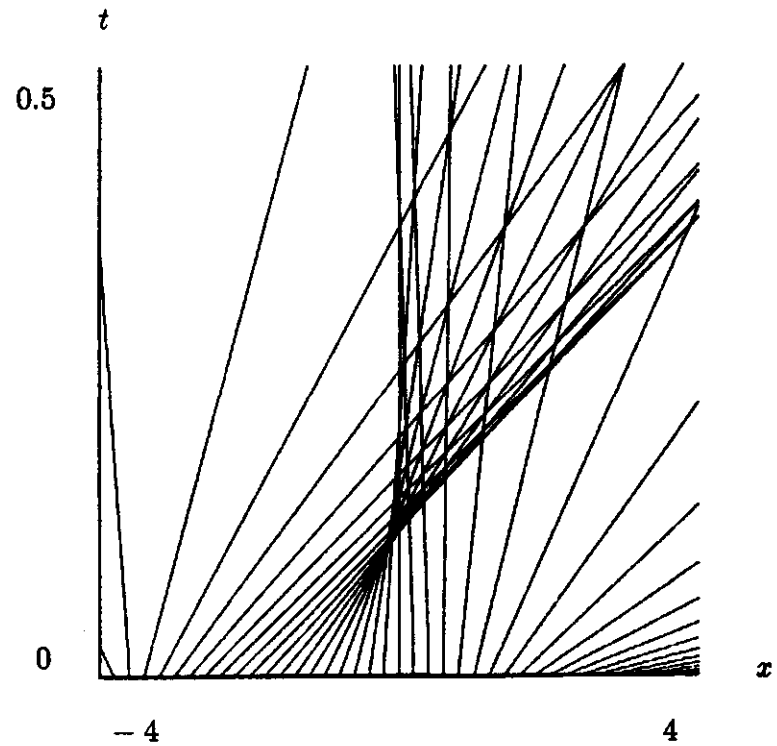

Fig. 1b. Rays in the (t,x)-space that intersect to form a cusp singularity. This happens when $s''(\gamma)$ becomes negative.

$$p(t,\gamma) = s'(\gamma), \tag{2.5b}$$

$$Q(t,\gamma) = s(\gamma) + (s'(\gamma))^2 t, \tag{2.5c}$$

$$J(t,\gamma) = 1 + 2s''(\gamma)t, \tag{2.5d}$$

$$w(t,\gamma) = z(\gamma)J^{-\frac{1}{2}}(t,\gamma). \tag{2.5e}$$

Thus, along the ray denoted by γ, the WKB-solution is

$$z(\gamma)(1 + 2s''(\gamma)t)^{-\frac{1}{2}} \exp[-k\{s(\gamma) + (s'(\gamma))^2 t\}]. \tag{2.6}$$

If $s''(\gamma) \geq 0$, the pattern of rays shows a simple covering of (t,x)-space by rays (Fig. 1a) and the WKB-solution is regular everywhere. If $s''(\gamma)$ becomes negative, the pattern of rays is typically as depicted in Fig. 1b. In that case, the expressions above show that the transport function w is singular at points where J vanishes (i.e. where rays intersect to form a caustic). Note that at

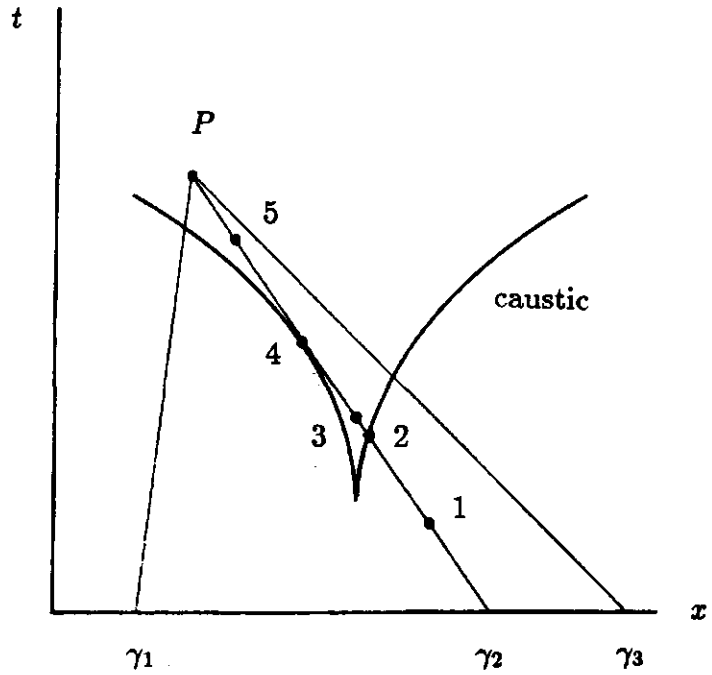

Fig. 2a. Every point P at the depicted side of the caustic (inside the cusp), is the point of intersection of three different real rays.

such points the eikonal function Q remains finite. It will be discussed how the WKB-method can be extended to a region where every point lies on three different rays, as in Fig. 2a. Moreover, we derive an asymptotic expansion that remains valid near caustics.

3. The exact solution of the diffusion problem

For an understanding of the phenomena that occur in the presence of caustics we study the integral that solves problem (2.1) exactly

$$v(t,x) = (k/2\pi)^{\frac{1}{2}} I(t,x), \tag{3.1a}$$

where

$$I(t,x) = \int_\Gamma g(\gamma,t,x) \exp[-kf(\gamma,t,x)]\, d\gamma, \tag{3.1b}$$

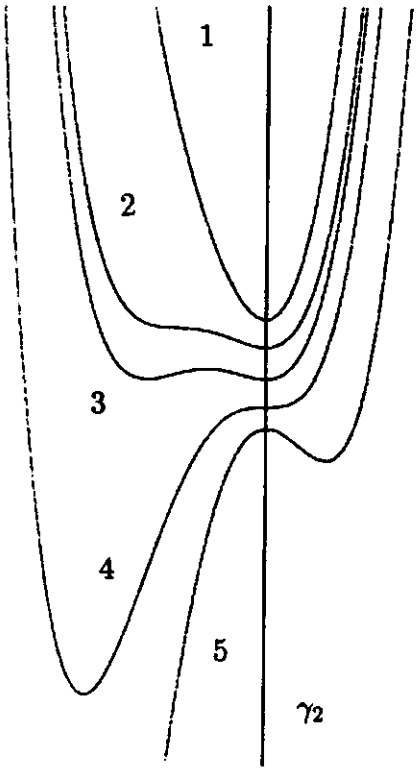

Fig. 2b. Form of the function f at the five positions on the ray γ_2 indicated in Fig. 2a. The left hand side minimum of 5 has not been depicted.

and

$$f = s(\gamma) + (\gamma - x)^2/4t, \quad g = (2t)^{-\frac{1}{2}} z(\gamma), \tag{3.1c}$$

and the integration interval Γ consists of the real line. For f and g sufficiently smooth the integral I can be expanded for $k \to \infty$ by the method of Laplace [2,4]. For given t and x the major contribution to the integral is from one or more of the local minima of f on Γ, that is from points $\gamma_i \in \Gamma$ satisfying

$$f_\gamma(\gamma_i, t, x) = 0, \tag{3.2a}$$

$$f_{\gamma\gamma}(\gamma_i, t, x) > 0. \tag{3.2b}$$

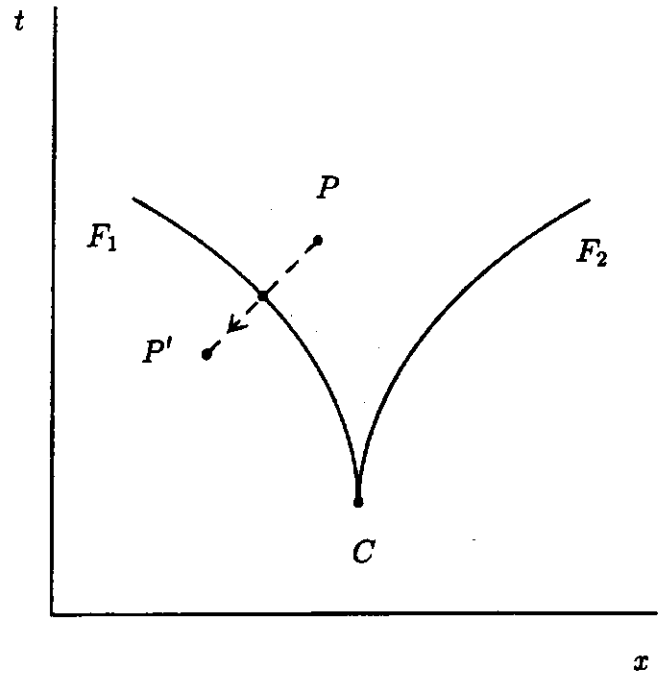

Fig. 3a. Point P passing through the fold curve F_1.

We find

$$I(t,x) \sim \sum_i (2\pi/kf_{\gamma\gamma}(\gamma_i,t,x))^{\frac{1}{2}} g(\gamma_i,t,x)\exp[-kf(\gamma_i,t,x)] \quad \text{as } k \to \infty, \tag{3.3}$$

where the summation extends over all γ_i satisfying (3.2). Typically, only one term in (3.3), viz. the one corresponding to the global minimum of f on Γ, contributes to I, while contributions of the other minima are negligible. With f defined as in the first equation of (3.1c), the conditions (3.2) become

$$s'(\gamma_i) + (\gamma_i - x)/2t = 0, \tag{3.4a}$$

$$s''(\gamma_i) + 1/2t > 0. \tag{3.4b}$$

Equation (3.4a) corresponds to (2.5a). Consequently, the ray denoted by γ_i may be identified with the locus of points in (t,x)-space, for which $f(\gamma,t,x)$ is stationary at $\gamma = \gamma_i$. Equation (3.4b) indicates that this stationary point is a local minimum, and thus gives an asymptotic contribution to I by the method of

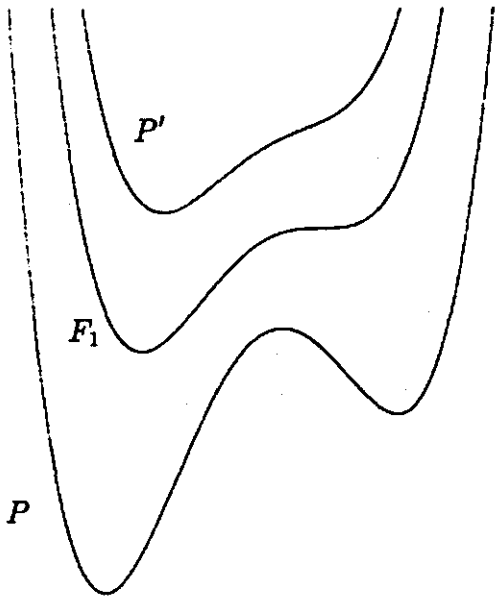

Fig. 3b. Coalescence of stationary points of f on F_1. The positions are indicated in Fig. 3a.

Laplace, as long as J is strictly positive along a ray, i.e. up to the moment that the tangent point with a caustic is reached. After passing this tangent point, the ray corresponds to a local maximum, see Fig. 2b, and does not contribute to the integral any more. Thus, from the three rays going through P in Fig. 2a, only γ_1 and γ_3 contribute to I. The contributions are given by (3.3) with $i = 1, 3$ and (3.1*b*). With (3.1*a*) this amounts to the same as (2.6) summed up for $\gamma = \gamma_1, \gamma_3$. The discussion which of the two contributions is dominant can be found in Section 5. The behaviour sketched above is different from that of the well-studied oscillatory integrals (which are of the form (3.1*b*) with f imaginary). In that case, application of the method of stationary phase yields asymptotic contributions from stationary points (thus local maxima included), and after touching a caustic a ray contributes as well.

For points P, P' in the (t, x)-plane, away from the caustic curves F_1, F_2, see Fig. 3a, expression (3.3) is a regular asymptotic expansion of I. However, when the point P passes through the caustic curve F_1 away from the point C, as indicated in Fig. 3a, the local minimum $f(\gamma_3, t, x)$ coalesces with the

local maximum $f(\gamma_2, t, x)$ and both stationary points disappear, see Fig. 3b. At the point of coalescence on F_1, $f_{\gamma\gamma} = 0$ and the corresponding expression (3.3) becomes singular. At the point C we have $f_{\gamma\gamma} = f_{\gamma\gamma\gamma} = 0$. The two minima $f(\gamma_1, t, x)$, $f(\gamma_3, t, x)$ and the intermediate maximum $f(\gamma_2, t, x)$ coalesce. In Section 4 we construct an asymptotic expansion of I for $k \to \infty$ that is uniformly valid in a region in the (t, x)-space containing caustic curves.

4. Uniform asymptotic expansion of the integral

In this section we study the uniform expansion for $k \to \infty$ of integrals of the type (3.1*b*) where we assume that f and g are known functions and that Γ is the real line. It is convenient to consider f as a function of γ, where t and x are merely parameters. We study the case that f has at most two local minima with respect to $\gamma \in \Gamma$, and where for some value of (t, x) these minima coalesce to form one minimum. Up to now we considered only real variables and functions. In order to apply the asymptotic theory of Chester, Friedman and Ursell [6] to the integral I, γ must be considered as a complex variable, and f and g as complex functions that are real for real γ. The parameters t and x remain real-valued. Moreover, f and g are assumed to be analytic functions of their arguments. Note that for all (t, x)-values there are always three complex stationary points γ_i of f, while not always three real stationary points γ_i of f exist. The typical way for three stationary points to coalesce, say at the point $\gamma = \gamma_c$ for $(t, x) = (t_c, x_c)$, is described as follows. In a neighbourhood of the point (t_c, x_c) in the (t, x)-parameter plane and for γ near γ_c, f has three stationary points γ_i:

$$f_\gamma(\gamma_i, t, x) = 0 \quad \text{for } i = 1, 2, 3. \tag{4.1}$$

By assumption, a stationary point distinct from the other two is simple, and if two (three) stationary points coalesce, it is into a stationary point of order two (three). This describes the cusp singularity. A merit of the asymptotic theory in [6] is that we can express the asymptotic solution of I in the presence of such a singularity, in terms of canonical integrals. This is done as follows. Near $\gamma = \gamma_c$, $(t, x) = (t_c, x_c)$ the cusp singularity can be brought in standard form by the change of variable from γ to ξ defined by [25]

$$f(\gamma, t, x) = \tfrac{1}{4}\xi^4 - a\xi^2 - b\xi + c \equiv P(\xi, a, b) + c, \tag{4.2}$$

where ξ is a function of γ, t, x and a, b, c are functions of t and x. For a description of the standard cusp we refer the reader to appendix A. Differentiation of expression (4.2) with respect to ξ gives

$$f_\gamma \frac{d\gamma}{d\xi} = \xi^3 - 2a\xi - b. \tag{4.3}$$

In order for (4.2) to be a one-to-one transformation, $d\gamma/d\xi$ must be different from zero or infinity. Consequently, the (complex) zeros γ_i of f_γ and the (complex) zeros of ξ_i of the right hand side of (4.3) must correspond

$$\gamma = \gamma_i \Longleftrightarrow \xi = \xi_i \quad \text{for } i = 1, 2, 3. \tag{4.4}$$

The zeros ξ_i can be expressed explicitly in terms of a and b, see appendix A. Insertion of (4.4) into (4.2) gives the therefore nonlinear equations

$$f(\gamma_i, t, x) = \tfrac{1}{4}\xi_i^4 - a\xi_i^2 - b\xi_i + c, \qquad i = 1, 2, 3. \tag{4.5}$$

When the values at the left hand side are known, these equations can be solved by an iterative method (not applicable near the caustic) or an algebraic method to obtain the functions a, b, c, see [12,10]. On the fold curve F_1 (where the roots γ_2 and γ_3 coalesce), the equations (4.5) imply

$$a = ((f_2 - f_1)/3)^{\frac{1}{2}}, \quad b = -(32/27)^{\frac{1}{2}}((f_2 - f_1)/3)^{\frac{3}{4}}, \quad c = (f_1 + 8f_2)/9, \tag{4.6}$$

with $f_2 = f_3$. On F_2, where γ_1 and γ_2 coalesce, similar formulas hold. We used the abbreviation f_i for $f(\gamma_i, t, x)$. At the cusp point C the expressions (4.6) reduce to

$$a = 0, \quad b = 0, \quad c = f_1 \ (= f_2 = f_3). \tag{4.7}$$

The integral I is written as

$$I(t, x) = \exp(-kc) \int g_0(\xi, t, x) \ \exp[-k(\tfrac{1}{4}\xi^4 - a\xi^2 - b\xi)] \ d\xi, \tag{4.8a}$$

where

$$g_0(\xi, t, x) = g(\gamma, t, x) \ \frac{d\gamma}{d\xi}. \tag{4.8b}$$

We assume that in a neighbourhood of $\gamma = \gamma_c$, $(t, x) = (t_c, x_c)$, g_0 has no zeros with respect to ξ. Straightforward expansion of g_0 in a power series in ξ near $\xi = 0$ results in a series for I in which successive terms do not decrease as

$k \to \infty$. In order to obtain an expansion that is asymptotic, we introduce the sequences of functions $\{g_i\}, \{h_i\}, \{p_i\}, \{q_i\}, \{r_i\}$, see [3]:

$$g_i = p_i + q_i\xi + r_i\xi^2 + (\xi^3 - 2a\xi - b)h_i, \tag{4.9a}$$

$$\frac{\partial h_i}{\partial \xi} = g_{i+1}, \tag{4.9b}$$

for $i = 0, 1, 2, \ldots, N$, where N is an arbitrary nonnegative integer, g_i, h_i are functions of ξ, t, x while p_i, q_i, r_i are functions of t and x. By repeated use of (4.9) in (4.8a) and integration by parts, we obtain the following uniform asymptotic expansion of (4.8a):

$$\begin{aligned} I(t,x) \sim \exp(-kc)\ \{&U_0(k^{\frac{1}{2}}a, k^{\frac{3}{4}}b)\sum_{i=0}^{N} p_i k^{-\frac{1}{4}-i} \\ &+U_1(k^{\frac{1}{2}}a, k^{\frac{3}{4}}b)\sum_{i=0}^{N} q_i k^{-\frac{1}{2}-i} \\ &+U_2(k^{\frac{1}{2}}a, k^{\frac{3}{4}}b)\sum_{i=0}^{N} r_i k^{-\frac{3}{4}-i}\}, \quad \text{as } k \to \infty, \end{aligned} \tag{4.10a}$$

where

$$U_i(a,b) = \int_{-\infty}^{+\infty} \xi^i\ \exp[-(\tfrac{1}{4}\xi^4 - a\xi^2 - b\xi)]\ d\xi, \quad i = 0, 1, 2, \tag{4.10b}$$

and the error of the expression between the brackets, made by truncation of the series, is of the order

$$O(k^{-N-1})\Big\{k^{-\frac{1}{4}}|U_0(k^{\frac{1}{2}}a, k^{\frac{3}{4}}b)| + k^{-\frac{1}{2}}|U_1(k^{\frac{1}{2}}a, k^{\frac{3}{4}}b)| + k^{-\frac{3}{4}}|U_2(k^{\frac{1}{2}}a, k^{\frac{3}{4}}b)|\Big\}. \tag{4.11}$$

The integration in (4.10b) is along the real axis (considered in the complex plane, this is from one valley at ∞ to another valley at ∞, as required by the theory). Thus, in the presence of a cusp singularity, the asymptotic solution for $k \to \infty$ of problem (2.1) is expressible in terms of the canonical integral $U_0(a,b)$ and, by

$$U_1(a,b) = \frac{\partial}{\partial b}U_0(a,b), \quad U_2(a,b) = \frac{\partial}{\partial a}U_0(a,b), \tag{4.12}$$

its first partial derivatives. Usually one is satisfied with the expansion (4.10a) with only $N = 0$. For $v(t,x)$ we then have

$$\begin{aligned} v(t,x) \sim (k/2\pi)^{\frac{1}{2}} \exp(-kc)\ \{&U_0(k^{\frac{1}{2}}a, k^{\frac{3}{4}}b)p_0 k^{-\frac{1}{4}} \\ &+U_1(k^{\frac{1}{2}}a, k^{\frac{3}{4}}b)q_0 k^{-\frac{1}{2}} \\ &+U_2(k^{\frac{1}{2}}a, k^{\frac{3}{4}}b)r_0 k^{-\frac{3}{4}}\}, \quad \text{as } k \to \infty. \end{aligned} \tag{4.13}$$

Henceforth we restrict ourselves to this case. Using the stationary points (4.4) in (4.9a) (with $i = 0$) and the expressions

$$\left(\frac{d\gamma}{d\xi}\right)_{\gamma_i} = \left(\frac{3\xi_i^2 - 2a}{f_{\gamma\gamma}(\gamma_i)}\right)^{\frac{1}{2}}, \quad i = 1, 2, 3, \tag{4.14}$$

which follow from (4.2) by two differentiations, we obtain the linear system

$$g(\gamma_i)\left(\frac{3\xi_i^2 - 2a}{f_{\gamma\gamma}(\gamma_i)}\right)^{\frac{1}{2}} = p_0 + \xi_i q_0 + \xi_i^2 r_0, \quad i = 1, 2, 3, \tag{4.15}$$

which can be solved for the functions p_0, q_0, r_0. In the limiting case that one or more stationary points coalesce, the right hand side of (4.14) is analytically well defined but numerically indeterminate. In such case it is sometimes convenient to use a so-called transitional approximation [11]. The canonical integral U_0 and its derivatives should be considered as special functions in the same way as Airy functions. Some properties of U_0 can be found in appendix B.

Example. Computations of the above kind can become quite involved (see worked out examples in [21,15]). In the present example we deal with functions that are already close to the standard form. Consider problem (2.1) with z identically one, and s given by the fourth degree polynomial

$$s(\gamma) = c_4\gamma^4 + c_3\gamma^3 + c_2\gamma^2 + c_1\gamma + c_0. \tag{4.16}$$

In order to satisfy condition (2.1c) we assume

$$c_4 > 0. \tag{4.17}$$

The second derivative of s is

$$s''(\gamma) = 12c_4\gamma^2 + 6c_3\gamma + 2c_2, \tag{4.18}$$

which is positive for $|\gamma|$ sufficiently large. The roots of $s''(\gamma) = 0$ are

$$\gamma_1 = \left(-c_3 - \sqrt{c_3^2 - 8c_2c_4/3}\right)/4c_4, \quad \gamma_2 = \left(-c_3 + \sqrt{c_3^2 - 8c_2c_4/3}\right)/4c_4, \tag{4.19}$$

which are real and distinct for

$$c_3^2 - 8c_2c_4/3 > 0. \tag{4.20}$$

We assume that (4.20) is satisfied. Then $s''(\gamma)$ is negative on the interval (γ_1, γ_2) and the corresponding rays get into a caustic curve, which is the cusp described above. The polynomial f given in (3.1c) with s as in (4.16) is brought into the standard form (4.2) by two transformations. By the transformation from γ to ρ defined by

$$\gamma = (4c_4)^{-\frac{1}{4}}\rho, \tag{4.21}$$

the coefficient in the term of degree four in the polynomial becomes $\frac{1}{4}$. Subsequently the transformation from ρ to ξ is applied, where

$$\rho = \xi + \tau, \tag{4.22}$$

and τ is determined such that the coefficient in the term of degree three in the polynomial becomes zero. As a result we find for $k \to \infty$:

$$v(t,x) = (k/4\pi t)^{\frac{1}{2}} h^{-\frac{1}{4}} \exp(-kc(t,x))\Big\{k^{-\frac{1}{4}} U_0(k^{\frac{1}{2}}a(t,x), k^{\frac{3}{4}}b(t,x)) + \\ + O(k^{-\frac{5}{4}})\Big\}, \tag{4.23}$$

with

$$\begin{aligned} a(t,x) &= \nu_0 - \nu_1/t, \\ b(t,x) &= \nu_2 + (\nu_3 + \nu_4 x)/t, \\ c(t,x) &= \nu_5 + (\nu_6 + \nu_7 x + x^2/4)/t, \end{aligned} \tag{4.24}$$

where the ν_i are expressed in the coefficients c_i as follows

$$\nu_0 = h^{-\frac{1}{2}}(3c_3^2/8c_4 - c_2), \quad (>0) \tag{4.25a}$$
$$\nu_1 = h^{-\frac{1}{2}}/4, \quad (>0) \tag{b}$$
$$\nu_2 = -2h^{-\frac{9}{4}}c_3^3 + 2h^{-\frac{5}{4}}c_3c_2 - h^{-\frac{1}{4}}c_1, \tag{c}$$
$$\nu_3 = h^{-\frac{5}{4}}c_3/2, \tag{d}$$
$$\nu_4 = h^{-\frac{1}{4}}/2, \quad (>0) \tag{e}$$
$$\nu_5 = -3h^{-3}c_3^4/4 + h^{-2}c_3^2c_2 - h^{-1}c_3c_1 + c_0, \tag{f}$$
$$\nu_6 = h^{-2}c_3^2/4, \quad (\geq 0) \tag{g}$$
$$\nu_7 = h^{-1}c_3/2, \tag{h}$$

and $h = 4c_4$. Inverting the first two equations in (4.24), we express t and x in a and b:

$$t = \nu_1/(\nu_0 - a), \quad x = (-\nu_3 + \nu_1(b - \nu_2)/(\nu_0 - a))/\nu_4. \tag{4.26}$$

In Fig. 1b we depicted rays for the case

$$s(\gamma) = \tfrac{1}{4}\gamma^4 + \gamma^3 - \gamma^2, \tag{4.27}$$

where

$$a(t,x) = \frac{5}{2} - \frac{1}{4t}, \quad b(t,x) = -4 + \frac{1+x}{2t}, \quad c(t,x) = \frac{1}{4}\left(-7 + \frac{1+2x+x^2}{t}\right). \tag{4.28}$$

5. Evaluation of the uniform asymptotic expansion near caustics

In the present section we discuss the application of the theory of the previous section to the diffusion problem of Section 2. Away from the caustic, the uniform asymptotic expansion (4.13) can be evaluated by the method of Laplace (we apply this method to (4.8a) with (4.9a)) to obtain

$$v(t,x) \sim \sum_i (3\xi_i^2 - 2a)^{-\frac{1}{2}}(p_0 + q_0\xi_i + r_0\xi_i^2)\ \exp[-k(P(\xi_i,a,b)+c)], \tag{5.1}$$

where the summation extends over the local minima of P with respect to ξ. We restrict ourselves to the global minima, since these yield the dominant contributions to v. Outside the cusp, only one local minimum exists, which therefore is global. Inside the cusp, two local minima ξ_1 and ξ_3 exist; away from the line $b = 0$, if $b > 0$ ($b < 0$), the minimum ξ_3 (ξ_1) is global; near $b = 0$, the minima ξ_1 and ξ_3 are of comparable magnitude and both should be accounted for. Thus, before a ray (say ξ_3) gets into a caustic curve (the fold F_1; the singularity near F_1 is described by integrals of incomplete Airy type with a boundary layer of order $O(k^{\frac{2}{3}})$, as follows from a simple version of the analysis in [15]), there is a region (extending from F_1 to a neighbourhood of $b = 0$) where it corresponds to a nonglobal minimum and where its contribution to v is negligible compared with that of the global minimum (ξ_1). Consequently, we may ignore the contribution of a ray shortly after it passes the line $b = 0$. Then, provided we are not too close to the cusp point, we never reach a vicinity of the fold curve to which the ray is tangent, and the contribution of the ray to v is always regular.

Next we study the region near $b = 0$, $a > 0$, where the rays ξ_1 and ξ_3, which correspond to the local minima of P ($P_\xi = 0, P_{\xi\xi} = 3\xi^2 - 2a > 0$), have

comparable contributions to v. The following analysis is valid for a not very small. Let $\xi_3 > 0$ and $\xi_1 = -\xi_3 - \epsilon$, where ϵ is a small positive number. The point of intersection (a_s, b_s) of the rays ξ_1 and ξ_3 is given by

$$a_s = (\xi_3^2 + \xi_3\xi_1 + \xi_1^2)/2 \approx \xi_3(\xi_3 + \epsilon)/2, \tag{5.2a}$$

$$b_s = -\xi_3^2\xi_1 - \xi_3\xi_1^2 \approx -\epsilon\xi_3^2, \tag{5.2b}$$

which is close to $b = 0$. At this point, we find for the function P in the exponential in (5.1):

$$\begin{aligned} P(\xi_1, a_s, b_s) &\approx \xi_3^3(-\xi_3/4 - 3\epsilon/2), \\ P(\xi_3, a_s, b_s) &\approx \xi_3^3(-\xi_3/4 + \epsilon/2). \end{aligned} \tag{5.3}$$

Forming the quotient of the exponentials in (5.1) we find

$$\exp[-k(P(\xi_3, a_s, b_s) + c)]/\exp[-k(P(\xi_1, a_s, b_s) + c)] = \exp(-k\xi_3^3 2\epsilon). \tag{5.4}$$

We discard the contribution from ξ_3 when this quotient is smaller than some value δ, $0 < \delta_{min} \leq \delta \leq 1$ (note δ is bounded below by some value δ_{min} and cannot be made arbitrary small). The corresponding value of ϵ is

$$\epsilon_\delta = -(2k\xi_3^3)^{-1} \log \delta. \tag{5.5}$$

For $\epsilon > \epsilon_\delta$ the contribution of ξ_3 is negligible with respect to that of ξ_1. It is instructive to state this result in terms of a_s and b_s. Using (5.2b) to express ϵ in terms of b_s and ξ_3, and using subsequently (5.2a) with $\epsilon = 0$ to express ξ_3 in terms of a_s, we find that the contribution of ξ_3 is negligible when $b_s < b_\delta$, where

$$b_\delta = k^{-1}(8a_s)^{-\frac{1}{2}} \log \delta, \tag{5.6}$$

This result shows how the width of the region where both ξ_1 and ξ_3 are important, decreases for increasing values of k and a_s.

From the discussion above it follows that away from the cusp point a valid asymptotic expansion is obtained from the dominant ray contribution(s). In a neighbourhood $\mathcal{N}$ of the cusp point where $a = O(k^{-\frac{1}{2}}), b = O(k^{-\frac{3}{4}})$, the expansion (4.10$a$) holds. The functions a, b, c and p_0, q_0, r_0 in this neighbourhood must be determined. Inside the cusp, the three real rays going through any given point can be constructed numerically by a boundary value method, see [23]. Three real values of Q (Section 2) are obtained, that, by identification of

(2.2) and (3.3) are the three values $f(\gamma_i, t, x)$ in Section 4, where we discussed the solution of equations (4.5) for a, b, c. The position *on* the cusp forms a limiting case that can be treated as well. Outside the cusp, the function f has one real and two complex conjugate stationary points. Accordingly, we have one real Q (associated with one real ray) and two complex conjugate Q-values (associated with two complex conjugate rays). The WKB-method of Section 2 produces only the one real ray and the associated real Q-value. Two more values of Q are required in (4.5) in order to determine a, b and c. Consequently, either we have to extend our ray method in Section 2 to obtain the complex conjugate values as well, or we must determine a, b and c outside the cusp in some other way. With respect to the latter possibility, we remark that since we need values of a, b, c in $\mathcal{N}$ only, these values can be obtained by extrapolation of nearby known values inside the cusp.

Similar remarks yield with respect to the determination of the functions p_0, q_0 and r_0. These follow from the linear system (4.15). We now assume the functions a, b, c and, consequently, ξ_i known in $\mathcal{N}$. For the same reason as above we only consider positions inside the cusp. There, two of the three rays going through any point P correspond to local minima of f. We rewrite the equations (4.15) for these rays as

$$w(\gamma_i, t, x)(3\xi_i^2 - 2a)^{\frac{1}{2}} = p_0 + q_0\xi_i + r_0\xi_i^2, \quad i = 1, 2, 3, \tag{5.7}$$

see (5.1). The values $w(\gamma_i, t, x)$ of the transport function can be obtained numerically by the ray method. The third ray going through P corresponds to a local maximum of f. Before reaching P, it touches a caustic, where w becomes infinite. Consequently, it is impossible to compute the transport function corresponding to that ray in P. This difficulty can be overcome as follows. Though w becomes infinite at a caustic, the left hand side of (4.15) remains finite since $d\gamma/d\xi$ does. We can compute values of the left hand side of (4.15) along the ray before it touches the caustic (we use the left hand side of expression (5.7) for this goal) and we use extrapolation to obtain values along this ray after touching the caustic. As a result we then have three linear equations that can be solved to obtain p_0, q_0 and r_0 in points P lying inside the cusp. Values of these functions in $\mathcal{N}$ outside the cusp can be obtained by extrapolation of the values inside the cusp.

An alternative way to determine the functions a, b, c is described as follows. Similarly as in [16], we can obtain differential equations for the functions

a, b, c and p_0, q_0, r_0. Therefore, the expression (4.13) for $v(t, x)$ is substituted into the diffusion equation (2.1a). Using the equations ($B2$) all second partial derivatives are written in terms of derivatives of lower than second order. This is easily seen, since it follows from ($B2$) that

$$U_{aa} = U + 2aU_a + bU_b. \tag{5.8}$$

The three highest order terms in k then vanish on setting

$$c_t + c_x^2 + 2a_x b b_x = 0, \tag{5.9a}$$

$$a_t - 2aa_x^2 - b_x^2 + 2a_x c_x = 0, \tag{5.9b}$$

$$b_t - a_x^2 b + 2b_x c_x - 4aa_x b_x = 0. \tag{5.9c}$$

The three next order terms in k vanish if

$$\begin{aligned} &2p_t + 4c_x p_x + 2(c_{xx} - a_x^2)p + 4a_x b q_x + 2(2a_x b_x + a_{xx} b)q + \\ &+ 2bb_x r_x + bb_{xx} r = 0, \end{aligned} \tag{5.10a}$$

$$\begin{aligned} &4b_x p_x + 2b_{xx} p + 2q_t + 4(c_x - 2aa_x)q_x + 2(c_{xx} - 2aa_{xx} + \\ &- 2a_x^2)q - 2(2ab_x + a_x b)r_x - (2ab_{xx} + 4a_x b_x + a_{xx} b)r = 0, \end{aligned} \tag{5.10b}$$

$$\begin{aligned} &4a_x p_x + 2a_{xx} p - 4b_x q_x - 2b_{xx} q + r_t + 2(c_x - 2aa_x)r_x + \\ &+ (c_{xx} - 2aa_{xx} - 3a_x^2)r = 0. \end{aligned} \tag{5.10c}$$

Here we dropped the subscripts of p_0, q_0 and r_0. The equations (5.9) are equivalent to the eikonal equation (2.3a) and the equations (5.10) form an improvement to the transport equation (2.3b) in the sense that they behave regular near the caustic. The computations above are too laborious to carry out by hand, and have been done by machine, using the formula manipulation package Macsyma. As remarked earlier, we are interested in the functions a, b, c and p_0, q_0, r_0 near the origin in the (a, b)-plane. A way to obtain these functions is to substitute their Taylor expansions at $a = b = 0$ into equations (5.9) and (5.10) above. The unknown Taylor coefficients are determined by using known values of these functions inside the cusp near $a = b = 0$, derived (with equations (4.5) and (4.15)) from numerical WKB-results.

6. Formal solution of the forward Kolmogorov equation

On the basis of the discussion in the preceding sections we are led to a formal approach to singularities (of which the cusp is one) arising in the asymptotic solution of the forward Kolmogorov equation with a small diffusion term. The problem is stated as follows. For $k \to \infty$ find the asymptotic solution of

$$\frac{\partial v}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(x)v) + \frac{1}{2k}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}(x)v), \tag{6.1a}$$

$$v(0,x) = v_0(x) \equiv z(x)\exp(-ks(x)), \quad z \geq 0, \tag{6.1b}$$

$$\int_\Gamma v(t,x)\, dx < \infty, \tag{6.1c}$$

where $t > 0$, $x_i \in (-\infty, +\infty)$, $i = 1, \ldots, n$, n is the spatial dimension (and consequently x a vector with components $x_1, \ldots, x_n$) and Γ extends over the entire R^n. Since the process in (6.1a) is time homogeneous we have the representation

$$v(t, x_1, \ldots, x_n) = \int_\Gamma v_0(\gamma_1, \ldots, \gamma_n) p(\gamma_1, \ldots, \gamma_n, t, x_1, \ldots, x_n)\, d\gamma_1 \ldots d\gamma_n, \tag{6.2}$$

where $p(\gamma_1, \ldots, \gamma_n, t, x_1, \ldots, x_n)$ is the transition probability (i.e. the conditional probability to be in $(x_1, \ldots, x_n)$, given that a time interval t before the position was $(\gamma_1, \ldots \gamma_n)$). The use of this integral in the case of simple stochastic processes (diffusion process, Ornstein-Uhlenbeck process) with known expressions for p, suggests the following representation for v in (6.1) to overcome singularities from intersecting rays:

$$v(t, x_1, \ldots, x_n) = (k/2\pi)^{\frac{n}{2}} \int_\Gamma g(\gamma_1, \ldots, \gamma_n, t, x_1, \ldots, x_n) \exp\left[-kf(\gamma_1, \ldots, \gamma_n, t, x_1, \ldots, x_n)\right] d\gamma_1 \ldots d\gamma_n. \tag{6.3}$$

Applying Laplace's method to the n-dimensional integral (6.3), a factor $(k/2\pi)^{-\frac{n}{2}}$ arises. We see from the initial condition (6.1b) that a factor in front of the exponential is of order k^0 (provided z is of this order). This explains the factor $(k/2\pi)^{\frac{n}{2}}$ in front of the integral in (6.3). As a generalization of (4.2) we change variables from γ_i to ξ_i as follows

$$f(\gamma_1, \ldots, \gamma_n, t, x_1, \ldots, x_n) = F(\xi_1, \ldots, \xi_m, a_1, a_2, \ldots) + c + Q(\xi_{m+1}, \ldots, \xi_n), \tag{6.4}$$

where F is the unfolding of the singularity in question, such as the cusp, the butterfly, etc. See for example [9] and related papers, or texts on catastrophe theory. The function Q is a sum of squares of its arguments representing the nonsingular part of f, by the Morse lemma. This latter part is not present in (4.2). The functions a_i (the number of these functions depends on the singularity) and c have arguments $t, x_1, \ldots, x_n$. The functions ξ_i $(i = 1, \ldots, n)$ depend on $\gamma_1, \ldots, \gamma_n, t, x_1, \ldots, x_n$. In much the same way as described in Section 4, we change variables from $\gamma_1, \ldots, \gamma_n$ to $\xi_1, \ldots, \xi_n$ and evaluate the resulting integral to arrive at a uniform asymptotic expansion of v in terms of new canonical integrals. In order for these integrals to be convergent (note Γ extends to infinity in all variables) not all singularities (as listed in for example [9]) can occur in F, and in Q the signs of the squares must be positive.

Example. In [22] we met a cusp singularity in the following problem. Determine the asymptotic solution of equation (6.1a) with $n = 2$ in stationary form (that is, with the left hand side equal to zero), where

$$\begin{aligned} &b_1(x) = x_1(1 - x_1) + x_2, \quad b_2(x) = -\beta x_2, \quad (\beta > 0, \beta \neq 1) \\ &a_{11} = a_{22} = 1, \quad a_{12} = a_{21} = 0. \end{aligned} \tag{6.5}$$

The deterministic system has the equilibrium points $(x_1, x_2) = (0, 0)$ and $(x_1, x_2) = (1, 0)$. The latter equilibrium is stable. The WKB-Ansatz (2.2), with the dependence on t omitted since we consider the stationary problem, leads to the system of ray equations

$$\frac{dx_1}{ds} = x_1(1 - x_1) + x_2 + p_1, \quad \frac{dx_2}{ds} = -\beta x_2 + p_2, \tag{6.6a}$$

$$\frac{dp_1}{ds} = (2x_1 - 1)p_1, \quad \frac{dp_2}{ds} = -p_1 + \beta p_2, \tag{6.6b}$$

$$\frac{dQ}{ds} = (p_1^2 + p_2^2)/2, \tag{6.6c}$$

where s is a variable along the rays and $p_i = \partial Q/\partial x_i$. The system (6.6$a$),(6.6$b$) has the equilibrium points

$$(x_1, x_2, p_1, p_2) = (1/2, -1/4(1 + \beta^2), -\beta^2/4(1 + \beta^2), -\beta/4(1 + \beta^2)),$$

$$(x_1, x_2, p_1, p_2) = (0, 0, 0, 0), \tag{6.7}$$

$$(x_1, x_2, p_1, p_2) = (1, 0, 0, 0).$$

Near the stable equilibrium $(x_1, x_2) = (1,0)$ of the deterministic system, it follows from a local analysis [22], that $Q(x) \approx \frac{1}{2}(x_1, x_2)P(x_1, x_2)^t$, where the matrix P is given by

$$P = \frac{2(1+\beta)}{\beta^2 + 2\beta + 2} \begin{pmatrix} 1+\beta & -1 \\ -1 & \beta^2 + \beta + 1 \end{pmatrix}. \tag{6.8}$$

We fixed $\beta = 3/2$. The system (6.6) of ordinary differential equations with the independent variable s has been solved numerically by an initial value routine. The initial points (for $s = 0$) were chosen on a circle with radius 0.01 around $(x_1, x_2) = (1,0)$. The corresponding initial values of p_1, p_2 and Q followed from the local analysis (by assumption, the initial condition for v is in the form $z(x)\exp(-ks(x))$ with $z(x)$ taken 1). A number of rays were chosen very close to the ray connecting $(x_1, x_2) = (1,0)$ with $(x_1, x_2) = (0,0)$. In the neighbourhood of $(x_1, x_2) = (0,0)$ part of those rays turn away to the right and intersect other rays. Figure 4 shows the rays and a cusp singularity.

Outside the cusp (and away from the cusp point), the WKB-method leads to an approximation of v for large k. The present paper describes how to use the WKB-solution to obtain an approximation of v inside and on the cusp (and away from the cusp point). Near the cusp point we have an expansion of v in terms of canonical integrals. That expansion is obtained similarly as described above. We make the Ansatz that the solution near the cusp point is represented by (6.3) with $n = 2$, where we leave out the dependence on t since we deal with a stationary problem. We change variables from γ_1, γ_2 to ξ_1, ξ_2 as follows

$$f(\gamma_1, \gamma_2, x_1, x_2) = \tfrac{1}{4}\xi_1^4 - a\xi_1^2 - b\xi_1 + c + \xi_2^2, \tag{6.9}$$

where the first three terms in the right hand side represent the unfolding of the cusp. The functions a, b and c depend on x_1, x_2 and the functions ξ_1 and ξ_2 depend on $\gamma_1, \gamma_2, x_1, x_2$. It turns out that the solution is of the asymptotic form

$$\begin{aligned} v(x_1, x_2) \sim (k/2\pi)^{\frac{1}{2}} \exp(-kc) \; \{ & U_0(k^{\frac{1}{2}}a, k^{\frac{3}{4}}b)p_0 k^{-\frac{1}{4}} \\ & + U_1(k^{\frac{1}{2}}a, k^{\frac{3}{4}}b)q_0 k^{-\frac{1}{2}} \\ & + U_2(k^{\frac{1}{2}}a, k^{\frac{3}{4}}b)r_0 k^{-\frac{3}{4}} \}, \quad \text{as } k \to \infty. \end{aligned} \tag{6.10}$$

The functions a, b, c and p_0, q_0, r_0 satisfy the following equations, similar to (4.5) and (4.15) respectively

$$f(\gamma_{1i}, \gamma_{2i}, x_1, x_2) = \tfrac{1}{4}\xi_{1i}^4 - a\xi_{1i}^2 - b\xi_{1i} + c, \qquad \text{i=1,2,3,} \tag{6.11}$$

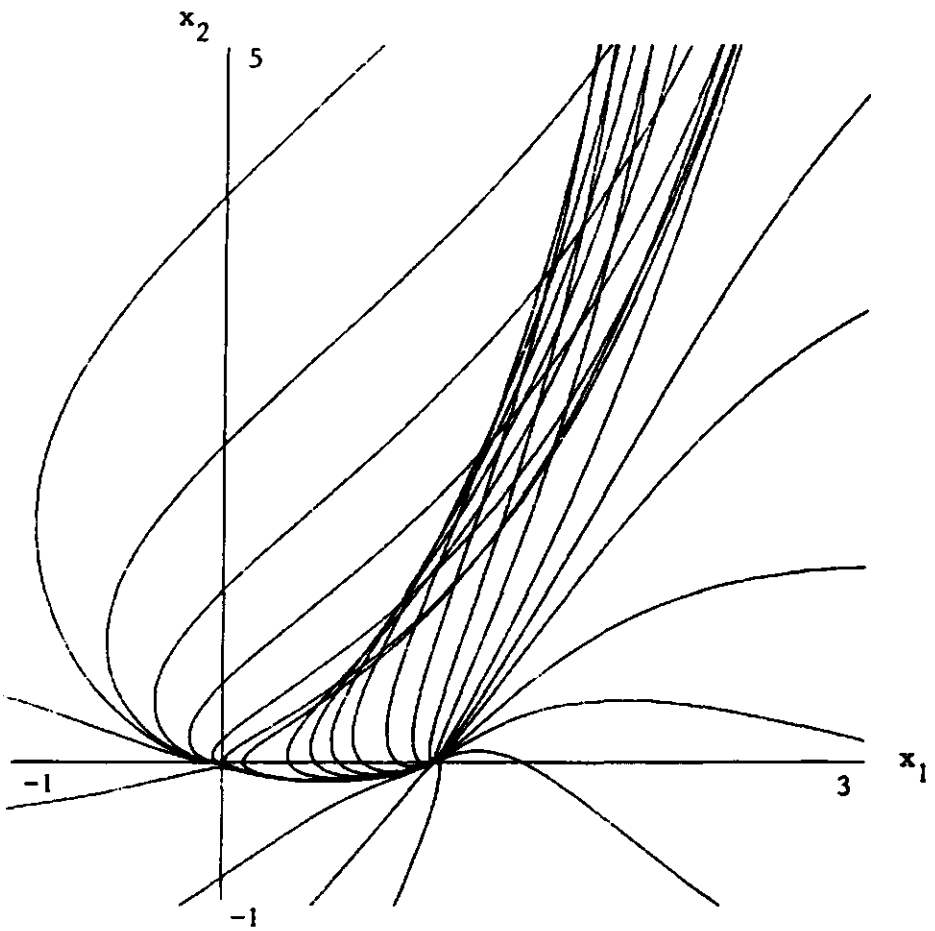

Fig. 4. A cusp singularity arising in the numerical solution of a stationary forward Kolmogorov equation.

and

$$g(\gamma_{1i},\gamma_{2i})\left[\frac{3\xi_{1i}^2-2a}{D(\gamma_{1i},\gamma_{2i})}\right]^{\frac{1}{2}} = p_0+q_0\xi_{1i}+r_0\xi_{1i}^2, \qquad \text{i=1,2,3,} \tag{6.12}$$

where

$$D(\gamma_1,\gamma_2) = f_{\gamma_1\gamma_1}f_{\gamma_2\gamma_2}-f_{\gamma_1\gamma_2}^2. \tag{6.13}$$

The index i in γ_{1i} and γ_{2i} denotes points where $\partial f/\partial\gamma_1 = 0, \partial f/\partial\gamma_2 = 0$, respectively. For a derivation of (6.11) and (6.12) see [8]. We proceed similarly

as in Section 5 to obtain the functions a, b, c and p_0, q_0, r_0 from data obtained by the numerical solution of the ray equations (6.6).

Acknowledgements. I appreciate the discussions I had with Johan Grasman. I wish to thank Nico Temme for bringing to my attention the literature on the asymptotic expansion of integrals and Ben Dijkhuis for carrying out computations with Macsyma.

Appendix A: The standard cusp

In standard form the cusp is given by

$$P(\xi, a, b) \equiv \tfrac{1}{4}\xi^4 - a\xi^2 - b\xi. \tag{A1}$$

Repeated differentiation yields

$$P_\xi = \xi^3 - 2a\xi - b, \tag{A2a}$$

$$P_{\xi\xi} = 3\xi^2 - 2a, \tag{A2b}$$

$$P_{\xi\xi\xi} = 6\xi, \tag{A2c}$$

$$P_{\xi\xi\xi\xi} = 6. \tag{A2d}$$

With these derivatives we define the following sets in the (a, b)-parameter plane. Lines satisfying $P_\xi = 0$:

$$b = \xi^3 - 2a\xi, \tag{A3}$$

correspond to rays, see Fig. 5. Along the line (A3), P is given by

$$P(\xi, a) = -\tfrac{3}{4}\xi^4 + a\xi^2. \tag{A4}$$

The points $P_\xi = P_{\xi\xi} = 0$:

$$27b^2 - 32a^3 = 0, \tag{A5}$$

form the caustic set. For $b \leq 0$ ($b \geq 0$) this set consists of the fold curve F_1 (F_2). The set $P_\xi = P_{\xi\xi} = P_{\xi\xi\xi} = 0$ consists of the cusp point C:

$$a = b = 0 \quad (= F_1 \cap F_2). \tag{A6}$$

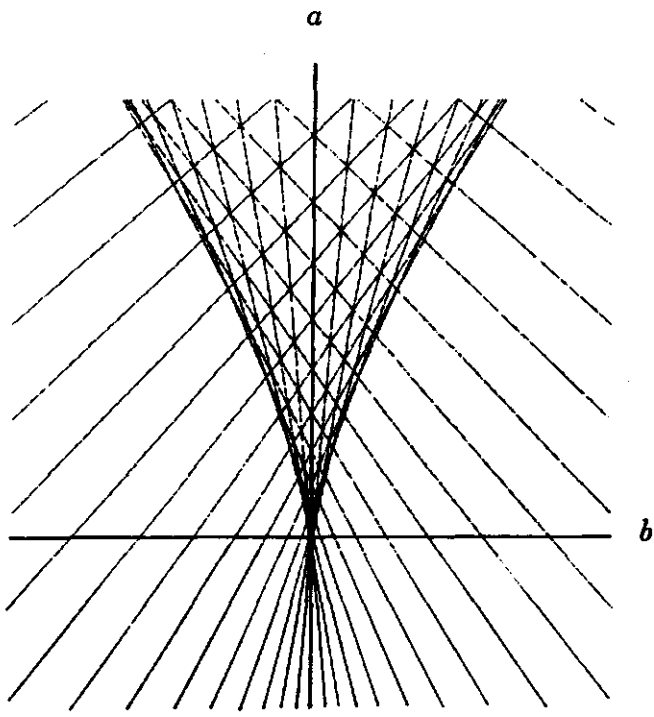

Fig. 5. Lines in the standard cusp that correspond with rays.

Fig. 6 shows the function $P(\xi, a, b)$ for various fixed values of a and b. As described in Section 4, the roots of $P_\xi = 0$ play an important role. These roots can be expressed explicitly in terms of a and b as follows.
For $27b^2 \leq 32a^3$ (inside or at caustic):

$$\begin{aligned} \xi_1 &= -2(2a/3)^{\frac{1}{2}} \sin(\pi/3 + \phi), \\ \xi_2 &= 2(2a/3)^{\frac{1}{2}} \sin\phi, \\ \xi_3 &= 2(2a/3)^{\frac{1}{2}} \sin(\pi/3 - \phi), \end{aligned} \tag{A7a}$$

where

$$\phi = \tfrac{1}{3} \arcsin[-\tfrac{1}{2} b (2a/3)^{-\frac{3}{2}}], \quad |\phi| \leq \pi/6. \tag{A7b}$$

For $a > 0$, $27b^2 \geq 32a^3$ we have the roots

$$\operatorname{sign}(b) 2(2a/3)^{\frac{1}{2}} \cosh\psi, \quad -\operatorname{sign}(b)(2a/3)^{\frac{1}{2}} \cosh\psi \pm i(2a)^{\frac{1}{2}} \sinh\psi, \tag{A8a}$$

where

$$\psi = \tfrac{1}{3} \operatorname{arccosh}[\tfrac{1}{2}|b|(2a/3)^{-\frac{3}{2}}]. \tag{A8b}$$

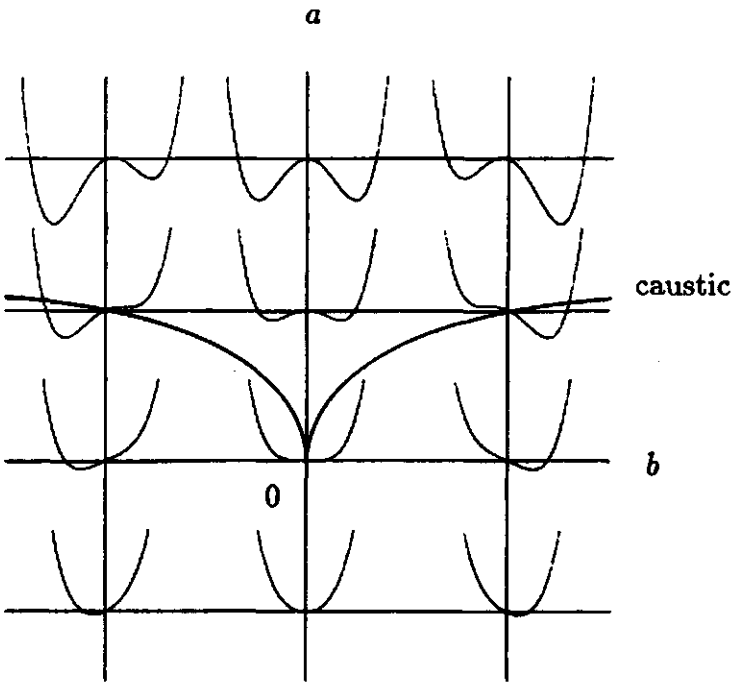

Fig. 6. Graph of the function $P(\xi, a, b)$ for various fixed values of a and b (indicated by the origin of each graph).

For $a = 0$ we have the roots

$$b^{\frac{1}{3}}, \quad \tfrac{1}{2}(-b)^{\frac{1}{3}} \pm i\tfrac{3}{4}(-b)^{\frac{2}{3}}, \tag{A9}$$

and when $a < 0$:

$$-2|2a/3|^{\frac{1}{2}} \sinh\theta, \quad |2a/3|^{\frac{1}{2}} \sinh\theta \pm i|2a|^{\frac{1}{2}} \cosh\theta, \tag{A10a}$$

where

$$\theta = \tfrac{1}{3}\mathrm{arcsinh}[-\tfrac{1}{2}b|2a/3|^{-\frac{3}{2}}]. \tag{A10b}$$

These formulas have been obtained using [20]. Thus, inside the caustic there are three real roots that are ordered as $\xi_1 < \xi_2 < \xi_3$. At F_1 (F_2) the roots ξ_2, ξ_3 (ξ_1, ξ_2) coalesce; at C ξ_1, ξ_2, ξ_3 coalesce. Outside the caustic one real and two complex roots exist. Apart from the formulas above, that one real root can also be obtained from Cardan's formula [1]:

$$\xi = [b/2 + \{(-2a/3)^3 + (b/2)^2\}^{\frac{1}{2}}]^{\frac{1}{3}} + [b/2 - \{(-2a/3)^3 + (b/2)^2\}^{\frac{1}{2}}]^{\frac{1}{3}}. \tag{A11}$$

For any value of a and b, the roots ξ_i of $P_\xi = 0$ are related by:

$$\begin{aligned} &\xi_1 + \xi_2 + \xi_3 = 0, \\ &\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1 = -2a, \\ &\xi_1\xi_2\xi_3 = b, \end{aligned} \tag{A12}$$

see [1].

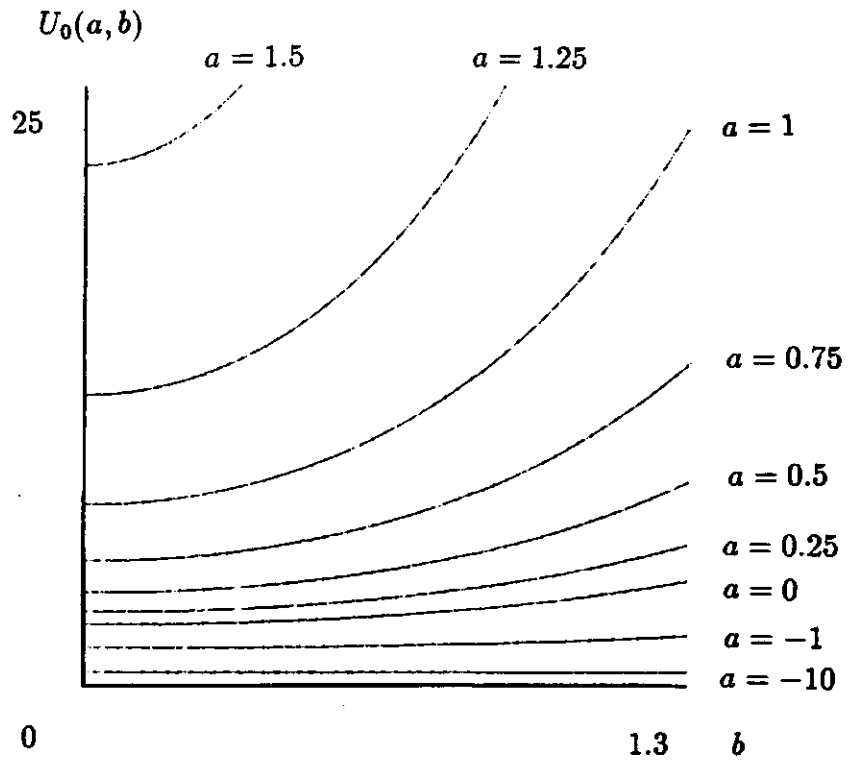

Fig. 7. Graph of the canonical integral $U_0(a,b)$ for various fixed values of a. Because of symmetry only positive values of b have been considered.

Appendix B: Properties of the canonical integral

Without derivation we mention some elementary properties of the canonical integral $U_0(a,b)$, which we denote for convenience by $U(a,b)$. These properties resemble those of the oscillatory counterpart of U, see for example [12]. The function U is symmetric with respect to b:

$$U(a,-b) = U(a,b). \tag{B1}$$

The following partial differential equations hold:

$$U_{bb} - U_a = 0, \tag{B2a}$$

$$U_{ab} - 2aU_b - bU = 0. \tag{B2b}$$

The derivatives of $U(a,b)$ at $(0,0)$ are given by

$$\frac{\partial^{m+n}U}{\partial a^m \partial b^n}(0,0) = \begin{cases} 0, & n \text{ odd}, \\ 2^{\frac{2m+n-1}{2}}\,\Gamma\left(\frac{2m+n+1}{4}\right), & n \text{ even}, \end{cases} \tag{B3}$$

which lead to the following Taylor series of $U(a,b)$ at (0,0):

$$U(a,b) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a^m b^{2n}}{m!(2n)!}\, 2^{\frac{2m+2n-1}{2}}\,\Gamma\left(\frac{2m+2n+1}{4}\right). \tag{B4}$$

The corresponding series for U_1 and U_2 are obtained by differentiation of (B4) with respect to b and a, respectively. The values of U and its first derivatives at (0,0) are

$$U(0,0) = 2^{-\frac{1}{2}}\,\Gamma(\tfrac{1}{4}) \approx 2.5637, \tag{B5a}$$

$$U_a(0,0) = 2^{\frac{1}{2}}\,\Gamma(\tfrac{3}{4}) \approx 1.7330, \tag{B5b}$$

$$U_b(0,0) = 0. \tag{B5c}$$

For $b = 0$ we have

$$U(a,0) = 2\sqrt{2}\,\exp(a^2/2)\,\mathrm{Ei}_4(-a/2), \tag{B6a}$$

$$= \begin{cases} \sqrt{-a}\,\exp(a^2/2)\,\mathrm{K}_{\frac{1}{4}}(a^2/2), & a \le 0, \\ \pi\sqrt{a/2}\,\exp(a^2/2)\left[\mathrm{I}_{-\frac{1}{4}}(a^2/2) + \mathrm{I}_{\frac{1}{4}}(a^2/2)\right], & a \ge 0, \end{cases} \tag{B6b}$$

where Ei is the Airy-Hardy integral [26], and K,I are modified Bessel functions. For large $|a|$ we have the expansions (use [1], p. 378, 9.7.2 and 9.7.1):

$$U(a,0) \sim \begin{cases} \sqrt{\pi/(-a)}\,\{1 - \frac{3}{16a^2} + \cdots\}, & a \to -\infty, \\ \sqrt{2\pi/a}\,\exp(a^2)\,\{1 + \frac{3}{16a^2} + \cdots\}, & a \to +\infty. \end{cases} \tag{B7}$$

Note that for $a \to -\infty$ there is a gradual decay to zero and for $a \to +\infty$ there is an exponentially fast growth. For some purposes it may be desirable to calculate $U(a,b)$ numerically. Using the symmetry of $U(a,b)$ with respect to b and the change of variable from ξ to y in (4.10b) (with $i = 0$), defined by $\exp(-\xi) = y$, we have

$$U(a,b) = \int_0^1 \exp[-\tfrac{1}{4}(\log y)^4 + a(\log y)^2]\{y^{b-1} + y^{-b-1}\}\, dy, \qquad (B8)$$

where the integration domain has turned into a finite interval. This form is suitable for a numerical integration routine. Note that the integrand equals 0 at $y = 0$ and 2 at $y = 1$. The canonical integral has been depicted in Fig. 7.

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SAMENVATTING

Analyse van het Uittreeprobleem voor Stochastisch Gestoorde Dynamische Systemen in Toepassingen

Een groot aantal verschijnselen dat optreedt in de biologie, mechanica, scheikunde en natuurkunde kan wiskundig gemodelleerd worden als een dynamisch systeem. Deze verschijnselen kunnen onderhevig zijn aan kleine toevalsfluctuaties. Indien deze fluctuaties van ondergeschikt belang zijn voldoet een deterministische beschrijving. In het andere geval dienen toevalsfluctuaties in de beschrijving van het verschijnsel opgenomen te worden. We hebben dan te maken met een stochastisch model. In een stochastisch model wordt onderscheid gemaakt tussen een systematisch gedeelte (het deterministische systeem) en een gedeelte bestaande uit toevalsfluctuaties. De toevalsfluctuaties spelen met name een belangrijke rol in de buurt van stabiele evenwichten van het deterministische systeem. Dit komt omdat een systeem lange tijd doorbrengt in de buurt van zulke evenwichten en toevalsfluctuaties gedurende deze tijd dus alle gelegenheid krijgen om zich te laten gelden. Dit aspect van stochastische modellen vormt het onderwerp van het proefschrift.

Stabiele evenwichten van een deterministisch systeem zijn bijvoorbeeld een puntaantrekker of een neutraal evenwichtspunt. Bij afwezigheid van toevalsfluctuaties is het dynamisch gedrag in de buurt van deze evenwichtspunten als volgt te beschrijven. Een systeem dat start in het aantrekkingsgebied van een puntaantrekker zal deze aantrekker uiteindelijk steeds dichter naderen. Een systeem dat start in de buurt van een zeker neutraal evenwicht zal een gesloten baan volgen om dit evenwicht heen. De aanwezigheid van kleine toevalsfluctuaties verandert het evenwichtsbeeld volledig. Als gevolg van deze fluctuaties wordt een begrensd gebied dat het evenwichtspunt bevat in eindige (maar soms

lange) tijd verlaten. Een maat voor de stabiliteit van een stochastisch systeem in de buurt van een stabiel evenwichtspunt is de verwachtingswaarde van de tijd die nodig is om uit zo'n gebied te ontsnappen. Bij een puntaantrekker moet men denken aan een verwachte ontsnappingstijd van de orde $K\exp(L/\epsilon)$, bij een neutraal evenwicht aan een verwachte ontsnappingstijd van de orde M/ϵ, waarin ϵ een kleine positieve parameter is die de intensiteit van de toevalsfluctuaties voorstelt en K, L en M constanten zijn. De toevalsfluctuaties worden gemodelleerd d.m.v. Gaussische witte ruis. Van practisch belang is ook de vraag waar op de rand het gebied verlaten wordt. In dit proefschrift richten wij ons hoofdzakelijk op de bepaling van statistische grootheden met betrekking tot de ontsnappingstijd en de plaats van ontsnapping. Wij gaan uit van *kleine* toevalsfluctuaties. Enerzijds is dit in veel gevallen een realistische aanname, anderzijds stelt dit ons in staat om voor bovengenoemde statistische grootheden asymptotische uitdrukkingen af te leiden. Hierbij is ϵ de kleine parameter.

Een model met een neutraal evenwicht komen we tegen in hoofdstuk 2. Het is beter om te spreken van een bijna-neutraal evenwicht want een kleine demping is toegestaan. Het gaat om een stijve staaf die aan een kant elastisch scharniert en aan de andere kant wordt belast. De belasting bestaat uit een constante deterministische kracht en een kleine stochastisch fluctuerende kracht. Beide krachten werken in een algemeen gehouden constante richting. Als gevolg van de stochastisch fluctuerende kracht kan de energie van het systeem toenemen en op een gegeven moment een kritieke waarde overschrijden. We beschouwen het gebied in de faseruimte dat het evenwichtspunt bevat en dat wordt begrensd door de gesloten kromme met energie gelijk aan de kritieke energie. De asymptotische methode waarvan wij ons in dit hoofdstuk bedienen om statistische uitdrukkingen met betrekking tot de ontsnappingstijd uit dit gebied af te leiden is goed gefundeerd, zie de desbetreffende referenties in hoofdstuk 1.

De hoofdstukken 3 en 5 beschrijven de dynamica van een systeem van biologische populaties met interactie. De faseruimte bestaat uit punten, waarvan de coördinaten de grootte (dichtheid) van de populaties weergeven. Deze ruimte wordt dus gevormd door het positieve orthant en de randen daarvan. We nemen aan dat dit systeem een deterministische puntaantrekker heeft in het inwendige van de faseruimte. We zijn geïnteresseerd in het verschijnsel dat het systeem als gevolg van toevalsfluctuaties een rand bereikt, m.a.w. dat een po-

pulatie uitsterft. De verwachtingswaarde van de tijd die hiervoor nodig is vormt een maat voor de persistentie van het ecologische systeem. Bovendien zijn we geïnteresseerd in welke rand waarschijnlijk het eerst bereikt wordt, m.a.w. welke populatie waarschijnlijk het eerst uitsterft. De desbetreffende statistische grootheden voldoen aan tweede orde partiële differentiaalvergelijkingen. Asymptotische uitdrukkingen worden verkregen door een WKB-aanname. Deze aanname leidt tot een stelsel eerste orde gewone differentiaalvergelijkingen (de stralenvergelijkingen). Dit stelsel wordt numeriek opgelost. Hoofdstuk 4 bespreekt enkele aspecten van deze numerieke integratie. Hoofdstuk 6 analyseert het soms waargenomen verschijnsel van elkaar snijdende stralen. Een voordeel van genoemde asymptotische methode is dat men haar kan toepassen op een grote klasse van stochastische problemen. Een nadeel van deze methode is het formele karakter.

Gezien het feit dat tal van verschijnselen zich laat beschrijven als dynamisch systeem en toevallig fluctuerende invloeden veelvuldig aanwezig zijn, mag men ervan uitgaan dat de practische toepasbaarheid van de gebruikte methoden en de afgeleide resultaten groot is.

C.V.

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