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Institute for Land and Water Management Research

FORMULAE FOR THE GROUNDWATER FLOW IN AREAS WITH SUB-IRRIGATION BY MEANS OF OPEN CONDUITS WITH A RAISED WATER LEVEL

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expressed by:

$$kh^2 = v_{phr}\, x^2 + a_1\, x + a_o \qquad (2)$$

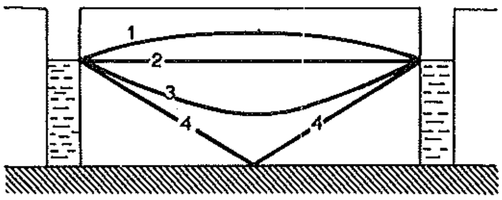

Fig. 1. Shape of the groundwater level in case of a mainly horizontal steady flow towards two parallel fully penetrating drains in an aquifer of homogeneous permeability

1 = ellipse 2 = straight line 3 = hyperbola
4 = two intersecting straight lines

In case the vertical axis of either ellipse or hyperbola is placed at x = 0, the integration constant a_1 vanishes. Besides, in stead of a_o, the elevation of the phreatic surface h(0) in the middle of the field may be used as a parameter.

$$\left[h(x)\right]^2 - \left[h(0)\right]^2 = \frac{v_{phr}\, x^2}{k} \qquad (3)$$

If there is a fixed level of the water in the open conduits, but v_{phr} is gradually increasing, finally h(0) will be located in the impermeable base and consequently the hyperbola will be degenerated into a pair of intersecting straight lines. The theoretical errors in the approximative equations (2) and (3) are very small (ERNST, 1963; fig. 1).

3. SOLUTION OF EQUATION (1) IN CASE OF A CONSTANT FLUX THROUGH THE PHREATIC SURFACE WITHOUT NEGLECT OF THE RADIAL FLOW

For those cases in which the radial flow cannot be neglected, two methods have been developed in the Netherlands. The first to be mentioned is the approach according to Hooghoudt in which the original shape of the equation (3), already known much earlier, may be recognized (HOOGHOUDT, 1940).

$$d^2 - (d + h_m - h_o)^2 = \frac{v_{phr}}{k}\left(\frac{L}{2}\right)^2 \qquad (4)$$

or:

$$- v_{phr} = \frac{4k(h_m - h_o)^2 + 8kd(h_m - h_o)}{L^2} \qquad (5)$$

h_m = elevation of the phreatic surface in the middle of the field

h_o = level of the water in the open conduits, in the drainage tubes, or in the infiltration tubes

d = thickness of the 'equivalent layer' (fig. 2)

L = spacing of the parallel open conduits

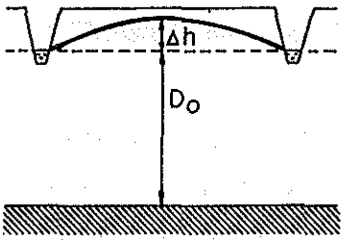

Fig. 2a. Actual situation with minimal thickness D_o of the aquifer near the open conduit and with a potential difference $\Delta h = h_m - h_o$ over the phreatic surface

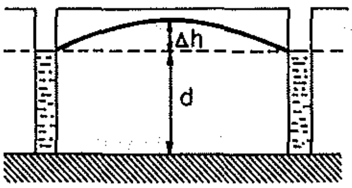

Fig. 2b. Imaginary situation with fully penetrating drains. Inflow, outflow, the potential difference Δh and the drainspacing L are equal to what is supposed in fig.2a. Consequently the 'equivalent layer' possesses a thickness d which is smaller than D_o in fig. 2a

Another formula for groundwater flow between parallel open conduits has been derived by introducing the conception 'radial resistace', or more precisely: 'the extra resistance within the area with radial flow' (see fig. 3 and ERNST, 1956, 1962):

$$h_o - h_m = v_{phr}\left(\frac{L^2}{8kD} + L\Omega\right) \qquad (6)$$

$$\frac{h_o - h_m}{D} = \frac{v_{phr}}{8k}\left(\frac{L}{D}\right)^2 \left(1 + \frac{8kD\Omega}{L}\right) \qquad (7)$$

D = (mean) thickness of aquifer

Ω = radial resistance

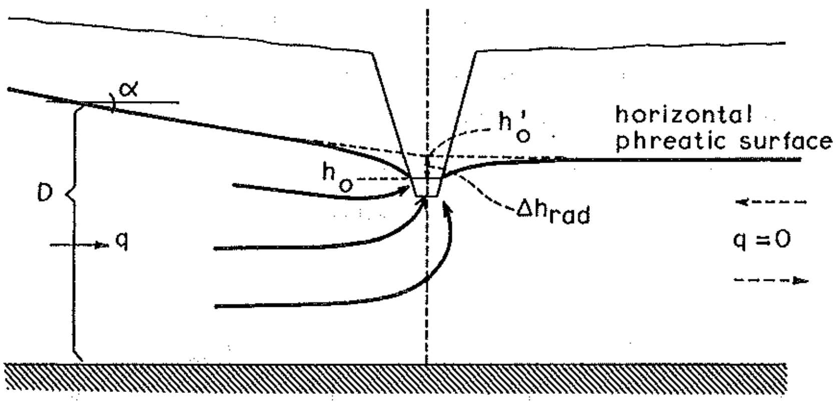

Fig. 3. Onesided flow towards a long straight drain within an aquifer of homogeneous permeability. Near the drain the radial component of the flow requires a potential difference Δh_{rad}

$\Delta h_{rad} = h_o' - h_o = q\Omega$

q = flow from lefthand side towards drain = kD tg α

If the soil is homogeneously permeable and there are no large differences in elevation of the phreatic surface the next formula may be applied for a calculation of the radial resistance:

$$\Omega = \frac{1}{\pi k} \ln \frac{D_o}{B_o} \qquad (8)$$

D_o = thickness of aquifer near the open conduit
B_o = (half-circular) wetted perimeter of the open conduit

From (5) and (6) and again for small $(h_m - h_o)$ so that $D' \approx D_o$ the following relation between d and Ω may be derived:

$$\frac{d}{D} = \frac{L}{L + 8kD\Omega} \qquad (8)$$

D in formula (6) has not necessarily to be considered as a constant, and Ω also may be varying in some way. The radial resistance has primarily to be considered dependent on q_0/kB_o, where q_o represents the flow intensity through the bottom of the open conduit (ERNST, 1962, fig. 28c). The variations in D and Ω may be neglected for reasons of simplification, and these quantities in (6) considered constant, the formula represents a linear relation between $h_o - h_m$ and v_{phr}.

Both the 'linear' relation according to formula (6) and the parabolaic relation according to formula (5) are shown in fig. 4. It is clear, that this parabolaic relation is also not completely correct; for in case of a very deep watertable with h_m at a depth d below h_o, the maximum value of v_{phr} cannot be obtained yet, since this will happen only in case h_m will be located in the impermeable base.

The correct relation is indicated in fig. 4 by a dotted line. It is quite conceivable in principle that this relation may be represented by formulae (6) or (7), provided D (mean value) and Ω are depending correctly on v_{phr} and possibly on other quantities. In case homogeneously permeable soils are concerned a reasonable approximation may be obtained by writing (7) for positive v_{phr} as follows:

$$\frac{h_o - h_m}{\bar{D}} = \frac{v_{phr}}{8k}\left\{\frac{L}{\bar{D}}\right\}^2 \left(1 + \frac{8kD_o\Omega_o}{L}\right) \tag{10}$$

$\bar{D}$ = mean value of the thickness of the layer = $D_o - \frac{1}{2}(h_o - h_m)$

Ω_o = radial resistance in case $q_o \to 0$

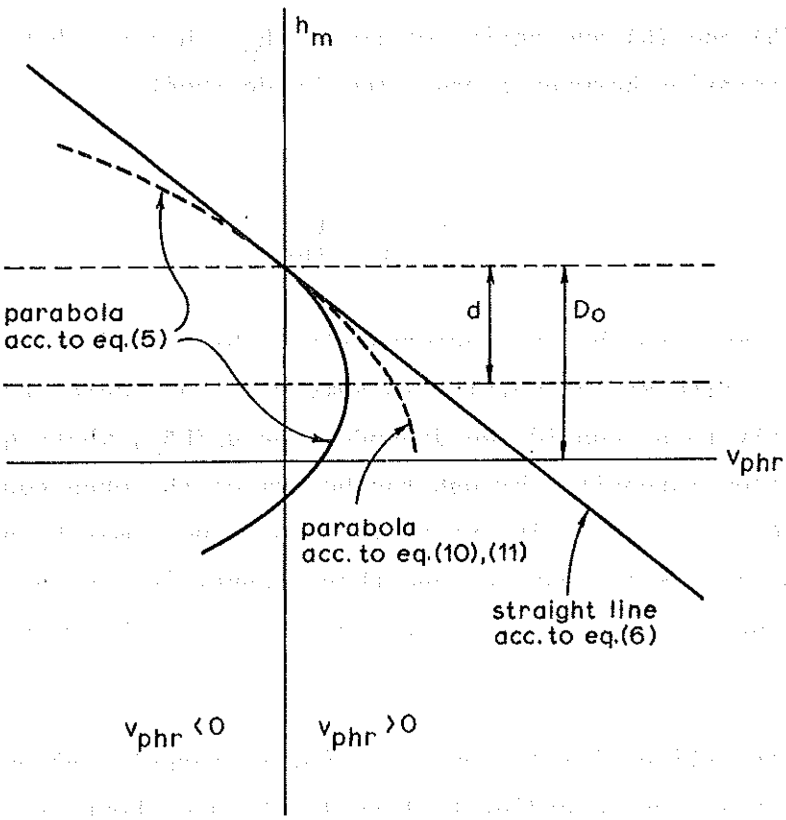

Fig. 4. Graphical representation of the relation between h_m and v_{phr} for steady states with v_{phr} constant for all x

h_m = level of the phreatic surface above the impermeable base and midway the drains

v_{phr} = flux through the phreatic surface (> 0 for upward direction)

h_o = level of open water

With the aid of (9) and (10), (5) can also be rewritten, which results in:

$$v_{phr} = \frac{8kd(h_o - h_m) - 4k\frac{d}{D_o}(h_o - h_m)^2}{L^2} \qquad (11)$$

Summarizing, it may be noted that for small positive and negative values of v_{phr}, the linear relation according to formula (6) will yield a good approximation. The parabolaic relation according to formula (5) can be recommended for all negative values of v_{phr}. For all positive values (provided these will not be so strongly positive as to make the phreatic surface reach the impermeable base over two parallel lines on both sides of x = 0) a good result may be expected by means of a modified parabola according to formula (11).

4. SUB-IRRIGATION FORMULAE IN CASE v_{phr} DEPENDS ON THE DEPTH OF THE PHREATIC LEVEL BELOW SOIL SURFACE OR BELOW BOTTOM OF ROOT ZONE

With a variable v_{phr} dependent on the groundwater level, the differential equation (1) will be particularly manageable in case the variation in the thickness of the aquifer will be small. In that case a constant value D can be taken and (1) passes into:

$$kD\,\frac{d^2h}{dx^2} = v_{phr} \qquad (12)$$

In the righthand part of (12) an expression has to be substituted indicating in which way v_{phr} is depending on h. Because here again equilibrium states are concerned, the vertical groundwater flow in the unsaturated zone from the phreatic surface upward to the lower side of the root zone will have a constant intensity independent of z. From this it follows immediately that very deep groundwater levels will be connected with very small positive values of v_{phr}. Very large values of v_{phr} can be expected in case the phreatic surface will be rather close to the bottom of the root zone and a sufficient degree of desiccation is maintained in the root zone. Actually, suchlike conditions are not likely to occur, since shallow groundwater levels will make the topsoil remain rather moist, so that the generated upward flow has to suffice only a bridging of the gap between the mean values of precipitation P and potential evaporation E_p. In the first approximation the relation between v_{phr} and h can be assumed to be as shown schematically in fig. 5a and b.

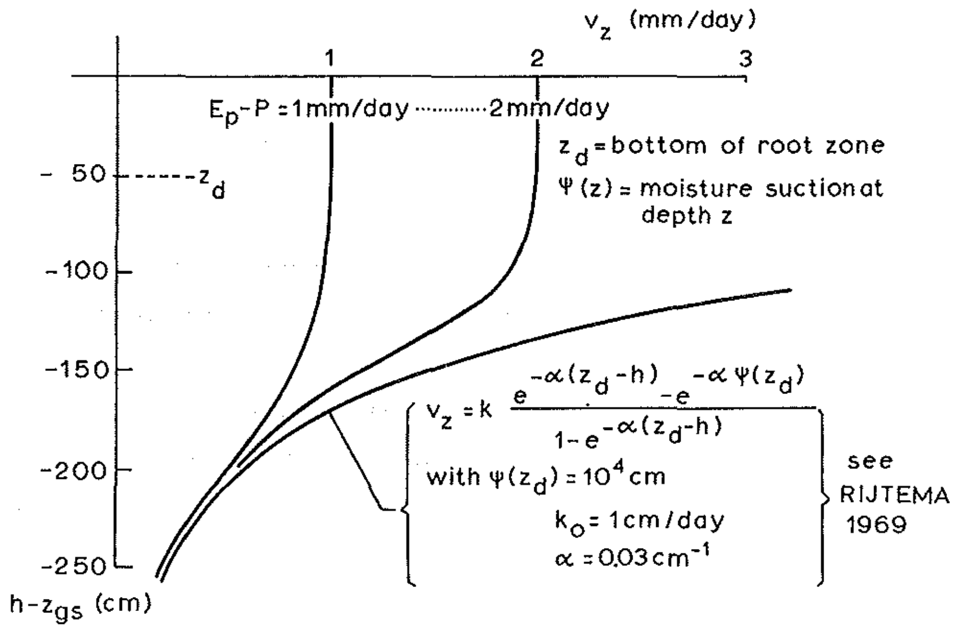

Fig. 5a. Graphical representation of the relation between the upward vertical flow v_z in the unsaturated zone and the depth $h - z_{gs}$ of the phreatic surface. Because of the steady state an equal value for vertical flow is valid for any value of z and consequently $v_z = v_{phr}$.

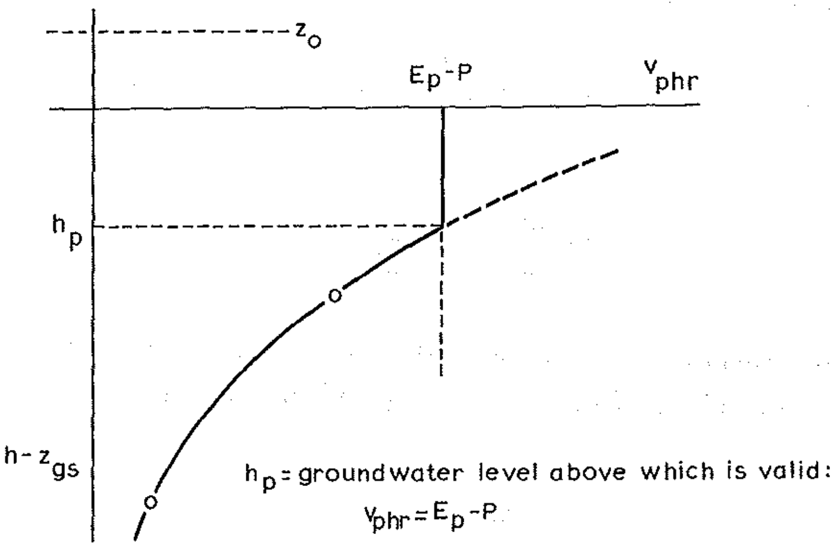

Fig. 5b. Simplification of fig. 5a

In order to obtain a simple solution of equation (12) fig. 5a has to be replaced by fig. 5b. This implies the assumption that in case of shallow groundwater levels, the upward flow will be equal to the difference between potential evaporation E_p and precipitation P. The potential evaporation E_p is often derived from the evaporation of open water by adding some correction factor ($E_p = gE_o$). For deepter groundwater levels the relation may assumed to be about hyperbolic.

In case h^* is understood to be the depth of the groundwater level with respect to an arbitrary level z_o and consequently:

$$h^* = z_o - h \tag{13}$$

a simple first approximation can be obtained by means of the next formula:

$$v_{phr} = \frac{a}{h^*} \tag{14}$$

The parameters z_o and a are not yet determined and this makes it possible to obtain agreement in two points of the equation (14) and the actual relation between v_{phr} and h.

Substitution of (13) and (14) into (12) yields the next equation:

$$kD \frac{d^2h^*}{dx^2} = -\frac{a}{h^*} \tag{15}$$

The next formula (16) is the solution of the differential equation (15). The value h_m^* belongs to x = 0 and h_i^* belongs to $\pm x_i$:

$$\frac{|x_i|}{h_m^*}\sqrt{\frac{a}{2kD}} = \int_0^{\sqrt{\ln \frac{h_m^*}{h_i^*}}} e^{-u^2}\,du = \frac{1}{2}\sqrt{\pi}\ \mathrm{erf}\left(\sqrt{\ln \frac{h_m^*}{h_i^*}}\right) \tag{16}$$

By means of differentiation of (16) an expression is found for the intensity q_i of the horizontal flow at the coordinate x_i:

$$q = kD\frac{dh^*}{dx} = kD\left(\frac{dx}{dh^*}\right)^{-1} \tag{17}$$

$$q(\pm x_i) = \pm\sqrt{2kDa\ \ln \frac{h_m^*}{h_i^*}} \tag{18}$$

For fixed values of h_m^* and x_i, the correponding value of h_i^* can be found with the aid of (16). Substitution of this value in (18) then yields a result for $q(x_i)$. A graphical representation of the formulae (16) and (18) is given in fig. 6.

A second empirical formula accounting for the relation between v_{phr} and h^*, that offers the possibility of a simple solution, is as follows:

$$v_{phr} = b_1 e^{-\frac{h^*}{b_2}} \qquad (19)$$

Here again the true relation and the approximation by eq. (19) may agree in 2 points to be chosen freely because of the two degrees of freedom:

$$b_1 e^{-\frac{z_o}{b_2}} \quad \text{en} \quad b_2$$

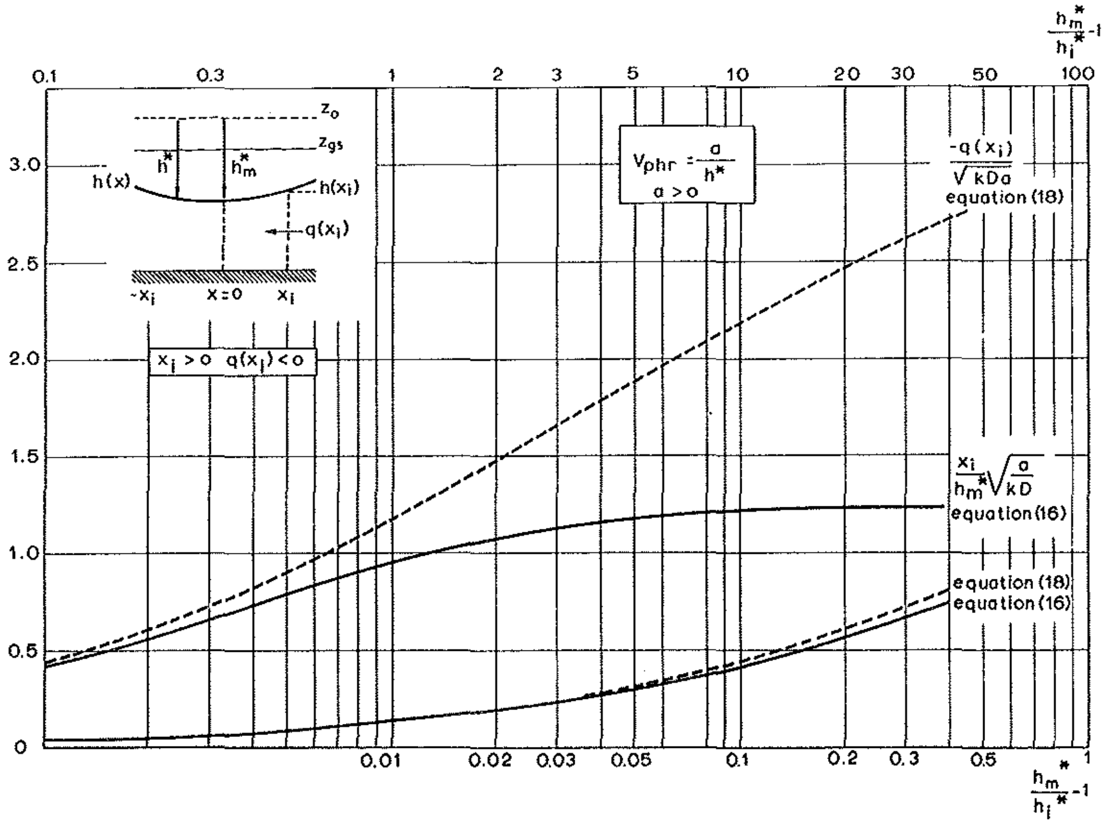

Fig. 6. Graphical representation of formulae (16) and (18).

Along the horizontal axis $\frac{h_m^x}{h_i^x} - 1$ has been plotted with 2 dimensionlesss scales for the upper and lower curves respectively.

Along the vertical axis there is only 1 dimensionless scale both for $\frac{-q(x_i)}{\sqrt{kDa}}$ and $\frac{x_i}{h_m^x}\sqrt{\frac{a}{kD}}$

Substitution of (13) and (19) in (12) will yield:

$$kD \frac{d^2h^*}{dx^2} = - b_1 \, e^{-\frac{h^*}{b_2}} \tag{20}$$

The solution of differential equation (20):

$$\frac{h_i - h_m^*}{2b_2} = \ln \cos\left(x_i \sqrt{\frac{b_1}{2kDb_2}} \, e^{-\frac{h_m^*}{2b_2}}\right) \tag{21a}$$

or:

$$x_i \sqrt{\frac{b_1}{2kDb_2}} = e^{\frac{h_m^*}{2b_2}} \arccos e^{\frac{h_i^* - h_m^*}{2b_2}} \tag{21b}$$

For the horizontal flow intensity is found:

$$q(\pm x_i) = kD\left(\frac{dh^x}{dx}\right)_{\pm x_i} = \mp \sqrt{2kDb_1b_2} \, e^{-\frac{h_m^*}{2b_2}} \, tg\left(x_i \sqrt{\frac{b_1}{2kDb_2}} \, e^{-\frac{h_m^*}{2b_2}}\right) \tag{22a}$$

or, after (21b) has been substituted in (22a):

$$q(\pm x_i) = \mp \sqrt{2kDb_1b_2} \, e^{-\frac{h_m^*}{2b_2}} \, tg\left(\arccos e^{\frac{h_i^* - h_m^*}{2b_2}}\right)$$

$$= \mp \sqrt{2kDb_1b_2\left(e^{-\frac{h_i^*}{b_2}} - e^{-\frac{h_m^*}{b_2}}\right)} \tag{22b}$$

These formulae have been represented graphically in fig. 7, 8a and 8b.

Finally, it should be remembered that formulae (14), ... (22) are holding only as long as $h^x > h_p^x$, that is as long as, accordingto either (14) or (19), $v_{phr} < E_p - P$. Generally in the case $h_o^x < h_p^x$, it will not be possible to indicate immediately, for instance by using (16) or (21), which value of x_p agrees with the given h_p^x, for h_m^x too will be unknown yet.

Additional relations, required for that reason, can be obtained from the boundary zones along the open conduits according to the following consideration. Symmetry is supposed in the following, which means that there are equal given values for either the level in the open conduits or the inflow. Since no radial flow is assumed to exist, possible variations in the spacing between the parallel open conduits or in the wetted perimeters will be of no significance.

In the boundary zones (see fig. 9) the maximal value for $v_{phr} = E_p - P$ is valid, and consequently the next equations may be given:

$$h_p^* - h_o^* = \frac{(E_p - P)\ (L - 2x_p)^2}{8kD} + \frac{L - 2x_p}{2kD}\, q(-\,x_p) \qquad (23)$$

$$-\,q_o = 2q(-\,x_p) + (E_p - P)\ (L - 2x_p) \qquad (24)$$

Dependent on the fact whether h_o^x or q_o will be given, it is equation (23) or eq. (24) which has to be used in order to eliminate the unknown $q(\pm\, x_p)$ from (18) or (22) after substitution of $i = p$. This leads to a relation in which x_p and h_m^x appear as unknowns. Elimination of x_p can be effectuated by substitution of either (16) or (21b) in the relation that has been derived before.

Though no fundamental difficulties are to be expected, rather complicated formulae are obtained, in which h_m^x being the only unknown, is implicitly present.

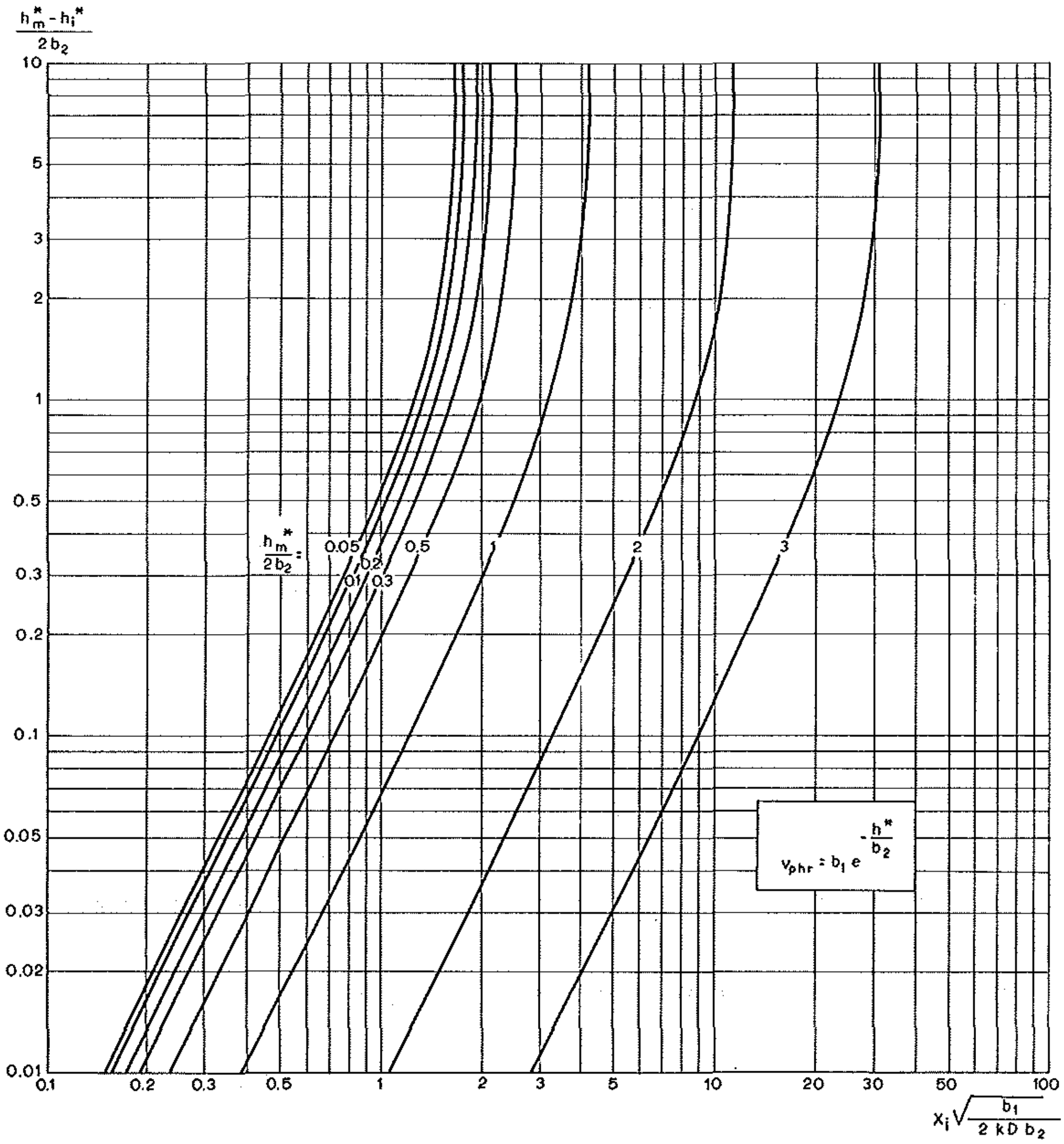

Fig. 7. Graphical representation for formula (21) with $x_i > 0$

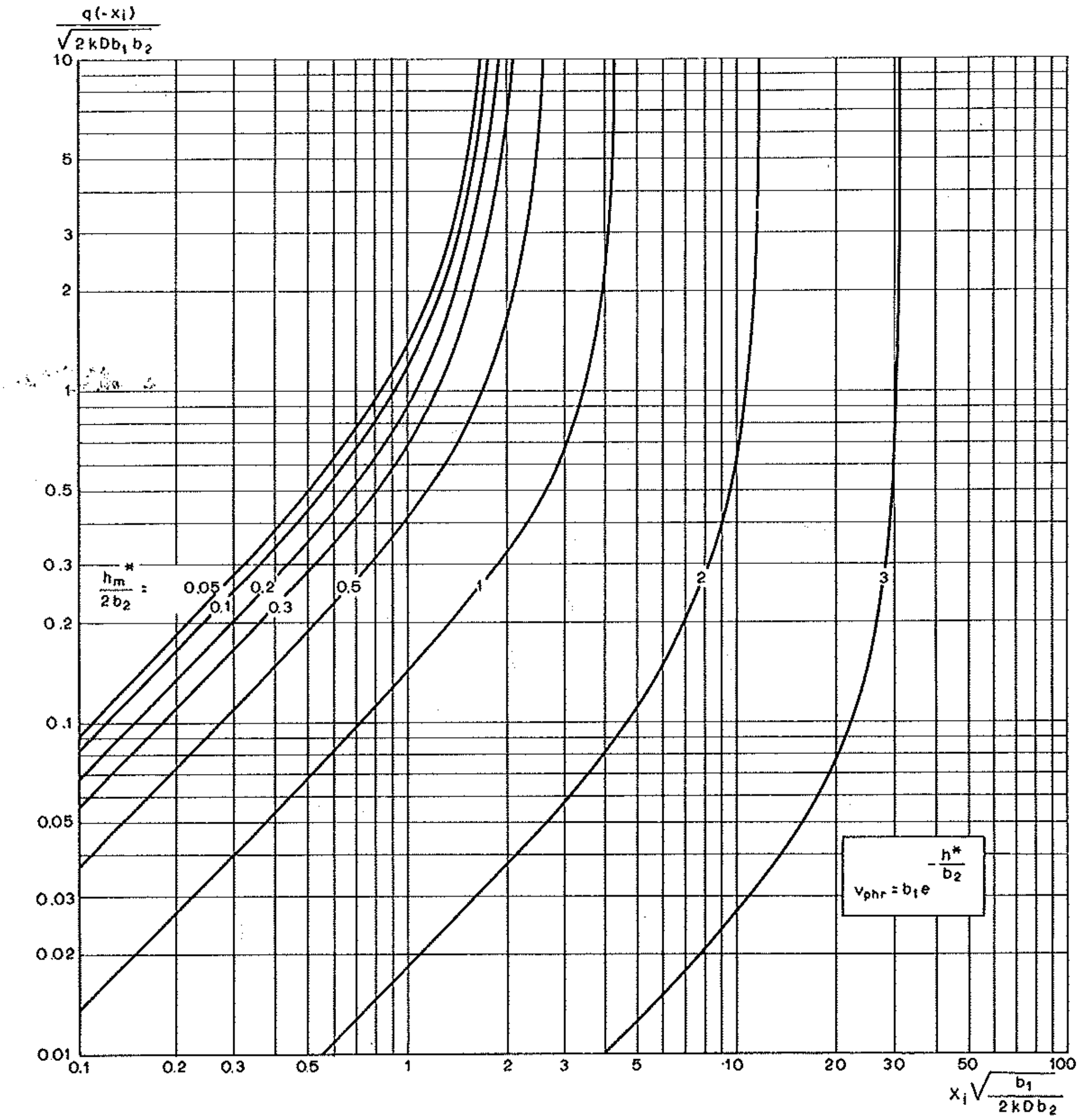

Fig. 8a. Graphical representation of formula (22a) with $x_i > 0$

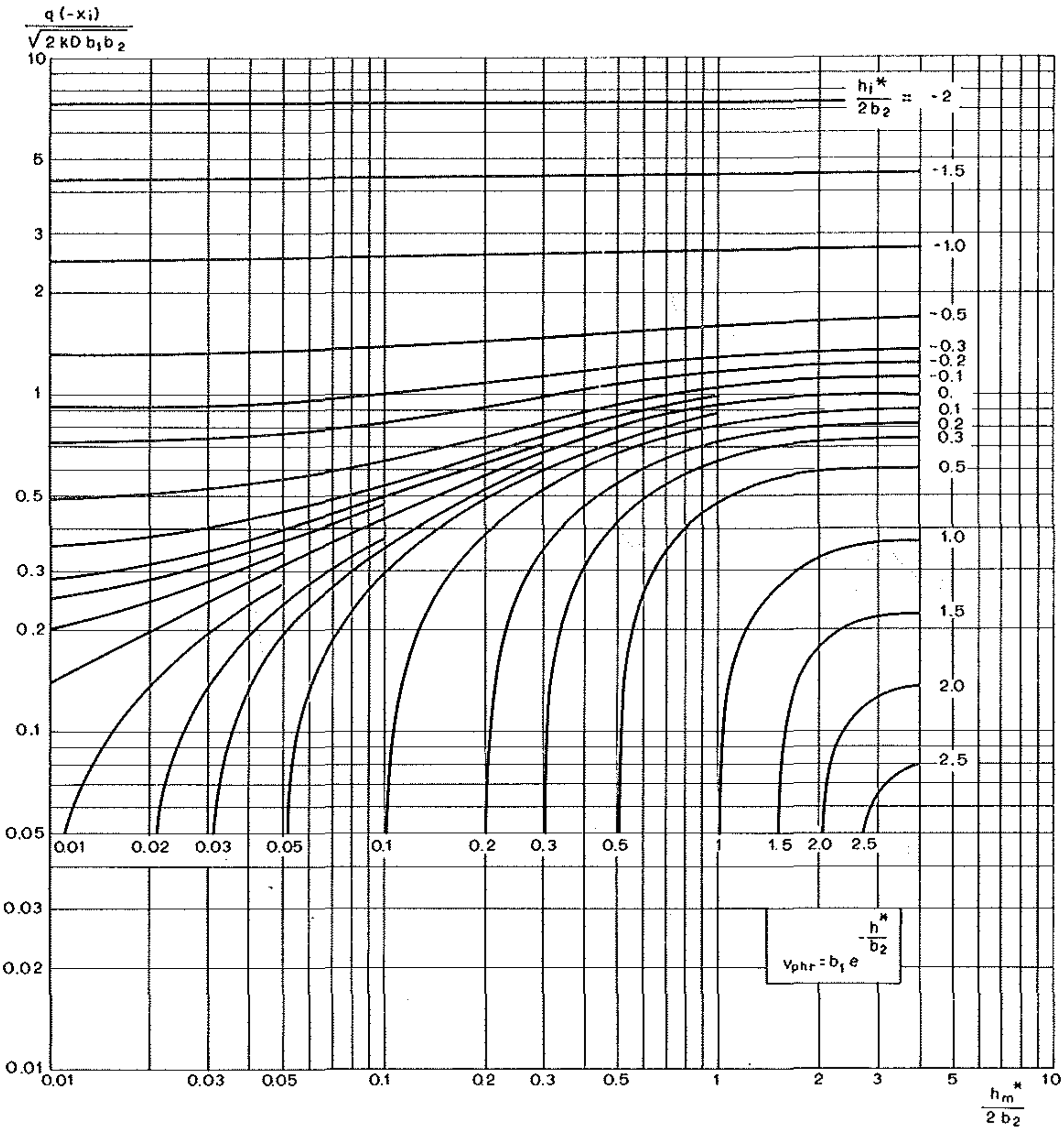

Fig. 8b. Graphical representation of formula (22b) with $x_i > 0$

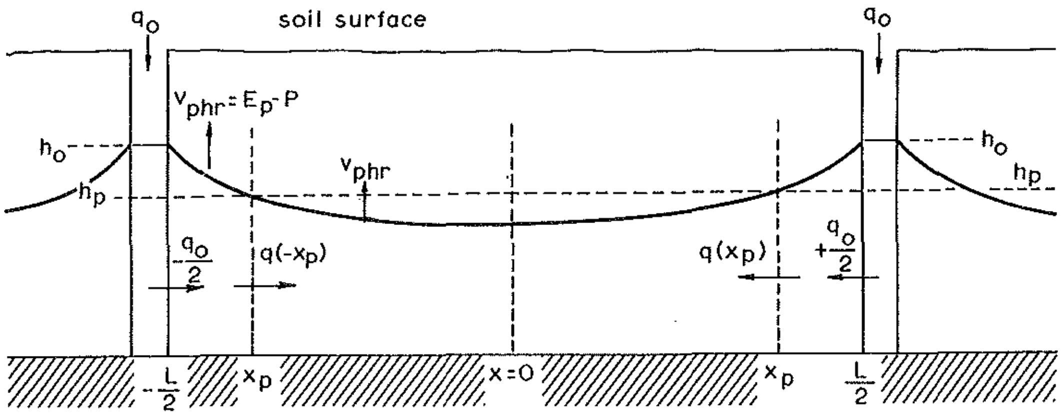

Fig. 9. Division of the area between two long straight fully penetrating drains according to the depth of the phreatic surface below soil surface.

For $-x_p < x < x_p$ it holds that $h < h_p$ and $v_{phr} < E_p - P$

For $x \gtrless \pm x_p$ it holds that $h > h_p$ and $v_{phr} \approx E_p - P$

Some remarks concerning the notation:

1. Flow to the right and upward flow are reckoned positive
2. In case of drainage the outflow is indicated by a positive q_o.
   In case of sub-irrigation the inflow is indicated by a negative q_o.
3. The arrows point into the actual direction of the flow. The symbol placed near an arrow sometimes bears a negative sign in order to make the quantity concerned to have a $\begin{cases}\text{positive}\\\text{negative}\end{cases}$ value in case the arrow points to the $\begin{cases}\text{right}\\\text{left}\end{cases}$ or $\begin{cases}\text{upwards}\\\text{downwards}\end{cases}$

For practical application the next approach might be preferable, especially since here the condition of symmetry is not necessary. At a given h_o^x for one of the open conduits, either an estimated value q_o can be introduced for this conduit or, at a given q_o , h_o^x may be estimated.

Substitution in (23) and (24) of the given and estimated values allows for the solution of $q(\pm x_p)$ and $L - 2x_p$. When $L - 2x_p$ is eliminated by substituting (24) into (23), a second degree equation remains with $q(\pm x_p)$ as only unknown, yielding the following solution:

$$q(\pm x_p) = \pm \frac{1}{2} \sqrt{q_o^2 - 8kD(E_p - P)\ (h_p^{\mathrm{x}} - h_o^{\mathrm{x}})} \tag{25}$$

A solution of $L-2x_p$ is now immediately obtainable by substitution of (25) into (24). The result for $q(\pm x_p)$ has to be substituted in (18) or in (22b), and so a value for h_m^{x} is obtained. Fig. 6 or fig. 8b may also be used for this purpose. By way of formula (16) or (21b) a value for x_p follows and together with the value for $L-2x_p$ already obtained before, an approximative value for L will be found.

It is not to be expected that at the first try this value for the drainspacing will be sufficiently similar to the actual value of L. In case of asymmetry, a check upon the deviation between calculated and given level in the second open conduit at a given spacing L will be required. A reiteration may be necessary to obtain a result of sufficient accuracy. In case the value found for L proves to be too high for instance, a lower value for either h_o^{x} or q_o should be introduced for the new estimation; should the first obtained value for L be too low, a reverse correction has to be applied.

5. SUB-IRRIGATION FORMULAE WHEN THE FUNCTIONS FOR v_{phr} FROM CHAPTER 4 ARE AGAIN VALID, HOWEVER WITHOUT NEGLECT OF THE RADIAL FLOW

When areas with radial flow are concerned, the application of formulae representing some relation between v_{phr} and h^{x} will give much more trouble. Fundamentally it is possible to solve this problem with the aid of a differential equation similar to (15) or (20) by adding some function of x to the denominator in the right hand term of (15) or to the exponent in the right hand term of (20). It is possible, however to prevent complications connected with this approach and to obtain a solution quite acceptable for practice and of a very good accuracy in case the areas with radial flow will possess a relatively small width with regard to the total spacing. The principle that has already been discussed in the previous chapter (division of the field into a number of strips and considering each strip separately) may be applied here again.

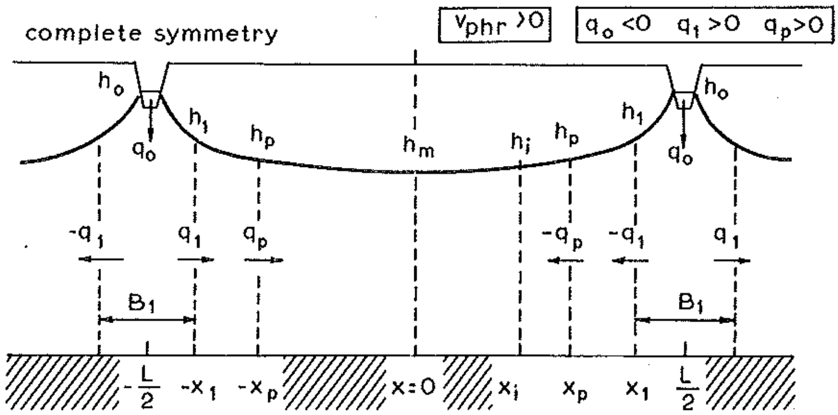

Fig. 10a. Division of the area between two partially penetrating drains for a case of complete symmetry with $v_{phr} > 0$. Lines of symmetry are the verticals through the open conduits ($x = \pm \frac{1}{2} L$) as well as midway between these open conduits ($x = 0$)

The width of the area, where the flow is mainly radial, will be indicated by B_1 (fig. 10a and b). In case the soil is homogeneously permeable a good approximation is obtained by taking B_1 equal to twice the thickness of the aquifer close to the open conduits. In stratified soils with a much less permeable upper layer, B_1 will be approximately equal to twice the thickness of this upper layer. In a first approximation, the following relation (eq. 26-31) can be derived for strip B_1 by assuming the extraction intensity v_{phr} to be constant (independent of x) and equal to the average of the actual extraction.

Provided there exists equality in flow towards both sides of an open conduit and there is no flow through the phreatic surface so that $q_o = 2q_1$ and $v_{phr} = 0$ (upper curve in centre of fig. 10b), there remains only 1 degree of freedom (variable inflow q_o) with a similarity in all possible shapes of the phreatic surface. Under these conditions the expression given by formula (26) can be used for the mean height of the phreatic level in the area with mainly radial flow.

The coefficient α_1 in this formula may be taken $\simeq 2/3$

$$h^*_{rad} = (1 - \alpha_1)\ h^*_o + \alpha_1 h^*_1 \approx \frac{1}{3}\ h^*_o + \frac{2}{3}\ h^*_1 \qquad (26)$$

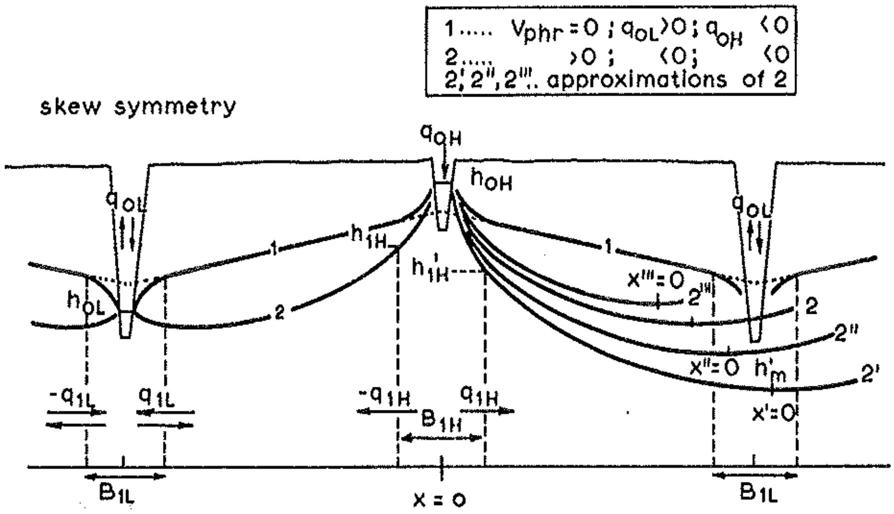

Fig. 10b. Shape of the phreatic surface in case of skew symmetry

In case $v_{phr} > 0$, which implies that $|q_o| > 2q_1$, it can also be shown that similarity exists as far as a constant ratio between q_o and q_1 is maintained. In such cases the value for α_1 should be a little higher. Under arbitrary conditions the constancy of α_2 cannot be sustained any longer. In view of the small variations that will be found in practice, fairly good results may be obtained by a general use of formula (26) with $\alpha_1 = 2/3$.

An reasonably accurate derivation of the mean value of v_{phr} for the radial flow area is much more difficult. Analogous to (26) the following expression might be used, in which case the value of the coefficient α_2 should be a little higher than α_1, but < 1 ($\alpha_2 \simeq 4/5$):

$$v_{phr,rad} = (1 - \alpha_2)\, v_{phr,\,0} + \alpha_2\, v_{phr,\,1} \simeq \frac{1}{5}\, v_{phr,\,0} + \frac{4}{5}\, v_{phr,\,1} \quad (27)$$

In some cases extreme simplification of the procedure may be expedient. To this end the main flux may be replaced by the flux at mean groundwater level, which may be expressed as follows:

$$v_{phr,\,rad} = v_{phr}(h^{*}_{rad}) \quad (28)$$

Using (26) and (14) or (19) the following approximative relation can be given immediately:

$$v_{phr,\,rad} = \frac{3a}{h^{x}_{o} + 2h^{x}_{1}} \quad (29a)$$

or

$$v_{phr,\,rad} = b_1\, e^{-\frac{h^{x}_{o} + 2h^{x}_{1}}{3b_2}} \quad (29b)$$

Another expression for $v_{phr,\,rad}$ is also available (for plus and minus signs, see fig. 9 and 10):

$$v_{phr,\,rad} = \frac{-\,q_o - 2q_1}{B_1} \quad (30)$$

Using again a constant extraction $v_{phr,\,rad}$ independent of x, the following formula can be derived for the horizontal flow within strip B_1:

$$h_1^* - h_o^* = - q_o \Omega - \frac{\frac{1}{2} q_o \cdot \frac{1}{2} B_1}{kD} - \frac{v_{phr,\ rad}\ B_1^2}{8kD}$$

$$\frac{h_1^* - h_o^*}{B_1} = - \frac{q_o}{kD}\left(\frac{1}{4} + \frac{kD\Omega}{B_1}\right) - \frac{v_{phr,\ rad}\ B_1}{8kD} = - \frac{q_o}{8kD}\left(1 + \frac{8kD\Omega}{B_1}\right) + \frac{q_1}{4kD} \quad (31)$$

In this way a sufficient description of the situation in the area with radial flow has been obtained. The number of operations still to be done depends on the level of h_p with respect to the unknown levels h_1 and h_m. In many cases it will not be possible to determine immediately which of the following three possibilities must be accepted:

a. $h_p^x > h_m$. Under these conditions a constant flux $v_{phr} = E_p - P$ is valid for all x. The procedure from chapters 2 and 3 can be maintained.

b. $h_1^x < h_p^x < h_m^x$. This level of h_p with respect to h_1 and h_m has been indicated in fig. 10a. For this case, some of the preceding formulae viz. (23), (24) and (29) require minor changes:

$$h_1^* - h_p^* = \frac{(E_p - P)\ (x_1 - x_p)^2}{2kD} + \frac{x_1 - x_p}{kD} q_p \quad (32)$$

$$q_1 - q_p = (E_p - P)(x_1 - x_p) \quad (33)$$

$$- q_o - 2q_1 = (E_p - P)\ B_1 \quad (34)$$

First and third member of formula (31 may be repeated unchanged:

$$\frac{h_1^* - h_o^*}{B_1} = \frac{q_o}{8kD}\left(1 + \frac{8kD\Omega}{B_1}\right) + \frac{q_1}{4kD} \quad (35)$$

The problem can now be described as follows. All conditions have to be known except for the 6 quantities q_o, q_1, q_p, h_1^x, h_m^x and x_p in case h_o^x will be given, or q_1, q_p, h_o^x, h_1^x, h_m^x and x_p in case q_o will be given.
For the calculation of these unknowns, sufficient relation have already been derived, viz. the formulae (16), (18), (32), (33), (34) and (35); formulae (16) and (18) may be replaced by (21) and (22). Because of the complexity of the formulae (16), (18), (21) and (22), like in the previous chapter no direct solution can be expected here and consequently a similar iterative method can be recommended.
With a given and an estimated value for h_o^x and q_o a solution for h_1^x and q_1 can be found with the aid of (34) and (35). These values, when substituted in (32) and (33) will yield results for q_p and $x_1 - x_p$. With the aid of either (18) or (22b) and substitution of $h_i^x = h_p^x$, a corresponding value for h_m^x will follow.
It is very unlikely that the deepest point of the phreatic surface will be found above h_p since such possibility would already have shown whilst checking up according to (a). Of course it may not be excluded, that the condition $h_1^x < h_p^x < h_m^x$ cannot be satisfied, and an calculation as described underneath under c has to follow.
In case for h_1^x, h_p^x and h_m^x a position in the correct sequence has actually been found, it is obvious that the calculation should be continued with the aid of formula (16), c.q. (21b).
In case of complete symmetry, it is possible to check immediately, by means of the x_p-value obtained from these formulae, how far the sum $B + 2(x_1 - x_p) + 2\, x_p$ will be still deviating from the given value of L.
In cases of skew symmetry, a more elaborate calculation will be necessary. Starting at one side (for instance where the conduit with the higher level is located as shown in centre of fig.10b) a first approximation of the phreatic surface can be obtained for a given and an estimated value of h_{OH} and q_{OH}, according to the method previously indicated. This initial approximation contains a minimum h_m' at which a new horizontal coordinate x can be put equal to zero. By such a transformation the formulae for the

symmetric case can be used again without any change.

Irrespective of the place of this minimum, the value of h' at the (left hand) border of the B_{1L}-strip ($2x = L - B_{1L}$) can now be calculated and also q'_{1L}. Whether the continuation within the B_{1L}-strip belongs to case (b) or case (c) depends on the position of h'_{1L}. For the calculation of h'_{OL} and q'_{OL} the formulae (35) and (29a) or (29b) or (34) may be used. Whatever may be the case, some reiterations will nearly always be required for obtaining an approximation of sufficient accuracy.

c. $h^x_p < h^x_1$. in this case only 4 unknowns have to be solved, viz. q_o, h^x_1, q_1, h^x_m in case h^x_o is given, or h^x_o, h_1, q_1, h^x_m in case q_o given. It is possible to produce immediately the solution of q_o and h^x_1 or h^x_o and h^x_1 with the aid of (19a) and (35) (second degree equation). In case (29b) and (35) have to be used, the solution will be a little more difficult. Immediate reading in a graphical representation may be recommended, as far as the limited accuracy of such graphs is not considered as an objection. Substitution of these results in (16) and (18) or in (21) and (22), however, produces once more such complicated equations that here again a similar iterative method, as recommended in previous cases, starting with a given and an estimated value for h^x_o and q_o (eventually for h^x_{OH} and q_{OH}), seems to be most suitable for practical application.

6. NON STEADY STATES WITH SPECIAL FUNCTIONS FOR THE FLOW IN THE UNSATURATED ZONE

In the rather complicated cases that will be discussed here, it is reasonable to divide the considered area, having two dimensions for the vertical cross-section and one dimension for the time, into a number of sub-areas, preferably equal parts if possible, in order to obtain a system of differences equations and to derive from these an (approximative) solution of the potentials and fluxes in question.

A suitable subdivision is shown in fig. 11. In the horizontal direction a division into parts of constant width Δx is taken, except near the open conduits where a radial flow component obliges to follow another procedure. Similar to the previous chapter, assumption of separate blocks of width B_1 would do here. For the sake of simplicity complete symmetry has been assumed to exist with respect to the open conduits so that at every open conduit two equal blocks of width $B_1/2$ may be distinguished.

In the vertical direction a division into a very large number of layers should be introduced when very accurate results should be required. This would imply moreover the necessity of working with very short time intervals. In order to simplify things, however, three layers only will be distinguished here, viz.:

1. A saturated zone from the impermeable basis up to a (horizontal) level z_o located at such a depth that z_o is always below the phreatic surface.
2. A partly unsaturated zone from z_o up to the bottom of the root zone z_d. In this layer the moisture suction Ψ increases from 0 up to Ψ_{sd} from the bottom to the top.
3. The rooted zone, with a moisture content w_u and a moisture suction Ψ_u, both considered independent of the vertical ordinate z, though dependent of the time t (RIJTEMA, 1969).

In case a difference in soil type is assumed for the first and second layer, the moisture content values w_u and w_{sd} will generally be different, though ever and anywhere $\Psi_u = \Psi_{sd}$ will be valid.

In order to be able to calculate the development of the situation with increasing time, some initial situation for a certain time t_0 has to be completely known, This should be understood in such a way that for all h_j and all $\Psi_{u,\,j}$ (see fig. 11) certain values have to be known. From these the situation for time t_1 should be calculated, and from this a next situation for t_2 etc.

As soon as the calculation for an arbitrary time t_i has been finished, this situation is considered as an initial situation with known values of $h_{j,\,i}$ and $\Psi_{u,\,j,\,i}$, for all x_j, which values are substituted in some system of formulae in a similar way as with previous values of t, in order to derive from this system the new $h_{j,\,i+1}$ and $\Psi_{u,\,j,\,i+1}$ for t_{i+1}.

In order to keep the system of formulae as simple as possible, time intervals $\Delta t = t_{i+1} - t_i$ will be kept constant as much as possible.

In case periods with rapid changes and slow changes can be distinguished, it may be advantageous to take Δt a total number of times bigger for periods with slow motion. In many cases for Δt will be chosen $\Delta t = 24$ hours or $\Delta t = 7$ days, in which cases there may be spoken of the situation for the previous day (week) and that for the next day (week) respectively.

a. Calculation of values for fluxes (q, v) and storage (w, W) from a complete set of given potential values (h, Ψ): equations (36) ... (45).

For all blocks with horizontal flow in the saturated zone (width of block Δx) the next two formulae may be used (for j only even integers, see fig. 11):

$$q_{j-1,\, i} = kD \frac{h_{j-2,\, i} - h_{j,\, i}}{\Delta x} \tag{36}$$

$$q_{j+1,\, i} = kD \frac{h_{j,\, i} - h_{j+2,\, i}}{\Delta x} \tag{37}$$

A simple expression can also be used for the vertical flow through the phreatic surface $v_{h,\, j,\, i}$:

$$v_{h,\, j,\, i}\, \Delta x = q_{j-1,\, i} - q_{j+1,\, i} \tag{38}$$

Determination of the vertical flux through the bottom of the root zone will be less simple. An approximating relation has been introduced to this end using a wellknown expression for the steady vertical flow in the unsaturated zone with moderate desiccation (RIJTEMA, 1965, 1969). Since here a flow in the lower part of the unsaturated layer is concerned, a subscript s (sublayer) is added to the quantities k_o and α:

$$\frac{v_{d,j,i} + v_{h,j,i}}{2} = k_{os} \frac{e^{-\alpha_s (z_d - h_{j,i})} - e^{-\alpha_s \Psi_{u,j,i}}}{1 - e^{-\alpha_s (z_d - h_{j,i})}} \tag{39}$$

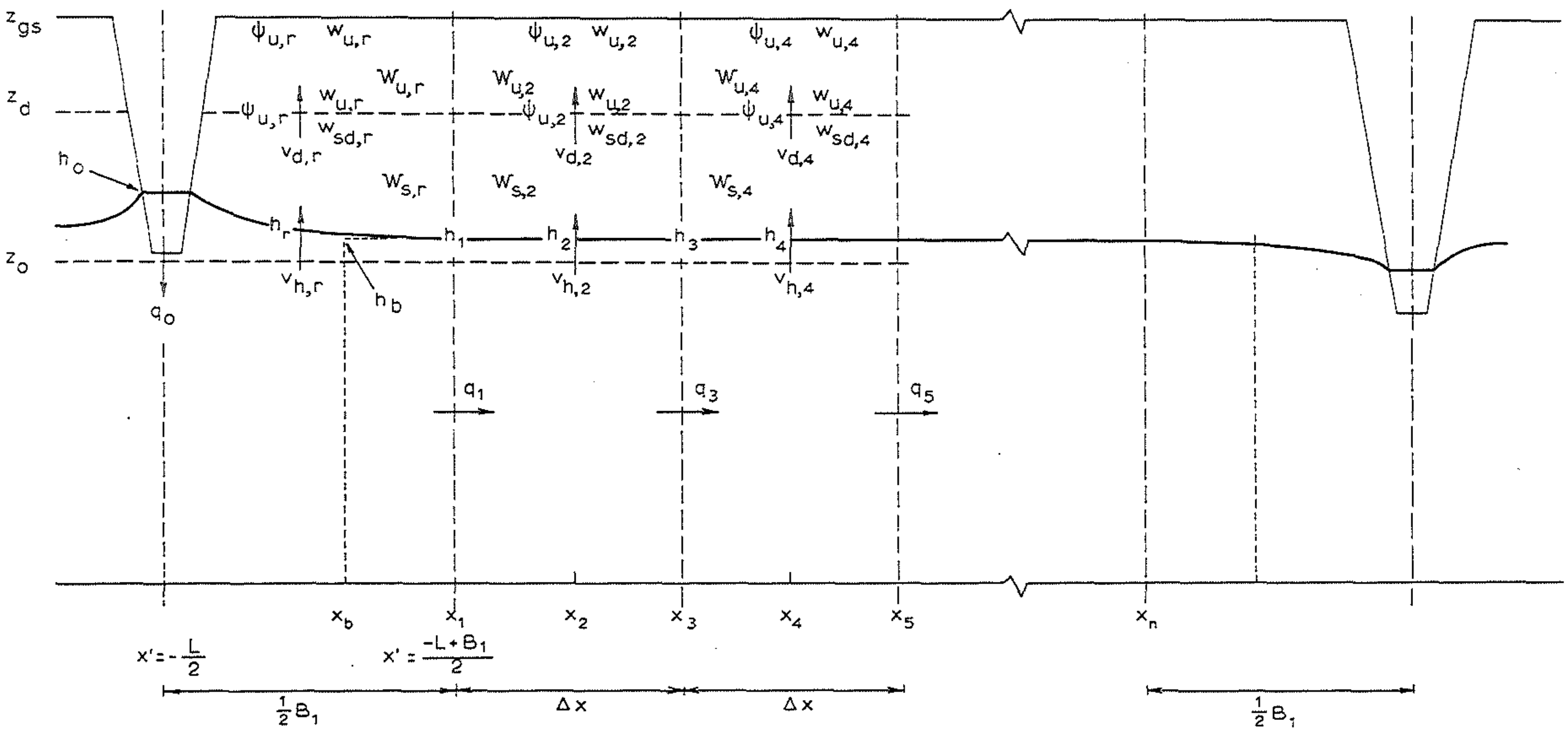

Fig. 11. Symbols, used for the calculation of the flow in the unsaturated zone.

$W_{u,\,4}$ and $W_{s,\,4}$ indicate the storage W in the upper layer and in the sub layer near x_4 respectively. On day i these quantities are indicated by $W_{u,\,4,\,i}$ and $W_{s,\,4,\,i}$

v_h = abbreviated notation for v_{phr}

r = subscript for area with radial flow

h = subscript for phreatic surface

With the aid of (36) ..... (39) it is possible to find the four quantities $q_{j-1,\,i}$, $q_{j+1,\,i}$, $v_{h,\,j,\,i}$ and $v_{d,\,j,\,i}$. For calculation of the storage (water content) at the beginning of the period concerned and the change in storage over that period the latter two quantities only are required. When determining the storage it will be necessary, besides having some knowledge of the quantities k_{os} and α_s in formula (39), to indicate more precisely the relation between moisture content w and moisture suction Ψ. Measurements specially done to this end may be used, or some data which can be found in literature for a great number of soil types (RIJTEMA, 1965, 1969).

Neglecting hysteresis effects two different relations $\Psi = f(w)$ may be applied for the upper layer and the middle layer. This may be expressed as follows, where f^{-1} represents the inverse function:

$$w_{u,j,i} = f_u^{-1}(\Psi_{u,j,i}) \tag{40}$$

$$w_{sd,j,i} = f_s^{-1}(\Psi_{sd,j,i}) \tag{41}$$

with

$$\Psi_{sd,j,i} = \Psi_{u,j,i} \tag{42}$$

For the moisture storage in the upper layer the following equation is valid:

$$W_{u,j,i} = w_{u,j,i}(z_{gs} - z_d) \tag{43}$$

z_{gs} = level of the soil surface
z_d = level of the bottom of the root zone

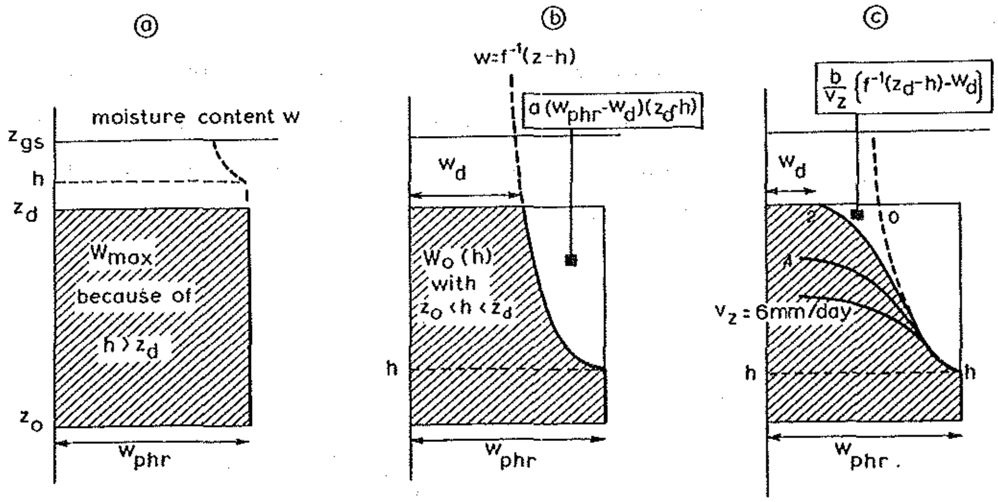

Fig. 12. Distribution of the moisture content in the zone between an arbitrary level z_o and the bottom of the root zone.

a. Complete saturation: W_{max} = maximal moisture storage between z_o and $z_d = w_{phr}(z_d - z_o)$

b. No flow in the unsaturated zone: $W_o(h)$ = moisture storage for the case that $v_z = 0$ and the groundwater level h is located between z_o and z_d

c. Moisture distribution in case of capillary rise, v_z being constant for all z. The moisture storage may now be indicated by $W_v(h)$, where the subscript v possesses the value of v_z for instance in mm day^{-1}

Deriviation of an exact formula for the storage in the intermediate layer (subscript s) will be difficult. Because in most soil types the moisture distribution curves for the vertical direction possess a certain similarity if there is no flow in the unsaturated zone ($v_z = 0$), the following equation (fig. 12b) may be written:

$$W_o(h) = w_{phr}(z_d - z_o) - a\{w_{phr} - f^{-1}(z_d - h)\}(z_d - h) \quad (44)$$

a 0.7 for clay and clay loam; 0.6 for sandy soils and basin clay soils; 0.4 for loam and fine sandy loam — derived from RIJTEMA, 1969

In case of a constant upward flow, a term can be added to (44) in order to take into account the changes in the moisture distribution curve (fig. 12c). Though this term is not very accurate, the equation (45a, b) will give a fairly better accuracy at variable v_{phr} than can be achieved with (44):

$$W_v(h) = w_{phr}(z_d - z_o) - a\{w_{phr} - f^{-1}(z_d - h)\}(z_d - h) - \frac{b}{v_z}\{f^{-1}(z_d - h) - w_d\} \quad (45a)$$

$b \approx 500$ to $1000\ mm^2/day$ (derived from RIJTEMA, 1969)

By using the notation introced in fig. 11 the equation (45a) turns over into an identical equation (45b):

$$W_{s,j,i} = W_{s,max} - a_s\{w_{phr} - f_s^{-1}(z_d - h_{j,i})\}(z_d - h_{j,i}) - \frac{b_s}{v_{d,j,i}}\{f_s^{-1}(z_d - h_{j,i}) - w_{sd,j,i}\} \quad (45b)$$

b. Calculation of values for t_{i+1} from data for t_i

For a calculation of the storage in both layers at t_{i+1}, the following equations can be used:

$$W_{u,j,i+1} = W_{u,j,i} + \delta W_{u,j,i} = W_{u,j,i} + P_i - E_{j,i} + v_{d,j,i} \quad (46)$$

$$W_{s,j,i+1} = W_{s,j,i} + \delta W_{s,j,i} = W_{s,j,i} + v_{h,j,i} - v_{d,j,i} \quad (47)$$

In the equation (46) and (47) some values for the precipitation P_i and the actual evaporation $E_{j,i}$ on the day concerned, have to be introduced, the actual evaporation can be put equal to the potential evaporation $E_p = gE_o$ (where E_o is the evaporation of open water, g is a coefficient dependent on the degree of density of the canopy, crop development and season) or it possesses a lower value. Accurate definition of the evaporation reduction proved to be rather difficult (RIJTEMA, 1965), for a first approximation, however, a dependency on the moisture stress $\Psi_{u,\ j,\ i}$ might be assumed (VISSER, 1963, 1968; BLOEM, 1966).

After certain values for P, E, W_u, W_s, v_d and v_h for the time t_i have been substituted in the formulae (46) and (47), the storage values $W_{u,\ j,\ i+1}$ and $W_{s,\ j,\ i+1}$ for the time t_{i+1} can be calculated. After substitution of these results into the next expressions, moisture content values (w_u, w_{sd}) and moisture suction values (Ψ) can be found for t_{i+1}.

$$w_{u,j,i+1} = \frac{W_{u,j,i+1}}{z_{gs} - z_d} \qquad (48)$$

$$\Psi_{sd,i,j+1} = \Psi_{u,j,i+1} = f_u(w_{u,j,i+1}) \qquad (49)$$

$$w_{sd,j,i+1} = f_s^{-1}(\Psi_{sd,i,j+1}) \qquad (50)$$

In the end formula (45b) is used once more to find also the groundwater level $h_{j,\ i+1}$ for the next day:

$$W_{s,j,i+1} = W_{s,max} - a_s\{w_{phr} - f_s^{-1}(z_d - h_{j,i+1})\}\ (z_d - h_{j,i+1}) - \\ - \frac{b_s}{v_{d,j,i+1}}\{f_s^{-1}(z_d - h_{j,i+1}) - w_{sd,j,i+1}\} \qquad (51)$$

However, it can be seen immediately, that equation (51) contains two yet unknown variables: $h_{j,\ i+1}$ and $v_{d,\ j,\ i+1}$. Provisionally an unchanged value $v_{f,\ j,\ i}$ can be used and if afterwords by means of the equations (36), (37), (35) it is found that there is not quite negligible difference between $v_{d,\ j,\ i}$ and $v_{d,\ j,\ i+1}$, the equation (51) may be used again with the new $v_{d,\ j,\ i+1}$-value in order to obtain a better approximation

c. Calculation for areas with symmetric radial flow

Formulae (36) ...... (51) are not simply transferable for the area with radial flow in the saturated zone, especially the formulae (36) ..... (39) require further consideration. Apart from the subscripts the formula (39) can be maintained. Since the width B_1 of this block may be relatively large and the values of $v_{d,r,i}$, $v_{h,r,i}$ and h may vary rather much, introduction of mean values in the left hand and right hand part of (39) will decrease the accuracy of this formula. In case of practical applications, however, this will not be of much importance.

$$\frac{v_{d,r,i} + v_{h,r,i}}{2} = k_{os} \frac{e^{-\alpha_s(z_d - h_{r,i})} - e^{-\alpha_s \Psi_{u,r,i}}}{1 - e^{-\alpha_s(z_d - h_{r,i})}} \tag{52}$$

Formula (38) changes only little in case of symmetry:

$$v_{h,r,i}\, B_1 = - q_{o,i} - 2q_{1,i} \tag{53}$$

Equation (54) in shape similar to (36) and (37) can be added to the preceding equations (in case $\Delta x > B_1$, x_b may be located at the opposite side of the open conduit without any objection, since h_b is only an imaginary quantity):

$$q_{1,i} = kD \frac{h_{b,i} - h_{2,1}}{\Delta x} \tag{54}$$

Now only one relation is still lacking. In case of symmetry with respect to the open conduit this can be met with the aid of (35), which is one of the formulae derived at the end of the previous chapter:

$$\frac{h_{o,i} - h_{1,i}}{B_1} = - \frac{q_{o,i}}{8kD}\left(1 + \frac{8kD\Omega}{B_1}\right) + \frac{q_{1,i}}{4kD} \tag{55}$$

It may be noticed additionally that still two more values for h_1 and h_b have to be added to the given values for h_r, h_2, h_4, ... To meet this, the next two formulae may be used:

$$h_{1,i} = \frac{h_{b,i} + h_{2,i}}{2} \tag{56}$$

$$h_{r,i} = \frac{h_{o,i} + 2h_{1,i}}{3} \tag{57}$$

Whereas formula (56) will be very accurate in case x is sufficiently small, the accuracy of the latter formula is dependent on the choice of the coefficients 1/3 and 2/3. This has been indicated already in the previous chapter when deriving formula (26).

d. Asymmetric radial flow

In case of asymmetry with respect to an open conduit, the number of unknown quantities in strip B_1 will be twice as large, except for h_o or q_o. This implies that, where in the symmetric case the six formulae (52), ... (57) are used, consequently, in an asymmetric case, 11, partly new formulae have to be introduced.

It will be clear that near an open conduit, where the flow at the left hand and the right hand side is not equal, formula (52), (54), (56) and (57) for the left hand and the right hand side may be redoubled by introducing positive and negative subscripts ($\pm$ 1, $\pm$ r, $\pm$ b, $\pm$ j, etc.) for right hand and left hand side of the open conduit respectively.

In the practical elaboration of such cases, where several open conduits are concerned and consequently a rather large number of equations is involved, it will probably be better to use a consecutive numbering to start at zero and continuing towards some positive integer. In such cases the meaning of the numbers should be explained by adding a complete schematic figure in which all numbers occur. The new formula (58) can be used as a substitute for the old formula (53):

$$(v_{h,-r,i} + v_{h,r,i}) \frac{B_1}{2} = - q_{o,i} - q_{-1,i} - q_{1,i} \tag{58}$$

Formula (55) will now be replaced by formulae (59) and (60):

$$\frac{2h_{o,i} - h_{-1,i} - h_{1,i}}{2B_1} = \frac{-q_{o,i}}{8kD}(1 + \frac{8kD\Omega}{B_1}) + \frac{q_{-1,i} + q_{1,i}}{8kD} \tag{59}$$

$$q_{1,i} - q_{-1,i} = 2kD \frac{h_{-1,i} - h_{1,i}}{B_1} \tag{60}$$

So, together with the redoubling of (52), (54), (56) and (57), 11 formulae are indeed available near each open conduit with asymmetric flow.

The formulae derived in this chapter make it possible, when there is a completely known situation at time t_i, to calculate the situation at the next time t_{i+1} with similar completeness. In practice the large number of equations might present some difficulties especially the reiterations in connection with equation (51).

In general a rather small constant value is taken for the size of the step $\Delta t = t_{i+1} - t_i$. In cases where the storage coefficient μ is constant, instability is known to occur beyond a certain limit viz. $\Delta t \geqq \frac{1}{2} \mu (\Delta x)^2 (kD)^{-1}$ (MILNE, 1953).

The resulting values found for h_r, Ψ_r, h_2, Ψ_2, h_4, Ψ_4, ... at time t_{i+1} may be considered as an initial situation for the next time interval Δt. After substitution in the previous formulae, the calculation can be repeated completely. This may be continued at will as far as a closed range of data for P_i and E_o (eventually for $h_{o,\,i}$ also) will be available.

Finally it may be noticed that a system of formulae, as has been derived previously, may also be used in case of steady states. In this case two suppositions have to be made:

a. h_o or $h_{o,\,i}$ will be given
   q_o or $q_{o,\,i}$ will be unknown

b. q_o or $q_{o,\,i}$ will be given
   h_o or $h_{o,\,i}$ will be unknown

A simple way to find a solution for a steady state groundwater flow problems is possible in case of symmetry, whether all open conduits are similar and the inflow of water is equal anywhere (complete symmetry) or higher and lower conduits will be alternating, all higher conduits then having an equal level and equal inflow, and all lower conduits an equal level and equal outflow (skew symmetry). In such cases it is possible to start with a given value for h_o (or q_o) at one of the open conduits, adding an estimated value for q_o (or h_o). All the sub-areas can now be elaborated succesively. By means of several reiterations a sufficient approximation of the correct result is pursued.

In case of complete symmetry such result will be obtained as soon as either $q = 0$ or the changing of the sign for q_j will be found to happen sufficiently near to the middle of the area involved.

In case of skew symmetry (see fig. 11 for instance) calculation may be started at the higher conduit. In the end it has to be checked whether the value that is found for h_o as q_o at the lower conduit is a sufficient approach to the actual value.

7. FORMULAE FOR SUB-IRRIGATION IN STRATIFIED AQUIFERS

In case of heterogeneous aquifers consisting of several horizontal layers with alternately large and small permeabilities, usually the assumption is made that there will be a horizontal flow in the layers with a large permeability and a vertical flow in the layers with a small permeability. Consequently in the case depicted in fig. 13a (a so called three layer problem because of the three layers in the saturated zone) the two following differential equations with two independent variables h and ϕ can be applied, except in areas near the open conduits where a radial flow may be dominating.

$$k_1 D_1 \frac{d^2h}{dx^2} = v_h - \frac{\phi_2 - \phi_1}{c} \qquad (61)$$

$$k_3 D_3 \frac{d^2h}{dx^2} = \frac{\phi_2 - \phi_1}{c} \qquad (62)$$

with $$c = \frac{D_2}{k_2} \qquad (63)$$

When there are four layers in the saturated zone (fig. 13b) the next three formulae with three independent variables h, ϕ_1 and ϕ_2 are valid:

$$\frac{\phi_1 - h}{c_1} = v_h \qquad (64)$$

$$k_2 D_2 \frac{d^2\phi_1}{dx^2} = \frac{\phi_1 - h}{c_1} - \frac{\phi_2 - \phi_1}{c_3} \qquad (65)$$

$$k_4 D_4 \frac{d^2\phi_2}{dx^2} = \frac{\phi_2 - \phi_1}{c_3} \qquad (66)$$

with $$c_1 = \frac{D_1}{k_1} \qquad c_3 = \frac{D_3}{k_3} \qquad (67)$$

Equations of this shape may also be used close to the open conduits where a more or less strong radial component in the groundwater flow exists, provided the continuation of the potential distribution in the top aquifer ($h(x)$ or $\phi_1(x)$) according to the previous formulae, is considered as an incorrect extrapolation which has to be indicated differently(e.g. $h'(x)$or $\phi_1'(x)$)and that the influence of the radial component will be accounted for by means of one of the two formulae hereafter (compare fig. 3), respectively valid for the three layer and the four layer problem:

$$\Delta h_{rad} = h_o' - h_o = q_o \, \Omega \qquad (68a)$$

$$\Delta\phi_{1,rad} = \phi_{1,0}' - \phi_{1,0} = q_o \, \Omega \qquad (68b)$$

In case symmetric steady states are concerned with a constant value for v_{phr} (independent of x and t), the derivation of a formula analogous to (6) and valid both for drainage and sub-irrigation will yield no fundamental trouble (ERNST, 1956, 1962). In order to take into account the radial flow component by means of a resistance value Ω, both for three- and four layer problems, equation (8) or similar equations for the radial resistance Ω may be used all the same (ERNST, 1963).

In case v_h is dependent on $z_{gs} - h$, however, the problem becomes much more complicated and this to such a degree that it is advisable to apply the same approach as discussed in chapter 6. It will be shown that this approach offers an opportunity to obtain a good approximation also for those cases where no symmetry exists.

Asymmetry exists wherever irregular differences in level of the ground surface are a natural cause of influent groundwater flow in the higher parts and effluent groundwater flow in the lower parts of the area. A suchlike situation is often expanding over large areas with a diameter of several kilometers. In a cross-section of such size the number of (parallel) open conduits may be rather large and in such cases complete execution of a calculation scheme similar to that discussed in the previous chapter would consequently lead to a much more extensive elaboration.

This is the reason that not only in cases with complete and skew symmetry, but also in case of asymmetry the elaboration will be limited preferably to one single strip between two parallel open conduits.

This implies that the distribution of influent and effluent seepage within the area should either be known or roughly estimated and that for the strip to be investigated, consequently 4 values have to supplied for inflow and outflow of groundwater by each of the two aquifers.

The formulae of chapter 6 remain applicable for the greater part. This is the case for formulae (39), ... (52), (56) and (57), though not completely for (36), ... (38), (53), ... (55) and (58), ... (60).

The equations have to be adjusted in such way that in case $h_{j,i}$-values are known for all x_j at the time t_i, from these data all other unknown quantities for the same time may be calculated.

This will be discussed for the three layer problem and for the four layer problem successively. As done in the previous chapter it will be assumed that the potentials and vertical fluxes have to be calculated at even values of j, the horizontal flow intensities q_1 and q_2 at odd values of j. A discussion of the calculation of the conditions at t_{i+1}, when a complete set of data for t_i is known, can be ommitted, as the equations (46) ... (51) eventually with some reiterations may be used again (see chapter 6c).

a. The three layer problem with given values for $h_{j,i}$ and h_o or q_o.

In this case (see fig. 13a) the formula (35) may be replaced by:

$$v_{h,j,i}\,\Delta x = q_{1,j-1,i} - q_{1,j+1,i} + v_{c,j,i}\,\Delta x \qquad (69)$$

In stead of formula (36) two similar relationsmay now be introduced viz. for the horizontal flow in the first and in the second aquifer respectively. For (37) no substitute will be given here, the difference between (36) and (37) being extant in the j-subscript only.

$$q_{1,j-1,i} = k_1 D_1 \frac{h_{j-2,i} - h_{j,i}}{\Delta x} \qquad (70)$$

$$q_{2,j-1,i} = k_3 D_3 \frac{\phi_{j-2,i} - \phi_{j,i}}{\Delta x} \qquad (71)$$

For the vertical flow in the second layer having a small permeability, there can be written:

$$v_{c,j,i} = \frac{\phi_{j,i} - h_{j,i}}{c} \qquad (72)$$

$$= \frac{q_{2,j-1,i} - q_{2,j+1,i}}{\Delta x} \qquad (73)$$

In all columns of width Δx, with nearly horizontal flow in the well- permeable layers, the 5 equations mentioned above are valid.

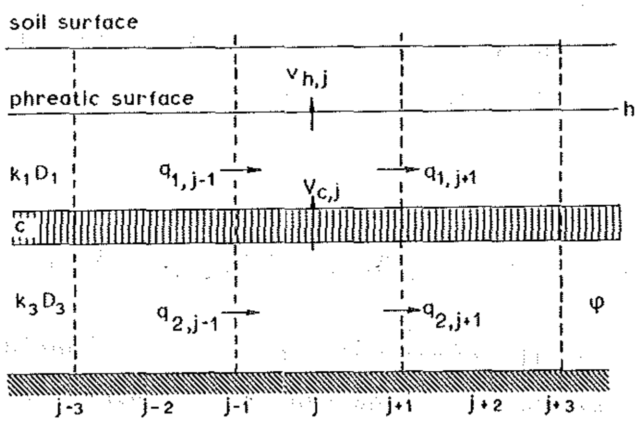

Fig. 13a. Schematization of the groundwater flow in an aquifer with three layers of different permeability below the phreatic surface. For the layers with a rather large transmissivity (k_1D_1 and k_3D_3) it is assumed that there is a mainly horizontal groundwater flow (q_1 and q_2), while for the intermediate layer a vertical flow is assumed (flux v_c and resistance $c = D_2/k_2$). Changes in the thickness D_1 with time will be neglected, but a dependency on x_j can be easily admitted

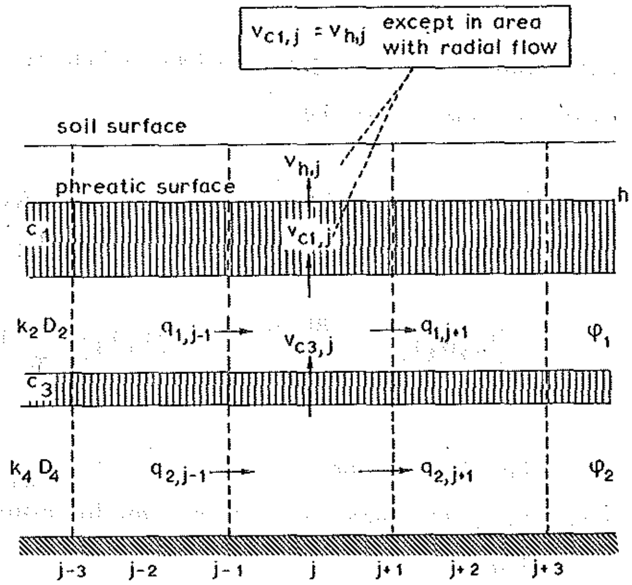

Fig. 13b. Representation similar to fig. 13a, but instead of three layers below the phreatic surface, four layers of alternately small and large permeability will be assumed in this case

In the columns with radial flow (width B_1) equation (53) has to be replaced by the next one (74). Provisionally the open conduit will be indicated by adding the subscript o (see left hand side of of fig. 14):

$$v_{h,r,i}\frac{B_1}{2} = -\frac{1}{2}q_{o,o,i} + q_{1,L,i} - q_{1,1,i} + v_{c,r,i}\frac{B_1}{2} \qquad (74)$$

In this case the given value $q_{1,\ L,\ i}$ is considered to be that inflow at the left hand side of the area concerned, that addition to the symmetric outflow of $q_{o,\ o}$ will produce the actual horizontal flow intensity in the top layer at the right hand side of conduit 0.

In stead of (54) the next two formulae may be used for the horizontal flow intensity in the two aquifers:

$$q_{1,1,i} = k_1D_1\frac{h_{b,i} - h_{2,i}}{\Delta x} \qquad (75)$$

$$q_{2,1,i} = k_3D_3\frac{\phi_{b,i} - \phi_{2,i}}{\Delta x} \qquad (76)$$

Formula (55) needs only to be provided with an extra term in the right hand side to remain valid:

$$\frac{h_{o,o,i} - h_{1,i}}{B_1} = \frac{-q_{o,o,i}}{8k_1D_1}\left(1 + \frac{8k_1D_1\Omega}{B_1}\right) + \frac{q_{1,1,i} - q_{1,L,i}}{4k_1D_1} + \frac{q_{1,L,i}}{2k_1D_1} =$$

$$= \frac{-q_{o,o,i}}{8k_1D_1}\left(1 + \frac{8k_1D_1\Omega}{B_1}\right) + \frac{q_{1,1,i} + q_{1,L,i}}{4k_1D_1} \qquad (77)$$

For the flow in the second aquifer in case $q_{2,\ i} = 0$, which means in case of symmetry, the next equation can be considered as a good first approximation:

$$(\phi_{o,i} - \phi_{1,i})/(\phi_{1,i} - \phi_{2,i}) = B_1/2\Delta x$$

Because of the given inflow $q_{2,\ L} = 0$ from outside the area, here also a term has to be added, so that the flow in the second layer

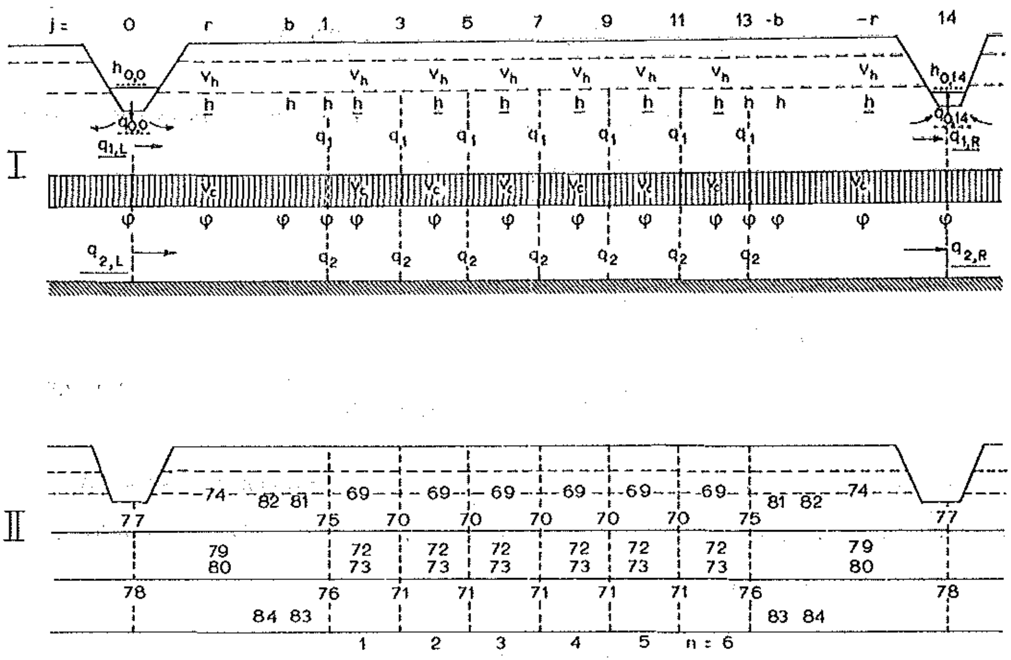

Fig. 14. Example of the dividing into rectangular elements of the area between two successive conduits with three different layers below the phreatic surface

I. Quantities underlined are given, quantities not underlined have to be calculated. One of the two quantities $h_{o,o}$ and $q_{o,o}$ must be given, the other must be calculated. The main of the region with a predominally horizontal direction for q_1 and q_2 is divided into 6 columns ($n = 6$) of 5 rectangular elements each. Number of unknowns = number of equations = $5n + 20 = 50$.

II. Numbers of the equations which have to be used for the calculation of the 50 unknown quantities under the open conduit may be represented by:

$$\phi_{o,i} - \phi_{1,i} - \frac{B_1}{2\Delta x}(\phi_{1,i} - \phi_{2,i}) = \frac{B_1 q_{2,L,i}}{4k_3 D_3} \tag{78}$$

For the vertical flow in the c-layer underneath the open conduit formulae (72) and (73) may be replaced by:

$$v_{c,r,i} = \frac{\phi_{r,i} - h_{r,i}}{c} \tag{79}$$

$$= (q_{2,L,i} - q_{2,1,i})\frac{2}{B_1} \tag{80}$$

Finally the 'irregular' limits x_b and x, should be considered, where the potentials have to be derived with the aid of formulae analogous to (56) and (57):

$$h_{1,i} = \frac{h_{b,i} + h_{2,i}}{2} \tag{81}$$

$$h_{r,i} = \frac{h_{o,o,i} + 2h_{1,i}}{3} \tag{82}$$

$$\phi_{1,i} = \frac{\phi_{b,i} + \phi_{2,i}}{2} \tag{83}$$

$$\phi_{r,i} = \frac{2\phi_{o,i} + \phi_{1,i}}{3} \tag{84}$$

The number of equations that has been obtained in this way is rather large. This is shown in the example given in fig. 14, where the area between two successive conduits has been divided into 6Δx-blocks (n = 6). From this figure the number of equations is immediately found to be equal to 5n + 20. In case n = 6 consequently 50 equations are obtained, all being of the first degree.

Because so many of these equations are similar and every equation has only a small number of unknowns, the solution will not be too difficult.

It can be seen immediately that the 8 unknowns v_h do occur only in the 8 equations (69) and (74), which therefore may be left out of consideration for the time being.

Next, the 7 unknowns q_1 may be calculated directly from (70) and (75). Similarly a direct calculation of $h_{o,\ o}$ or $q_{o,\ o}$, $h_{o,\ 14}$, or $q_{o,\ 14}$, h_b, h_1, h_{13} and h_{-b} is possible with the aid of (77), (81) and (82).

In this way both the number of equations and the number of unknowns have been reduced to 29. For in the remaining equations viz. (71), (72), (73), (76), (78), (79), (80), (83) and (84) all v_c, all q_2 and all ϕ are still acting as unknowns.

Next all v_c and all q_2 may be eliminated by combining (72) with (73) and (79) with (80). The equations (71) and (76) can be substituted in the other equations. Similarly ϕ_b, ϕ_1, ϕ_{13} and ϕ_{-b} may be eliminated with the aid of (83) and (84).

By this treatment finally 10 equations remain in which 10 unknown potentials occur viz. ϕ_o, ϕ_r, ϕ_2, ϕ_4, ϕ_6, ϕ_8, ϕ_{10}, ϕ_{12}, ϕ_{-r}, ϕ_{14} (number = n + 4).

As soon as all quantities in the saturated zone have been calculated in this way, it will be possible to proceed with the situation in the unsaturated zone and calculate the change towards the next day. For that purpose formulae can be used which are similar to those given in chapter 6. For each x_j given values for $\Psi_{u,\ j}$, h_j and $v_{h,\ j}$ are needed. The latter quantity has been obtained as a final result of the elaboration of the 50 linear equation mentioned above.

b. The four layer problem

In case of the four-layer problem (fig. 13b) the number of equations is increasing only little. The equations of the three layer problem may be used once more, though mostly other subscripts have to be introduced in this case. Moreover two new equations are required for the vertical flow in the upper layer with small permeability. In order to show this clearly all the equations will be written here in full shape and with the correct subscripts:

$$v_{c1,j,i} = \frac{\phi_{1,j,i} - h_{j,i}}{c_1} \tag{85}$$

$$= \frac{q_{1,j-1,i} - q_{1,j+1,i}}{\Delta x} + v_{c3,j,i} \tag{86}$$

$$q_{1,j-1,i} = k_2 D_2 \frac{\phi_{1,j-2,i} - \phi_{1,j,i}}{\Delta x} \tag{87}$$

$$q_{2,j-1,i} = k_4 D_4 \frac{\phi_{2,j-2,i} - \phi_{2,j,i}}{\Delta x} \tag{88}$$

$$v_{c3,j,i} = \frac{\phi_{2,j,i} - \phi_{1,j,i}}{c_3} \tag{89}$$

$$v_{c3,j,i} = \frac{q_{2,j-1,i} - q_{2,j+1,i}}{\Delta x} \tag{90}$$

$$v_{c1,r,i} = \frac{\phi_{1,r,i} - h_{r,i}}{c_1} \tag{91}$$

$$= (q_{1,L,i} - \frac{1}{2} q_{o,o,i} - q_{1,1,i}) \frac{2}{B_1} + v_{c3,r,i} \tag{92}$$

$$q_{1,1,i} = k_2 D_2 \frac{\phi_{1,b,i} - \phi_{1,2,i}}{\Delta x} \tag{93}$$

$$q_{2,1,i} = k_4 D_4 \frac{\phi_{2,b,i} - \phi_{2,2,i}}{\Delta x} \tag{94}$$

$$\frac{h_{o,o,i} - \phi_{1,1,i}}{B_1} = -\frac{q_{o,o,i}}{8k_1 D_1}(1 + \frac{8k_1 D_1 \Omega}{B_1}) + \frac{q_{1,1,i} + q_{1,L,i}}{4k_2 D_2} \tag{95}$$

$$\phi_{2,o,i} - \phi_{2,1,i} - \frac{B_1}{2\Delta x}(\phi_{2,1,i} - \phi_{2,2,i}) = \frac{B_1 q_{2,L,i}}{4k_4 D_4} \tag{96}$$

$$v_{c3,r,i} = \frac{\phi_{2,r,i} - \phi_{1,r,i}}{c_3} \tag{97}$$

$$= (q_{2,L,i} - q_{2,1,i}) \frac{2}{B_1} \tag{98}$$

$$\phi_{1,1,i} = \frac{\phi_{1,b,i} + \phi_{1,2,i}}{2} \tag{99}$$

$$\phi_{1,r,i} = \frac{h_{o,o,i} + 2\phi_{1,1,i}}{3} \tag{100}$$

$$\phi_{2,1,i} = \frac{\phi_{2,b,i} + \phi_{2,2,i}}{2} \tag{101}$$

$$\phi_{2,r,i} = \frac{2\phi_{2,o,i} + \phi_{2,1,i}}{3} \tag{102}$$

Fig. 15 shows as an example the division of a four layer aquifer. The number of Δx-strips is taken equal so that in fig. 14 ($n = 6$). The number of unknowns and the number of equations amounts to $6n + 22 = 58$. In this case no difference has to be made between v_{c1} and v_h, for vertical flow through the layer with resistance c_1 and the phreatic surface respectively.

The elimination of v_{c1}, v_{c3}, q_1 and q_2 makes clear how much unknowns and equations will remain. For $n = 6$ it will be found that 18 equations ($2n + 6$) will remain, every equation with 3 or 4 unknowns and totally 10 unknown ϕ_2 and 8 unknown ϕ_1.

It is not very difficult to solve these linear equations even when they are in a rather large number. The matrix of coefficients shows a regular distribution, which makes it possible to apply the following artifice. One equation is selected, in which no more than three unknowns occur. By taking for two of these unknowns, arbitrary, though not too improbable values, the third unknown may be solved. This result may be substitured in a next equation in which, besides the unknowns already used, only one more unknown occurs. This latter unknown may now easily be calculated, next be substituted in a fourth equation etc. In this way a range of values is obtained, which range will not be correct however,because these values do satisfy only ($2n + 4$) equations.

A suchlike calculation has to be done three times over with a set of trial values for ever the same two unknowns. By making a linear combination of these three estimations a correct solution can be obtained. For it is possible to multiply the 3 sets of trial values with α, β and $1 - \alpha - \beta$ respectively and demand that the linear combination containing 2 unknown coefficients α and β must satisfy also the 2 remaining equations.

The coefficients α and β can be solved easily from two equations of the first degree.

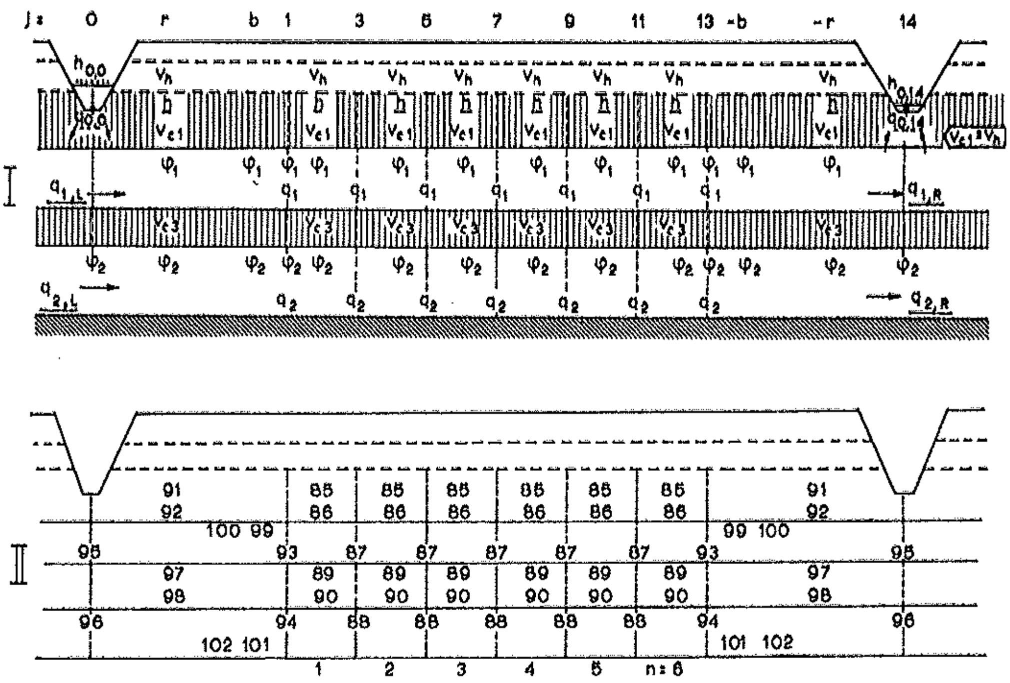

Fig. 15. Representation similar to fig. 14, but with a four layer aquifer in stead of a three layer aquifer.
For explication of symbols and numbers see caption of fig. 14. Number of unknowns = 6n + 22 = 58

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FREQUENTLY USED SYMBOLS

B_o = wetted perimeter of open conduit
B_1 = width of area with radial flow
c = resistance to vertical flow = D/k
D = thickness of (homogeneous) layer
d = thickness of equivalent layer
E = evaporation
h = elevation of phreatic surfase (hydraulic head)
h_m = h midway between drains
h_o = level of open water
k = permeability
L = drainspacing
P = precipitation
q = intensity of horizontal flow = - kD dh/dx
q_1 = q in upper aquifer
q_2 = q in lower aquifer
q_o = discharge or recharge through wet perimeter of open conduit
t = time
v = velocity, flux
v_{phr} = flux through phreatic surface (chapters 1-5)
v_h = flux through phreatic surface (chapters 6-7)
W = moisture storage in unsaturated zone
w = moisture content in unsaturated zone
x = horizontal coordinate
z = vertical coordinate
z_d = depth of bottom of root zone
ϕ = hydraulic head in deep aquifer
Ψ = moisture suction (matric potential)
Ω = resistance to radial flow
Δh = $h_m - h_o$