COMMISSIE VOOR HYDROLOGISCH ONDERZOEK T.N.O.
COMMITTEE FOR HYDROLOGICAL RESEARCH T.N.O.

VERSLAGEN EN MEDEDELINGEN NO. 10
PROCEEDINGS AND INFORMATIONS NO. 10

STEADY FLOW OF GROUND WATER TOWARDS WELLS

COMPILED BY THE
HYDROLOGISCH COLLOQUIUM

VERSL. MEDED. COMM. HYDROL. ONDERZ. T.N.O. NO. 10 – THE HAGUE – 1964

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CONTENTS

1. FOREWORD

Hydrology provides to a large extent the basis for the solution of water-supply problems and of questions of water control in general. A stream of questions on new problems emenates from the many and important fields to which it is being applied. The great progress made by hydrological science in the last few decades has been born out of this interaction between science and the practical application thereof.

In 1942 a small group of Dutch hydrologists set up a centre of discussion called the "Hydrologisch Colloquium". Problems especially in the field of ground-water flow are regularly discussed in this group, experiences in the application of new calculation methods are exchanged and new approaches in this field are studied.

Gradually the need for a comprehensive publication on the flow of ground water into wells made itself felt. It was decided to prepare a publication which would meet the needs of the many hydrologists concerned with this subject. Efforts were made to complete the texts of lectures presented at the colloquium and to cover any problems of ground-water flow into wells so far not discussed in the group. The result is this compendium, which aims at reviewing present knowledge in this field, presenting it in such way that it can be used in practice.

The Committee for Hydrological Research of the Central Organization for Applied Scientific Research in The Netherlands (T.N.O.) welcomes the opportunity to publish this compendium and thus make it more widely available.

COMMITTEE FOR HYDROLOGICAL RESEARCH
OF THE CENTRAL ORGANIZATION FOR
APPLIED SCIENTIFIC RESEARCH

2. INTRODUCTION

Nearly everyone dealing with the theoretical or practical problems of ground-water flow through porous formations – or more specifically, with the flow of a liquid through a porous medium – will encounter the problem of flow into a well. This problem which in principle concerns cylindrical-symmetrical or spherical flow or something in between the two, has many aspects.

Its theoretical treatment is generally based on certain simplified flow patterns. The selection of the mathematical procedure to be followed will depend on the simplifications introduced, on the solvability of the resulting equations and on the accuracy required. The mathematical operation depends further on the question whether in course of time changes in the flow pattern, in the porous medium or in the flowing liquid itself have to be taken into account.

Several of these aspects, together with material on this subject encountered in literature, have been discussed in sessions of the Hydrologisch Colloquium. The papers which have been presented in the sessions, and the proceedings of the discussions, however, give only a fragmentary picture of the knowledge of this subject possessed by the members of the Colloquium. A handbook containing all the information at present available on ground-water flow into wells does not exist either; literature on this subject is widely dispersed.

Yet a synopsis of the theory required in the first place to tackle actual flow problems and in the second place to develop further the theory itself, is urgently needed. The Hydrologisch Colloquium has, therefore, decided to publish such a synopsis, presuming that many colleagues in this country and abroad would be glad to have a compendium on the computation of ground-water flow into wells.

The Colloquium is glad to say that the Committee for Hydrological Research T.N.O. has expressed its willingness to bear the cost of publishing this work. The Colloquium expresss its gratitude for this valuable aid.

In compiling the material for the present publication the Hydrologisch Colloquium aimed at collecting the theory on ground-water flow into wells, in so far as is required for practical problems and in such a way

1°. that anyone somewhat familiar with differential and integral calculus will be able to verify the development of the equations;

2°. that the resulting formulas can immediately be used for practical work and do not involve the consultation of other literature.

The various cases of flow into wells will first be dealt with in a simple manner, in order to arrive at approximate formulas sufficiently accurate for the solution of most actual problems. Subsequently, the approximations and simplifications introduced will, if necessary, be discussed further.

The present work concerns permanent flow only; the Colloquium hopes to deal with non-permanent flow in another publication. Numerical and graphical methods fall outside the scope of this compendium and are not discussed.

The decimal system has been chosen for the division into chapters and paragraphs. Diagrams, figures and formulas bear the number of the paragraph in which they occur, a serial number being added.

The compiling and editing of the material for the publication was entrusted to a small committee. Many other members of the Colloquium, however, also contributed to the work. It is not possible to mention all of them. One exception, however, should be made for the member representing the Netherlands Mathematical Centre. This member took it upon himself to look after the mathematical side of the compendium. The Colloquium is much indebted to him for his most important contribution.

The Hydrologisch Colloquium is aware of the fact that a publication like this cannot be complete and infallible. The Colloquium therefore requests readers to bring to the notice of its secretary [1]) anything which in their opinion might contribute to the improvement of the compendium. As new subjects will be worked out in the coming years, a second, improved and enlarged, edition may have to be published at a later date.

PUBLICATION COMMITTEE OF THE
HYDROLOGISCH COLLOQUIUM

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[1]) c/o Committee for Hydrological Research T.N.O., P.O. Box 297, The Hague, Netherlands.

3. FUNDAMENTAL TERMS AND LAWS PROPERTIES OF WATER AND SOIL SYMBOLS

3.1. System of units

A coherent system of units is used throughout this publication, adapted to the needs of each individual problem. Accordingly, the day is generally used as the unit of time and the metre as the unit of length. When numerical values are given for the various quantities, they are as far as possible expressed both in units of the Giorgi system and in the commoner units.

3.2. Viscosity of water

The kinematic viscosity of water, ν, is expressed in Stokes or in m^2/day.
1 St = 1 cm^2/sec = 8.64 m^2/day

Temp., in °C	+0	5	10	15	20
Visc., in 10^{-2} Stokes	1.79	1.52	1.31	1.14	1.01
Visc., in m^2/day	0.152	0.133	0.113	0.098	0.087

3.3. Density (specific mass) and specific weight of water

The density ρ is expressed in kg of mass per m^3.

Temp., in °C	+0	4	5	10	15	20
Density, in kg/m^3	999.868	1000	999.992	999.727	999.126	998.230

Specific weight γ is obtained by multiplying density by the acceleration of gravity, g (= 9.81 m/sec^2):

$$\gamma = \rho g$$

3.4. Phreatic ground water and phreatic surface

Phreatic ground water is the coherent mass of water in a water-bearing stratum where, above that ground water, air is present in the pores that is in free connection with the atmosphere.

3.4.

The phreatic surface is the plane in the mass of ground water where the hydrostatic pressure equals the atmospheric pressure. The mass of ground water also extends a little above that plane, due to capillary action. The soil below the phreatic surface is generally fully saturated with water.

In the present publication it will be assumed that the phreatic surface constitutes the upper boundary of the coherent mass of ground water and that the soil below that surface is fully saturated with water.

3.5. Confined ground water

This is the ground water in a water-bearing formation above and below which there are impervious or semi-pervious layers, provided the ground-water mass extends up to the upper confining layer.

If both confining layers are impervious, the ground water is termed completely confined; otherwise it is semi-confined.

3.6. Filtration velocity of ground water

The filtration velocity or Darcy velocity v is defined as the amount of ground water passing a unit area perpendicular to the direction of flow per unit of time.

The filtration velocity is only a term used for purposes of calculation. The actual mean velocity of the water in the pores of the soil is much higher, dependent on pore volume and the structure of the soil.

3.7. Potential of ground water

The potential or piezometric head φ of ground water at a certain point is the elevation to which the water would rise in an open tube sunk to the point in question, the elevation being measured from an arbitrarily chosen plane of reference.

Hence

$$\varphi = z + h,$$

where

z = the elevation of the point considered above the plane of reference;
h = the height of the water column in the observation tube.

This relation can also be expressed as:

$$\varphi = z + \frac{p - p_0}{\rho g},$$

where
p = hydrostatic pressure of the ground water at that point;
p_0 = atmospheric pressure;
ρ = density of the ground water.

The plane of reference is often chosen in such a way that in the undisturbed condition of flow

$$\varphi = 0.$$

3.8. Pore volume, effective porosity and storage

The total porosity of a soil, β, is the aggregate volume of the pores or interstices in a unit volume of soil.

If a volume of soil V contains an amount of solid material equal to V_s units of volume the porosity of the soil

$$\beta = \frac{V - V_s}{V}.$$

The effective porosity, β_e, i.e. the pore volume which is effective with respect to flow, is that part of the total volume in which the water is free to move. It is the pore volume after deduction of the space occupied by sorption water and air, both expressed as a ratio to the total soil volume.

The effective porosity of saturated sand normally lies between 0.2 and 0.3; in clay β_e can be very small as clay minerals are able to retain much water on the surface of the particles. If the soil also contains air or other gases, β_e becomes still smaller.

The storage coefficient, β_s or more usually S, is that part of the volume which is effective with respect to storage. It can be defined as the amount of water taken into (or released from) storage per unit surface area of the aquifer if the phreatic surface rises (or drops) by a unit of length.

S, like β_e, is also smaller than the total porosity; for when the phreatic surface drops, part of the water is retained on the soil particles. When the water table rises air can be trapped in the filling pores; the storage coefficient for rising water, therefore, is often smaller than that for a dropping water level.

3.9. Homogeneous isotropy and anisotropy of the permeability of the soil

Isotropy with respect to permeability means that the permeability of the soil at a certain point has the same value for any direction of flow. In anisotropic soils this is not so.

3.9.

In the case of homogeneous isotropy, the permeability of the soil has the same value at any point in the aquifer and is independent of the direction of flow; in the case of homogeneous anisotropy the permeability to flow in a certain direction has the same value at any point in the aquifer.

3.10. DARCY'S LAW

Darcy's Law (or the law of linear resistance) applies to the movement of ground water through a porous soil. For flow through a homogeneous isotropic medium this law can be worded as follows: The filtration velocity v (or Darcy velocity) in the direction of flow is proportional to the gradient of the potential head in that direction, $\frac{d\varphi}{ds}$.

So:

$$v = -k\frac{d\varphi}{ds}.$$

Since it is obvious that the positive directions of v and s should preferably be the same, a minus sign in the formula is needed because the liquid always flows in the direction of decreasing potential.

In this publication k is termed the permeability coefficient of the soil. Strictly speaking this is not correct, k also depending on the viscosity of the liquid. Since, however, the compendium is purposely restricted to the flow of ground water of constant temperature and constant composition, the effect of viscosity may justifiably be ignored.

In soils of anisotropic permeability different values of k have to be used for different directions:

$$v_x = -k_x\frac{\partial\varphi}{\partial x},$$

$$v_y = -k_y\frac{\partial\varphi}{\partial y},$$

$$v_z = -k_z\frac{\partial\varphi}{\partial z}.$$

So it can be stated that, for ground water, the permeability coefficient depends solely on the structure of the porous medium, i.e. the size and shape of the pores. Therefore the obvious thing would be to try and find a relation between k and the factors determining the size and the shape of the pores. These factors are:

1°. grain-size distribution;
2°. porosity;
3°. shape of the grains;
4°. arrangement and orientation of the grains.

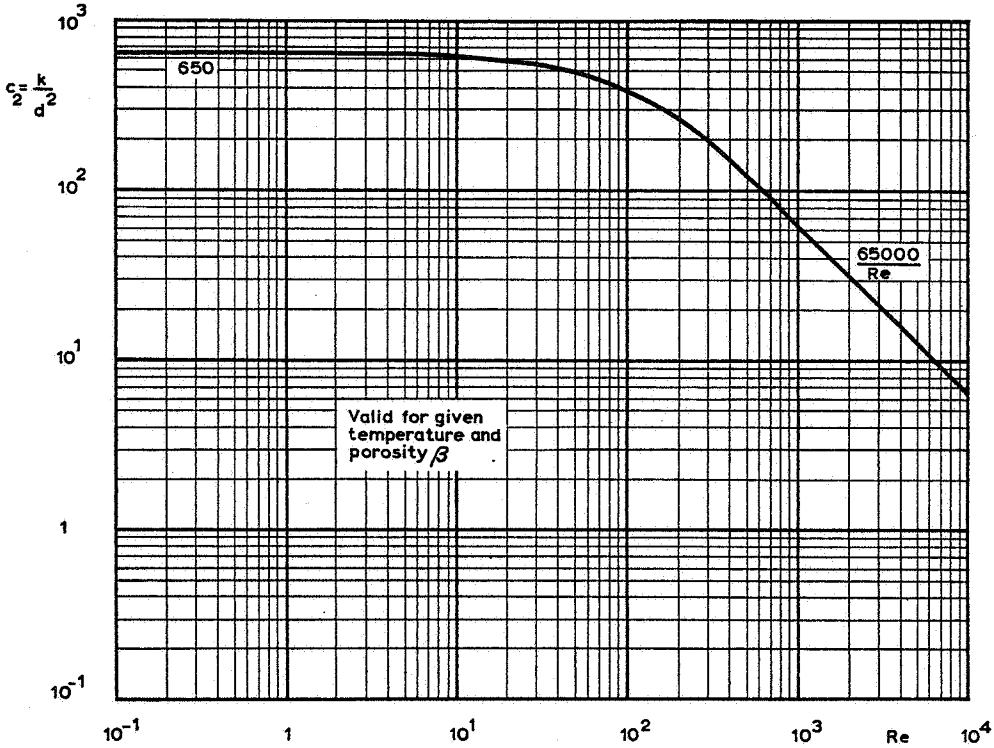

Fig. 3.10.–1

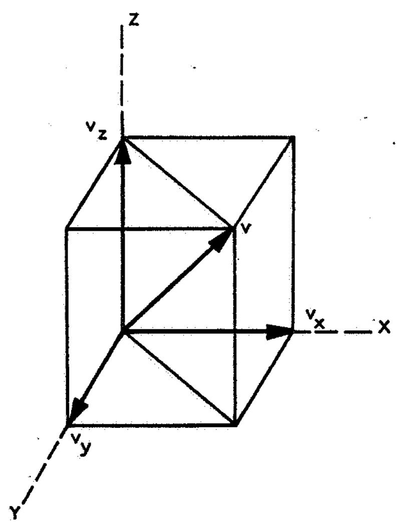

Fig. 3.10.–2

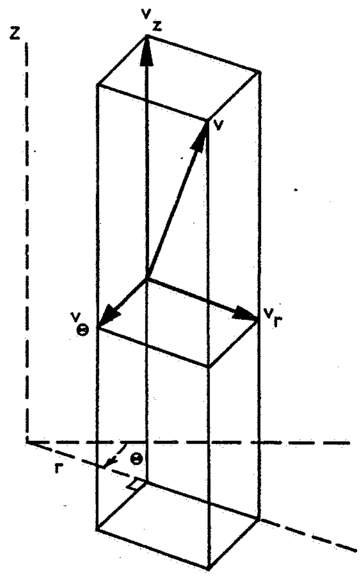

Fig. 3.10.–3

It is clear that the effect of the last two factors cannot easily be expressed numerically. The grain-size distribution also presents difficulties in this respect. That explains why a reliable and generally valid formula has not yet been found, in spite of attempts to do so on the part of many investigators. The formulas found can only be considered as sufficiently reliable for granular media with fairly uniform, rounded grains that are not too fine (i.e. not loam, silt, etc.).

Most of the formulas developed are of the following form:

$$k = C_1 \frac{\beta^3}{(1-\beta)^2} d^2,$$

where d is a characteristic grain size of the medium, β is the porosity and C_1 is a constant that depends on the shape and arrangement of the grains. For rounded grains of fairly uniform size this relation can be written as

$$k = C_2 \, d^2,$$

in which C_2 is largely determined by the porosity β.

If for d, the characteristic grain size, the 10% diameter d_{10} is chosen, i.e. the grain size that exceeds the size (diameter) of 10% of the material, and if k is expressed in m/day and d_{10} in mm, we have according to Allen Hazen

$$C_2 = 400 \text{ to } 1200, \text{ with an average of } 1000.$$

Darcy's Law, the law of linear resistance, only holds good as long as the accelerations of the particles of the liquid in the pores of the medium are so small that their influence can be ignored. This depends on the velocity of the flow, the viscosity of the liquid and the grain-size distribution of the medium. Indicative in this respect is the dimensionless Reynolds number, as expressed by the formula

$$Re = \frac{v \, d}{\nu},$$

where

v = filtration velocity as defined above;

d = a characteristic grain size of the medium;

ν = kinematic viscosity of the liquid.

There are various methods for the determinations of d. Practically all of them show that Darcy's Law holds good as long as the Reynolds number does not exceed some value between 1 and 10.

The range of validity of Darcy's Law appears from the graph in figure 3.10.–1, giving the relation between the Reynolds number and the coefficient C_2.

For $Re < 1$, C_2 is constant; so the permeability coefficient k is constant, too, and the resistance encountered by the liquid is proportional to the velocity of the flow, in other words Darcy's Law holds good.

For $Re > 1000$, C_2 decreases in inverse proportion to Re, i.e. to v. Now the resistance is proportional to v^2 and the flow is completely turbulent.

For Re between 1 and 1000, the resistance is proportional to the velocity to a power between 1 and 2.

Remark

For flow through pipes the change from laminar flow into turbulent flow occurs when Re amounts to 1000 or more, i.e. a much higher value than in the case of ground-water flow (in the formula for Re for flow through pipes, d stands for the diameter of the pipe).

The explanation of this difference may be found in the following:

1°. In the formula for Re for ground-water flow, v stands for the filtration velocity; the maximum actual velocity of the water in the pores, however, is more than 10 times as high, according to some investigators.

2°. The pore channels or interstices through which the water flows have a most irregular shape; under these conditions the flow readily looses contact with the walls of the pores or interstices.

It may be desirable to introduce the velocity component in a certain direction into many of the computations. To that end Darcy's Law has to be expressed in rectangular or cylindrical co-ordinates.

In rectangular co-ordinates (see fig. 3.10.–2) this law reads:

$$v_x = -k\frac{\partial\varphi}{\partial x},$$

$$v_y = -k\frac{\partial\varphi}{\partial y},$$

$$v_z = -k\frac{\partial\varphi}{\partial z}.$$

In cylindrical co-ordinates (see fig. 3.10–3) the law reads:

$$v_r = -k\frac{\partial\varphi}{\partial r},$$

$$v_z = -k\frac{\partial\varphi}{\partial z},$$

$$v_\Theta = -\frac{k}{r}\frac{\partial\varphi}{\partial\Theta}.$$

If φ depends only on r, we have

$$v_r = -k\frac{\partial\varphi}{dr}.$$

For soils with anisotropic permeability, different values of k have to be introduced for different directions.

3.11. Law of continuity

For a general derivation of the law of continuity the reader is referred to the literature on that subject.

In its general form the law can be written as follows (for an infinitely small element of the aquifer with lengths of side dx, dy and dz):

$$\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = -\frac{\partial}{\partial t}(\rho\beta),$$

where v_x, v_y and v_z are the three velocity components parallel to the separate axes;

ρ = density of the liquid;
β = porosity of the medium.

Because of the low compressibility of water, ρ may be regarded as constant. For permanent flow the above formula then becomes:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0.$$

If the permeability of the medium is the same in any direction (homogeneous isotropic), v_x, v_y and v_z in the above equation may be replaced by

$$-k\frac{\partial\varphi}{\partial x}, \quad -k\frac{\partial\varphi}{\partial y} \quad \text{and} \quad -k\frac{\partial\varphi}{\partial z}.$$

This yields the well-known Laplace equation:

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial x^2} = 0.$$

The equation of continuity expressed in cylindrical co-ordinates is derived from the relation

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = \frac{1}{r}\frac{\partial}{\partial r}(r\, v_r) + \frac{1}{r}\frac{\partial v_\Theta}{\partial \Theta}$$

and reads:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\, v_r) + \frac{1}{r}\frac{\partial v_\Theta}{\partial \Theta} + \frac{\partial v_z}{\partial z} = 0.$$

The Laplace equation now developes into:

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\Theta^2}+\frac{\partial^2\varphi}{\partial z^2}=0.$$

The various flow problems to be discussed in this compendium all concern symmetrical (cylindrical or spherical) cases, which generally makes it preferable not to develop the law of continuity from the general equation but to introduce this law in a simpler form according to the problem in question.

3.12. TRANSMISSIBILITY OF A WATER-BEARING STRATUM

In many cases of flow through a water-bearing stratum, the product of the permeability coefficient k of the soil and the thickness H of the layer will turn out to be an important factor in the computation or analysis of flow. This product, kH, can be regarded as characteristic of a water-bearing stratum and is termed its transmissibility or transmissivity T.

If the water-bearing stratum is of a stratified nature, i.e. that the permeability coefficient is not a constant along the vertical axis but shows variations, the transmissibility T becomes $\sum_{i=1}^{n} T_i$ (see 5.3.2.).

The quantities k, H and T, together with a few other quantities characterizing the water-bearing stratum and the adjoining strata, are termed formation constants.

3.13. HYDRAULIC RESISTANCE OF A SEMI-PERVIOUS LAYER

The treatment of this subject is based on a simplification of the common case of two water-bearing strata separated by a layer of limited thickness and of relatively small permeability (fig. 3.13.–1). The flow – or leakage – through this separating layer, a flow which will be practically vertical, is not analysed in detail, but the vertical filtration velocity in the layer is assumed to be proportional to the difference between the potential heads φ_0 and φ_1 respectively above and below the layer:

$$v=\frac{\varphi_1-\varphi_0}{c} \qquad (1)$$

The quantity c, which has the dimension of time, is termed the hydraulic resistance of the layer. Like k, H and T, it is a formation constant. Sometimes in English-American literature the name leakage coefficient or leakance is used for $\frac{1}{c}$. This name may lead to confusion with the term leakage factor (see 3.14.) and is therefore not recommended.

Generally v, φ_1, φ_0 and c are functions of the horizontal co-ordinates. In the following, however, c is assumed to be a constant, unless stated to be otherwise.

If the thickness of the semi-pervious layer is H' and its permeability coefficient for vertical flow is k', we can write

$$c = \frac{H'}{k'} \tag{2}$$

since from (1) it follows that

$$v = \frac{H'}{c} \times \frac{\varphi_1 - \varphi_0}{H'} \approx -\frac{H'}{c} \times \frac{\partial \varphi}{\partial z}.$$

Comparison of this relation with the equation of Darcy's Law (3.10.) leads to formula (2).

In nature the variations in H' and k' of a semi-pervious layer are often larger than the variations in c, so that the resistance of a layer is a more workable parameter in practical problems than the individual quantities H' and k'.

Non-uniform resistance of a semi-pervious layer may often cause difficulties in the analysis or in the computation of ground-water flows. At spots with a lower resistance the flow or leakage through the layer will be stronger compared with the leakage elsewhere; the leakage will be more or less concentrated at the weak spots. Consequently, as far as they contribute to the determination of the flow pattern, these spots are of much greater importance than their area might lead one to expect; even a small gap in a semi-pervious layer may cause a large increase in leakage.

As mentioned above it will be assumed in the following chapters that semi-pervious layers occurring in the problems discussed, have uniform resistance. Therefore the formulas developed are based on this assumption.

The formulas will be used in practice for the solution of actual flow problems, e.g. for the analysis of the flow of ground water in formations where the semi-pervious layers most probably have non-uniform resistance. When used for the determination of the formation constants, such analysis will produce a value for the resistance of the (non-uniform) semi-pervious layer which is termed the "effective resistance". The hydrological meaning of "effective resistance" can be made clear as follows: if the actual, non-uniform semi-pervious layer was replaced in the field by a uniform layer having a resistance equal to the effective resistance found, the hydrological quantities (heads, yields, etc.) at the points of observation used for the analysis would remain practically unchanged.

The effective resistance of a non-uniform layer is not a constant factor. If the flow pattern changes owing to some cause or other, weak spots which previously contri-

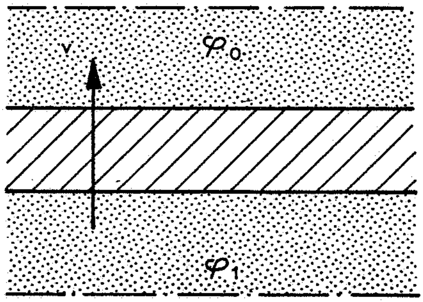

Fig. **3.13.-1**

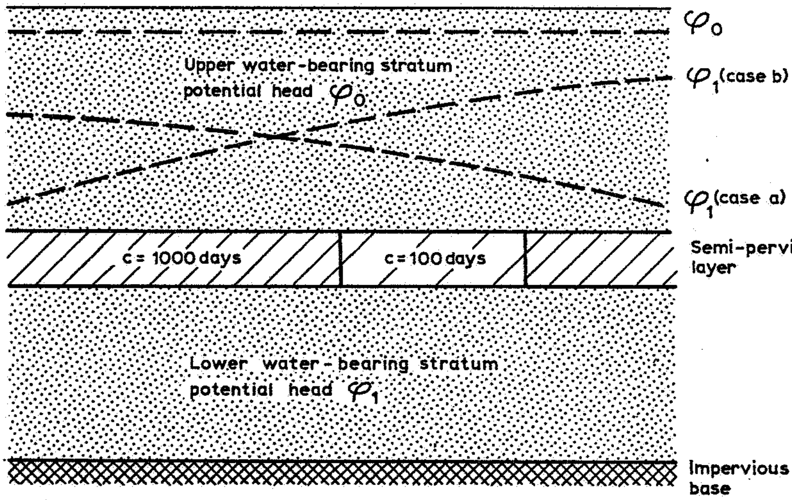

Fig. **3.13.-2.**

buted little to the total leakage may now attract relatively much larger quantities of water, thus altering the effective resistance.

This is illustrated in figure 3.13.–2, showing two water-bearing strata separated by a non-uniform semi-pervious layer. Two cases of flow, (a) and (b), are represented. The average difference in head, $\varphi_0 - \varphi_1$, is the same in both cases. In case (a), however, the difference in head at the weak spot is much greater than in case (b); that is to say that in case (a) much more water will percolate through the weak spot and the total leakage over the area considered will be considerably larger than in case (b). Now if the resistance of the semi-pervious layer is computed by means of formula 3.13.–(1), using the average difference in potential head and the total leakage over the area under consideration, one would find a much smaller value for the resistance in case (a) than in case (b).

Fortunately, however, the variations in a ground-water flow are generally fairly small, so that the changes in the effective resistance of a semi-pervious layer produced by these variations are small compared with uncertainties and inaccuracies in the value of the resistance due to other causes.

3.14. Leakage factor of a semi-confined (or leaky) water-bearing stratum

The leakage factor, λ, determines the distribution of the leakage into the aquifer or, in other words, it determines the origin of the water withdrawn from a well tapping the aquifer.

$$\lambda = \sqrt{kHc} = \sqrt{Tc},$$

where k is the permeability coefficient, H the thickness and T the transmissibility of the aquifer and c the resistance of the semi-pervious covering layer. Like k, H, T and c, also λ is a formation constant. For further details see 5.1.2.1.

3.15. Symbols and dimensions of the principle quantities

Quantity	Symbol	Dimension
Co-ordinates,		
– horizontal	x, y	L
– vertical	z	L
– horizontal in cases of axial symmetry	r	L
– special values	x_0, r_w, r_∞, R	L
Rate of flow through a cross section of the aquifer per unit width	q	$L^2 T^{-1}$

Quantity	Symbol	Dimension
Yield; total discharge	Q	$L^3 T^{-1}$
Thickness of a water-bearing stratum	H	L
Thickness of a semi-pervious layer	H'	L
Grain size (diameter)	d	L
Effective (specific) grain size	d_m, d_s	L
Permeability coefficient of a water-bearing stratum	k	LT^{-1}
Permeability coefficient of a semi-pervious layer	k'	LT^{-1}
Transmissibility coefficient of a water-bearing stratum	$T = k H$	L^2T^{-1}
Hydraulic resistance of a semi-pervious layer	$c = \frac{H'}{k'}$	T
Leakage factor of a water-bearing stratum	$\lambda = \sqrt{Tc}$	L
Density (specific mass)	ρ	ML^{-3}
Acceleration of gravity	g	LT^{-2}
Specific weight	$\gamma = \rho g$	$ML^{-2}T^{-2}$
Dynamic viscosity	η	$ML^{-1}T^{-1}$
Kinematic viscosity	$\nu = \frac{\eta}{\rho}$	L^2T^{-1}
Elevation of the phreatic surface above an impervious base	h	L
Ground-water potential, piezometric head	φ	L
Drawdown of the ground-water head	s	L
Velocity of filtration	v	LT^{-1}
Average velocity in the pores	v_p	LT^{-1}
Total porosity	β	–
Effective porosity with respect to flow	β_e	–
Effective porosity with respect to storage; storage coefficient	S, β_s	–
Gradient (of the piezometric level)	i	–
Slope of an impervious layer	I	–
Replenishment of ground water by rainfall, per unit of area and per unit of time	P	LT^{-1}

3.15.

4. ELEMENTARY CASES OF FLOW

4.1. Spherical equipotential planes

The following equation discribes the radial flow of water abstracted from a spherical surface (spherical or point sink) situated in an aquifer of infinite extent with homogeneous isotropic permeability.

$$Q = F v_r = 4\pi r^2 \left(-k \frac{d\varphi}{dr}\right) = -4\pi k r^2 \frac{d\varphi}{dr},$$

in which Q is the flow across a spherical surface of radius r, the centre of which coincides with the centre of the sink (see fig. 4.1.–1).

This is the Darcy equation.

Since s and v are taken positive in the same direction (3.10), Q becomes negative on abstracting water.

According to the continuity equation Q is independent of r, so

$$Q = Q_0 = \text{constant}.$$

The Darcy equation now becomes

$$\frac{-Q_0 dr}{4\pi k r^2} = d\varphi.$$

And after integration

$$\frac{+Q_0}{4\pi k r} = \varphi - C \quad \text{or}$$

$$\varphi = \frac{Q_0}{4\pi k r} + C \tag{1}$$

The boundary condition defines the value of C. When choosing for instance the reference plane of the potential at such a level that $\varphi = 0$ when $r = \infty$, it appears that $C = 0$.

The case of radial flow towards a spherical or point sink is almost exclusively theoretical; it can be used successfully to solve difficult problems that can be approximated by the superposition of a number of cases of such flow (see e.g. 4.3.).

4.2. Cylindrical equipotential planes

The radial flow of water towards a linear sink or a cylindrical well of infinite depth, situated in a porous medium of infinite extent and of homogeneous isotropic permeability can be described by

$$q = 2\pi r v = -2\pi r k \frac{d\varphi}{dr}, \tag{1}$$

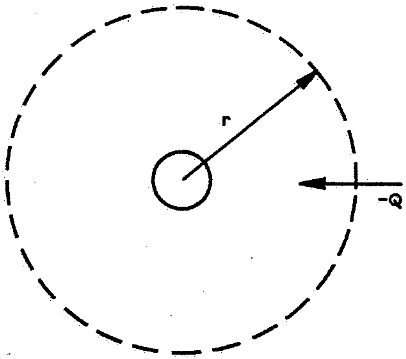

Fig. 4.1.-1

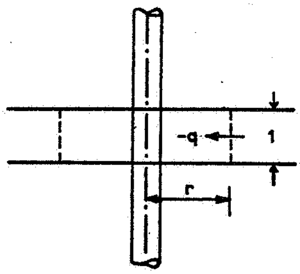

Fig. 4.2.-1

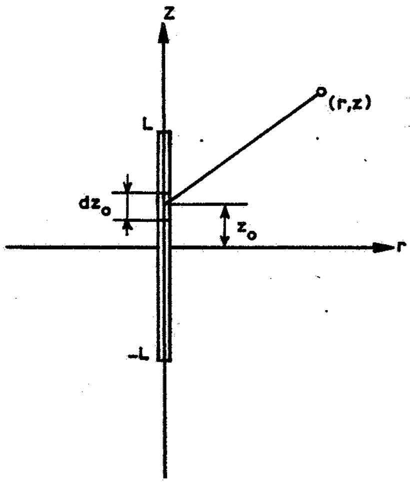

Fig. 4.3.-1

where q is the amount of water that flows across a cylinder of radius r and unit height, the axis of which coincides with the axis of the well (see fig. 4.2.–1).
This is the Darcy equation.

According to the continuity equation q is independent of r, so

$$q = q_0 = \text{constant.}$$

The Darcy equation now becomes

$$d\varphi = -\frac{q_0\, dr}{2\pi kr}. \tag{2}$$

After integration

$$\varphi = -\frac{q_0}{2\pi k} \ln r = C. \tag{3}$$

The boundary condition again determines the value of C.

4.3. Confocal ellipsoids as equipotential planes

The problem of the flow of water towards a cylindrical well of limited depth in a porous medium of infinite extent can be solved by making use of the potential and streamline distributions of a linear sink of limited length.

The potential $d\varphi$ at the point (r, z) caused by a point sink with abstraction dq situated at the point $r = 0$, $z = z_0$ (cylindrical co-ordinates) can (see fig. 4.3.–1), according to 4.1., be expressed by the following formula:

$$d\varphi = \frac{dq}{4\pi k} \frac{1}{\sqrt{(z_0 - z)^2 + r^2}}. \tag{1}$$

By placing point sinks, having discharges totaling the yield of the well Q_W, at equal distances between $z = -L$ and $z = L$, the total potential can be determined by substituting $dq = \frac{Q_W}{2L}\, dz_0$ in formula (1).

After integration to z_0 between $z_0 = -L$ and $z_0 = L$, it follows that

$$\varphi = \frac{Q_W}{8\pi k L} \int_{-L}^{L} \frac{dz_0}{\sqrt{(z - z_0)^2 + r^2}} = \frac{Q_W}{8\pi k L} \ln \left[z_0 - z + \sqrt{(z_0 - z)^2 + r^2} \right]_{z_0 = -L}^{z_0 = L} =$$

$$= \frac{Q_W}{8\pi k L} \ln \frac{L - z + \sqrt{(L - z)^2 + r^2}}{-L - z + \sqrt{(L + z)^2 + r^2}}. \tag{2a}$$

As

$$L - z + \sqrt{(L-z)^2 + r^2} = \frac{r^2}{z - L + \sqrt{(L-z)^2 + r^2}}$$

and

$$-L - z + \sqrt{(L+z)^2 + r^2} = \frac{r^2}{z + L + \sqrt{(z+L)^2 + r^2}},$$

instead of (2*a*) is valid

$$\varphi = \frac{Q_W}{8\pi k L} \ln \frac{z + L + \sqrt{(z+L)^2 + r^2}}{z - L + \sqrt{(z-L)^2 + r^2}} = \tag{2b}$$

$$= \frac{Q_W}{8\pi k L} \ln \frac{(z + L + \sqrt{(z+L)^2 + r^2})\,(L - z + \sqrt{(L-z)^2 + r^2})}{r^2} \tag{2c}$$

Formulas (2a), (2b) and (2c) are of course identical, but if r approaches zero, formula (2a) only means something when $z < -L$, formula (2b) only when $z > L$, while formula (2c) is only the right formula for φ if $-L < z < L$.

Some characteristics of these potential and streamline distributions will now be worked out.

a. The potential distribution (2) can be regarded as that of a linear sink, with z between $-L$ and $+L$ and an abstraction of water $\frac{Q_W}{2L}$ per unit of length. This appears from the derivation given and also from the following fact.

The radial component of the Darcy velocity is, according to (2),

$$v_r = -k\frac{\partial\varphi}{\partial r} = \frac{Q_W}{8\pi L}\frac{1}{r}\left[\frac{L+z}{\sqrt{(L+z)^2 + r^2}} + \frac{L-z}{\sqrt{(L-z)^2 + r^2}}\right].$$

The discharge $dq\,(z_0)$ across the area of a part of a cylinder of radius r around the z-axis and between $z = z_0$ and $z = z_0 + dz_0$ is

$$2\pi r\, v_r\,(r, z_0)\, dz_0 = \frac{Q_W dz_0}{4L}\left[\frac{L+z_0}{\sqrt{(L+z_0)^2 + r^2}} + \frac{L-z_0}{\sqrt{(L-z_0)^2 + r^2}}\right].$$

When r approaches zero,

$\sqrt{(L+z_0)^2 + r^2}$ approaches $/L + z_0/$,

$\sqrt{(L-z_0)^2 + r^2}$ approaches $/L - z_0/$, so

$dq\,(z_0)$ approaches $\frac{Q_W dz_0}{4L}\left[\frac{L+z_0}{/L+z_0/} + \frac{L-z_0}{/L-z_0/}\right].$

That means $dq\,(z_0)$ approaches 0 when $z_0 > L$,

it approaches $\frac{Q_W}{2L}\, dz_0$ when $-L < z_0 < L$,

and it approaches 0 again when $z_0 < -L$. (3)

4.3.

So $\frac{Q_W}{2L}$ flows into the line sink per unit of length, between $z = -L$ and $z = +L$.

b. The potential (2) satisfies the potential equation

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} = 0.$$

This can be proved by direct substitution and by superposition (the elementary potentials (1) satisfy the potential equation).

c. The equipotential planes in the flow pattern described by (2), can be found as follows.

Supposing $\sqrt{(z+L)^2 + r^2} = R_1$ and $\sqrt{(z-L)^2 + r^2} = R_2$ (see fig. 4.3.–2) then $R_1{}^2 - R_2{}^2 = 4\,z\,L$, and together with e.g. (2a)

$$\varphi = \frac{Q_W}{8\pi kL} \ln \frac{4L^2 - R_1{}^2 + R_2{}^2 + 4LR_2}{-4L^2 - R_1{}^2 + R_2{}^2 + 4LR_1} =$$

$$= \frac{Q_W}{8\pi kL} \ln \frac{(R_2 - R_1 + 2L)(R_2 + R_1 + 2L)}{(R_2 - R_1 + 2L)(R_2 + R_1 - 2L)} = \frac{Q_W}{8\pi kL} \ln \frac{R_1 + R_2 + 2L}{R_1 + R_2 - 2L} \quad (4)$$

(unless $R_1 = R_2 + 2L$, which means $r = 0$ and $z > L$; however, in that case formula (2a) means nothing). Formula (2b) or (2c) gives the same result, of course.

From formula (4) it follows that φ is constant on the surfaces where $R_1 + R_2$ is constant, i.e. on rotation-ellipsoids with foci at the points $z = L$ and $z = -L$.

Supposing $R_1 + R_2 = 2m$ $(m > 1)$ and $\sqrt{m^2 - L^2} = n$, the equation for this ellipsoid is

$$\frac{z^2}{m^2} + \frac{r^2}{n^2} = 1 \quad (5)$$

(m = semi long-axis, n = semi short-axis) and the value of the potential on this ellipsoid is

$$\varphi = \frac{Q_W}{8\pi kL} \ln \frac{m + L}{m - L} \quad (6)$$

When m approaches L, n approaches zero. The ellipsoids become very slender and approach their limit, viz. the line along which the line sink is situated.

d. The following approximation applies to the potential at a great distance from the well.

Taking $\sqrt{r^2 + z^2} = R$ and supposing $R \gg L$, we have

$$R_1 = \sqrt{R^2 + 2Lz + L^2} = R\sqrt{1 + \frac{2Lz}{R^2} + \frac{L^2}{R^2}} = R\left(1 + \frac{Lz}{R^2} + \frac{L^2}{2R^2} - \frac{L^2 z^2}{2R^4} + \ldots\right),$$

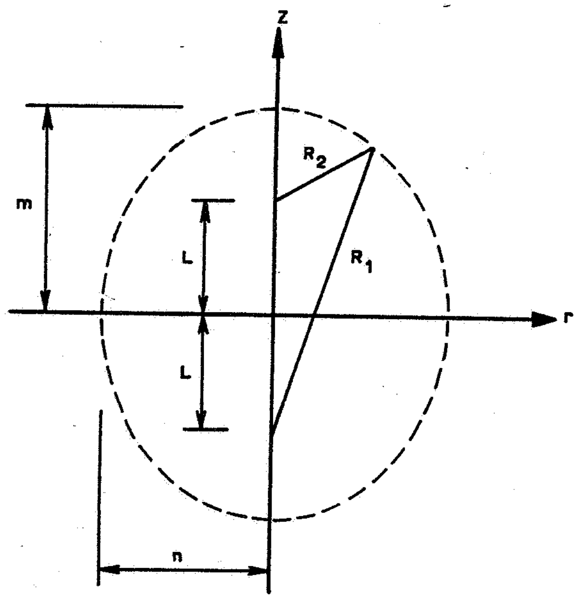

Fig. 4.3.-2

and by analogy

$$R_2 = R\left(1 - \frac{Lz}{R^2} + \frac{L^2}{2R^2} - \frac{L^2z^2}{2R^4} + \ldots\right),$$

$$\text{so } (R_1 + R_2) = 2R\left(1 + \frac{L^2}{2R^2} - \frac{L^2z^2}{2R^4} + \ldots\right).$$

Using formula (4),

$$\varphi = \frac{Q_W}{8\pi kL} \ln \frac{1 + \dfrac{2L}{R_1 + R_2}}{1 - \dfrac{2L}{R_1 + R_2}} = \frac{Q_W}{4\pi kL}\left[\frac{2L}{R_1 + R_2} + \frac{1}{3}\left(\frac{2L}{R_1 + R_2}\right)^3 + \ldots\right] =$$

$$= \frac{Q_W}{4\pi kR}\left[\frac{1}{1 + \dfrac{L^2}{2R^2} - \dfrac{L^2z^2}{2R^4} + \ldots} + \frac{1}{3}\frac{L^2}{R^2}\cdot\frac{1}{1 + \ldots} + \ldots\right] =$$

$$= \frac{Q_W}{4\pi kR}\left[1 - \frac{1}{6}\frac{L^2}{R^2}\left(1 - \frac{3z^2}{R^2}\right) + \ldots\right]. \qquad (7)$$

In the first approximation φ is the potential of a point sink with a discharge Q_W (in other words, the ellipsoidal equipotential surfaces more and more closely approach spheres).

e. The following holds good for the potential close to the line sink. When $-L < z < L$ and r is small with regard to $L + z$ as well as to $L - z$ (i.e. that the distance from the point r, z to the axis of the line sink is small with regard to the distances from r, z to the ends of the line sink) it follows that

$$L \pm z + \sqrt{(L \pm z)^2 + r^2} = 2(L \pm z)\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{r^2}{(L \pm z)^2}}\right] =$$

$$= 2(L \pm z)\left[1 + \frac{r^2}{4(L \pm z)^2} + \ldots\right],$$

and with formula (2c)

$$\varphi = \frac{Q_W}{8\pi kL} \ln\left[\frac{4(L^2 - z^2)}{r^2}\left\{1 + \frac{r^2}{4(L + z)^2} + \ldots\right\}\left\{1 + \frac{r^2}{4(L - z)^2} + \ldots\right\}\right] =$$

$$= \frac{Q_W}{4\pi kL}\left[\ln\frac{2L}{r} + \frac{1}{2}\ln\left(1 - \frac{z^2}{L^2}\right) + \frac{r^2}{8(L + z)^2} + \frac{r^2}{8(L - z)^2} + \ldots\right] \qquad (8)$$

When r is small with regard to L and z is not too close to $\pm L$, the first term dominates and the potential depends but slightly on z. This is the reason why the potential and streamline distribution of the flow into a line sink can be used as an approximation of the potential and streamline distribution of the flow

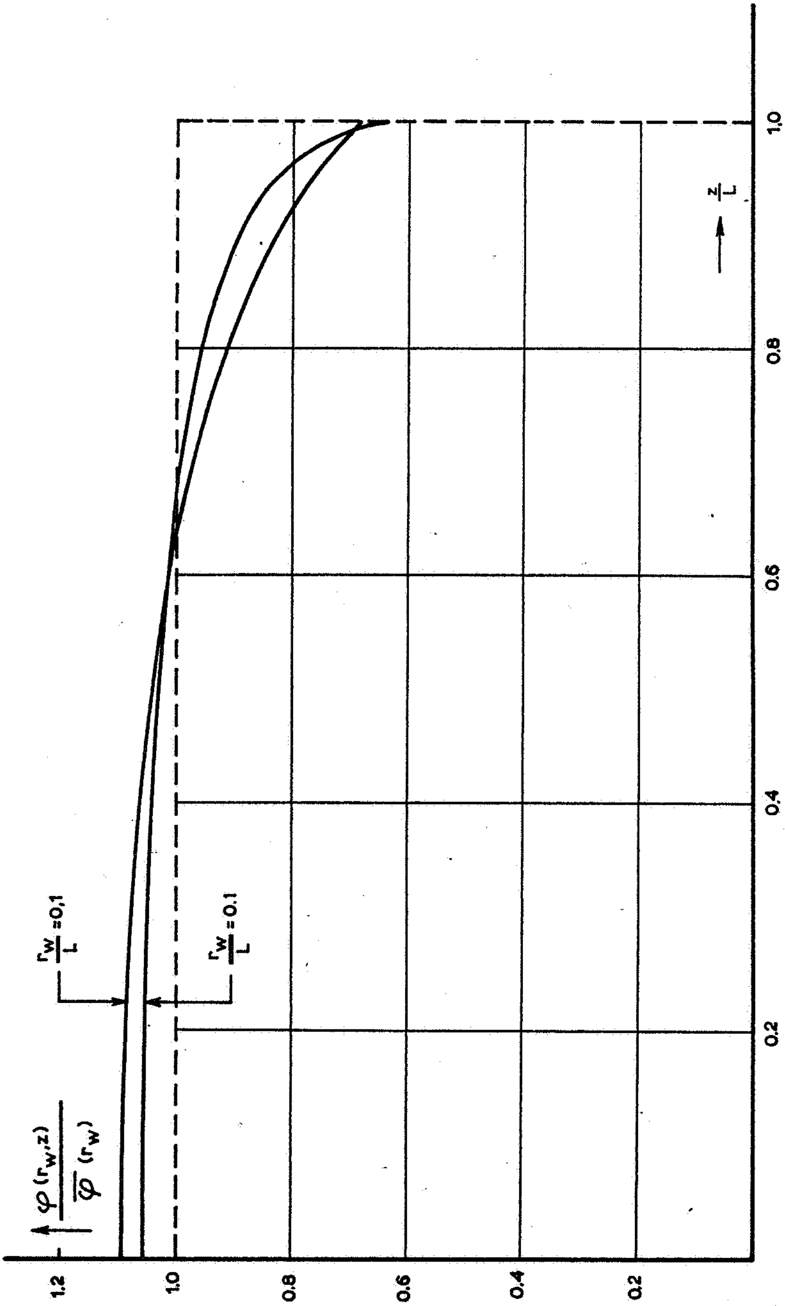
$\frac{\varphi(r_w, z)}{\overline{\varphi}(r_w)}$
1.2
1.0
0.8
0.6
0.4
0.2
$\frac{r_w}{L} = 0,1$
$\frac{r_w}{L} = 0.1$
0.2
0.4
0.6
0.8
1.0
$\frac{z}{L}$

Fig. 4.3.-3

towards a cylindrical screen, of which the radius r_W is small with regard to the height $2L$ and of which the potential along the screen surface is constant.

f. The mean value of $\varphi(r, z)$ on the cylinder $r = r_W$, $-L < z < L$, is

$$\bar{\varphi}(r_W) = \frac{1}{2L} \int_{-L}^{L} \varphi(r_W, z)\, dz =$$

$$= \frac{Q_W}{4\pi k L} \left[\ln \frac{4L}{r_W} + \ln \frac{1}{2} \left(1 + \sqrt{1 + \frac{r_W^2}{4L^2}} \right) - \sqrt{1 + \frac{r_W^2}{4L^2}} + \frac{r_W}{2L} \right]. \qquad (9)$$

In figure 4.3.–3 the value $\dfrac{\varphi(r_W, z)}{\bar{\varphi}(r_W)}$ is plotted as a function of $\dfrac{z}{L}$ (wherein $0 < z < L$) for $\dfrac{r_W}{L} = 0.1$ and $\dfrac{r_W}{L} = 0.01$.

The figure shows that the potential produced by the line sink on most of the cylinder $r = r_W$, $-L < z < L$ approaches the mean value reasonably closely. Consequently a line sink between $z = -L$ and $z = L$ with a discharge Q_W is a reasonable approximation for a slender cylindrical screen of the same height and having the same discharge. Then the mean value of the potential produced by the line sink on the cylinder $r = r_W$, $-L < z < L$, can be taken as an approximation of the potential at the surface of the screen.

Using (9) a good approximation of this mean value when $r_W \ll L$ is found to be

$$\bar{\varphi}(r_W) = \frac{Q_W}{4\pi k L} \left[\ln \frac{4L}{r_W} - 1 \right]. \qquad (10)$$

Remark

When $r_W \ll L$, $-L < z < L$ and z is not too close to $\pm L$, formula (8) leads to the following approximation:

$$\varphi(r_W, z) = \frac{Q_W}{4\pi k L} \left[\ln \frac{2L}{r_W} + \frac{1}{2} \ln \left(1 - \frac{z^2}{L^2} \right) \right] = \frac{Q_W}{8\pi k L} \ln \frac{4(L^2 - z^2)}{r_W^2}. \qquad (11)$$

Taking the mean value of $\varphi(r_W, z)$ for all values of z between $-L$ and $+L$, formula (11) leads back to formula (10) which was a good approximation of the mean of the exact value of the potential on the cylinder $r = r_W$, $-L < z < L$. This means that it is permissible to introduce approximations before the mean values are computed, in spite of the fact that approximation (11) is poor when z approaches $+L$ or $-L$. This property will be used in 5.4.2.

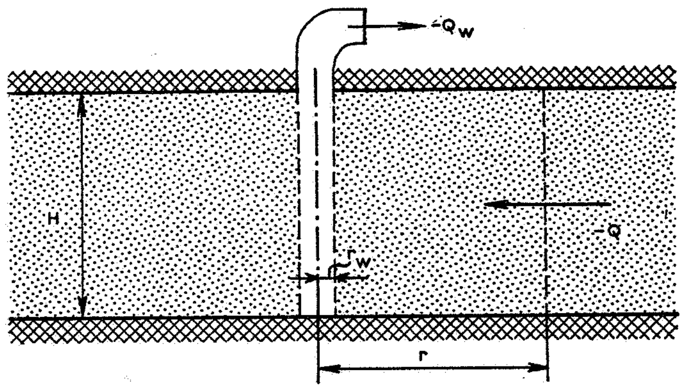

Fig. 5.1.1.–1

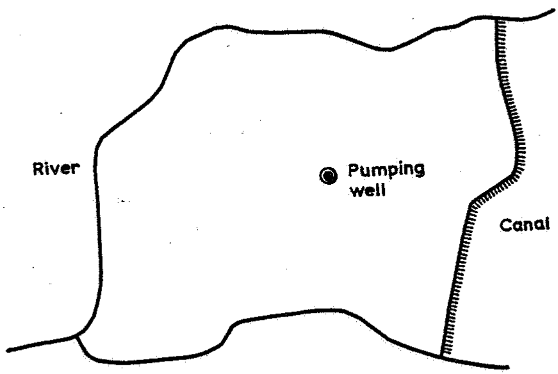

Fig. 5.1.1.–2.

5. CYLINDER-SYMMETRICAL FLOW (OR TWODIMENSIONAL RADIAL FLOW)

5.1. Well in a confined homogeneous and isotropic aquifer of constant thickness

In a confined aquifer of constant thickness, the flow of water to a fully penetrating well takes place along parallel planes. The flow is twodimensional and when described in plane polar co-ordinates uni-directional.

In the immediate vicinity of a partially penetrating well, however, there is also a velocity component perpendicular to the above mentioned planes and the flow is threedimensional (compare 5.4.). Beyond a certain distance from the well, however, this velocity component is so small as to be negligible and again the flow can be considered as being twodimensional.

5.1.1. Well in an aquifer between two impervious layers (completely confined aquifer)

In an aquifer, bounded at top and bottom by entirely impervious layers (fig. 5.1.1.–1) the flow pattern around a fully penetrating well may be considered as part of the flow pattern described in 4.2. for a cylindrical well of infinite length. If H is the aquifer thickness, the equations 4.2.–(1), (2) and (3) can now be rewritten as:

$$Q = Q_W = qH = -2\pi r H k \frac{d\varphi}{dr}, \tag{1}$$

$$d\varphi = \frac{-Q_W}{2\pi k H} \frac{dr}{r}, \tag{2}$$

$$\varphi = \frac{-Q_W}{2\pi k H} \ln r + C. \tag{3}$$

With the well in the centre of a circular island with radius R, the potential φ at the boundary of the island is constant at φ_R. Substitution of that boundary condition gives as the value of the integration constant

$$C = \varphi_R + \frac{Q_W}{2\pi k H} \ln R.$$

The complete solution is then

$$\varphi = \varphi_R + \frac{Q_W}{2\pi k H} \ln \frac{R}{r} \tag{4}$$

(with Q_W negative for ground-water abstraction).

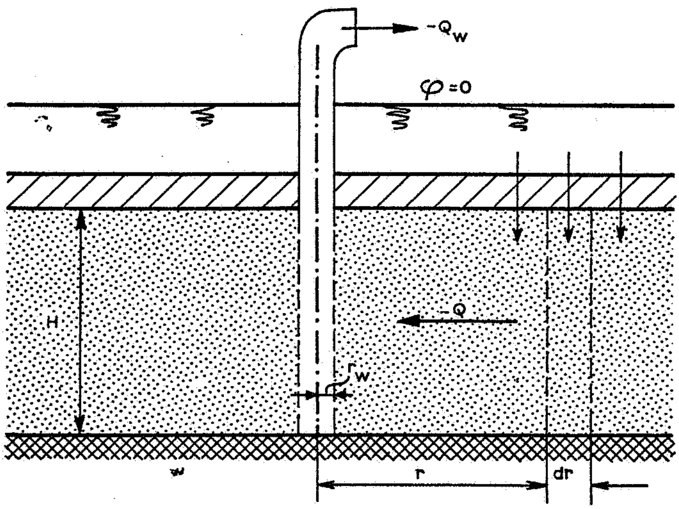

Fig. **5.1.2.1.–1**

The boundary condition $\varphi_R = 0$ at $R = \infty$ gives no sensible solution because it leads to $\varphi = (\pm)\infty$.

A circular island with a well at the centre is the exception rather then the rule. A fully confined aquifer is also rare; in most cases some leakage across the upper and lower bounding planes of the aquifer occurs. Nevertheless formula (4) can be used for a well in a confined aquifer with, at certain distances round about the well, boundaries (rivers, canals, etc., see fig. 5.1.1.–2) where the ground-water potential is constant, i.e. not influenced by the ground-water abstraction. If the area within these boundaries is not too large, the leakage from above or from below is small compared with the lateral inflow of water into the aquifer and may safely be ignored. The value of R to be substituted in formula (4) is found by idealizing the irregular outline of the area by a circle which is concentric with the well.

The drawdown is only proportional to the logarithm of R and so variations in the real distance from the well to the perimeter of the area under consideration have but little influence (compare 5.2.1.). For a large distance from the well to the centre of the enclosed area, however, substituting a circle concentric with the well for the real outline is not feasible. Chapter 6 gives the values of R to be inserted in the drawdown formula (4) to satisfy other situations of the well with regard to open water.

5.1.2. Well in an aquifer covered by a semi-pervious layer (leaky or semi-confined aquifer)

5.1.2.1. *The leakage through the covering layer is proportional to the drawdown in the aquifer*

Figure 5.1.2.1.–1 shows a fully penetrating well in a confined aquifer resting on an impervious base and overlain by a semi-pervious layer. Above the semi-pervious layer phreatic water with a constant level at datum line is assumed to exist. The assumption that the flow through the aquifer is horizontal, as stated in 5.1., is now only permissible when the recharge from above is small compared with the lateral flow of water. This is the case when

$$\frac{H}{\lambda} \ll 1,$$

in which λ is the leakage factor of the water-bearing formation (see 3.14.). This condition is dealt with in greater detail in section 5.3.1.

For radial flow the Darcy equation is

$$Q = 2\pi r H v = -2\pi r k H \frac{d\varphi}{dr}. \tag{1}$$

The continuity equation may be drawn up by considering the water balance of an elemental cylindrical shell of inner radius r and thickness dr, which is co-axial with the well.

$$Q(r) - Q(r+dr) = 2\pi r\, dr \frac{\varphi(r) - 0}{c} \quad \text{or} \quad -\frac{dQ}{dr} = \frac{2\pi r}{c} \varphi(r). \tag{2}$$

From (1) and (2) it follows that

$$\frac{2\pi r}{c} \varphi(r) = 2\pi k H \frac{d}{dr}\left(r \frac{d\varphi}{dr}\right) \quad \text{or} \quad \frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr} - \frac{\varphi}{kHc} = 0.$$

Assuming $r = x\sqrt{kHc} = x\lambda$ we get

$$\frac{d\varphi}{dr} = \frac{1}{\lambda}\frac{d\varphi}{dx} \text{ and } \frac{d^2\varphi}{dr^2} = \frac{1}{\lambda^2}\frac{d^2\varphi}{dx^2}.$$

Hence

$$\frac{d^2\varphi}{dx^2} + \frac{1}{x}\frac{d\varphi}{dx} - \varphi = 0. \tag{3}$$

This is the modified Bessel equation of zero order (see Appendix), the general solution of which is

$$\varphi = A\, I_0(x) + B\, K_0(x), \tag{4}$$

in which A and B are integration constants to be determined from the boundary conditions. Using the original variable r instead of x equation (4) becomes

$$\varphi(r) = A\, I_0\left(\frac{r}{\lambda}\right) + B\, K_0\left(\frac{r}{\lambda}\right). \tag{4a}$$

The first boundary condition is

$$r = \infty, \varphi = 0.$$

Since with $x \to \infty$, $I_0(x) \to \infty$ and $K_0(x) \to 0$, it follows that $A = 0$, so that

$$\varphi(r) = B\, K_0\left(\frac{r}{\lambda}\right). \tag{5}$$

The second boundary condition is given by the discharge of the well, Q_w, which must equal the inflow Q_r at $r = r_w$, the well radius. According to (1) and (4) the rate of ground-water flow is equal to

$$Q = -2\pi k H r \frac{d\varphi}{dr} = -2\pi k H\, B \frac{r}{\lambda} K_0'\left(\frac{r}{\lambda}\right).$$

Since $K_0'(x) = -K_1(x)$ (see Appendix), we can write

$$Q = 2\pi k H\, B \frac{r}{\lambda} K_1\left(\frac{r}{\lambda}\right),$$

giving as boundary condition

$$Q_w = 2\pi k H\, B \frac{r_w}{\lambda} K_1\left(\frac{r_w}{\lambda}\right).$$

5.1.2.1.

From this equation we get the value of the integration constant

$$B = \frac{Q_W}{2\pi kH} \frac{1}{\frac{r_W}{\lambda} K_1\left(\frac{r_W}{\lambda}\right)}$$

and the formula for the potential distribution becomes

$$\varphi(r) = \frac{Q_W}{2\pi kH} \frac{K_0\left(\frac{r}{\lambda}\right)}{\frac{r}{\lambda} K_1\left(\frac{r_W}{\lambda}\right)}. \tag{6}$$

In practice r_W is very much smaller than λ. Now for small values of x (see Appendix)

$$x\, K_1(x) \sim 1,$$

with an error of less than 1‰ when $x < 0.02$. With a very close approximation, therefore, formula (6) can be simplified to

$$\varphi(r) = \frac{Q_W}{2\pi kH} K_0\left(\frac{r}{\lambda}\right), \tag{7}$$

making the potential distribution independent of the well radius.

For small values of x, the infinite series

$$K_0(x) = \left\{1 + \left(\frac{x}{2}\right)^2 + \frac{1}{2!2!}\left(\frac{x}{2}\right)^4 + \ldots\right\} \ln\frac{1.123}{x} +$$

$$+ \left(\frac{x}{2}\right)^2 + \frac{1+\frac{1}{2}}{2!2!}\left(\frac{x}{2}\right)^4 + \frac{1+\frac{1}{2}+\frac{1}{3}}{3!3!}\left(\frac{x}{2}\right)^6 + \ldots\ldots$$

may be approximated by

$$\ln\frac{1.123}{x}.$$

The error introduced by this approximation is less than 5% when $x<0.33$, less than 1% when $x<0.18$ and less than 1‰ when $x<0.06$. In the vicinity of the well $r \ll \lambda$. So the above approximation may be substituted in (7). This gives us the formula for the potential distribution in the vicinity of the well,

$$\varphi(r) \sim \frac{Q_W}{2\pi kH} \ln\frac{1.123\,\lambda}{r},$$

from which the potential at the well face, $r = r_W$, can be derived:

$$\varphi_W = \frac{Q_W}{2\pi kH} \ln\frac{1.123\,\lambda}{r_W}. \tag{8}$$

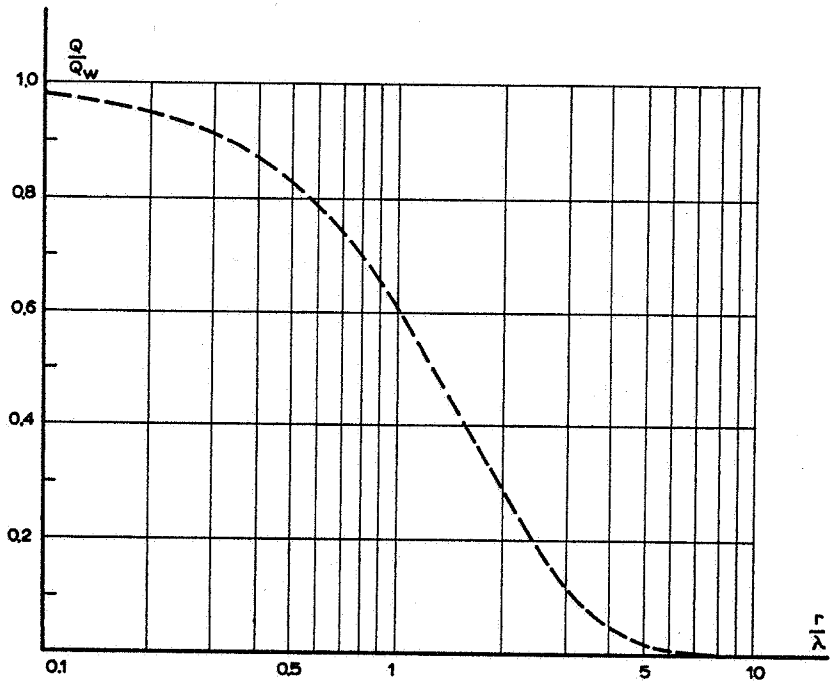

Fig. 5.1.2.1.–2

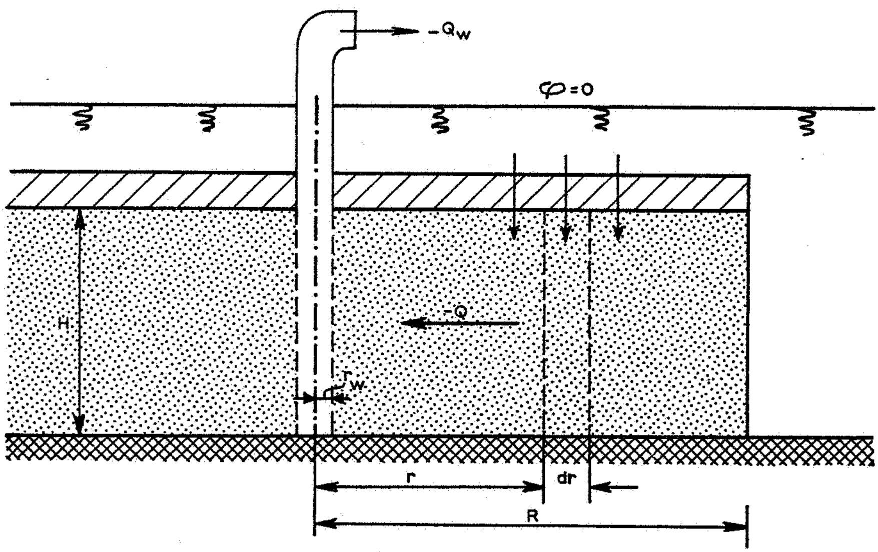

Fig. 5.1.2.2.–1

The rate of flow can be calculated from equation (7), and is

$$Q = -2\pi kHr\frac{d\varphi}{dr} = -Q_W\frac{r}{\lambda}K_0'\left(\frac{r}{\lambda}\right) \doteq Q_W\frac{r}{\lambda}K_1\left(\frac{r}{\lambda}\right).$$

Hence

$$\frac{Q}{Q_W} = \frac{r}{\lambda}K_1\left(\frac{r}{\lambda}\right). \tag{9}$$

Equation (9) is graphically represented in figure 5.1.2.1.–2. It shows that the ratio $\frac{Q}{Q_W}$ drops rapidly with increasing values of $\frac{r}{\lambda}$. The leakage factor λ may be considered as a true measure of the well's sphere of influence. With $r = 4\lambda$ for instance, $\frac{Q}{Q_W}=0.050$, meaning that only 5% ot the well's discharge originates from infiltration outside an area with radius $r = 4\lambda$.

A description of the application of the formulas given above for the determination of the formation constants (transmissibility kH, leakage factor λ, hydraulic resistance c) by the analysis of pumping tests will be given in chapter 7.

5.1.2.2. *The leakage through the covering layer is proportional to the drawdown in the aquifer, the drawdown being zero at a certain distance from the well (well in the centre of an island)*

The conditions mentioned in the heading are shown in figure 5.1.2.2.–1. The equations of Darcy and of continuity are the same as those given in 5.1.2.1. Formula (4) in that section is again the general solution here:

$$\varphi(r) = A\,I_0\left(\frac{r}{\lambda}\right) + B\,K_0\left(\frac{r}{\lambda}\right), \tag{1}$$

in which A and B are integration constants to be determined from the boundary conditions. The first boundary condition now is

$$r = R,\ \varphi = 0,$$

from which it follows that

$$A = -B\frac{K_0\left(\frac{R}{\lambda}\right)}{I_0\left(\frac{R}{\lambda}\right)},$$

which, substituted in (1), gives us

$$\varphi(r) = B\left\{K_0\left(\frac{r}{\lambda}\right) - \frac{K_0\left(\frac{R}{\lambda}\right)}{I_0\left(\frac{R}{\lambda}\right)}I_0\left(\frac{r}{\lambda}\right)\right\}. \tag{2}$$

The second boundary condition is again found to be

$$r = r_W,\ Q = Q_W.$$

Since

$$Q = -2\pi r\, kH \frac{d\varphi}{dr}$$

and $\frac{d\varphi}{dr}$ follows from (2), it is seen that

$$Q = -2\pi\, kH\, B \left\{ \frac{r}{\lambda} K_0'\left(\frac{r}{\lambda}\right) - \frac{K_0\left(\frac{R}{\lambda}\right)}{I_0\left(\frac{R}{\lambda}\right)} \frac{r}{\lambda} I_0'\left(\frac{r}{\lambda}\right) \right\}.$$

Since, according to the Appendix,

$K_0'(x) = -K_1(x)$ and $I_0'(x) = I_1(x)$, we get

$$Q = 2\pi\, kH\, B \left\{ \frac{r}{\lambda} K_1\left(\frac{r}{\lambda}\right) + \frac{K_0\left(\frac{R}{\lambda}\right)}{I_0\left(\frac{R}{\lambda}\right)} \frac{r}{\lambda} I_1\left(\frac{r}{\lambda}\right) \right\}. \tag{3}$$

The boundary condition, therefore, becomes

$$Q_W = 2\pi\, kH\, B \left\{ \frac{r_W}{\lambda} K_1\left(\frac{r_W}{\lambda}\right) + \frac{K_0\left(\frac{R}{\lambda}\right)}{I_0\left(\frac{R}{\lambda}\right)} \frac{r_W}{\lambda} I_1\left(\frac{r_W}{\lambda}\right) \right\}.$$

As $r_W \ll \lambda$ and as for small values of x, $xK_1(x) \sim 1$, $xI_1(x) \sim x^2 + \ldots \sim 0$, the value of the integration constant B may with close approximation be written as

$$B = \frac{Q_W}{2\pi kH}.$$

Introducing this into (2) gives us

$$\varphi(r) = \frac{Q_W}{2\pi kH} \left\{ K_0\left(\frac{r}{\lambda}\right) - I_0\left(\frac{r}{\lambda}\right) \frac{K_0\left(\frac{R}{\lambda}\right)}{I_0\left(\frac{R}{\lambda}\right)} \right\}. \tag{4}$$

Here again the potential distribution is independent of the well radius r_W if it is small compared with λ.

Introducing the integration constant B into equation (3) gives us

$$\frac{Q}{Q_W} = \frac{r}{\lambda} \left\{ K_1\left(\frac{r}{\lambda}\right) + \frac{K_0\left(\frac{R}{\lambda}\right)}{I_0\left(\frac{R}{\lambda}\right)} I_1\left(\frac{R}{\lambda}\right) \right\}. \tag{5}$$

5.1.2.2.

For $$K_1(x)\,I_0(x) + K_0(x)\,I_1(x) = -\{I_0(x)\,K_0'(x) - I_0'(x)\,K_1(x)\} = \frac{1}{x},$$
this equation gives for $r = R$
$$\frac{Q_R}{Q_W} = \frac{1}{I_0\left(\frac{R}{\lambda}\right)}.$$

With the help of this formula it is possible to determine which part of the well discharge originates from seepage from above and which part flows in horizontally across the outer boundary of the area under consideration. The latter part is of little importance when $R \gg \lambda$ (for instance $\frac{R}{\lambda} = 4.65$, $I_0(4.65) = 20$, Q_R is only 5% of the well discharge Q_W), while for $R \ll \lambda$ nearly all the water flows in laterally across the outer boundary of the area (for instance $\frac{R}{\lambda} = 0.44$, $I_0(0.44) = 1.05$, $Q_R = 0.95\,Q_W$).

Under these extreme conditions formula (4) for the potential decrease may be simplified still more as follows:

For $R \gg \lambda$, the second term in (4) is small relative to the first (compare the series representation of $I_0(x)$ and $K_0(x)$ for large values of x in the Appendix), except when r is nearly R, but then both terms are small. Formula (4) then approaches formula 5.1.2.1.–(7) for a well in a semi-confined aquifer of infinite extent.

For $R \ll \lambda$, all Bessel functions in (4) may be written as the series
$$K_0\left(\frac{r}{\lambda}\right) = \left(1 + \frac{r^2}{4\lambda^2} + \ldots\right) \ln \frac{1.123\lambda}{r} + \frac{r^2}{4\lambda^2} + \ldots.$$
$$I_0\left(\frac{r}{\lambda}\right) = 1 + \frac{r^2}{4\lambda^2} + \ldots\ .$$

After substitution we get
$$\varphi = \frac{Q_W}{2\pi kH}\left[\left(1 + \frac{r^2}{4\lambda^2} + \ldots\right) \ln \frac{R}{r} - \frac{R^2 - r^2}{4\lambda^2} + \ldots\right].$$

If the first-order terms, too, are ignored we get formula 5.1.1.–(4) for a well in a fully confined aquifer. Measured in terms of the leakage factor, the outer boundary of the area is now situated so close to the well, that the leakage is negligible compared with the inflow across the outer boundary. Here, with close approximation the semi-pervious covering layer may be considered impermeable.

5.1.3. Well in an aquifer between two semi-pervious layers

As is shown in figure 5.1.3.–1 the well is set in an aquifer bounded at top and base by semi-pervious layers. Overlying the upper semi-pervious layer is another water-

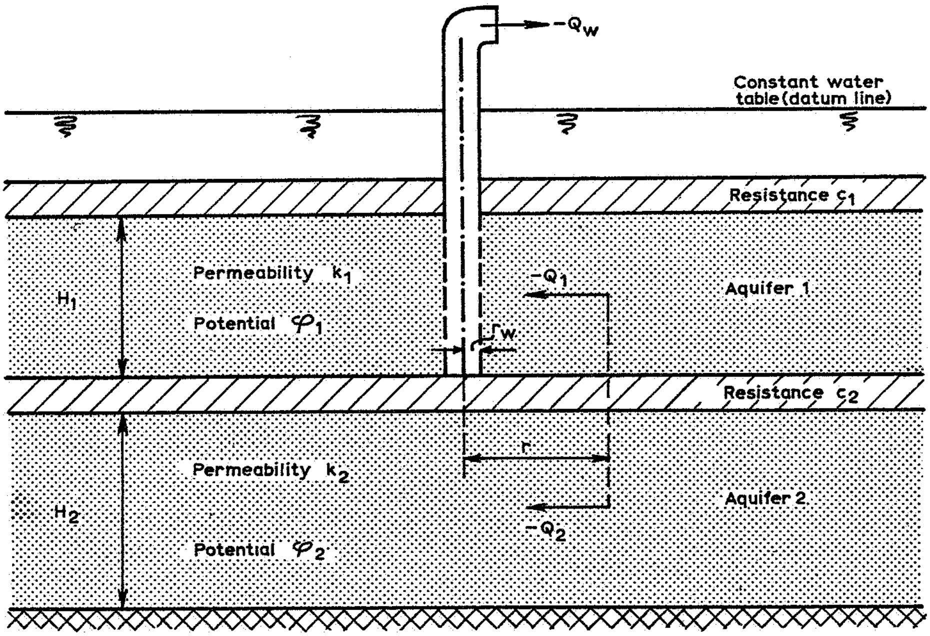
-Qw
Constant water
table(datum line)
Resistance c1
Permeability k1
Potential φ1
-Q1
Aquifer 1
H1
rw
Resistance c2
r
Permeability k2
H2
Aquifer 2
-Q2
Potential φ2

Fig. **5.1.3.–1**

bearing stratum, in which there is phreatic water at a constant level; below the lower semi-pervious layer there is ground water with a potential head that is affected by ground-water abstraction by the well. The recharging of the aquifer in this case does not only occur form the phreatic water above it, as in 5.1.2.1., but also from the deeper ground water below it.

On the same assumptions as those given in 5.1.2.1. the Darcy equations are

aquifer 1: $Q_1 = 2\pi r H_1 v_1 = -2\pi r k_1 H_1 \dfrac{d\varphi_1}{dr}$; (1)

aquifer 2: $Q_2 = 2\pi r H_2 v_2 = -2\pi r k_2 H_2 \dfrac{d\varphi_2}{dr}$. (2)

The equations of continuity are

aquifer 1: $dQ_1 = 2\pi r dr \left(\dfrac{0-\varphi_1}{c_1} + \dfrac{\varphi_2-\varphi_1}{c_2}\right)$ or $\dfrac{dQ_1}{dr} = -2\pi r\left(\dfrac{\varphi_1}{c_1} + \dfrac{\varphi_1-\varphi_2}{c_2}\right)$; (3)

aquifer 2: $dQ_2 = 2\pi r dr \dfrac{\varphi_1-\varphi_2}{c_2}$ or $\dfrac{dQ_2}{dr} = -2\pi r \dfrac{\varphi_2-\varphi_1}{c_2}$. (4)

Eliminating Q_1 from (1) and (3) gives

$$k_1 H_1 \left(\frac{d\varphi_1}{dr} + r\frac{d^2\varphi_1}{dr^2}\right) = r\left(\frac{\varphi_1}{c_1} + \frac{\varphi_1-\varphi_2}{c_2}\right).$$

With $\alpha_1 = \dfrac{1}{k_1 H_1 c_1}$, $\alpha_2 = \dfrac{1}{k_2 H_2 c_2}$ and $\beta_1 = \dfrac{1}{k_1 H_1 c_2}$,

this equation simplifies to

$$\frac{d^2\varphi_1}{dr^2} + \frac{1}{r}\frac{d\varphi_1}{dr} = \alpha_1 \varphi_1 + \beta_1(\varphi_1 - \varphi_2). \tag{5}$$

In the same way the elimination of Q_2 from (2) and (4) gives

$$\frac{d^2\varphi_2}{dr^2} + \frac{1}{r}\frac{d\varphi_2}{dr} = \alpha_2(\varphi_2 - \varphi_1). \tag{6}$$

It is obvious to try and find solutions for equations (5) and (6) similar to those given in 5.1.2.1.:

$$\varphi_1(r) = a\, K_0(r/\lambda_1) + b\, K_0(r/\lambda_2)$$
$$\varphi_2(r) = c\, K_0(r/\lambda_1) + d\, K_0(r/\lambda_2) \tag{7}$$

As the function $K_0(x)$ satisfies the equation

$$\frac{d^2 K_0(x)}{dx^2} + \frac{1}{x}\frac{d K_0(x)}{dx} = K_0(x)$$

(the modified equation of Bessel; see Appendix), the set of functions (7) will be a solution of the simultaneous differential equations (5) and (6), if the coefficients are chosen in such a way that they satisfy the following four equations.

$$a/\lambda_1^2 = (\alpha_1 + \beta_1)\, a - \beta_1 c, \tag{8}$$
$$b/\lambda_2^2 = (\alpha_1 + \beta_1) b - \beta_1 d, \tag{9}$$
$$c/\lambda_1^2 = -\alpha_2 a + \alpha_2 c, \tag{10}$$
$$d/\lambda_2^2 = -\alpha_2 b + \alpha_2 d. \tag{11}$$

If $1/\lambda_1^2$ and $1/\lambda_2^2$ (with $\lambda_1^2 < \lambda_2^2$) are the two positive roots of the quadratic equation

$$\begin{vmatrix} \alpha_1 + \beta_1 - 1/\lambda^2 & -\beta_1 \\ -\alpha_2 & \alpha_2 - 1/\lambda^2 \end{vmatrix} = 0 \qquad \text{or}$$

$$\alpha_1\alpha_2\lambda^4 - (\alpha_1 + \alpha_2 + \beta_1)\lambda^2 + 1 = 0, \tag{12}$$

then equations (8) and (10) like equations (9) and (11) are linearly dependent. For this special choice of λ_1^2 and λ_2^2, the set of equations (8) to (11) is equivalent to

$$c = \frac{\alpha_2\lambda_1^2}{\alpha_2\lambda_1^2 - 1}\, a, \quad d = \frac{\alpha_2\lambda_2^2}{\alpha_2\lambda_2^2 - 1}\, b.$$

Substitution of these values for c and d in (7) gives

$$\varphi_1(r) = a\, K_0(r/\lambda_1) + b\, K_0(r/\lambda_2),$$

$$\varphi_2(r) = \frac{\alpha_2\lambda_1^2}{\alpha_2\lambda_1^2 - 1}\, a\, K_0(r/\lambda_1) + \frac{\alpha_2\lambda_2^2}{\alpha_2\lambda_2^2 - 1}\, b\, K_0(r/\lambda_2).$$

With $K_0(\infty) = 0$ these equations satisfy the boundary conditions $\lim_{r\to\infty} \varphi_1(r) = 0$ and $\lim_{r\to\infty} \varphi_2(r) = 0$. The boundary conditions for $r \to 0$ demand that

for aquifer 1: $Q_W = -2\pi k_1 H_1 \left[r \dfrac{d\varphi_1}{dr} \right]_{r=r_W}$ from (1), and

for aquifer 2: $0 = -2\pi k_2 H_2 \left[r \dfrac{d\varphi_2}{dr} \right]_{r=0}$ from (2),

in which $-Q_W$ is the yield of the well and r_W the radius of the well. With $K_0(x) = \ln \dfrac{1.123}{x}$ (that holds good for small values of x) it is possible to satisfy these equations by using constants a and b determined as follows:

$$a = \frac{(\alpha_2\lambda_1^2 - 1)\lambda_2^2}{\lambda_1^2 - \lambda_2^2} \frac{Q_W}{2\pi k_1 H_1},$$

$$b = \frac{-(\alpha_2\lambda_2^2 - 1)\lambda_1^2}{\lambda_1^2 - \lambda_2^2} \frac{Q_W}{2\pi k_1 H_1}.$$

Substitution of these values gives the complete solution

$$\varphi_1(r) = \frac{Q_W}{2\pi k_1 H_1} \frac{1}{\lambda_1^2 - \lambda_2^2} \{(\alpha_2\lambda_1^2 - 1)\lambda_2^2 K_0(r/\lambda_1) - (\alpha_2\lambda_2^2 - 1)\lambda_1^2 K_0(r/\lambda_2)\},$$

$$\varphi_2(r) = \frac{Q_W}{2\pi k_1 H_1} \frac{\alpha_2\lambda_1^2\lambda_2^2}{\lambda_1^2 - \lambda_2^2} \{K_0(r/\lambda_1) - K_0(r/\lambda_2)\},$$

5.1.3.

in which

$$\begin{matrix}\lambda_1^2\\ \lambda_2^2\end{matrix} = \frac{2}{\alpha_1 + \alpha_2 + \beta_1 \pm \sqrt{(\alpha_1 + \alpha_2 + \beta_1)^2 - 4\alpha_1\alpha_2}},$$

$$\alpha_1 = \frac{1}{k_1 H_1 c_1}, \quad \alpha_2 = \frac{1}{k_2 H_2 c_2}, \quad \beta_1 = \frac{1}{k_1 H_1 c_2}.$$

5.1.4. WELL IN AN AQUIFER BELOW TWO SEMI-PERVIOUS LAYERS

The hydro-geological profile is the same as that shown in figure 5.1.3.–1. The fully penetrating well screen, however, is now set in the lower aquifer, which is recharged only from above by leakage through the lower semi-pervious layer. With the point of abstraction placed thus, however, the Darcy and continuity equations, as given in 5.1.3., remain unchanged. Using the same notations as were applied there, the general solution may again be expressed as

$$\varphi_1(r) = a\,K_0(r/\lambda_1) + b\,K_0(r/\lambda_2), \quad \varphi_2(r) = \frac{\alpha_2\lambda_1^2}{\alpha_2\lambda_1^2 - 1}\,a\,K_0(r/\lambda_1) + \frac{\alpha_2\lambda_2^2}{\alpha_2\lambda_2^2 - 1}\,b\,K_0(r/\lambda_2).$$

This solution satisfies the boundary conditions for $r \to \infty$,

$\lim\limits_{r\to\infty} \varphi_1(r) = 0$ and $\lim\limits_{r\to\infty} \varphi_2(r) = 0$.

For $r \to 0$, the boundary conditions are now

for aquifer 1: $\quad 0 = -2\pi k_1 H_1 \left[r \dfrac{d\varphi_1}{dr}\right]_{r=0},$

and

for aquifer 2: $\quad Q_W = -2\pi k_2 H_2 \left[r \dfrac{d\varphi_2}{dr}\right]_{r=r_W}$

Substituting the values for φ_1 and φ_2 taken from the general solution, and using the relation $K_0(x) = \ln \dfrac{1.123}{x}$ (that holds good for small values of x), the constants a and b can be calculated as

$$a = -b = -\frac{Q_W}{2\pi k_2 H_2}\,\frac{(\alpha_2\lambda_1^2 - 1)(\alpha_2\lambda_2^2 - 1)}{\alpha_2(\lambda_1^2 - \lambda_2^2)},$$

or in view of (12) in 5.1.3.,

$$a = -b = \frac{Q_W}{2\pi k_2 H_2}\,\frac{\beta_1\lambda_1^2\lambda_2^2}{\lambda_1^2 - \lambda_2^2}.$$

Substitution of these values gives the complete solution

$$\varphi_1(r) = \frac{Q_W}{2\pi k_2 H_2}\,\frac{\beta_1\lambda_1^2\lambda_2^2}{\lambda_1^2 - \lambda_2^2}\{K_0(r/\lambda_1) - K_0(r/\lambda_2)\},$$

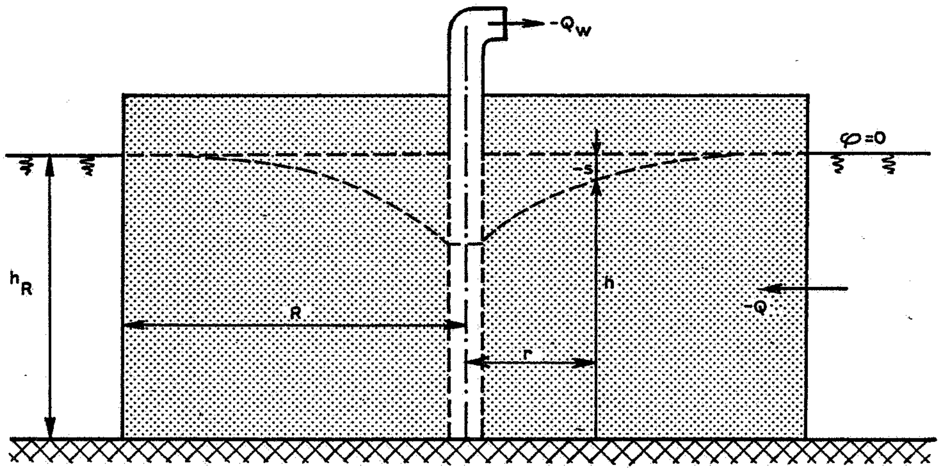

Fig. 5.2.1.–1

$$\varphi_2(r) = \frac{Q_W}{2\pi k_2 H_2} \frac{1}{\lambda_1^2 - \lambda_2^2} \{-(\alpha_2\lambda_2^2 - 1)\lambda_1^2 K_0(r/\lambda_1) + (\alpha_2\lambda_1^2 - 1)\lambda_2^2 K_0(r/\lambda_2)\},$$

with

$$\begin{matrix}\lambda_1^2 \\ \lambda_2^2\end{matrix} = \frac{2}{\alpha_1 + \alpha_2 + \beta_1 \pm \sqrt{(\alpha_1 + \alpha_2 + \beta_1)^2 - 4\alpha_1\alpha_2}}, \quad \alpha_1 = \frac{1}{k_1 H_1 c_1}, \quad \alpha_2 = \frac{1}{k_2 H_2 c_2}, \quad \beta_1 = \frac{1}{k_1 H_1 c_2}.$$

5.2. WELL IN A HOMOGENEOUS ISOTROPIC AQUIFER WITH PHREATIC GROUND WATER

In the phreatic surface the hydrostatic pressure is equal to the atmospheric pressure (3.04); therefore, the potential in the phreatic surace is equal to the height of this surface above the plane of reference chosen.

In this chapter just as in 5.1. the vertical components of the velocity will be ignored. Consequently, there is a uniform potential distribution along any vertical in the aquifer.

5.2.1. WELL IN THE CENTRE OF A CIRCULAR AREA WITH A FIXED POTENTIAL ALONG ITS BOUNDARY (WELL IN THE CENTRE OF A CIRCULAR ISLAND)

A fully penetrating well is supposed to be placed in the centre of the circular island, postulated in figure 5.2.1.–1. Such situation practically never occurs. Nevertheless this set-up can be applied if at certain distances round about the well the aquifer has boundaries (rivers, canals etc.) where the ground-water potential remains constant (see 5.1.1.).

The set-up represented in the figure and the formulas obtained below do not give an exact picture of the situation in the immediate vicinity of the well, because they are based on some simplifying suppositions. E.g. the influence of the seepage zone existing along the well screen above the water level, and the influence of the ground-water movement in the capillary zone have been ignored. The influence of the vertical velocity components which reach their maximum in the vicinity of the well has been ignored too.

Let the potential measured with respect to the height of the ground-water table before pumping and at a distance r from the well be φ. Then $\varphi = h - h_R = s$ (s negative for falling and positive for rising ground-water level). From $\varphi = h - h_R$ it follows that $\frac{d\varphi}{dr} = \frac{dh}{dr}$.

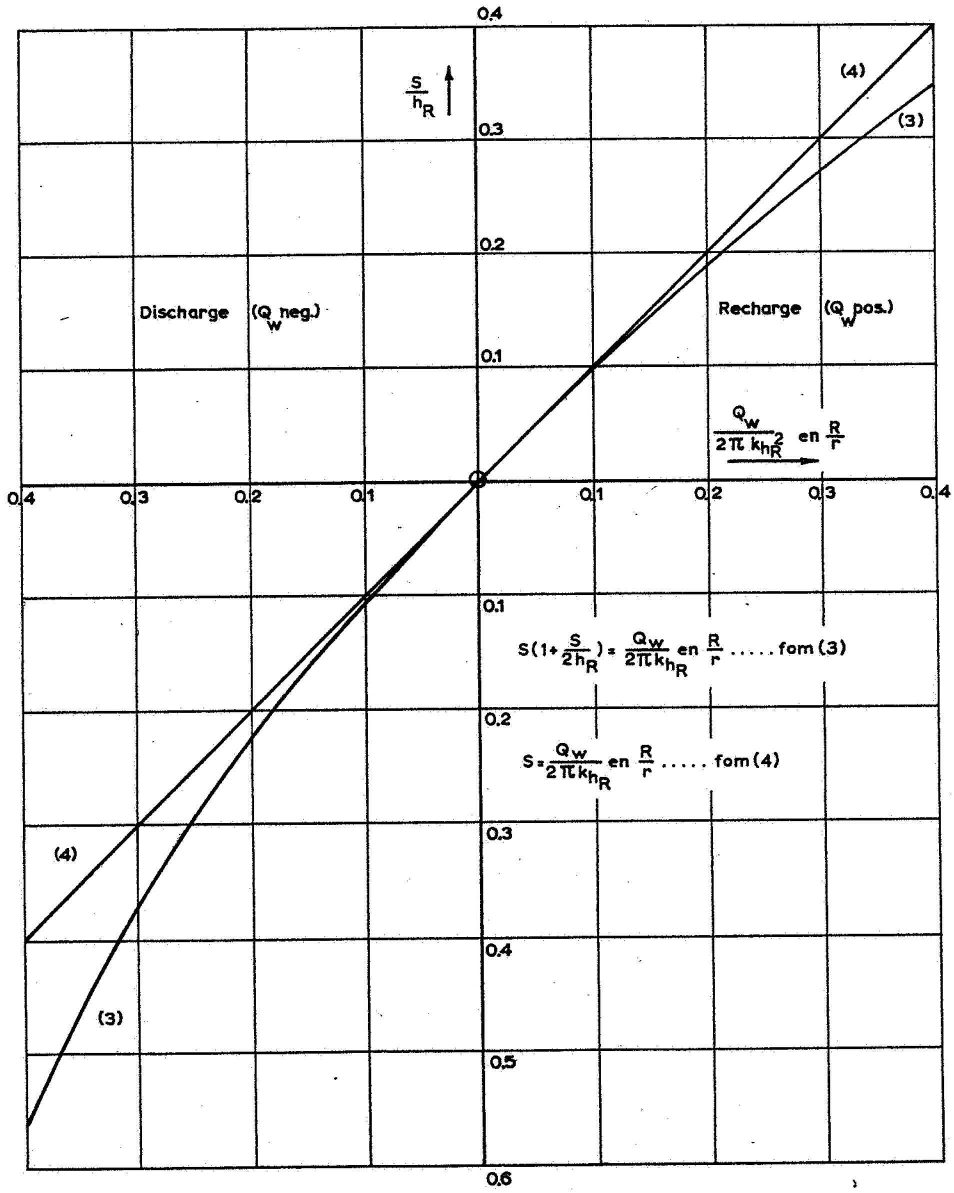
0,4
0,3
0,2
0,1
0,1
0,2
0,3
0,4
0,5
0,6
s/h_R
(4)
(3)
Discharge (Q_w neg.)
Recharge (Q_w pos.)
Q_w / 2π k h_R² en R/r
0,4 0,3 0,2 0,1 0,1 0,2 0,3 0,4
S(1+ S/2h_R) = Q_w/2πkh_R en R/r ..... fom (3)
S = Q_w/2πkh_R en R/r ..... fom (4)
(4)
(3)

Fig. 5.2.1.–2

Let the quantity of water flowing across the cylindrical surface of radius r and height h (see fig. 5.2.1.–1) be Q.

Then $Q = -2\pi r\, kh \dfrac{d\varphi}{dr}$,

$$= -2\pi r\, kh \frac{dh}{dr} \quad \text{(Darcy equation)}.$$

A second condition is;

$$Q = Q_W \; (Q \text{ independent of } r; \text{ equation of continuity}).$$

From the Darcy equation and the equation of continuity it can be deduced that

$$-\frac{Q_W}{2\pi k}\frac{dr}{r} = h\,dh\,,$$

or after integration

$$-\frac{Q_W}{2\pi k}\ln r = \tfrac{1}{2}h^2 + C.$$

If the radius of the island is R, the boundary condition becomes $r = R$, $h = h_R$, which gives

$$c = \frac{-Q_W}{2\pi k}\ln R - \tfrac{1}{2}h_R^2$$

and for h as a function of r

$$h_R^2 - h^2 = \frac{-Q_W}{\pi k}\ln\frac{R}{r} \qquad (1)$$

(Q_W is negative when water is being withdrawn). Written in this form the formula is known as the formula of Dupuit.

When the change in the phreatic level with respect to the original ground-water level is termed s and is introduced into (1) this formula becomes

$$s^2 + 2\,sh_R = \frac{Q_W}{\pi k}\ln\frac{R}{r} \qquad (2)$$

or

$$s\left(1 + \frac{s}{2\,h_R}\right) = \frac{Q_W}{2\pi k h_R}\ln\frac{R}{r}\,. \qquad (3)$$

If s is small with respect to h, i.e. the thickness of the water-bearing stratum is practically constant, formula (3) can be simplified to

$$s = \frac{Q_W}{2\pi k h_R}\ln\frac{R}{r} = \frac{Q_W}{2\pi k H}\ln\frac{R}{r}\,,$$

H being the thickness of the aquifer.

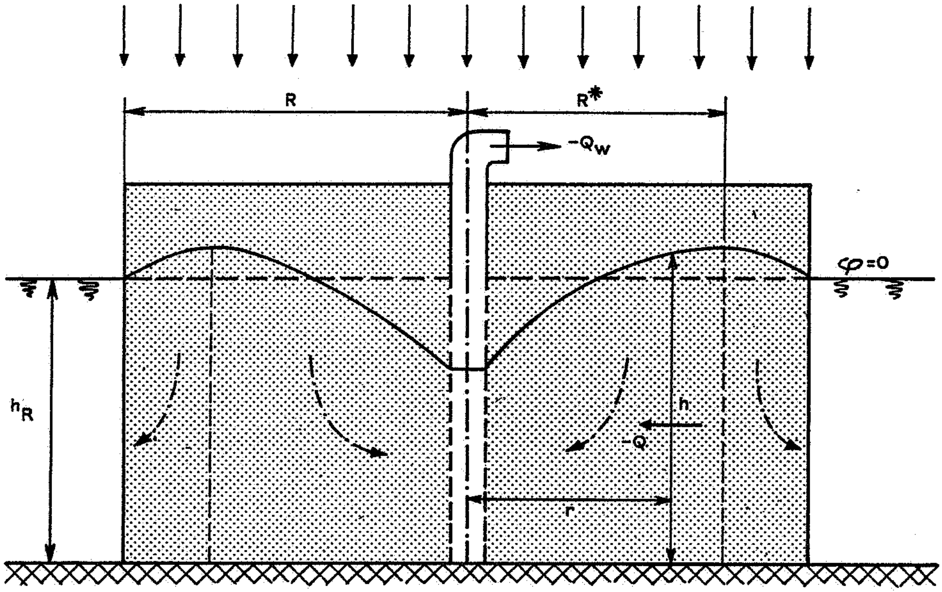

Fig. 5.2.2.–1

This formula, sometimes called the formula of Thiem, is identical with the formula arrived at in 5.1.1. for a well in confined ground water on a circular island.

The admissibility of using this simplified formula is illustrated in figure 5.2.1.–2. Here the dimensionless values of $\frac{s}{h_R}$ and $\frac{Q_W}{2\pi k h_R^2} \ln \frac{R}{r}$ are plotted against each other The graph shows the error introduced when the formula of Thiem is used.

5.2.2. Well in the centre of a circular area with a fixed potential along its boundary and replenished by rainfall

The theory of the flow of phreatic ground water into a fully penetrating well in the centre of a circular island as described in 5.2.1. will often have to be used for areas where there are also other ground-water movements, in particular those caused by precipitation. The flow pattern existing in such a situation is shown in figure 5.2.2.–1.

The derivation of the formulas is based on the principle of steady ground-water flow implying also a constant replenishment of the ground water by precipitation, as shown in figure 5.2.2.–1.

The Darcy and continuity equations become

$$Q = -2\pi r\, k\, h \frac{dh}{dr}, \qquad (1)$$

$$dQ = 2\pi r\, dr\, P, \qquad (2)$$

where P is the replenishment of the ground water by precipitation (depth of water per unit of time).

Integration of (2) gives

$$Q = \pi r^2 P + C_1. \qquad (3)$$

When $r = r_W$ (r_W being the radius of the well), $Q = Q_W$. Hence $Q_W = \pi r^2_W P + C_1$ or $C_1 = Q_W - \pi r_W{}^2 P$.

The term $\pi r_W{}^2 P$ in this equation, representing as it does the precipitation on the top of the well itself, is very small and may be ignored.

From (3) we obtain

$$Q = \pi r^2 P + Q_W \qquad (4)$$

(Q_W is negative when water is being withdrawn).

The differential equation of the ground-water flow now becomes

$$-\left[\frac{P}{2k} r^2 + \frac{Q_W}{2\pi k}\right] \frac{dr}{r} = h\, dh,$$

After integration

$$-\frac{P}{4k}r^2 - \frac{Q_W}{2\pi k}\ln r = \tfrac{1}{2}h^2 + C_2,$$

C_2 can be deduced from the boundary conditions $h = h_R$ for $r = R$, R being the radius of the circular area.

Hence
$$c_2 = -\frac{P}{4k}R^2 - \frac{Q_W}{2\pi k}\ln R - \tfrac{1}{2}h_R^2, \tag{5}$$

Therefore the values of h as a function of r are shown by

$$\frac{P}{2k}(R^2 - r^2) + \frac{Q_W}{\pi k}\ln\frac{R}{r} = (h^2 - h_R^2). \tag{6}$$

This equation can also be obtained by superposition of the drawdown curve of a single well on the curve for the phreatic table produced by precipitation only.

Introduction of s, indicating the deviation of the phreatic level from the ground-water level which occurs when $P = 0$ and $Q_W = 0$, gives the following relation between s and r

$$s^2 + 2\,sh_R = \frac{P}{2k}(R^2 - r^2) + \frac{Q}{\pi k}\ln\frac{R}{r}$$

or

$$s\left(1 + \tfrac{1}{2}\frac{s}{h_R}\right) = \frac{P}{4kh_R}(R^2 - r^2) + \frac{Q}{2\pi kh_R}\ln\frac{R}{r}, \tag{7}$$

The left subsection and the second term of the right subsection of equation (7) correspond with equation (3) from 5.2.1.

When $Q_W = 0$ equations (6) and (7) give the relation between s and r representing the phreatic level caused by precipitation only.

For small values of $\frac{s}{h}$ equation (7) can be simplified into

$$s = \frac{P}{4\,kh_R}(R^2 - r^2) + \frac{Q_W}{2\pi\,kh_R}\ln\frac{R}{r}, \tag{8}$$

In the case of the flow pattern sketched in figure 5.2.2.–1 it is important to differentiate between the area in which the height of the phreatic level is affected by the withdrawal by the well, and the area in which the ground water runs off towards the well. The former area, of radius R, is the circular area having a fixed potential along its boundary. The latter area, of radius R^*, is the area from which water is withdrawn by the well (area of origin of the water withdrawal).

It may be clear that the value of R^* has real significance only when $R^* < R$. The magnitude of R^* can be deduced directly from the equation of continuity

$$\pi R^{*2}P = -Q_W \quad \text{or} \quad R^* = \sqrt{-\frac{Q_W}{\pi P}},$$

in which Q_W is always negative.

5.2.2.

5.3. The influence of the vertical velocity component of the flow towards fully penetrating wells

Throughout 5.1. and 5.2. the influence of the vertical velocity component has been neglected. Consequently, the potential and the velocity of filtration can only be regarded as functions of the radial co-ordinate r and not of the vertical co-ordinate z.

Clearly, this supposition can only be approximately correct. With semi-confined ground water there is recharging of the aquifer from the semi-pervious layers and in the case of phreatic water the upper boundary of the aquifer is not horizontal. Consequently, vertical velocity components must appear in such cases. Also for partially penetrating wells vertical velocity components must obtain in the vicinity of the well.

It will be shown in this paragraph under what conditions the results of 5.1. and 5.2. for fully penetrating wells give good approximations. The case of partially penetrating wells will be discussed in 5.4.

When the vertical velocity components are taken into account, the question arises whether the permeability is the same vertically and horizontally (anisotropy, see 3.09.). Besides, there are situations in which the aquifer consists of a number of layers differing in permeability.

In 5.3.1. the influence of the vertical velocity components will be investigated for the case of a homogeneous aquifer. There it will be supposed that the permeability in all horizontal directions is the same, but differing from the vertical permeability. This form of anisotropy is the most common.

As a rule it is the consequence of a certain structure of the soil originating from its formation. During sedimentation, for instance, scaly particles have a tendency to lie flat. Consequently the vertical permeability will be smaller than the permeability in a horizontal direction.

The special case of an aquifer of homogeneous isotropic permeability is included in the results of 5.3.1.

In 5.3.2. the case of a horizontal stratified (i.e. a non-homogeneous) aquifer is discussed.

5.3.1. Homogeneous permeability of the aquifer (isotropic and anisotropic formations)

For a fully penetrating well in an aquifer with confined ground water the ground-water movement is exactly horizontal. The results of 5.1.1., therefore, hold good without approximation. This is also true for an aquifer of anisotropic permeability; in this situation vertical permeability plays no part.

In the case of semi-confined ground water, vertical velocity components will occur in consequence of the recharging of the aquifer through the semi-pervious covering layer. It will be shown in 5.3.1.1. that the influence of the vertical velocity components can be neglected if

$$\frac{k}{k_z}\left\{\frac{H}{\lambda}\right\}^2 = \frac{H}{k_z c} = \frac{k_z' H}{k_z H'} \ll 1. \tag{1}$$

Here H is the thickness of the aquifer, k the horizontal permeability, k_z the vertical permeability and c, H' and $k_z' = H'/c$ are resp. the resistance, thickness and permeability in a vertical direction of the semi-pervious covering layer. The leakage factor λ is given by the formula

$$\lambda = \sqrt{kHc}.$$

Under the conditions mentioned above, the results of 5.1.2. also hold good if $k_z \neq k$.

If the conditions of (1) cannot be fulfilled, the calculation necessarily becomes more complicated. The calculation (not given here) is carried out just like the general calculation for a partially penetrating well in an aquifer with semi-confined ground water (given in 5.4.3.).

Vertical velocity components also occur in the case of phreatic water. In 5.3.1.2. it is shown that the results of 5.2.1. are valid in all cases in which the gradient of the phreatic surface is small. Moreover, the exact validity of the formula of Dupuit

$$Q_W = \frac{-\pi k (h_R^2 - h_W^2)}{\ln (R / r_W)}$$

will also appear.

This formula gives the discharge of the well Q_W as a function of h_R (the height of water level along the boundary of the circular island) and h_W (the height of water level in the well).

5.3.1.1. *Semi-confined ground water*

In this paragraph the situation as in 5.1.2.1. is discussed. The other cases of 5.1.2., 5.1.3. and 5.1.4. can be treated in the same way.

If the horizontal permeability is k and the vertical permeability is k_z, the Darcy formula for cylinder-symmetrical ground-water movement gives

$$v_r = -k\frac{\partial\varphi}{\partial r}, \quad v_z = -k_z\frac{\partial\varphi}{\partial z}. \tag{1}$$

The equation of continuity is

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,v_r) + \frac{\partial v_z}{\partial z} = 0. \tag{2}$$

Provided the plane $z = 0$ is taken to coincide with the impervious base, these equations are valid for

$$r_w < r < \infty,\ 0 < z < H.$$

The boundary conditions are

$$z = 0, \quad r_w < r < \infty: \quad v_z = 0, \tag{3}$$

$$z = H, \quad r_w < r < \infty: \quad v_z = \frac{1}{c}\varphi, \tag{4}$$

$$r = r_w, \quad 0 < z < H: \quad \varphi = \varphi_w, \tag{5}$$

$$r = \infty, \quad 0 < z < H: \quad \varphi = 0. \tag{6}$$

Compare 3.13. with the boundary condition (4).

It is possible to solve the set of equations and conditions numbered (1) to (6) exactly and to compare the solution arrived at with the solution given in 5.1.2.1. (see the calculations given in 5.4.3.).

It is simpler, however, to examine the question as to what inaccuracies are involved when substituting the equations in 5.1.2.1. for the equations (1) to (6).

The total flow across a cylinder of radius r and height H, which is co-axial with the well is

$$Q_r = 2\pi\, r \int_0^H v_r(r, z)\, dz. \tag{7}$$

According to (1)

$$Q_r = -2\pi\, k\, r \int_0^H \frac{\partial\varphi(r, z)}{\partial r} dz = -2\pi\, k\, r\, \frac{d}{dr}\int_0^H \varphi(r,z)\, dz = -2\pi\, k\, H\, r\, \frac{d\Phi}{dr}, \tag{8}$$

in which

$$\Phi(r) = \frac{1}{H}\int_0^H \varphi(r, z)\, dz. \tag{9}$$

$\Phi(r)$ = the average value of the potential on the cylindrical surface of radius r. Further it follows from (7) and (2) that

$$\frac{dQ}{dr} = 2\pi \int_0^H \frac{\partial}{\partial r}(r\, v_r)\, dz = -2\pi r \int_0^H \frac{\partial v_z}{\partial z}\, dz = -2\pi\, r \Big[v_z \Big]_{z=0}^{z=H}.$$

This expression also follows directly from the equation of continuity.

In consequence of the boundary conditions (3) and (4) we find that

$$\frac{dQ}{dr} = -\frac{2\pi r}{c}\varphi(r, H). \tag{10}$$

The boundary conditions of the function $\Phi(r)$ are

$$r = r_W,\ \Phi = \varphi_W;\ r = \infty,\ \Phi = 0. \tag{11}$$

Comparison with 5.1.2.1. shows that the above equations (8), (10) and (11) are the same as the equations 5.1.2.1.–(1) and –(2) and the boundary conditions mentioned in 5.1.2.1., when in the right subsection of (10) the potential $\varphi(r, H)$ at the upper boundary of the aquifer is replaced by the average potential $\Phi(r)$. Therefore the approximation used in 5.1.2.1. is acceptable if

$\Phi(r) - \varphi(r, H)$ is small in comparison with $\Phi(r)$ or $\varphi(r, H)$.

An estimation of the difference between $\Phi(r)$ and $\varphi(r, H)$ can be found as follows.

Partial integration of (9) gives:

$$\Phi(r) = \frac{1}{H}\left[z\varphi(r, z)\right]_{z=0}^{z=H} - \frac{1}{H}\int_0^H z\frac{\partial\varphi}{\partial z}\,dz = \varphi(r, H) + \frac{1}{k_z H}\int_0^H z\, v_z\, dz,$$

with the boundary conditions

and

$$z = 0, \quad v_z = 0$$

$$z = H, \quad v_z = \frac{1}{c}\varphi(r, H).$$

It is reasonable to suppose that for

$$0 < z < H,$$

always $0 < v_z < \frac{1}{c}\varphi(r, H)$.

In this case, when $\varphi(r, H) > 0$ we get

$$0 < \int_0^H z\, v_z\, dz < \frac{\varphi(r, H)}{c}\int_0^H z\, dz = \frac{1}{2c} H^2 \varphi(r, H).$$

Hence

$$0 < \frac{\Phi(r) - \varphi(r, H)}{\varphi(r, H)} < \frac{H}{2k_z c}. \tag{12}$$

The same result will be arrived at when $\varphi(r, H) < 0$. From this it follows that the approximations given in 5.1.2.1. hold good if

$$\frac{H}{k_z c} = \frac{H\,k'z}{H'\,k_z} = \frac{k}{k_z}\left(\frac{H}{\lambda}\right)^2 \ll 1 \tag{13}$$

(with $c = \frac{H'}{k_z'}$, $\lambda = \sqrt{kHc}$; see 3.13. and 3.14.).

5.3.1.1.

Mostly $H \ll \lambda$; therefore the result of 5.1.2.1. always holds good (also when $k_z \neq k$), if k_z is not so much smaller than k that (13) is no longer valid.

5.3.1.2. *Phreatic ground water*

In this paragraph the situation of 5.2.1. (well in the centre of a circular island) is dealt with.

Darcy's Law and the equation of continuity are again as follows

$$v_r = -k\frac{\partial \varphi}{\partial r}, \quad v_z = -k_z\frac{\partial \varphi}{\partial z}, \tag{1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r, v_r) + \frac{\partial v_z}{\partial z} = 0. \tag{2}$$

These equations now hold good for

$$r_w < r < R, \quad 0 < z < h_r,$$

in which h_r is the height of the phreatic surface above the impervious layer at the base of the aquifer.

$$\text{For } z = 0, \quad v_z \text{ is also } 0, \tag{3}$$

but when $z = h_r$ two boundary conditions are obtained (the function h_r on the other hand now is one of the unknown quantities here). The first of these conditions shows that in the phreatic surface the water pressure is equal to the atmosferic pressure. In consequence of the choice of $z = 0$ as the plane of reference (compare 3.07.),

$$\varphi(r, h_r) = h_r. \tag{4}$$

The second boundary condition is a kinematic one showing that no liquid can pass the phreatic surface. Therefore, the velocity is tangential to the phreatic surface (fig. 5.3.1.2.–1). So $v_z \cos i - v_r \sin i = 0$.

Because $\tan i = \frac{dh}{dr}$ it follows that the boundary condition is

$$v_z(r, h_r) = v_r(r, h_r)\frac{dh}{dr}. \tag{5}$$

A seepage surface at the boundaries of the area can make the boundary conditions more complicated.

If along the circumference of the area ($r = R$, see fig. 5.3.1.2.–2) water enters

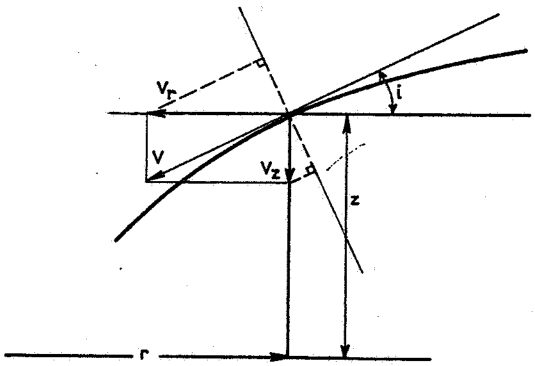

Fig. **5.3.1.2.–1**

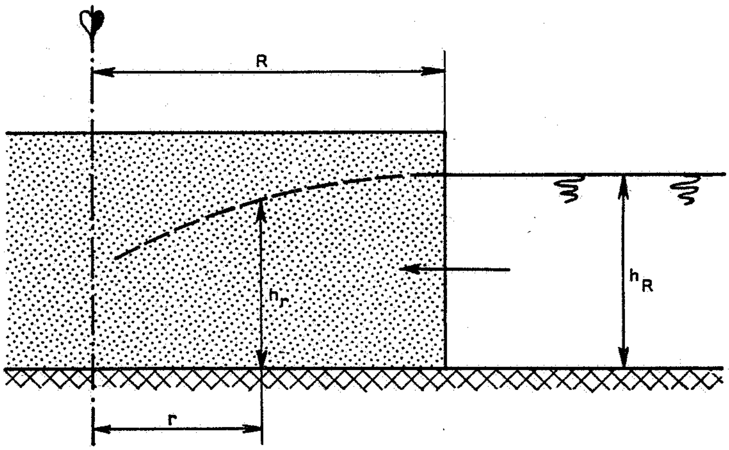

Fig. **5.3.1.2.–2**

the aquifer, no difficulty crops up; the boundary condition here is

$$\varphi(R, z) = h_R \text{ if } 0 \leqslant z \leqslant h_R \tag{6}$$

(the pressure in the surface water along this boundary is the hydrostatic pressure).

Along the innermost boundary, i.e. at the well face, where ground water leaves the aquifer, a seepage surface will as a rule exist (fig. 5.3.1.2.-3). In this case the height of the phreatic surface h_{rw} is greater than the height h_w of the water level in the well. Between h_{rw} and h_w the ground water flows out under atmospheric pressure. So the boundary conditions here are

$$\begin{aligned} &\varphi(r_w, z) = h_w && \text{for } 0 \leqslant z \leqslant h_w, \\ &\varphi(r_w, z) = z && \text{for } h_w \leqslant z \leqslant h_{rw}. \end{aligned} \tag{7}$$

The completely formulated problem given here is very complicated, in consequence of the circumstance that the phreatic surface (a boundary of the area where the equations (1) and (2) apply) is unknown. Therefore an exact analytical solution has not yet been found. With the help of the following trick however, the transition to the simple method described in 5.2.1. can be devised and an approximation of the error involved can be arrived at.

The total discharge across a vertical cylinder of radius r can be expressed as

$$Q = 2\pi r \int_0^{h_r} v_r(r, z)\, dz = -2\pi k r \int_0^{h_r} \frac{\partial \varphi}{\partial z}(r, z)\, dz \tag{8}$$

The continuity equation gives

$$Q = \text{constant} = Q_w \tag{9}$$

(this can also be derived from the continuity equation (2) with the boundary conditions (3) and (5)).

Now consider the function

$$\Phi_r = \int_0^{h_r} \varphi(r, z)\, dz - \tfrac{1}{2} h_r^2. \tag{10}$$

This gives (the upper limit of the integral also depending from r):

$$\frac{d\Phi_r}{dr} = \int_0^{h_r} \frac{\partial \varphi}{\partial r}(r, z)\, dz + \varphi(r, h_r)\frac{dh_r}{dr} - h_r \frac{dh_r}{dr} = \int_0^{h_r} \frac{\partial \varphi}{\partial r}(r, z)\, dz$$

(in consequence of (4)).

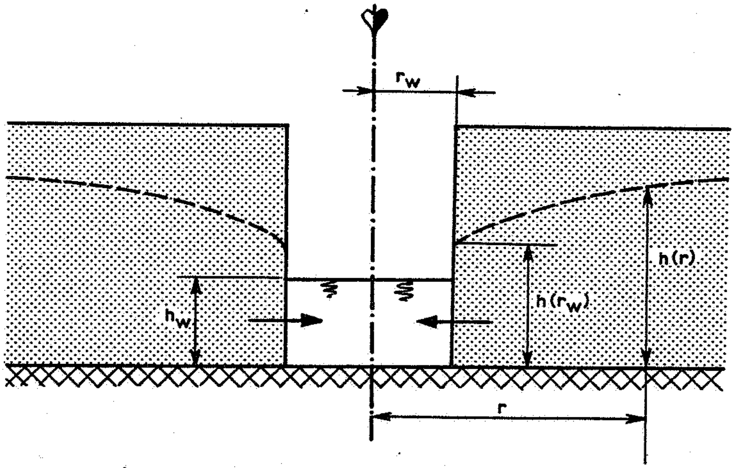

Fig. 5.3.1.2.–3

From (8) and (9) it follows, therefore, that

$$\frac{d\Phi_r}{dr} = \frac{-Q_W}{2\pi k r}. \tag{11}$$

Formally this is the same equation as that which was arrived at in 5.1.1. for completely confined ground water.

It appears that the values of the function Φ_r – which in general is only a magnitude for calculation – along the vertical boundaries are quite simple to derive from the boundary conditions for $\varphi(r,z)$. In consequence of (6) and (10) it is seen that along the outer vertical boundary of the aquifer where the ground water is replenished,

$$\Phi_R = \int_0^{h_R} h_R\, dz - \tfrac{1}{2} h_R^2 = \tfrac{1}{2} h_R^2. \tag{12}$$

In consequence of (7) and (10) it is seen that at the face of the well, where ground water leaves the aquifer,

$$\Phi_{r_W} = \int_0^{h_W} h_W\, dz + \int_{h_W}^{h_{rW}} z\, dz - \tfrac{1}{2} h^2_{rW} = \tfrac{1}{2} h_W^2. \tag{13}$$

Equation (11) with boundary condition (12) gives after integration

$$\Phi_r = \tfrac{1}{2} h_R^2 + \frac{Q_W}{2\pi k} \ln \frac{R}{r} \tag{14a}$$

and from (13) it then follows

$$Q_W = -\pi k \frac{h_R^2 - h_W^2}{\ln (R/r_W)}. \tag{14b}$$

From this it is clear that the formula of Dupuit (see 5.2.1.) giving Q_W as a function of h_R, h_W, R and r_W is correct and independent of the influence of the vertical velocity components (or of a different value of k_z).

To find the relation between Φ_r and the height of the phreatic surface h_r a partial integration will be carried out in (10).

$$\Phi_r = \Big[z\,\varphi(r, z)\Big]_{z=0}^{z=h_r} - \int_0^{h_r} z \frac{\partial \varphi}{\partial z}(r, z)\, dz - \tfrac{1}{2} h_r^2 =$$

$$= \tfrac{1}{2} h_r^2 + \frac{1}{k_z} \int_0^{h_r} z\, v_z(r, z)\, dz. \tag{15}$$

From this it follows that if the vertical velocity components can be neglected $\Phi_r = \frac{1}{2}h_r^2$, so that (14) turns into formula (1) of 5.2.1. To estimate the error in this calculation it will be supposed, just like in 5.3.1.1., that $v_z(r, z)$ lies between $v_z(r, 0) = 0$ and $v_z(r, h_r)$ for any value of r.

The latter value of v_z can be determined from the boundary conditions (4) and (5). Differentiation to r of equation (4) which holds good for all values of r between r_w and R, results (in combination with Darcy's Law) in

$$-\frac{1}{k} v_r(r, h) - \frac{1}{k_z} v_z(r, h) \frac{dh_r}{dr} = \frac{dh_r}{dr}.$$

Together with (5) it follows that

$$\frac{v_z(r, h_r)}{k_z} = -\frac{\frac{k}{k_z}\left(\frac{dh}{dr}\right)^2}{1+\frac{k}{k_z}\left(\frac{dh}{dr}\right)^2}.$$

So $v_z(r, h_r)$ is always negative.

In consequence of the supposition that $v_z(r, h_r) < v_z(r, z) < 0$ it follows that

$$0 < -\frac{1}{k_z}\int_0^{h_r} z\, v_z(r, z)\, dz < \tfrac{1}{2} h_r^2 \frac{\frac{k}{k_z}\left(\frac{dh}{dr}\right)^2}{1+\frac{k}{k_z}\left(\frac{dh}{dr}\right)^2}.$$

Therefore

$$0 < \frac{\frac{1}{2} h_r^2 - \Phi_r}{\frac{1}{2} h_r^2} < \frac{\frac{k}{k_z}\left(\frac{dh}{dr}\right)^2}{1+\frac{k}{k_z}\left(\frac{dh}{dr}\right)^2}$$

or after a small transformation,

$$0 < \frac{h_r - \sqrt{2\Phi_r}}{\sqrt{2\Phi_r}} < \left\{\sqrt{1+\frac{k}{k_z}\left(\frac{dh}{dr}\right)^2} - 1\right\} < \frac{k}{2k_z}\left(\frac{dh}{dr}\right)^2. \qquad (16)$$

This formula shows that if $\frac{1}{2} h_r^2$ is substituted for Φ_r – which implies that the exact but physically meaningless formula (14) is turned into the approximate formula (1) from 5.2.1. – the percentual error will be small if

$$\frac{k}{k_z}\left(\frac{dh}{dr}\right)^2 \ll 1. \qquad (17)$$

5.3.1.2.

If k_z is not much smaller than k, this condition (17) indicates, that the gradient of the phreatic surface $\frac{dh}{dr} = \tan i$ must be small. In the direct vicinity of the well this condition will certainly not be fulfilled; it can be proved that wherever there is a seepage zone, $\frac{dh}{dr} \approx \infty$ on the periphery of the well.

Nevertheless the following example will show that a short distance from the seepage zone the approximation will already be applicable.

Consider for example the situation in which

$$R = 100 \text{ m}, \; h_R = 5 \text{ m}, \; r_W = 0{,}5 \text{ m and } h_W = 0 \text{ m}.$$

From (14*a*) and (14*b*) it follows that

$$Q_W = \frac{-k \pi h^2_R}{\ln (R/r_W)},$$

$$\Phi r = \tfrac{1}{2} h^2_R \frac{\ln (r/r_W)}{\ln (R/r_W)}.$$

From $h \sim \sqrt{2\Phi}$ the values of h and $\frac{dh}{dr}$ can now be arrived at by approximation.

The following values are obtained:

r	Φ	h	$\frac{dh}{dr}$
$r =$ 1 m	$\Phi =$ 1.63 m²	$h =$ 1.83 m	$\frac{dh}{dr} =$ 1.29
= 3 m	= 4.22 m²	= 2.91 m	= 0.27
= 10 m	= 7.07 m²	= 3.76 m	= 0.063
= 30 m	= 9.46 m²	= 4.39 m	= 0.018
= 100 m	= 12.50 m²	= 5.00 m	= 0.005

The fourth member of (16) is now calculated for two cases, viz., $k/k_z = 1$ and $k/k_z = 10$.

The following values are obtained:

	$(k/k_z = 1)$	$(k/k_z = 10)$
for $r =$ 1 m	0.83 (0.64)	8.30 (3.19)
= 3 m	0.036	0.36 (0.32)
= 10 m	0.0020	0.020
= 30 m	0.00016	0.0016
= 100 m	0.00001	0.0001

The value of the third member of (16) has been added in parentheses if it differs significantly from the value of the fourth member. The error proves to be less than 5% in the case of $k/k_z = 1$ for $r > 2.5$ m. and in the case of $k/k_z = 10$ for $r > 6.5$ m.

The total discharge of the well in this case is found to be $Q_W = -14.8$ m³/day, which is independent of the value of k_z and of the approximation $h \sim \sqrt{2\,\Phi}$.

Remark 1

For $k_z \to 0$ the ground-water flow is exactly horizontal and the height of the phreatic surface is equal to h_R everywhere (if $h_W < h_R$). Then there is a seepage zone on the well face between $z = h_R$ and $z = h_W$. In this case the function Φ_r has of course nothing to do with the height of the phreatic surface, but formula (14) for the total yield of the well still holds good.

Remark 2

The situation described in 5.2.2. (well in phreatic water in the centre of a circular island with precipitation and a constant potential along the boundary) can be dealt with in the same way. Here it will be discovered that the formula

$$Q_W = -2\pi\, k \frac{\frac{1}{2}\,(h^2{}_R - h^2{}_W) + \dfrac{P}{4\,k}\,(R^2 - r^2{}_W)}{\ln\,(R/r_W)}\,,$$

remains strictly applicable, whilst formula 5.2.2.–(6),

$$h^2 = h^2{}_R + \frac{Q_W}{\pi k}\ln\,(R/r) + \frac{P}{2k}\,(R^2 - r^2)\,,$$

constitutes a good approximation if

$$P \ll k_z \text{ and}$$

$$\frac{k}{k_z}\left(\frac{dh}{dr}\right)^2 \ll 1$$

(the first condition will nearly always be fulfilled, because P is of the order of 0.001 m/day).

5.3.2. Horizontal stratification of the aquifer

An aquifer often has a more or less pronounced horizontal stratification; layers of coarse or fine sand alternate with less permeable strata of leam and clay. In the following paragraphs the influence of such stratification is considered in some simple cases. For the sake of simplicity it is assumed that the stratification is quite horizontal, so that the horizontal and the vertical permeability (k and k_z respectively) only depend on the ordinate z. This dependence may be discontinuous (clearly distinct layers) or continuous. In the latter case Darcy's Law reads

$$\left.\begin{aligned} v_r(r,z) &= -\,k(z)\,\frac{\partial\varphi(r,z)}{\partial r} \\ v_z(r,z) &= -k_z(z)\,\frac{\partial\varphi(r,z)}{\partial z} \end{aligned}\right\} \qquad (1)$$

Towards a fully penetrating well in a confined aquifer the flow will be quite horizontal. So the potential at a certain point only depends on the distance r between that point and the axis of the well. The total flow Q through a cylindrical surface of radius r can be found from (1) to be

$$2\pi r\int_0^H v_r(r,z)\,dz = -2\pi r\frac{d\,\varphi(r)}{dr}\int_0^H k(z)\,dz = -2\pi r\,\bar{k}\,H\frac{d\varphi}{dr}\,, \qquad (2)$$

where $\bar{k}$ is the mean horizontal permeability:

$$\bar{k} = \frac{1}{H}\int_0^H k(z)\,dz. \qquad (3)$$

Formula (2) is the same as the corresponding formula for isotropic permeability (see 5.1.1.), provided $\bar{k}$ is substituted for k.

From this it becomes clear that the characteristic constant for a confined aquifer is

$$\bar{k}H = \int_0^H k(z)\,dz\,. \qquad (4a)$$

If the aquifer consists of n layers of thickness H_i (with $\sum_{i=1}^{n} H_i = H$), the permeability of each layer being constant, it will be found that

$$\bar{k}H = \sum_{i=1}^{n} k_i\,H_i \qquad (4b)$$

The term $\bar{k}H$, which stands for the transmissibility of the aquifer is equal to the sum of the transmissibilities of the respective layers. In a more general case (formula 4*a*) the sum is replaced by the integral.

In treating the case of a fully penetrating well in a leaky artesian aquifer the method described in 5.3.1. can be used.

5.3.2.

For the mean potential $\Phi(r)$ (see formula (9) in 5.3.1.) the function

$$\Phi(r) = \frac{\int_o^H k(z)\,\varphi(r,z)\,dz}{\int_o^H k(z)\,dz}$$

may be introduced.

Then

$$Q(r) = -2\pi\,\bar{k}\,Hr\frac{d\Phi}{dr}$$

$$\frac{dQ(r)}{dr} = -\frac{2\pi r}{c}\varphi(r,H)$$

(see formulas (8) and (10) in 5.3.1.1.).

Substituting $\Phi(r)$ for $\varphi(r,H)$ gives the equations in 5.1.2.1. with $\overline{kH}$ appearing instead of kH. So here again transmissibility is one of the formation constants of the aquifer.

$\varphi(r,H)$ may be replaced by $\Phi(r)$, under a condition resulting from the same considerations as those given in 5.3.1.1.

It transpires that the formulas of 5.1.2.1. may be used – with $\overline{kH}$ instead of kH – provided

$$\tfrac{1}{2}\left(\frac{H}{\lambda}\right)^2\frac{k_{max}}{k_{z\,min}} \ll 1, \text{ where } \lambda - \sqrt{\overline{k}Hc} = \text{leakage factor of the aquifer,}$$

$$k_{max} = \text{maximum value of } k(z),$$
$$k_{min} = \text{minimum value of } kz(z)$$

This condition is a generalisation of condition (13) in 5.3.1.1. It is evident that the maximum value of k and the minumum value of k_z are important; a layer with an extremely large value of k acts as a „short-circuit"; nearly all the water reaching such a layer will flow through it towards the well and the lower layers will hardly get any at all. Similarly the layers underlying a stratum of low k_z - value will play practically no part in the transport of water. In these cases the approximation given in 5.3.1.1. cannot be expected to be a good one. If in the latter case, the value of k_z is so small that the layer can be looked upon as semi-permeable, the methods given in 5.1.3. may be used.

The mathematical treatment of partially penetrating wells and of phreatic ground water in stratified aquifers is fraught with difficulties. Generally speaking, a rather good approximation may be obtained by putting $\overline{kH}$ in the formulas instead of kH.

5.3.2.

If, however, the k-value of certain layers is very great or that of k_z very small, one should proceed very cautiously. An extreme case of the latter type is dealt with in 5.1.3. All the same it will be noticed that the semi-permeable layers of clay and loam often do not occur uninterruptedly over large areas, but consist of banks. Then the mean value of vertical permeability over a large area will not be too small. Provided no very extensive clay bank underlies the well, the flow pattern will not deviate from the case of homogeneous permeability as much as it might be expected from above.

5.4. The effect of partial penetration of the well

In the cases, dealt with in the foregoing paragraphs of chapter 5, it was always assumed that the well penetrated the aquifer throughout its thickness H (fully penetrating well). Such an assumption will often not be justified especially considering wells for water-supply purposes. The sandbeds from which the water is drawn should be very thick and have a good permeability. Its thickness will make it difficult for the aquifer to be penetrated completely. The drawdown necessary to get a certain rate of discharge from a long well may be smaller than that which must be applied to a shorter well, yet the saving on expenditure on energy which can be obtained often does not balance the higher costs of a deeper well. Somewhere an economic optimum may be found; as a rule it can be found by not penetrating the aquifer completely.

In highly permeable (coarse) sandbeds a higher velocity of flow into the well will be permissible, so deep penetration of the aquifer will not be necessary, a condition conductive to the avoidance of clogging.

In 4.3. it has been shown that a slender cylindrical well can be replaced reasonably well by a line sink of the same length and yield, provided that half the length of the well is more than ten times its radius. This condition will always be fulfilled in practice. The diameter of drilled, gravel-packed wells will be a few decimeters; the length will be several meters at least.

In the following paragraphs methods will be discussed whereby the effect of partial penetration of the well can be calculated. The method given in 5.4.1. is suitable for determining the above-mentioned effect at some distance from the well. In this connection, it is of special importance that as a rule the influence of partial penetration appears to be negligible at distances from the well of more than twice the thickness of the aquifer.

When formation constants have to be deduced from drawdowns produced by a partially penetrating well, difficulties in interpretation can often be avoided by taking the level readings at such distances from the well that the effect of the partial penetration can be neglected. In aquifers of high transmissibility, however, it will be difficult to produce at large distances from the well potential drops large enough to be measured with sufficient accuracy. In that case the use of drawdowns measured at shorter distances cannot be avoided. Then it is necessary to know the ratio of the potential drop measured in the field of the partially penetrating well, to the potential drop which would have been caused if the well had been a fully penetrating one.

Paragraph 5.4.2. deals especially with the potential drop on the well screen, making use of the formulas from 4.3. The difference between the potential on a partially penetrating well screen and the potential on a fully penetrating well screen, having the same radius is found to be

$$\varphi_{\mathrm{w\,partially}} - \varphi_{\mathrm{w\,fully}} = \frac{Q_{\mathrm{W}}}{2\pi k H} \frac{1-\delta}{\delta} \left\{ \ln \frac{4H}{r_{\mathrm{W}}} - F(\delta, \varepsilon) \right\}$$

(for the meaning of the symbols δ and ε, see fig. 5.4.2.–2). The function $F(\delta, \varepsilon)$ has been tabulated (see page 82).

It may be useful to point out that in practice one should not expect to achieve accurate results when calculating the potential on the well screen. As is the case with a fully penetrating well, the determination of r_{W} is difficult. Then again, it becomes necessary to determine not only kH but H, too. It often proves difficult to determine the thickness, especially in an aquifer of great thickness and irregular stratification. It is, therefore, reasonable to give simplified formulas, though they give less accurate results. From these formulas

$\varphi_{\mathrm{w\,part}} - \varphi_{\mathrm{w\,fully}}$ turns out to be equal to

$$\frac{Q_{\mathrm{W}}}{2\pi k H} \frac{1-\delta}{\delta} \ln \frac{\alpha L}{r_{\mathrm{W}}}.$$

α has been tabulated as a function of δ and ε (see page 93).

In 5.4.3. it will be shown that as a rule the effect of partial penetration of a well in general does not depend on whether the aquifer is replenished from overlying or underlying layers or not at all.

Homogeneous anisotropy of the aquifer generally has a limited effect, too, in connection with partial penetration, as is demonstrated in 5.4.4.

5.4.1. Partially penetrating line sink in a confined aquifer

A confined aquifer of thickness H is assumed to be bounded by a cylindrical surface (radius R) where potential $\varphi = 0$.

In the vertical axis there is a line sink, having a total yield of Q_{W}. The lower and upper ends of the sink are assumed to be at distances a and b respectively from the lower boundary of the aquifer, where the vertical co-ordinate $z = 0$ (see fig. 5.4.1.–1).

The potential $\varphi(r, z)$ – in cylindrical co-ordinates – will have to satisfy the partial differential equation

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} = 0\,, \tag{1}$$

with boundary conditions

$$\text{for } z = 0 \qquad \frac{\partial \varphi}{\partial z} = 0, \tag{2}$$

$$\text{for } z = H \qquad \frac{\partial \varphi}{\partial z} = 0, \tag{3}$$

$$\text{for } r = R \qquad \varphi = 0, \tag{4}$$

and for $r \to 0$ (compare formula 4.3.–(3))

$$\lim_{r \to 0}\left(-2\pi\, k\, r \frac{\partial \varphi}{\partial r}\right) = \begin{cases} 0 \text{ for } 0 \leqslant z \leqslant a, \\ \dfrac{Q_w}{L} \text{ for } a < z < b, \\ 0 \text{ for } b \leqslant z \leqslant H, \end{cases} \tag{5}$$

L being equal to $b - a$, the length of the sink (or in practice the well screen).

By separating variables, functions $\varphi(r, z)$ can be found, which have the form

$$\varphi(r, z) = F(r) \times G(z) \tag{6}$$

and satisfy differential equation (1) and boundary conditions (2), (3) and (4).

Substituting (6) in (1) and dividing by $F \times G$ gives us the equation

$$\frac{1}{F}\left(\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr}\right) = -\frac{1}{G}\frac{d^2G}{dz^2}\,.$$

The left-hand part of this equation does not depend on z, neither does the right-hand part depend on r, so both parts have to equal a constant, expressed as $\frac{p^2}{H^2}$. Respectively F and G will have to satisfy differential equations

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} - \frac{p^2}{H^2}F = 0 \tag{7}$$

and

$$\frac{d^2G}{dz^2} + \frac{p^2}{H^2}G = 0\,. \tag{8}$$

If $p = 0$, the general solution of (7) is $F(r) = A_1 \ln r + A_2$, and of (8) it is $G(z) = B_1 z + B_2$, to which formulas the function $\varphi_0(r, z) = F(r)\, G(z)$ belongs.

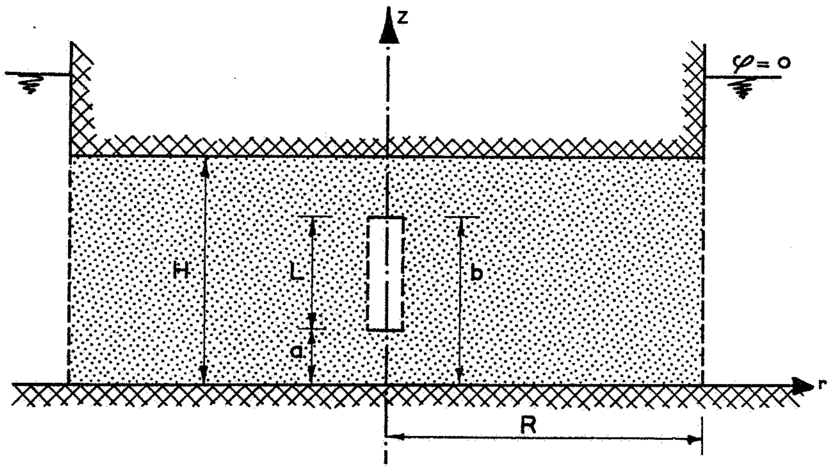

Fig. 5.4.1.–1

This function only satisfies the boundary conditions, if $A_2 = -A_1 \ln R$ and $B_1 = 0$.

$$\text{Hence} \quad \varphi_0(r, z) = C_0 \ln \frac{R}{r}, \quad C_0 \text{ being a constant.}$$

If $p \neq 0$, the general solution of (7) will be $F(r) = A_1 K_0\left(\frac{pr}{H}\right) + A_2 I_0\left(\frac{pr}{H}\right)$, K_0 and I_0 being the modified Bessel functions of zero order (see Appendix); and the general solution of (8) will be $G(z) = B_1 \sin\frac{pz}{H} + B_2 \cos\frac{pz}{H}$. The function $\varphi(r, z)$ belonging to this solution only satisfies the boundary conditions if

$$B_1 = 0, \qquad -pB_2 \sin p = 0$$

and

$$A_1 K_0\left(\frac{pR}{H}\right) + A_2 I_0\left(\frac{pR}{H}\right) = 0.$$

This indicates that $B_1 = B_2 = 0$ and therefore $\varphi(r, z) = 0$, unless $\sin p = 0$ or $p = \pi, 2\pi, 3\pi \ldots$. etc. Only if $p = n\pi$, $n = 1, 2, 3 \ldots$ etc. can solutions of the above-mentioned form be arrived at. These solutions are

$$\varphi_n(r, z) = C_n \left\{ K_0\left(\frac{n\pi r}{H}\right) - \frac{K_0\left(\frac{n\pi R}{H}\right)}{I_0\left(\frac{n\pi R}{H}\right)} I_0\left(\frac{n\pi r}{H}\right) \right\} \cos\frac{n\pi z}{H},$$

C_n being an arbitrary constant.

It may now be assumed that the potential $\varphi(r, z)$ can be expressed by

$$\varphi(r, z) = C_0 \ln\frac{R}{r} + \sum_{n=1}^{\infty} C_n \left\{ K_0\left(\frac{n\pi r}{H}\right) - \frac{K_0\left(\frac{n\pi R}{H}\right)}{I_0\left(\frac{n\pi R}{H}\right)} I_0\left(\frac{n\pi r}{H}\right) \right\} \cos\frac{n\pi z}{H}. \qquad (9)$$

If the series converges satisfactorily, this function will satisfy differential equations (1) and boundary conditions (2), (3) and (4). It can be proved that the constants $C_0, C_1, C_2 \ldots$. etc. can be chosen in such a way that boundary condition (5) will be satisfied as well.

From (9) it follows that

$$-2\pi k r \frac{\partial\varphi}{\partial r} = 2\pi k \left[C_0 - \sum_{n=1}^{\infty} C_n \frac{n\pi r}{H} \left\{ K_0'\left(\frac{n\pi r}{H}\right) - \frac{K_0\left(\frac{n\pi R}{H}\right)}{I_0\left(\frac{n\pi R}{H}\right)} I_0'\left(\frac{n\pi r}{H}\right) \right\} \cos\frac{n\pi z}{H} \right].$$

5.4.1.

As $\lim_{x\to 0} x\, K_0'(x) = -\lim_{x\to 0} x\, K_1(x) \approx = -1$,
and

$\lim_{x\to 0} x\, I_0'(x) = \lim_{x\to 0} x\, I_1(x) = 0$

it follows that

$$\lim_{x\to 0}\left(-2\pi k\, r\frac{\partial\varphi}{\partial r}\right) = 2\pi k\left[C_0 + \sum_{n=1}^{\infty} C_n \cos\frac{n\pi z}{H}\right]. \tag{10}$$

If (5) and (10) are compared it will be seen that the constants $C_0, C_1, C_2 \ldots.$ etc. for $0 \leqslant z \leqslant H$ must satisfy the equation

$$C_0 + \sum_{n=1}^{\infty} C_n \cos\frac{n\pi z}{H} = f(z)$$

in which $f(z)$ is a discontinuous function defined as

$$f(z) = \begin{cases} 0 & \text{for } 0 \leqslant z < a, \\ \dfrac{Q_W}{2\pi k\, L} & \text{for } a < z < b, \\ 0 & \text{for } b < z \leqslant H. \end{cases}$$

The theory of Fourier series shows that in fact these quations can be satiesfied if

$$C_0 = \frac{1}{H}\int_0^H f(z)\, dz = \frac{Q_W}{2\pi k H},$$

$$C_n = \frac{2}{H}\int_0^H f(z)\cos\frac{n\pi z}{H}\, dz = \frac{Q_W}{\pi k L H}\int_a^b \cos\frac{n\pi z}{H}\, dz = \frac{Q_W}{n\pi^2 k L}\left(\sin\frac{n\pi b}{H} - \sin\frac{n\pi a}{H}\right)$$

$n = 1, 2, 3 \ldots.$ etc.

By substituting these values of C in (9) we get

$$\varphi(r, z) = \frac{Q_W}{2\pi k H}\left[\ln\frac{R}{r} + \frac{2H}{\pi L}\sum_{n=1}^{\infty}\left(\sin\frac{n\pi b}{H} - \sin\frac{n\pi a}{H}\right)\cdot \cos\frac{n\pi z}{H}\left\{K_0\left(\frac{n\pi r}{h}\right) - \frac{K_0\left(\frac{n\pi R}{H}\right)}{I_0\left(\frac{n\pi R}{H}\right)} I_0\left(\frac{n\pi r}{H}\right)\right\}\right] \tag{11}$$

5.4.1.

If $a = 0$, $b = H$ (i.e. $L = H$) (11) becomes the formula for a fully penetrating well according to 5.1.1., having a capacity of Q_W:

$$\varphi_{\text{fully}} = \frac{Q_W}{2\pi kH} \ln \frac{R}{r}.$$

So the difference between the potential of a partially penetrating well and of a fully penetrating well can be expressed as

$$\varphi_{\text{part}} - \varphi_{\text{fully}} = \frac{Q_W}{2\pi kH} \cdot \frac{2H}{\pi L} \sum_{n=1}^{\infty} \frac{1}{n} \left(\sin \frac{n\pi b}{H} - \sin \frac{n\pi a}{H} \right) \cdot$$

$$\cdot \cos \frac{n\pi z}{H} \left\{ K_0\left(\frac{n\pi r}{H}\right) - \frac{K_0\left(\frac{n\pi R}{H}\right)}{I_0\left(\frac{n\pi R}{H}\right)} I_0\left(\frac{n\pi r}{H}\right) \right\}.$$

The function K_0 decreases rapidly as the argument increases ($K_0(\pi) = 0{,}0296$, $K_0(2\pi) = 0.00092$). So the second term between {} is of little significance if R is large compared to H. If $R > 2H$ a good approximation is

$$\varphi_{\text{part}} - \varphi_{\text{fully}} = \frac{Q_W}{2\pi kH} \cdot \frac{2H}{\pi L} \sum_{n=1}^{\infty} \frac{1}{n} \left(\sin \frac{n\pi b}{H} - \sin \frac{n\pi a}{H} \right) \cos \frac{n\pi z}{H} K_0\left(\frac{n\pi r}{H}\right). \tag{12}$$

As was mentioned in 5.4. in certain circumstances it may be useful to know the ratio $\frac{\varphi_{\text{part}}}{\varphi_{\text{fully}}}$.

If $R > 2H$ then

$$\frac{\varphi_{\text{part}}}{\varphi_{\text{fully}}} = 1 + \frac{2H}{\pi L \ln \frac{R}{r}} \sum_{n=1}^{\infty} \frac{1}{n} \left(\sin \frac{n\pi b}{H} - \sin \frac{n\pi a}{H} \right) \cos \frac{n\pi z}{H} K_0\left(\frac{n\pi r}{H}\right).$$

So far in this chapter it has been assumed that the well was located at the centre of a cylindrical aquifer with radius R. However, the right-hand part of formula (12) does not depend on R and practically equals 0 if $r > 2H$. Therefore it seems obvious (and it can be conclusively proved) that formula (12) is a good approximation for the difference $\varphi_{\text{part}} - \varphi_{\text{fully}}$ of a linear well at the middle of an aquifer of arbitrary shape and having arbitrary boundaries, provided that the shortest distance between the well and any part of the boundary is greater than $2H$.

In 4.3. it has been shown that the potential of a partially penetrating linear well will closely approximate the potential of a slender cylindrical partially penetrating well. Therefore the formulas given above will also apply to such a well.

To recapitulate:

If the shortest distance from the axis of a slender cylindrical partially penetrating well to any point of the boundary of an aquifer is greater than twice the thickness of the aquifer, the difference between the potential of this well and the potential of a fully penetrating well having the same capacity can be expressed as in formula (12). At distances from the well greater than twice the thickness of the aquifer the above-mentioned difference is practically zero.

5.4.2. The additional drawdown in a well caused by partial penetration

The formula given in 5.4.1. for the difference between the respective potentials of a partially penetrating line sink and a fully penetrating line sink which have equal total discharge and the same potential at a great distance, is not fit for computations in the immediate neighbourhood of the axis. In the following pages, therefore, an alternative expression for this difference will be determined by means of the method of images.

An approximation will be derived from this expression giving the difference between the potential on the face of a partially penetrating well and the potential on the face of a fully penetrating well with the same radius and discharge. Since the potential of a fully penetrating well in completely confined ground water does not depend on the radius of the well, we may consider a line sink between $z = 0$, and $z = H$ instead of a fully penetrating well.

According to 4.3., in an infinitely large aquifer we see that, for the potential of a line sink with discharge Q_w along the line $r = 0$ between $z = z_1$ and $z = z_2$ $(z_1 < z_2)$

$$\varphi = \frac{Q_w}{4\pi k} f(r, z; z_1, z_2) \text{ in which}$$

$$(r, z; z_1, z_2) = \frac{1}{z_2 - z_1} \ln \frac{z - z_1 + \sqrt{(z - z_1)^2 + r^2}}{z - z_2 + \sqrt{(z - z_2)^2 + r^2}} = \tag{1a}$$

$$= \frac{1}{z_2 - z_1} \ln \frac{z_2 - z + \sqrt{(z_2 - z)^2 + r^2}}{z_1 - z + \sqrt{(z_1 - z)^2 + r^2}} = \tag{1b}$$

$$= \frac{1}{z_2 - z_1} \ln \frac{(z_2 - z + \sqrt{(z_2 - z)^2 + r^2})(z - z_1 + \sqrt{(z - z_1)^2 + r^2})}{r^2} \tag{1c}$$

(compare formulas (2a), (2b) and (2c) in 4.3.). These formulas are equivalent, but for small r (2*a*) is the most useful one for $z > z_2$; (2b) for $z < z_1$ and (2c) for $z_1 < z < z_2$.

The difference between the respective potentials of a partially penetrating line sink between $z = a$ and $z = b$ $(0 \leqslant a < b \leqslant H)$ and a fully penetrating line sink between $z = 0$ and $z = H$, each with discharge Q_W in a layer of completely confined ground water between $z = 0$ and $z = H$, may be found in the following manner

(compare formulas (2*a*), (2*b*), and (2*c*) in 4.3.).

If the boundary conditions $v_z = 0$ or $\frac{\partial \varphi}{\partial z} = 0$ for $z = 0$ and $z = H$ are ignored,

the term $$\frac{Q_W}{4\pi k}\{f(r, z; a, b) - f(r, z; 0, H)\}$$

expresses the afore-mentioned difference, since it satisfies the differential equation as well as the conditions that between $z = a$ and $z = b$ there should be a line sink with total discharge Q_W and between $z = 0$ and $z = H$ a line sink with a total discharge of $-Q_W$.

Now if a well lying between $Z = -a$ and $z = -b$ and having a discharge of Q_W is added to a well lying between $z = 0$ and $z = -H$ having a discharge of $-Q_W$, it follows from considerations of symmetry that no water flows accross the plane $z = 0$ (see fig. 5.4.2.–1)

In other words, the expression

$$\frac{Q_W}{4\pi k}\{f(r, z; a, b) + f(r, z; -b, -a) - f(r, z; 0, H) - f(r, z; -H, 0)\}$$

satisfies both the potential equation and the boundary conditions for $z = 0$. To satisfy the boundary condition for $z = H$ the images with respect to the plane $z = H$ of the line sinks mentioned so far must be added. That is, line sinks with discharge $+Q_W$ must be added between $z = 2H-b$ and $z = 2H-a$ and between $z = 2H+a$ and $z = 2H+b$, and line sinks with discharge $-Q_W$ between $z = H$ and $z = 2H$ and between $z = 2H$ and $z = 3H$.

Then, however, the boundary condition in which $z = 0$ is no longer satisfied. In order to fulfill this boundary condition the images with respect to the plane $z = 0$ of the previously added line sinks must be added, too.

Next the images with respect to the plane $z = H$ of these last few line sinks have also to be added, etc. In this manner we obtain an infinite series of line sinks which is symmetrical with respect to both the planes $z = 0$ and $z = H$, so that the difference in potential desired, which fulfills both boundary conditions, may be expressed as

$$\varphi_{\text{fully penetrating}} - \varphi_{\text{partially penetrating}} = \frac{Q_W}{4\pi k}\sum_{n=-\infty}^{\infty}\{f(r, z; a + 2nH, b + 2nH) + \quad (2)$$

$$f(r, z; -b + 2nH, -a + 2nH) - f(r, z; 2nH, (2n+1)H) - f(r, z; (2n-1)H, 2nH)\}\cdot$$

5.4.2.

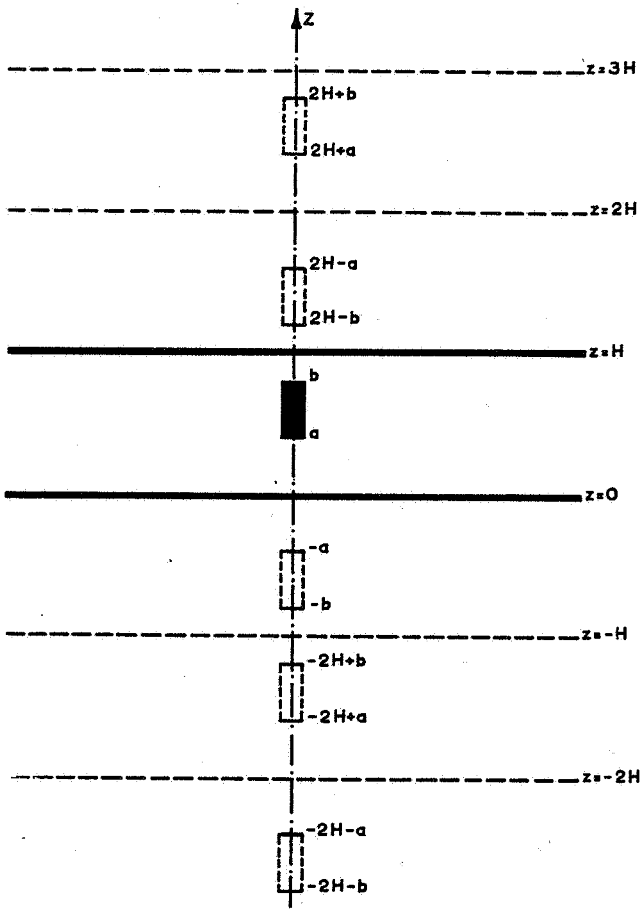

Fig. **5.4.2.–1**

The more or less intuitive reasoning followed above can be made mathematically exact by proving that the series (2) converges uniformly within the range $r > 0$, $0 \leqslant z \leqslant H$. This follows from the fact that for large values of $|n|$ we have $|a_n| < C|n|^{-3}$ where a_n stands for the general term of equation (2), and C is a constant independent of r and z.

It can then be seen that the sum of the series

1°. satisfies the potential equation,

2°. fulfills the boundary conditions for $z = 0$ and $z = H$,

3°. approaches zero for $r \to \infty$ (the terms of the series then all tend to zero)

4°. consists of only two line sinks in the range $0 \leqslant z \leqslant H$, viz. the line sink with discharge Q_w between $z = a$ and $z = b$ and the line sink with discharge $-Q_w$ between $z = 0$ and $z = H$.

It follows from the circumstance that both expressions satisfy the same differential equation and the same boundary conditions that (2) respresents the same function as (12) in 5.4.1. It can also be shown that said equation and boundary conditions can be satisfied by one function only.

As an alternative we can expand the right-hand side of (2), which is a periodical function of z with period $2H$, in a Fourier series. The result of this procedure is the right-hand side of 5.4.1.–(12).

In order to evaluate the right-hand side of (2) we replace $\sum\limits_{n=-\infty}^{\infty}$ by $\lim\limits_{N\to\infty} \sum\limits_{n=-N}^{N}$.

Since the sink between $z = 0$ and $z = H$ and its several images adjoin, we may replace their sum by the potential of a single line sink between $z = -(2N + 1)H$ and $z = (2N + 1)H$ with discharge $-2(2N + 1)Q_w$.

In other words

$$\sum_{n=-N}^{N} \{ f(r, z; 2nH, (2n + 1)H) + f(r, z; (2n - 1)H, 2nH) \} =$$

$$\frac{1}{H} \ln \frac{\{(2N+1)H - z + \sqrt{((2N+1)H-z)^2 + r^2}\}\{(2N+1)H + z + \sqrt{((2N+1)H+z)^2+r^2}\}}{r^2} =$$

$$= \frac{2}{H} \ln N + \frac{2}{H} \ln \frac{4H}{r} + \frac{1}{H} \ln \frac{(2N+1)H - z + \sqrt{((2N+1)H-z)^2+r^2}}{4NH} +$$

$$+ \frac{1}{H} \ln \frac{(2N+1)H + z + \sqrt{((2N+1)H+z)^2+r^2}}{4NH}. \qquad (3)$$

The last two terms of (3) approach zero as $N \to \infty$, so that equation (2) becomes

$$\varphi_{\text{part}} - \varphi_{\text{fully}} = -\frac{Q_W}{2\pi k H} \ln \frac{4H}{r} + \frac{Q_W}{4\pi k} \lim_{N\to\infty} \left[\sum_{n=-N}^{N} \{ f(r, z; a + 2nH, b + 2nH) + f(r, z; -b + 2nH, -a + 2nH) \} - \frac{2}{H} \ln N \right] \quad (4)$$

If r is small with respect to H simplifications may be introduced. It follows from (1*a*) and (1*b*) that a fair approximation is

$$f(r, z; z_1, z_2) = \frac{1}{z_2 - z_1} \ln \frac{z - z_1}{z - z_2} \text{ for } z > z_2 \text{ and } r \ll z - z_2,$$

$$f(r, z; z_1, z_2) = \frac{1}{z_2 - z_1} \ln \frac{z_2 - z}{z_1 - z} \text{ for } z < z_1 \text{ and } r \ll z_1 - z.$$

Substituting these approximations in (4), except in the terms derived from the line sink and from the first images in the upward and downward directions, we get (if $L = b - a$, the height of the partially penetrating well)

$$\sum_{n=-N}^{N} \{ f(r, z; a + 2nH, b + 2nH) + f(r, z; -b + 2nH, -a + 2nH) \} \approx$$
$$\approx f(r, z; a, b) + f(r, z; -b, -a) + f(r, z; 2H - b, 2H - a) +$$
$$+ \frac{1}{L} \sum_{n=1}^{N} \ln \frac{b + 2nH - z}{a + 2nH - z} + \frac{1}{L} \sum_{n=1}^{N} \ln \frac{z - a + 2nH}{z - b + 2nH} + \frac{1}{L} \sum_{n=2}^{N} \ln \frac{-a + 2nH - z}{-b + 2nH - z} +$$
$$+ \frac{1}{L} \sum_{n=1}^{N} \ln \frac{z + b + 2nH}{z + a + 2nH}. \quad (5)$$

Having introduced these simplifications, the limits in (4) can be expressed in what is known as the *Gamma-function* (a table of values to be found in JAHNKE-EMBDE-LÖSCH [1]) and other publications). For $u > -1$ it is seen that

$$\lim_{N\to\infty} \left\{ u \ln N - \sum_{n=1}^{N} \ln \left(1 + \frac{u}{n}\right) \right\} = \ln \Gamma(1 + u) \cdot$$

From this it follows that e.g.

$$\lim_{N\to\infty} \left\{ \frac{1}{L} \sum_{n=1}^{N} \ln \frac{b + 2nH - z}{a + 2nH - z} - \frac{1}{2H} \ln N \right\} =$$
$$= \frac{1}{L} \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \ln \left(1 + \frac{b - z}{2nH}\right) - \frac{b - z}{2H} \ln N \right\} +$$
$$- \frac{1}{L} \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \ln \left(1 + \frac{a - z}{2nH}\right) - \frac{a - z}{2H} \ln N \right\} = -\frac{1}{L} \ln \frac{\Gamma\left(1 + \dfrac{b - z}{2H}\right)}{\Gamma\left(1 + \dfrac{a - z}{2H}\right)}.$$

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[1]) JAHNKE – EMBDE – LÖSCH, Tables of higher functions, B. G. Teubner, Stuttgart, 1960.

Treating the other terms of (5) in a similar manner, the following approximation applicable to $r \ll H$ is found:

$$\varphi_{\text{part}} - \varphi_{\text{fully}} = -\frac{Q_W}{2\pi kH} \ln \frac{4H}{r} +$$

$$+ \frac{Q_W}{4\pi kL} \ln \frac{(z - a + \sqrt{(z-a)^2 + r^2})(b - z + \sqrt{(b-z)^2 + r^2})}{r^2} +$$

$$+ \frac{Q_W}{4\pi kL} \ln \frac{(z + b + \sqrt{(z+b)^2 + r^2})(2H - a - z + \sqrt{(2H - a - z)^2 + r^2})}{(z + a + \sqrt{(z+a)^2 + r^2})(2H - b - z + \sqrt{(2H - b - z)^2 + r^2})} +$$

$$- \frac{Q_W}{4\pi kL} \ln \frac{\Gamma\left(1 + \frac{z-a}{2H}\right) \cdot \Gamma\left(1 + \frac{b-z}{2H}\right) \cdot \Gamma\left(2 - \frac{z+a}{2H}\right) \cdot \Gamma\left(1 + \frac{b+z}{2H}\right)}{\Gamma\left(1 - \frac{z-a}{2H}\right) \cdot \Gamma\left(1 - \frac{b-z}{2H}\right) \cdot \Gamma\left(1 + \frac{z-a}{2H}\right) \cdot \Gamma\left(2 - \frac{b+z}{2H}\right)} \qquad (6)$$

Remark

For $a = 0$ and $b = H$ the right-hand side of (6) becomes

$$-\frac{Q_W}{2\pi kH} \ln \frac{4H}{r} + \frac{Q_W}{4\pi kH} \ln \frac{(2H - z + \sqrt{(2H-z)^2 + r^2})(H + z + \sqrt{(H+z)^2 + r^2})}{r^2} +$$

$$- \frac{Q_W}{4\pi kH} \ln \frac{\Gamma\left(2 - \frac{z}{2H}\right) \cdot \Gamma\left(\frac{3}{2} + \frac{z}{2H}\right)}{\Gamma\left(1 - \frac{z}{2H}\right) \cdot \Gamma\left(\frac{1}{2} + \frac{z}{2H}\right)},$$

which may by means of the formula $\Gamma(1 + u) = u\Gamma(u)$ be reduced to

$$\frac{Q_W}{4\pi kH} \ln \frac{(2H - z + \sqrt{(2H-z)^2 + r^2})(H + z + \sqrt{(H+z)^2 + r^2})}{4(2H - z)(H + z)}.$$

For $r \ll H$ this equals zero (for $0 \leqslant z \leqslant H$) in the approximation used here.

Now consider the values of $\varphi_{\text{part}} - \varphi_{\text{fully}}$ on the cylinder $r = r_W$, $a < z < b$, in which $r_W \ll L$ (for instance $r_W \leqslant 0.05L$). The second term in (6) is the potential of a line sink between $z = a$ and $z = b$ in an infinite aquifer. This term becomes large if $r \ll L$, and it is then largely independent of z (see 4.3.). The next term remains limited for small r, unless $a = 0$ and z is close to 0. In that case, however, this term may be combined with the preceding one to form the potential of a single line sink between $z = -b$ and $z = b$. This potential is regular near $z = 0$.

For the fourth term of (6) the situation is analogous. The last term of (6) does not

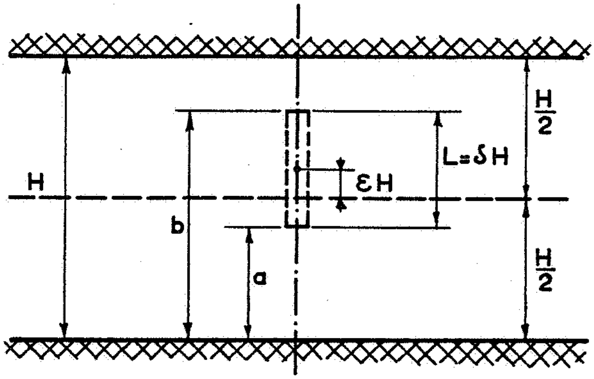


Fig. 5.4.2.–2

depend on r and is certainly limited if $a < z < b$. It then turns out that the function given by (6) is reasonably constant on the circumference of a slender cylinder between $z = a$ and $z = b$, as it is in the case dealt with in 4.3.

The average value between $z = a$ and $z = b$ of the right-hand side of (6) with $r = r_W$ may therefore be regarded as a reasonable approximation of the difference between the potentials on the respective boundaries of a cylindrical partially penetrating well and a cylindrical fully penetrating well ($\varphi_{W\,part} - \varphi_{W\,fully}$) with the same discharge Q_W, the same radius r_W and the same potential distribution at a great distance.

This average value will now be computed. Since $r_W \ll L$ the second term of (6) may be replaced by

$$\frac{Q_W}{4\pi kL} \ln \frac{4(z-a)(b-z)}{r_W^2},$$

when taking averages as was shown towards the end of 4.3.

On the same grounds the third term of (6) may be replaced by

$$\frac{Q_W}{4\pi kL} \ln \frac{(z+b)(2H-a-z)}{(z+a)(2H-b-z)}.$$

Next, using the formula $\Gamma(u+1) = u\,\Gamma(u)$ the second and third terms of (6) may be combined with the fourth term, giving

$$\varphi_{W\,part} - \varphi_{W\,fully} = \frac{Q_W}{2\pi kH} \frac{1}{L} \int_a^b \left[\frac{H-L}{L} \ln \frac{4H}{r_W} + \right.$$

$$\left. - \frac{H}{2L} \ln \frac{\Gamma\left(\frac{z-a}{2H}\right) \cdot \Gamma\left(\frac{b-z}{2H}\right) \cdot \Gamma\left(1 - \frac{z+a}{2H}\right) \cdot \Gamma\left(\frac{b+z}{2H}\right)}{\Gamma\left(1 - \frac{z-a}{2H}\right) \cdot \Gamma\left(1 - \frac{b-z}{2H}\right) \cdot \Gamma\left(\frac{z+a}{2H}\right) \cdot \Gamma\left(1 - \frac{b+z}{2H}\right)} \right] dz. \qquad (7)$$

The following quantities are now introduced (see fig. 5.4.2.–2)

$\delta = \dfrac{L}{H}$, the relative length of the screen

$\varepsilon = \dfrac{a+b-H}{2H}$, the relative eccentricity of the screen

(in which, of course, since $0 \leqslant a < b \leqslant H$: $0 < \delta \leqslant 1$ and $-\frac{1}{2}(1-\delta) \leqslant \varepsilon \leqslant \frac{1}{2}(1-\delta)$).

Moreover the quantities $a = (\frac{1}{2} + \varepsilon - \frac{1}{2}\delta)H$ and $b = (\frac{1}{2} + \varepsilon + \frac{1}{2}\delta)H$ and the function $H(x)$ which for $-\frac{1}{2} \leqslant x < \frac{1}{2}$ is defined as $H(x) = H(-x) = \int_0^x \ln \frac{\Gamma(\frac{1}{2} - u)}{\Gamma\frac{1}{2}(+u)} du$ (8)
are introduced.

Formula (7) may then be modified to give

$$\varphi_{\mathrm{w\,part}} - \varphi_{\mathrm{w\,fully}} = \frac{Q_{\mathrm{w}}}{2\pi kH} \cdot \frac{1-\delta}{\delta} \left\{ \ln \frac{4H}{r_{\mathrm{w}}} - F(\delta, \varepsilon) \right\} \qquad (9)$$

in which

$$F(\delta, \varepsilon) = \frac{1}{\delta(1-\delta)} \left[2H(\tfrac{1}{2}) - 2H(\tfrac{1}{2} - \tfrac{1}{2}\delta) + 2H(\varepsilon) - H(\varepsilon - \tfrac{1}{2}\delta) - H(\varepsilon + \tfrac{1}{2}\delta) \right].$$

Some values of the function $H(x)$ are:

x	$H(x) = H(-x)$	x	$H(x) = H(-x)$
0	0	0.25	0.12881
0.05	0.00492	0.30	0.19003
0.10	0.01978	0.35	0.26703
0.15	0.04491	0.40	0.36403
0.20	0.08093	0.45	0.49016
0.25	0.12881	0.50	0.68850

Some values of $F(\delta, \varepsilon) = F(\delta, -\varepsilon)$ are:

$\varepsilon \rightarrow$ / $\delta \downarrow$	0	0.05	0.10	0,15	0.20	0.25	0.30	0.35	0.40	0.45
0.1	4.298	4.297	4.294	4.287	4.276	4.259	4.232	4.184	4.084	3.605
0.2	3.809	3.806	3.797	3.781	3.756	3.716	3.650	3.525	3.116	
0.3	3.586	3.581	3.566	3.537	3.490	3.425	3.276	2.893		
0.4	3.479	3.471	3.445	3.395	3.312	3.165	2.786			
0.5	3.447	3.433	3.388	3.302	3.145	2.754				
0.6	3.479	3.455	3.374	3.208	2.786					
0.7	3.586	3.538	3.370	2.893						
0.8	3.809	3.688	3.116							
0.9	4.298	3.605								

For $\delta \rightarrow 1$ (fully penetrating well) $F(\delta, e)$ approaches infinity, but only logarithmically, so that $(1 - \delta)\, F(\delta, \varepsilon)$ and hence also the right-hand side of (9) approaches zero.

5.4.2.

Remark 1

In 6.2.6. it will be shown that the potential on the boundary of a fully penetrating well may in many cases be expressed as

$$\varphi_W = \frac{Q_W}{2\pi kH} \ln \frac{R_{eq}}{r_W}$$

in which R_{eq} (the equivalent radius) is a quantity that depends on the geo-hydrological properties of the area under consideration and on the location of the well.

For a partially penetrating well it is seen that analogically

$$\varphi_W = \frac{Q_W}{2\pi kH}\left[\ln \frac{R_{eq}}{r_W} + \frac{1-\delta}{\delta}\left\{\ln \frac{4H}{r_W} - F(\delta, \varepsilon)\right\}\right]$$

Figure 5.4.2.–3 shows the behaviour of the term in square brackets in the above formula, i.e. of the quantity $\frac{\varphi_W}{Q_W} \cdot 2\pi kH$ as a function of δ if $\varepsilon = 0$ (well in the centre of the aquifer) and if $\varepsilon = \frac{1}{2} - \frac{1}{2}\delta$ (well on the upper or lower boundary of the aquifer) for

$$r_W = 0.25 \text{ m}$$
$$H = 40 \text{ m}$$
$$R_{eq} = 200 \text{ m}$$

Remark 2

It can be proved that $2H(\frac{1}{2} - \frac{1}{2}x) + 2H(\frac{1}{2}x) = H(\frac{1}{2} - x) + H(\frac{1}{2}) - x(1 - x) \ln 2$, from which it may concluded that

$$F(\delta, 0) = F(\delta, \tfrac{1}{2} - \tfrac{1}{2}\delta) + \ln 2.$$

The meaning of this formula becomes clear if a well is considered, extending to the upper boundary of the aquifer ($\varepsilon = \frac{1}{2} - \frac{1}{2}\delta$). Introducing images with respect to this upper boundary, a well of twice the original height is obtained as it were standing in the centre of an aquifer of thickness $2H$ and having a discharge of $2Q_W$.

Therefore the loss of potential head of a well extending to the upper or lower boundary of the aquifer exceeds that of a well situated symmetrically in the aquifer by

$$\frac{Q_W}{2\pi kH} \frac{1-\delta}{\delta} \ln 2.$$

Remark 3

The treatment given above is approximately analogous to that given by Muskat [1]). By increasing somewhat the discharge per unit length of the line sinks towards the

[1]) The flow of homogeneous fluids through porous media, p. 263–276. Physics 2, 1932 p. 329.

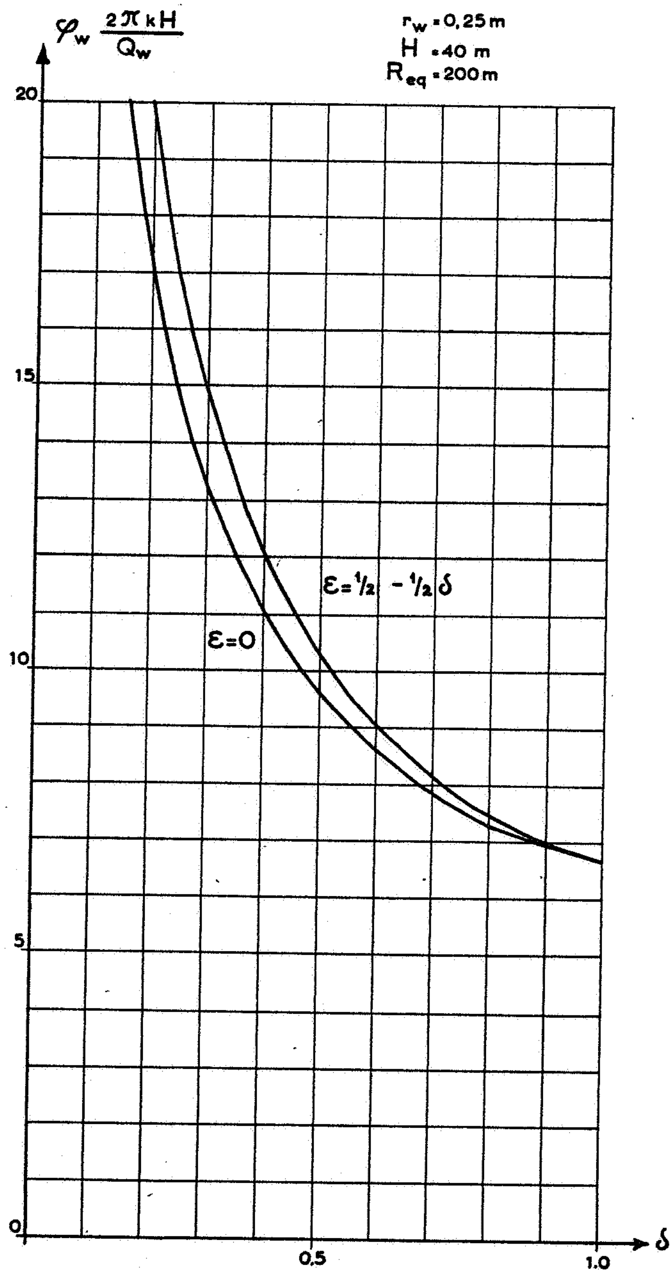

$\varphi_w \frac{2\pi kH}{Q_w}$
$r_w = 0,25$ m
$H = 40$ m
$R_{eq} = 200$ m
20
15
10
5
0
0,5
1.0
δ
ε=½ - ½δ
ε=0

Fig. 5.4.2.-3

extremities, MUSKAT tries to enforce closer observance of the boundary condition that φ = constant on the cylindrical screen. This has to be done numerically.

On the basis of several elaborated examples (in which $\varepsilon = \frac{1}{2} - \frac{1}{2}\delta$) MUSKAT concludes that a fair approximation may be obtained by taking the function $F(\delta, \frac{1}{2} - \frac{1}{2}\delta)$ to be

$$F(\delta, \tfrac{1}{2} - \tfrac{1}{2}\delta) = \frac{1}{2(1-\delta)} \ln \frac{\Gamma(\frac{1}{8}\delta)\,\Gamma(7/8\delta)}{\Gamma(1-\frac{1}{8}\delta)\,\Gamma(1-7/8\delta)} .$$

The numerical differences with respect to the values given in the above section are comparatively small (less than 8%).

Remark 4

The difference between the treatment given in the above section and that given by DE GLEE [2]) chiefly consists in the following points:

1°. DE GLEE takes of the function $f(r, z; a, b)$ the value obtained on the equipotential ellipsoid which has the same area as the curved surface of the cylindrical screen of the well.

2°. The images of the line sinks are replaced by point sinks at the centre of the image, and the influence of these sinks on the potential head at the centre of the original well is computed. This procedure introduces an error of several percent for the first-order image; for the higher order images the discrepancy is smaller.

The consequence of DE GLEE's assumptions is that the cases $a \to 0$ and $b \to H$ have no continuous limits in his results.

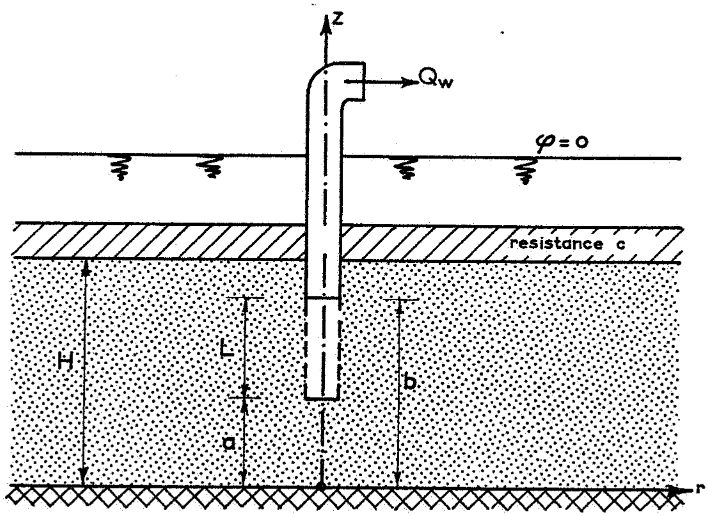
The numerical differences between the correction found by DE GLEE and that obtained in the above section are of the order of magnitude of 10%.

5.4.3. LINE SINK IN SEMI-CONFINED GROUND WATER

Consider an aquifer of semi-confined groundwater H in height, the upper boundary of which is a layer of low permeability (see fig. 5.4.3.–1). Above this layer there is water at a constant potential of $\varphi = 0$. Along a vertical line sink of discharge Q_W is situated. Its upper and lower extremities are at distances a and b respectively from the lower boundary of the aquifer. Horizontally the aquifer is supposed to be infinite. This gives

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial z^2} = 0 , \qquad (1)$$

[2]) Thesis Delft 1930.



5.4.3.–1

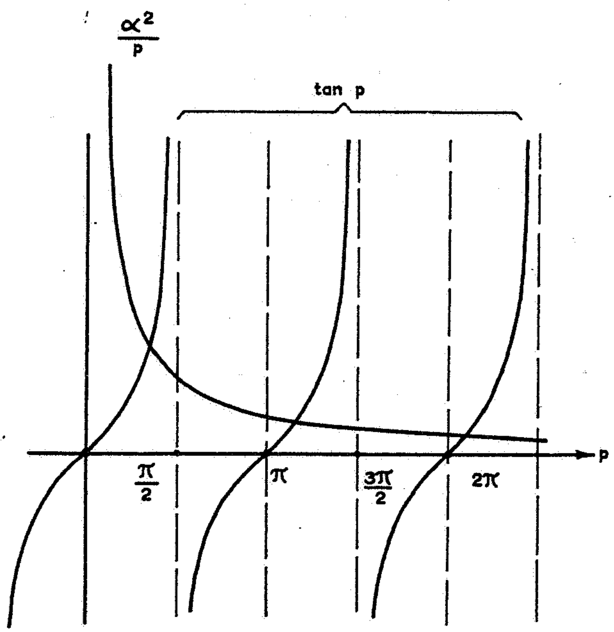


Fig. 5.4.3.–2

with the boundary conditions

$$z = 0 \qquad \frac{\partial \varphi}{\partial z} = 0\,, \tag{2}$$

$$z = H \qquad -k\frac{\partial \varphi}{\partial z} = \frac{\varphi}{c}\,, \tag{3}$$

$$r = \infty \qquad \varphi = 0, \tag{4}$$

$$r \to 0 \qquad \lim_{r \to 0}\left(-2\pi k r \frac{\partial \varphi}{\partial r}\right) = \begin{cases} 0 & \text{for } 0 \leqslant z < a\,, \\ \dfrac{Q_{\mathrm{w}}}{L} & \text{for } a < z < b\,, \\ 0 & \text{for } b < z \leqslant H\,, \end{cases} \tag{5}$$

in which $L = b - a$.

As was the case in 5.4.1. it is possible with the help of the method of separation of variables to find functions $\varphi(r, z)$ which satisfy the equations (1), (2), (3) and (4) and which take the form

$$\varphi(r, z) = F(r) \times G(z). \tag{6}$$

The equations for F and G again turn out to be

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} - \frac{p^2}{H^2}F = 0\,, \tag{7}$$

$$\frac{d^2G}{dz^2} + \frac{p^2}{H^2}G = 0\,. \tag{8}$$

For $p = 0$ the general solution of (8) is

$$G = B_1 z + B_2.$$

However, no solution of the type under consideration exists for $p = 0$, because the boundary conditions (2) and (3) are fulfilled only if $B_1 = 0$ and $B_1 + \dfrac{B_1 H + B_2}{kc} = 0$, which implies that $B_1 = B_2 = 0$.

For $p \neq 0$ the general solution of (7) and (8) is

$$F = A_1 K_0\left(\frac{pz}{H}\right) + A_2 I_0\left(\frac{pz}{H}\right),$$

$$G = B_1 \sin\frac{pz}{H} + B_2 \cos\frac{pz}{H}.$$

The boundary conditions (2), (3) and (4) are fulfilled if

$$B_1 = 0 \text{ and } B_2\left(\frac{kp}{H}\sin p - \frac{1}{c}\cos p\right) = 0\,, \tag{9}$$

$$A_2 = 0$$

From (9) it follows that $B_2 = 0$ (which means that no solution exists), unless

$$p \tan p = \alpha^2. \tag{10}$$

$$\text{in which } \alpha = \sqrt{\frac{H}{kc}} = \frac{H}{\lambda} \text{ and } \lambda = \sqrt{kHc}.$$

The transcendental equation (10) has an infinite number of positive roots $p_0, p_1, p_2, \ldots$ etc. in which

$$n\pi < p_n < (n + \tfrac{1}{2})\pi \qquad \text{(see fig. 5.4.3.–2)}$$

For these values of p, then, we have functions of the type (6), which fulfill (1), (2), (3) and (4), viz.

$$\varphi_n(r, z) = C_n K_0\left(\frac{p_n r}{H}\right) \cos \frac{p_n z}{H}.$$

The function

$$\varphi(r, z) = \sum_{n=0}^{\infty} C_n K_0\left(\frac{p_n r}{H}\right) \cos \frac{p_n z}{H} \tag{11}$$

according to the foregoing satisfies (1), (2), (3) and (4), provided there is sufficient convergence of the series. An attempt may now be made to select such values for the constants C_n that the function also fulfills the boundary condition (5). From (11) it follows that

$$\lim_{r\to 0}\left(-2\pi k r \frac{\partial \varphi}{\partial r}\right) = -2\pi k \lim_{r\to 0} \sum_{n=0}^{\infty} C_n \left(\frac{p_n r}{H}\right) K_0'\left(\frac{p_n r}{H}\right) \cos \frac{p_n z}{H} =$$

$$= 2\pi k \sum_{n=0}^{\infty} C_n \cos \frac{p_n z}{H}, \tag{12}$$

so that the following relation must hold good.

$$\sum_{n=0}^{\infty} C_n \cos \frac{p_n z}{H} = f(z), \qquad 0 \leqslant z \leqslant H, \tag{13}$$

$$\text{with } f(z) = \begin{cases} 0 & \text{for } 0 \leqslant z < a, \\ \dfrac{Q_w}{2\pi k L} & \text{for } a < z < b, \\ 0 & \text{for } b < z \leqslant H. \end{cases}$$

It can be proved that the elaboration (13) is possible. The coefficients C_n can be evaluated by using what is known as the orthogonality property of the functions $\cos \frac{p_n z}{H}$:

$$\int_0^H \cos \frac{p_n z}{H} \cos \frac{p_m z}{H} dz = \begin{cases} 0 \text{ for } n \neq m; \\ \dfrac{H}{2}\left(1 + \dfrac{\alpha^2}{p_m^2 + \alpha^4}\right) \text{ if } n = m. \end{cases}$$

That the above formulas have this property can be proved by the fact that the numbers p_n and p_m are roots of the transcendental equation (10).

On multiplying (13) by a factor $\cos \frac{p_n z}{H}$ and integrating between 0 and H it is seen that the term with $n = m$ only contributes to the result, so that

$$\frac{H}{2}\left(1 + \frac{\alpha^2}{p_m^2 + \alpha^4}\right) C_m = \int_0^H f(z) \cos \frac{p_m z}{H}\, dz = \frac{Q_w}{2\pi k L} \int_a^b \cos \frac{p_m z}{H}\, dz =$$

$$\frac{Q_w H}{2\pi k L} \frac{1}{p_n} \left(\sin \frac{p_m b}{H} - \sin \frac{p_m a}{H}\right),$$

from which it follows that

$$C_m = \frac{Q_w}{2\pi k H} \frac{2H}{p_m L} \frac{\sin \frac{p_m b}{H} - \sin \frac{p_m a}{H}}{1 + \frac{\alpha^2}{p_m^2 + \alpha^4}}.$$

It is then seen that the potential

$$\varphi(r, z) = \frac{Q_w}{2\pi k H} \frac{2H}{L} \sum_{n=0}^{\infty} \frac{1}{p_m} \frac{\sin \frac{p_m b}{H} - \sin \frac{p_m a}{H}}{1 + \frac{\alpha^2}{p_m^2 + \alpha^4}} \cos \frac{p_m z}{H} K_0\left(\frac{p_m r}{H}\right).$$

This expression may be greatly simplified in the same way as it was done when dealing with the fully penetrating well, viz. by supposing that the leakage factor λ of the aquifer is large as compared with its thickness; this implies that $\alpha^2 \ll 1$.

The roots p_m of the transcendental equation (10) then approach integer multiples of π. We have

$$p_0 = \alpha\left(1 - \frac{1}{6}\alpha^2 + \ldots.\right),$$

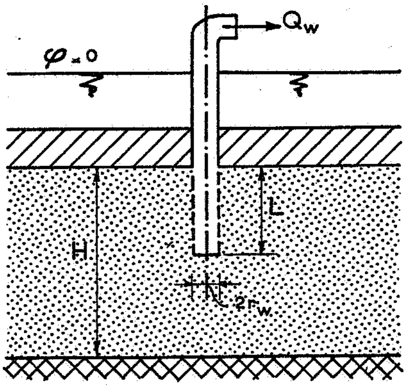
$$p_n = n\pi\left(1 + \frac{\alpha^2}{n^2\pi^2} + \ldots\right), \text{ in which } n \geqslant 1.$$

From this it follows that

$$C_0 = \frac{Q_w}{2\pi k H} \frac{2H}{p_0 L} \frac{\frac{p_0}{H}(b - a) + \ldots}{1 + 1 + \ldots.} \approx \frac{Q_w}{2\pi k H},$$

$$C_n = \frac{Q_w}{2\pi k H} \frac{2H}{\pi L} \frac{1}{n(1 + \ldots)} \frac{\sin \frac{n\pi b}{H} - \sin \frac{n\pi a}{H} + \ldots.}{1 + \ldots}$$

$$\approx \frac{Q_w}{2\pi k H} \frac{2H}{\pi L} \frac{1}{n}\left(\sin \frac{n\pi b}{H} - \sin \frac{n\pi a}{H}\right);$$



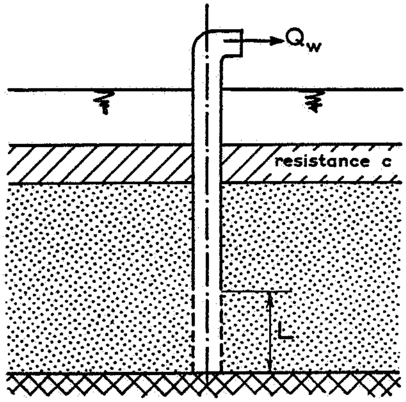


Fig. 5.4.3.–3

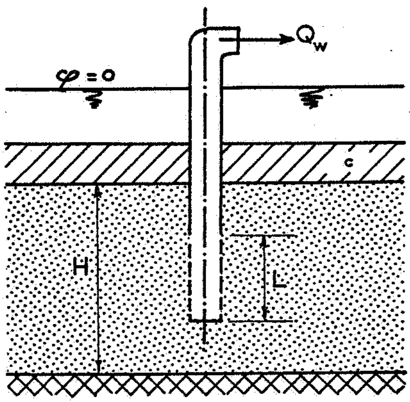


Fig. 5.4.3.–4

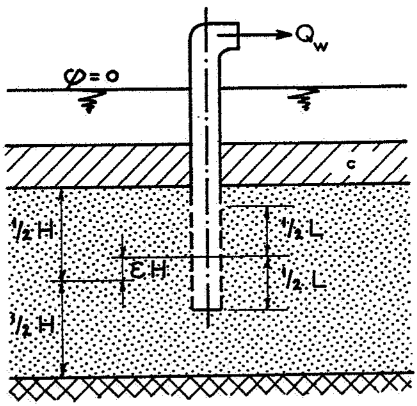


Fig. 5.4.3.–5

$$\cos p_0 z = 1 - \tfrac{1}{2}p_0^2 z^2 \approx 1, \quad K_0\left(\frac{p_0 r}{H}\right) \approx K_0\left(\frac{r}{\lambda}\right);$$

$$\cos p_n z \approx \cos\frac{n\pi z}{H}, \quad K_0\left(\frac{p_n r}{H}\right) \approx K_0\left(\frac{n\pi r}{H}\right),$$

so that the following approximation is obtained:

$$\varphi(r, z) = \frac{Q_W}{2\pi kH}\left\{K_0\left(\frac{r}{\lambda}\right) + \frac{2H}{\pi L}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\frac{n\pi b}{H} - \sin\frac{n\pi a}{H}\right)\cdot\cos\frac{n\pi z}{H}\,K_0\left(\frac{n\pi r}{H}\right)\right\}. \quad (14)$$

The neglected higher terms are all smaller by an order α^2 than the terms preserved, and the approximation given here is therefore essentially the same as that in the treatment of the fully penetrating well.

From (14) the formula for the fully penetrating well can be derived by taking $a = 0$ and $b = H$. We may therefore also write

$$\varphi_{\text{part}} - \varphi_{\text{fully}} = \frac{Q_W}{2\pi kH}\cdot\frac{2H}{\pi L}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\frac{n\pi b}{H} - \sin\frac{n\pi a}{H}\right)\cdot\cos\frac{n\pi z}{H}\cdot K_0\left(\frac{n\pi r}{H}\right).$$

This formula is exactly the same as the corresponding formula for completely confined ground water (formula 5.4.1.–(12)), which is understandable also on physical grounds: if $\lambda \gg H$, the leakage through the layer of low permeability in the region $r < 2H$ constitutes only a fraction of Q_W. In this case then we have almost completely confined ground water. Outside the region $r < 2H$ the influence of the partial penetration of the well is therefore negligible. From this it follows that if $\lambda \gg H$ the conclusions and results of paragraphs 5.4.1. and 5.4.2. also apply to the case of semi-confined ground water.

Remark 4 in 5.4.2. refers to the formulas developed by De Glee. If the screen of a partially penetrating well in a semi-confined aquifer is located just below the covering layer or just above the base of the aquifer (fig. 5.4.3.–3), De Glee's formula for the potential on the well face can be written as

$$\varphi_W = \frac{Q_W}{4\pi k}\left[\frac{2}{L}\ln\frac{\pi L}{2r_W} + \frac{2}{H}\left\{-\ln\frac{2H}{\lambda} + 0.216\right\}\right].$$

This equation is applicable if

$$\frac{L}{r_W} > 10, \quad 1.3\,L < H \text{ and } H < \lambda.$$

On working this out we get

$$\varphi_W = \frac{Q_W}{2\pi kL}\ln\frac{\pi L}{2r_W} + \frac{Q_W}{2\pi kH}\ln\frac{0.621\,H}{\lambda} \quad (15)$$

The potential on the face of a fully penetrating well in a semi-confined aquifer can be expressed approximately as

$$\varphi_w = \frac{Q_w}{2\pi kH}\left(\ln\frac{\lambda}{r_w} + 0.1159\right) = \frac{Q_w}{2\pi kH}\ln\frac{1.122\,\lambda}{r_w} \tag{16}$$

After some elaboration equations (15) and (16) give the following expression for the extra drawdown on the face of the well due to the partial penetration of the well.

$$\Delta\varphi_w = \varphi_{w\,\text{part}} - \varphi_{w\,\text{fully}} = \frac{Q_w}{2\pi kH}\ln\left(\frac{\pi L}{2\,r_w}\right)^{\frac{H}{L}}\frac{0.5525\,r_w}{H}.$$

This can be converted into

$$\Delta\varphi_w = \frac{Q_w}{2\pi kH}\frac{1-\delta}{\delta}\ln\frac{aL}{r_w}, \tag{17}$$

in which

$$\delta = \frac{L}{H} \text{ and } a = \left(\frac{\pi}{2}\right)^{\frac{1}{1-\delta}}(0.5525\delta)^{\frac{\delta}{1-\delta}}.$$

Some values of a are given in the following table.

δ	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
a	1.330	1.196	1.013	0.883	0.776	0.682	0.590	0.492	0.366

Since quantity a multiplied by $\frac{L}{r_w}$ (which is > 10) occurs in equation (17) as a logarithm, even a fair approximation of quantity a yields sufficiently accurate results. By writing $a = 1.2 - \delta$, equation (17) becomes

$$\Delta\varphi_w = \frac{Q_w}{2\pi kH}\frac{1-\delta}{\delta}\ln\frac{(1.2-\delta)L}{r_w}. \tag{18}$$

If the well screen is located exactly in the middle of the aquifer (see fig.5.4.3.–4), equation (18) is converted into

$$\Delta\varphi_w = \frac{Q_w}{2\pi kH}\frac{1-\delta}{\delta}\ln\frac{(1.2-\delta)L}{2\,r_0}, \tag{19}$$

because of the symmetrical conditions obtaining in the vicinity of the well.

This relation holds good if

$$\frac{1/2L}{r_w} > 10 \text{ and } 1.3\,L < H.$$

If the well screen is located eccentrically (see fig. 5.4.3.–5) the extra drawdown on the well face due to partial penetration can be expressed as

$$\Delta \varphi_W = \frac{Q_W}{2\pi k H} \frac{1-\delta}{\delta} \ln \frac{\alpha H}{r_W}, \tag{20}$$

in which $\delta = \frac{L}{H}$ and α is a function of δ and ε.

Some values of α are given in the table below, the figures in which have been derived from the table for the function $F(\delta, \varepsilon)$ given in 5.4.2.

$\varepsilon\rightarrow$ $\delta\downarrow$	0	0.05	0.1	0.15	0.20	0.25	0.30	0.35	0.40	0.45
0.1	0.54	0.54	0.55	0.55	0.56	0.57	0.59	0.61	0.67	1.09
0.2	0.44	0.44	0.45	0.46	0.47	0.49	0.52	0.59	0.89	
0.3	0.37	0.37	0.38	0.39	0.41	0.43	0.50	0.74		
0.4	0.31	0.31	0.32	0.34	0.36	0.42	0.62			
0.5	0.25	0.26	0.27	0.29	0.34	0.51				
0.6	0.21	0.21	0.23	0.27	0.41					
0.7	0.16	0.17	0.20	0.32						
0.8	0.11	0.13	0.22							
0.9	0.06	0.12								

5.4.4. THE INFLUENCE OF HOMOGENEOUS ANISOTROPY OF THE AQUIFER

If a linear well, placed in a leaky artesian aquifer is being considered, the anisotropic stratum is assumed to have a coefficient of horizontal permeability k and of vertical permeability k_z.

(A linear sink placed in a confined anisotropic aquifer can be similarly traited.)

Using the same notations as in 5.4.3. the Darcy equations are found to be

$$v_r = -k \frac{\partial \varphi}{\partial r} \qquad v_z = -k_z \frac{\partial \varphi}{dz} \quad .$$

and the equation of continuity is

$$\frac{1}{r} \frac{\partial}{\partial r} (r\, v_r) + \frac{dv_z}{\partial z} = 0\,.$$

From this it follows that the differential equation in φ is

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{k_z}{k} \frac{\partial^2 \varphi}{\partial z^2} = 0$$

with boundary conditions

$$\text{for}\quad z=0\qquad v_z=0\quad\text{or}\quad \frac{\partial\varphi}{\partial z}=0\,,$$

$$\text{for}\quad z=H\qquad v_z=\frac{\varphi}{c}\quad\text{or}\quad \frac{\partial\varphi}{\partial z}+\frac{1}{k_z c}\varphi=0\,,$$

$$\text{for}\quad r=\infty\qquad \varphi=0\,,$$

$$\text{for}\quad r\to 0\qquad \lim_{r\to 0} 2\pi r v_r=\begin{cases}0 & \text{for } 0\leqslant z<a\,,\\ \dfrac{Q_w}{L} & \text{for } a<z<b\,,\\ 0 & \text{for } b<z\leqslant H\,,\end{cases}$$

or

$$\lim_{r\to 0}\left(-r\frac{\partial\varphi}{\partial r}\right)=\begin{cases}0 & \text{for } 0\leqslant z<a\,,\\ \dfrac{Q_w}{2\pi kL} & \text{for } a<z<b\,,\\ 0 & \text{for } b<z\leqslant H\,.\end{cases}$$

New variables may be introduced, such as

$$\bar r=\sqrt{\frac{k_z}{k}}\,r\,,\qquad \bar v_r=\sqrt{\frac{k}{k_z}}\,v_z\,,\qquad \bar v_z=\frac{k}{k_z}\,v_z\,,\qquad \bar c=\frac{k_z}{k}\,c\,.$$

The equations given above then change into

$$\bar v_r=-k\frac{\partial\varphi}{d\bar r}\,,\quad \bar v_z=-k\frac{\partial\varphi}{\partial z}\,,\quad \frac{\partial^2\varphi}{\partial\bar r^2}+\frac{1}{\bar r}\frac{\partial\varphi}{\partial\bar r}+\frac{\partial^2\varphi}{dz^2}=0$$

with boundary conditions:

$$z=0\quad :\quad \bar v_z=0\quad\text{or}\quad \frac{\partial\varphi}{\partial z}=0\,,$$

$$z=H\quad :\quad \bar v_z=\frac{\varphi}{\bar c}\quad\text{or}\quad \frac{\partial\varphi}{\partial z}+\frac{1}{k\bar c}\varphi=0\,,$$

$$\bar r=\infty\quad :\quad \varphi=0\,,$$

$$\bar r=0\quad :\quad \lim_{\bar r\to 0} 2\pi\,\bar r\,\bar v_r=\begin{cases}0 & \text{for } 0\leqslant z<a\,,\\ \dfrac{Q_w}{L} & \text{for } a<z<b\,,\\ 0 & \text{for } b<z\leqslant H\,,\end{cases}$$

or

$$\lim_{\bar r\to 0}\left(-\bar r\frac{\partial\varphi}{\partial\bar r}\right)=\begin{cases}0 & \text{for } 0\leqslant z<a\,,\\ \dfrac{Q_w}{2\pi k\,L} & \text{for } a<z<b\,,\\ 0 & \text{for } b<z\leqslant H\,.\end{cases}$$

5.4.4.

Now in the new variables the same equations appear as applied to an isotropic aquifer. So it may be concluded that all the results reached in the foregoing paragraphs remain applicable if expressed in these new variables.

For instance

$$\varphi(\bar{r}, z) = \frac{Q_W}{2\pi kH}\left\{K_0\left(\frac{\bar{r}}{\bar{\lambda}}\right) + \frac{2H}{\pi L}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\frac{n\pi b}{H} - \sin\frac{n\pi a}{H}\right)\cdot\cos\frac{n\pi z}{H}\cdot K_0\left(\frac{n\pi\bar{r}}{H}\right)\right\}$$

$$\text{where } \bar{\lambda} = \sqrt{k\bar{c}H} = \sqrt{\frac{k_z}{k}}\lambda .$$

In the original variables this can be expressed as

$$\varphi(r,z) = \frac{Q_W}{2\pi kH}\left\{K_0\left(\frac{r}{\lambda}\right) + \frac{2H}{\pi L}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\frac{n\pi b}{H} - \sin\frac{n\pi a}{H}\right)\cdot\cos\frac{n\pi z}{H}\cdot K_0\left(\frac{n\pi r}{H}\cdot\sqrt{\frac{k_z}{k}}\right)\right\} \quad (1)$$

and

$$\varphi_{W\,part} - \varphi_{W\,fully} = \frac{Q_W}{2\pi kH}\cdot\frac{1-\delta}{\delta}\left\{\ln\frac{4H}{\bar{r}_W} - F(\delta,\varepsilon)\right\} =$$

$$= \frac{Q_W}{2\pi kH}\frac{1-\delta}{\delta}\left\{\ln\frac{4H}{r_W} + \tfrac{1}{2}\ln\frac{k}{k_z} - F(\delta,\varepsilon)\right\} \quad (2)$$

The conditions of validity of (1) and (2) are that for equation (1)

$$\bar{\lambda} \gg H \quad \text{or} \quad \lambda \gg \sqrt{\frac{k}{k_z}}H \quad (3)$$

(if $k_z < k$ this is a more stringent condition than in the case of an isotropic aquifer; see also discussion in 5.3.1.1.) and for equation (2) condition (3) also has to be satisfied, moreover

$$\bar{r}_W \ll L \quad \text{or} \quad r_W \ll \sqrt{\frac{k}{k_z}}L \quad (4)$$

(if $k_z < k$ this condition is less stringent than in the case of an isotropic aquifer).

At a distance R from the well in a confined aquifer, the condition

$$R > 2\sqrt{\frac{k_z}{k}}H \quad (5)$$

has to be fulfilled if the simplified formulas are to be applicable. If $k_r < k$ this condition is more stringent than the condition for an isotropic aquifer.

If conditions (3) and (4) together and (5) and (4) together are satisfied, formula (2) gives the extra loss of head caused by the partial penetration of an anisotropic aquifer by a well. The term

$$\frac{1}{2}\ln\frac{k}{k_z}$$

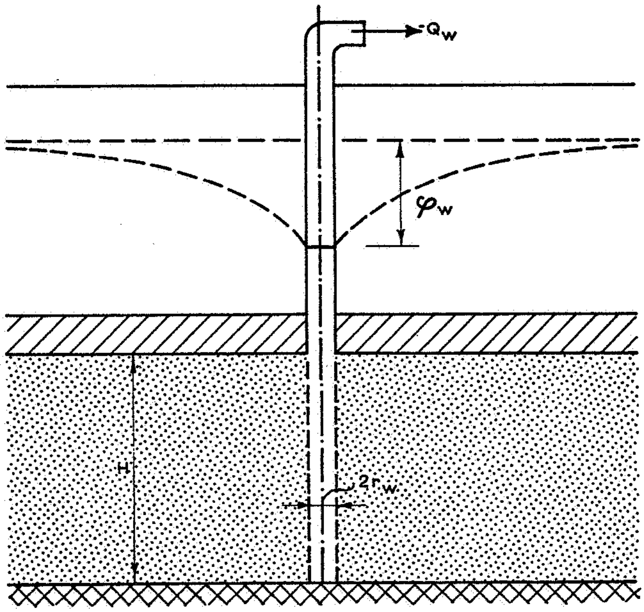


Fig. 5.5.–1

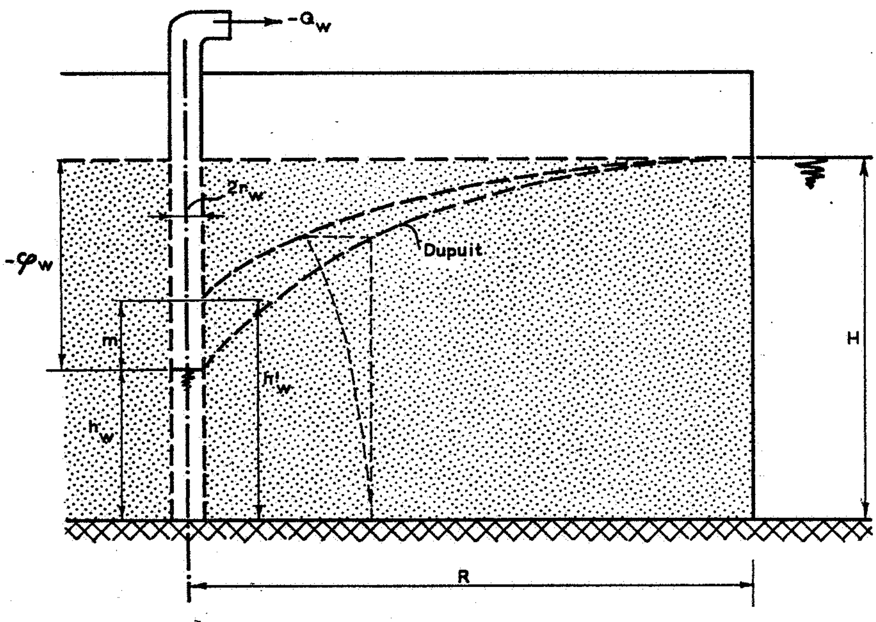


Fig. 5.5.–2

expresses the influence of anisotropy. If $k_z < k$ the extra loss of head is greater than when $k_z = k$ (isotropy), because of the greater resistance to vertical flow.

Generally the other terms of (2) are of more importance; the influence of anisotropy therefore is only important when k_z is much smaller than k.

Remark

If (in an extreme case) $k_z = 0$ conditions (3), (4) and (5) cannot be satisfied. Vertical flow has become impossible and the aquifer has become confined, having a thickness equal to the length of the well.

5.5. Maximum yield of a well

The maximum yield of a well is determined primarily by the largest possible drawdown at the well screen. For a fully penetrating well in confined ground water (see fig. 5.5.–1) the drawdown at the well screen can be expressed

$$\varphi_W = -\frac{Q_W}{2\pi kH} \ln \frac{R}{r_W} \qquad (1)$$

in which $-Q_W$ = the yield of the well,
r_W = the outer radius of the well screen,
kH and R are geo-hydrological constants.

As is evident from the formula the yield Q_W in a given geo-hydrological situation can be increased by raising the drawdown φ_W and by increasing the diameter of the well. The yield is proportional to φ_W; however, it increases but slightly as the radius r_W is increased (because r_W is after the log. symbol).

If the top of the (confined or semi-confined) aquifer lies far below the piezometric level, the maximum yield of a well of a given diameter is not determined by the largest possible lowering of the potential at the well face, but by the extent to which it is technically feasible to abstract ground water from a great depth. In some cases of ground-water abstraction from consolidated rock formations drawdowns in the wells of scores of metres have occurred.

It should be emphasised that φ_W is the drawdown at the outer face of the well screen.

The drawdown inside the well is larger in consequence of the resistance of the screen and as a result of the loss due to friction caused by the upward flow in the well.

The frictional resistance is proportional to the square of the yield of the well. In consequence of the gradual increase in roughness of the well casing due to incrustation, etc., the frictional resistance will increase in course of time. The resistance of the

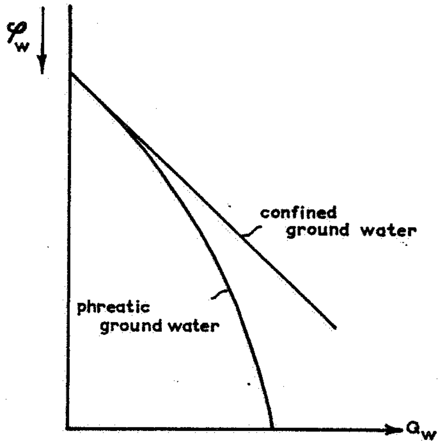


Fig. 5.5.–3

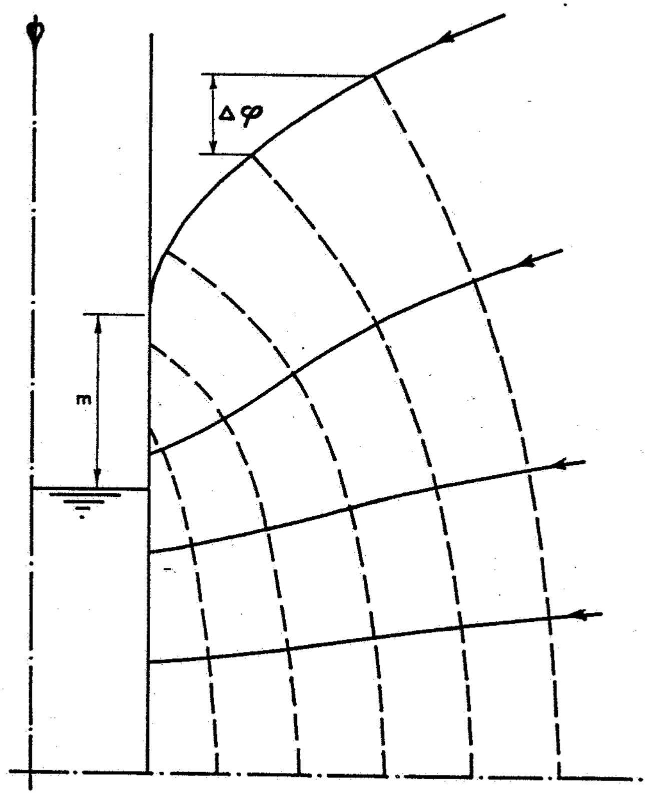


Fig. 5.5.–4

screen is proportional to the yield of the well to some power between 1 and 2. In a well screen of good construction this resistance will be negligible at first. As time goes by, however, it may reach a high figure, due to clogging of the well screen.

For a well in phreatic ground water (as well as for a well in confined ground water provided the water table in that well is drawn down below the top of the aquifer) the formula of Dupuit is applicable (see fig. 5.5.–2). This formula is (see 5.2.1.).

$$H^2 - h^2_W = \frac{Q_W}{\pi k} \ln \frac{R}{r_W} \tag{2}$$

in which h_W is the height of the water table at the outer face of the well screen, Q_W the yield of the well, r_W the outer radius of the well screen and k, H and R geo-hydrological constants.

Under given geo-hydrological conditions the yield Q_W can be increased again by increasing the drawdown and by raising the well's diameter. However, the increase in Q_W by increasing the drawdown at the well face is much smaller for this case than for that of confined ground water (see fig. 5.5.–3).

If the depth of the impervious base below the ground-water table is not too great, the well can be pumped quite dry. Then $h_W = 0$ and the maximum yield is expressed by the formula

$$Q_{W\,max} = \frac{\pi k H^2}{\ln \dfrac{R}{r_W}} \tag{3}$$

It should be borne in mind (see 5.3.1.2.) that formula (2) may not be used for computation of the drawdown at the well face. This formula is based on the assumption that the streamlines are parallel. In the vicinity of a well in phreatic ground water, however, only the streamlines immediately above the impervious base of the aquifer are parallel. Higher up the streamlines are markedly curved. Therefore, the formula of Dupuit only gives correct values for the piezometric level of the ground water along this impervious base and (provided the resistance of the screen and the frictional losses in the well are not taken into account) the correct water table in the well itself.

In the vicinity of the well the phreatic level in the aquifer is higher than the water level in the well itself. So there is a seepage zone on the face of the well (*m* in fig. 5.5.–2). The existence of such a zone can be proved with the aid of the potential theory. In the absence of a seepage zone the phreatic level and the vertical outer face of the well screen, should meet each-other at an angle of 90°, the former being a streamline and the latter an equipotential line.

It appears, however, that without a seepage zone this condition can not be fulfilled;

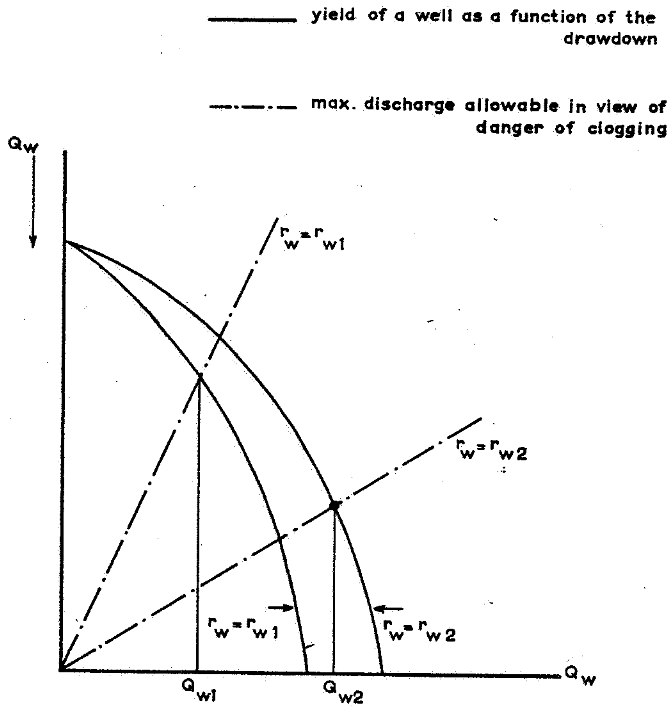

yield of a well as a function of the drawdown
max. discharge allowable in view of danger of clogging
Qw
rw = rw1
rw = rw2
rw = rw1
rw = rw2
Qw1
Qw2
Qw

Fig. 5.5.–5

because it is then impossible to construct a network of streamlines and equipotential lines which satisfies the requirements of the potential theory.

If, however, a seepage zone is supposed to exist, the network of streamlines and equipotential lines satisfying the potential theory is easy to construct (see fig. 5.5.-4).

With the help of the potential theory it can also be proved that the phreatic level must be tangential to the seepage face.

It should be noted that although formula (2) does not give correct values for the drawdown at the well face, formula (3) gives a good approximation of the maximum yield of the well.

The maximum yield of a well will be limited in practise not only by the greatest possible drawdown of the potential, but also by the desirability of preventing the well screen from clogging. Greater withdrawal from a well implies a greater velocity of the ground-water flow towards it and an increase in the risk of fine particles moving into the direction of the well screen.

An other danger is, that if the drawdown of the potential is excessif, the chemical equilibrium of the ground water will be upset, and deflocculation of lime and iron will occur.

The greatest velocity which can be permitted without the risk of transporting fine soil particles, depends on the distribution and the granular structure of the soil, the chemical properties of the ground water etc. A precise calculation of this velocity cannot be determinated accurately. GROSS worked out some figures for wells used for the winning of drinking water.

The following formula can be derived from these figures, taken from actual examples

$$v_{max} = 170\, d_{40} \tag{5}$$

in which, v_{max} = the maximum permissable velocity according to Darcy outside the outer wall of the well screen in m/day and d_{40} the grain size in mm, which will not be attained by 40% of the soil particles.

SICHARDT gives the following formula for temporary wells used for a temporary drawdown of the ground-water table

$$v_{max} < 20\sqrt{k}. \tag{6}$$

For permanent wells for the winning of drinking water SICHARDT gives

$$v_{max} < 10\sqrt{k}. \tag{7}$$

In both formulas (6) and (7) v_{max} and k are expressed in m/day.

Introducing the relation given by ALLAN HAZEN (see 3.10.), $k = (400 \text{ a } 1200)\, d_{10}^2$, we find for permanent wells

$$v_{max} = (200 \text{ a } 350)\, d_{10}.$$

Here d_{10} is the 10% grain size in mm. v_{max} is expressed in m/day.

This formula agrees well with the formula of GROSS.

Experience in the Netherlands, however, has taught that the maximum velocities arrived at when using this formula, are still to high and lead to rapid clogging.

The maximum permissible velocity should not exceed half of the value calculated with the formulas of GROSS and SICHARDT.

At the beginning of this chapter it was stated that for a given drawdown of the water table in the well, the yield can be increased slightly by increasing the well radius; this applies both to confined ground water and phreatic ground water. If, however, the risk of clogging is taken into account in the case of confined ground water, the maximum permissable yield is directly proportional to the well radius.

In the case of phreatic water the increase of the permissable yield of the well with increasing well radius is somewhat smaller, in consequence of the decrease of the saturaded thickness of the aquifer in the vicinity of the well for greater withdrawals (see fig. 5.5.–5).

6. NON-CYLINDER-SYMMETRICAL FLOWS COMPOSITE FLOW PATTERNS

6.1. General equations and methods

In this chapter a number of more complicated cases of flow are considered, occurring when the well is not in the centre of a circular area, or when there are several wells operating simultaneously. In such cases we generally have to replace the ordinary differential equations used in 5.1., by partial differential equations (see 6.1.1.).

In most cases, however, it is possible, to built up the potential field from the simple radially-symmetrical potential fields of 5.1., by applying the method of superposition (see 6.1.2.) and the method of images (see 6.1.3.).

We do not propose to give a complete account of every possible situation. In 6.2. and 6.3. we only consider a number of special cases, but many other cases can be solved by the same methods.

Furthermore, only fully penetrating wells in an aquifer with isotropic permeability are dealt with. With the help of the considerations of 5.3. and 5.4. corrections can easily be made for the influence of partial penetration (which is generally of importance only in a small area around the well) and of anisotropy or stratification of the aquifer by referring to 5.3. and 5.4.

It is shown that the case of phreatic water can be expressed by equations of the same type as those describing the case of completely confined ground water. Consequently in 6.2. and 6.3. we therefore consider the cases of completely and semi-confined ground water only.

6.1.1. General equations

As a rule when the flow in an aquifer is not exactly radial (as a result of the presence of more than one well or of boundaries other than a circular canal concentric with the well) two horizontal co-ordinates are needed to express the velocity field and the potential field.

a. In the case of completely confined ground water between parallel confining layers the flow is exactly horizontal (at least in the case of fully penetrating wells, or at some distance from partially penetrating wells).

Here only the two horizontal components v_x and v_y of the Darcy velocity are of importance.

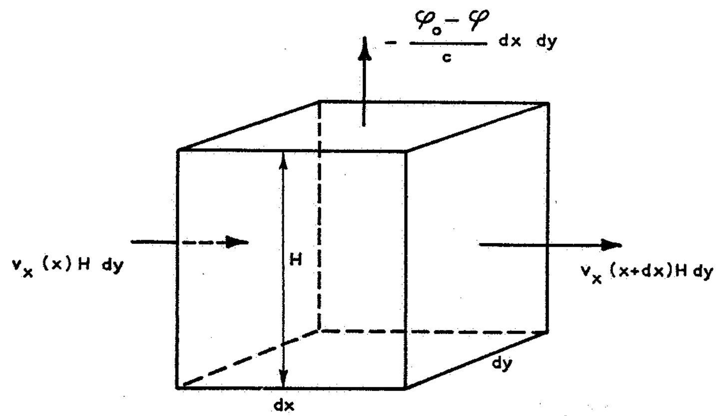


Fig. 6.1.1.–1

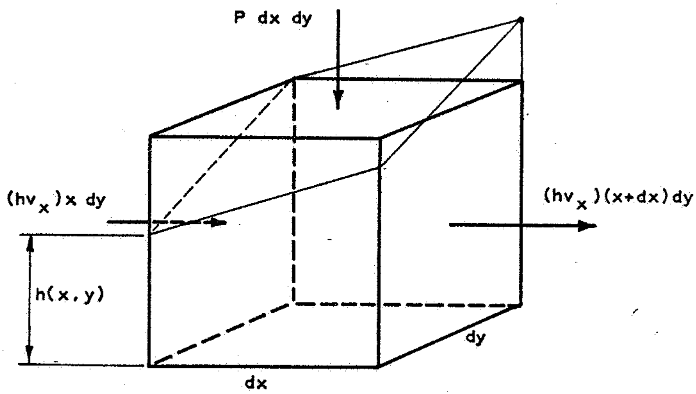


Fig. 6.1.1.–2

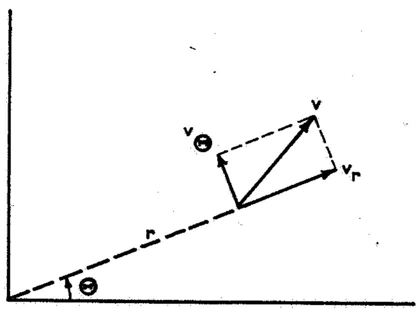


Fig. 6.1.1.–3

We then have the equation of continuity

$$\frac{\partial v_x}{\partial_x} + \frac{\partial v_y}{\partial_y} = 0 \qquad (1a)$$

and Darcy's Law

$$v_x = -k\frac{\partial \varphi}{\partial x}, \; v_y = -k\frac{\partial \varphi}{\partial y}. \qquad (2a)$$

The combination of (1a) and (2a) gives

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\varphi^2 \partial}{\partial y^2} = 0. \qquad (3a)$$

b. In the case of semi-confined ground water (between a lower horizontal impermeable layer and an upper semi-impermeable layer parallel to it) we shall always suppose the leakage factor $\lambda = \sqrt{kHc}$ (see 5.1.2.) to be large with respect to the thickness H of the aquifer. In that case the potential and the horizontal velocity components may (in a good approximation) be regarded as being independent of the vertical component z (see 5.3.1.1.; the discussions given there can be taken as generalisations).

Here the equation of continuity (1a) has to be modified because of leakage through the semi-permeable layer. Supposing the leakage to be proportional to the drop in potential, having regard to the net flow through a parallelepiped with bottom area $dx\, dy$ (fig. 6.1.1.–1) we get

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} - \frac{\varphi_0 - \varphi}{cH} = 0. \qquad (1b)$$

In this equation $\varphi_0(x, y)$ is the potential above the semi-permeable layer. Darcy's Law again is

$$v_x = -k\frac{\partial \varphi}{\partial x}, \; v_y = -k\frac{\partial \varphi}{\partial y}. \qquad (2b)$$

If $\varphi_0 = 0$, which we usually assume to be the case in the following considerations, the combination of (1b) and (2b) gives us

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} - \frac{1}{\lambda^2} \cdot \varphi = 0 \qquad (3b)$$

c. In the case of phreatic water above a horizontal impermeable layer we also suppose that the potential and the horizontal velocity components are approximately independent on the vertical co-ordinate. A criterion for this is that the gradient of the phreatic plane must be small (see 5.3.2.).

A consideration of an elementary parallelepiped (fig. 6.1.1.–2) gives us the following equation of continuity

$$\frac{\partial}{\partial_x}(hv_x) + \frac{\partial}{\partial y}(hv_y) - P = 0, \qquad (1c)$$

in which P means the average rainfall, and $h(x, y)$ the height of the phreatic plane above the impermeable layer. From the above assumptions it follows that we have for the potential

$$\varphi(x, y) = h(x, y).$$

Therefore Darcy's Law for this case reads

$$v_x = -k\frac{\partial h}{\partial x}, \quad v_y = -k\frac{\partial h}{\partial y}. \tag{2c}$$

From (1c) and (2c) we obtain

$$\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right) = -\frac{P}{k}$$

or

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}(\tfrac{1}{2}h^2) = -\frac{P}{k}. \tag{3c}$$

Apart from the right-hand member this equation is identical with (3a), if in the latter we replace φ by $\frac{1}{2}h^2$.

Instead of the equations (1), (2) and (3) in rectangular co-ordinates we can also use the corresponding formulas in plane polar co-ordinates r and Θ. The velocity components will then be designated by v_r and v_Θ (fig. 6.1.1.– 3). Darcy's Law becomes

$$v_r = -k\frac{\partial \varphi}{\partial r}, \quad v_\Theta = -k\frac{1}{r}\frac{\partial \varphi}{\partial \Theta}.$$

Instead of (3b) we find

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \Theta^2} - \frac{1}{\lambda^2}\cdot\varphi = 0 \text{ (see 3.11.).} \tag{3b*}$$

The other equations are analogous.

It is also useful to consider expressions for the total discharge across a "vertical surface" (a cylinder with vertical rulings).

In case of confined ground water we find for the total discharge through the vertical cylinder intersecting the $x\,y$ plane along a curve C between the points A and B (fig. 6.1.1.–4)

$$Q_{AB} = H\int_A^B v_n ds = -kH\int_A^B \frac{\partial \varphi}{\partial n} ds \tag{4a}$$

In this formula v_n is the component of the Darcy velocity in the direction of the normal n of the curve, and $\frac{\partial\varphi}{\partial n}$ is the partial derivative of φ in this direction.

In case of phreatic water we have according to (2c)

$$Q_{AB} = \int_A^B h\, v_n\, ds = -k \int_A^B h \frac{\partial h}{\partial n} ds = -k \int_A^B \frac{\partial}{\partial n} (\tfrac{1}{2}h^2) ds \tag{4c}$$

In addition to the differential equations there are boundary conditions. In the following the vertical boundaries of the aquifer are assumed to be of the same type throughout their height (they are not partially penetrating wells nor partially penetrating ditches, etc.).

In the case of completely confined ground water the most important boundary condition is one that gives the potential along a vertical boundary of the aquifer. Since as a rule the potential will be the same along connected parts of the boundary (see fig. 6.1.1.–5: top view of an aquifer with differing boundary conditions) it follows that $\varphi = \varphi_0$.

When the aquifer is bounded by a vertical impermeable wall, the boundary condition is

$$v_n = 0,$$

v_n being the component of the Darcy velocity normal to the boundary. As a result of Darcy's Law this can also be written as

$$\frac{\partial\varphi}{\partial n} = 0$$

(derivative of φ in the direction n perpendicular to the wall).

A further possibility is that the potential of the well is not given but its total discharge Q_W. In such a case the potential has the same value all along the cylindrical face of the well (along this vertical boundary the aquifer is in contact with open water), but this value is not known (and as a rule it should not be given together with Q_W).

In the case of phreatic water analogous boundary conditions may occur. Along a vertical boundary at which the aquifer is in free contact with open water of head H we have the boundary condition

$$h = H, \text{ or } \tfrac{1}{2}h^2 = \tfrac{1}{2}H^2$$

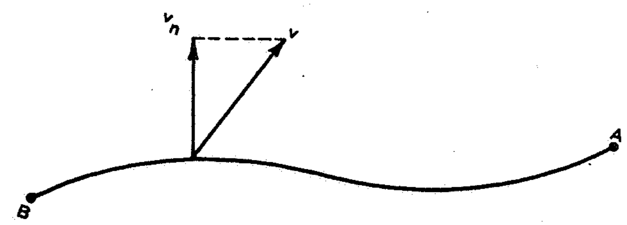


Fig. **6.1.1.–4**

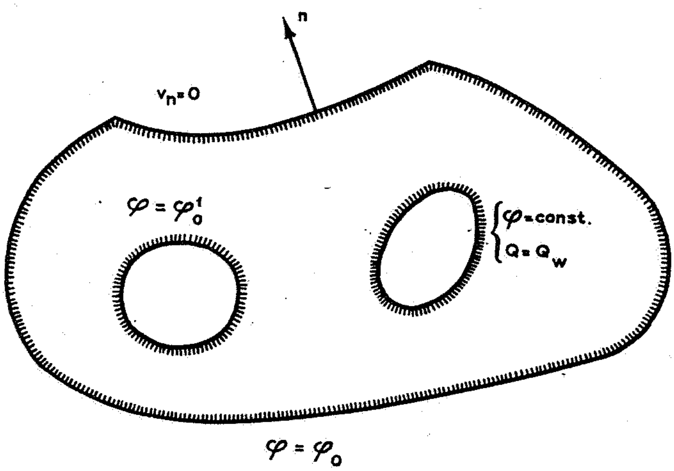


Fig. **6.1.1.–5**

(this also applies when we make allowance for a seepage surface, provided we bear in mind that the function $\frac{1}{2}h^2(x, y)$ gives a poor approximation for the height of the phreatic surface in the neighbourhood of the wall (see 5.3.1.3.)).

For an impermeable wall we have $v_n = 0$, or because of Darcy's Law either $\frac{\partial h}{\partial n} = 0$, or $\frac{\partial}{\partial n}(\frac{1}{2}h^2) = 0$.

Again the total discharge of a well may be given, while the height of the phreatic surface at the well face may be constant, but not given.

From the preceding account it is evident that the case of phreatic water (with $P=0$) is mathematically equivalent to that of completely confined ground water when we take the term $\frac{1}{2}h^2$ to be a "potential". This also applies to the computations of the discharge, provided that in this case we replace φH (not φ) by $\frac{1}{2}h^2$ (see (4a) and (4c)). It is for this reason that in the following account we shall mostly consider the cases of completely and semi-confined ground water only.

6.1.2. Superposition

The general equations (3), which were formulated in 6.1.1. are linear in the functions φ and $\frac{1}{2}h^2$ respectively. Moreover equations (3a) and (3b) are homogeneous in φ (the equation does not contain a known term); (3c) is not homogeneous, unless $P = 0$. The same thing applies to the boundary conditions, which are also linear and may or may not be homogeneous.

Now obviously, when two functions φ_1 and φ_2 satisfy the same homogeneous linear equation, the latter is also satisfied by $c_1\varphi_1 + c_2\varphi_2$, in which c_1 and c_2 are arbitrary constants. Moreover, when φ_0 is a solution of a non-homogeneous equation and φ_1 a solution of the corresponding homogeneous equation, the non-homogeneous equation is also satisfied by $\varphi_0 + c\varphi_1$, in which c is an arbitrary constant.

These properties of linear equations enable fields of flow to be superposed.

If for instance we have a non-homogeneous differential equation and several non-homogeneous boundary conditions, we may proceed as follows. First we solve the solution φ_1 in the non-homogeneous differential equation, supposing all boundary conditions to be homogeneous. Next we solve the solution φ_2 in the homogeneous differential equation, supposing one boundary condition to be non-homogeneous and all the others homogeneous, etc. The addition of $\varphi_1, \varphi_2, \ldots$ then yields the desired result.

If for instance we have a number of wells with given discharges in an aquifer, bounded by a wall where the potential is equal to zero, we can determine successively the potential fields for the cases in which all wells except one have been stopped, and then superpose these fields.

If the potentials should be given instead of the discharges, we may proceed in an analogous manner.

The superposition method is valuable mainly because in practise the radii of the wells are always small as compared with the other horizontal lengths, such as the distances between the wells and the distances from the boundaries of the aquifer. If this were not the case, the flow field of a well with discharge Q_1 would be influenced by the presence of a neighbouring well with discharge Q_2, even if we take $Q_2 = 0$ (homogeneous boundary condition), because the potential must be constant at the face of the second well. In such a case computation of the potential field of the first well is considerably complicated by the presence of the second well.

A different situation obtains when the distance between the wells is large. Here two cases must be distinguished; in one the potentials at the faces of the wells are known (and constant), while in the other the total discharge per well is given. We shall confine ourselves in what follows to aquifers bounded by vertical boundaries where the potential is equal to zero. The treatment may be generalized in a simple way to make it applicable to cases in which either the potential or the flow at the vertical boundary of the aquifer is a known function. However, since such a case seldom occurs in practice, this generalization will not be given.

Let us consider two wells and compute the potential field of each well separately, assuming the other well to be absent. Now if we find that the effect of one well at the location of the other well is negligible in comparison with the given potential at the face of the latter, we may add the two fields together, the sum being a very good approximation to the composite field created by the two wells.

When the aquifer contains several wells, the individual fields may be added together, provided that at the location of each well the sum of the fields of the other wells is negligible. In some cases this latter sum may not be neglected, although the field of each of the wells considered separately may be negligible.

If it turns out that the field of the one well at the location of the other is not negligible, we can find the composite field of the wells by means of a mathematical trick.

Let φ_1^w and φ_2^w be the respective potentials at the faces of two wells, the centres of which are P_1 and P_2 respectively. Let $\varphi_1(P)$ and $\varphi_2(P)$ be the potentials at a point P if only the first or the second of the wells is present respectively. We now assume that

6.1.2.

the potential $\varphi(P)$ at P which results from the presence of both wells may be expressed as

$$\varphi(P) = A_1\, \varphi_1(P) + A_2\, \varphi_2(P). \tag{5}$$

Now consider a point P on the face of the first well. For this point we have

$$\varphi(P) = \varphi_1(P) = \varphi_1{}^{w}.$$

According the assumption mentioned before, the field of the second well on the face the first one is constant, and we may attribute to this field the value it has in the centre of the first well. This gives

$$\varphi_1{}^{w} = A_1\, \varphi_1{}^{w} + A_2\, \varphi_2(P_1)$$

and in an analogous manner

$$\varphi_2{}^{w} = A_1\, \varphi_1(P_2) + A_2\, \varphi_2{}^{w}.$$

Solving A_1 and A_2 from these two equations and substituting the values thus found in (5), we get

$$\varphi(P) = \frac{[\varphi_1{}^{w} - \varphi_2(P_1)]\, \varphi_2{}^{w}\, \varphi_1(P) + [\varphi_2{}^{w} - \varphi_1(P_2)]\, \varphi_1{}^{w}\, \varphi_2(P)}{\varphi_1{}^{w}\, \varphi_2{}^{w} - \varphi_1(P_2)\, \varphi_2(P_1)} \tag{6}$$

We can find the composite field by using this formula if the fields of the separate wells are known.

Although we have assumed that $\varphi_1(P_2)$ is not negligible compared to $\varphi_2{}^{w}$, nor $\varphi_2(P_1)$ compared to $\varphi_1{}^{w}$, the product $\varphi_1(P_2)\, \varphi_2(P_1)$ will usually be negligible compared to the product $\varphi_1{}^{w}\, \varphi_2{}^{w}$. This gives us the simplified formula

$$\varphi(P) = \left[1 - \frac{\varphi_2(P_1)}{\varphi_1{}^{w}}\right] \varphi_1(P) + \left[1 - \frac{\varphi_1(P_2)}{\varphi_2{}^{w}}\right] \varphi_2(P)\cdot \tag{6a}$$

From (6) or (6a) it is evident that the superposition method may still be applied to fields in which the assumptions made before still hold good, provided that the fields computed for each well separately are multiplied by a suitably chosen factor.

The foregoing reasoning may be generalized obviously for the case of more than two wells. In such a case we must assume that along the face of each well the sum of the fields of the other wells is constant. Here again cases may occur in which this assumption is not satisfied, although it does hold for the field of each of the other wells separately.

When the total discharge per well is given the superposition method may be applied without correction, provided that along the face of each well the sum of the fields of the other wells may be regarded as constant. In this case the total flow towards any one well will not be influenced by the potential field of the others. For completely

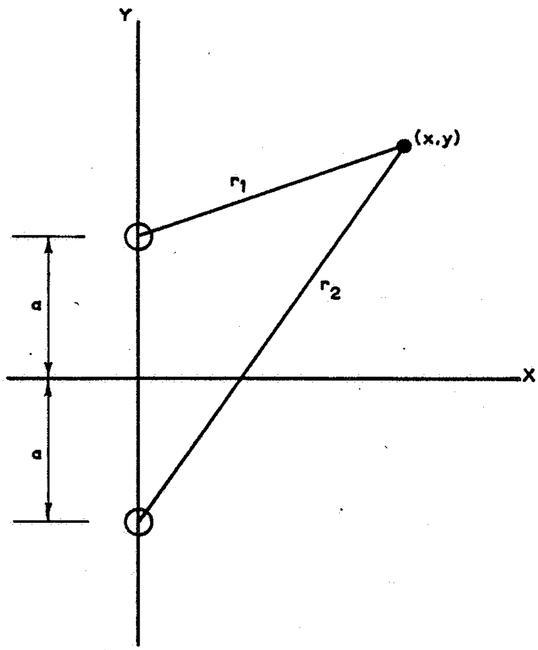


Fig. **6.1.2.–1**

confined ground water (and thence also for phreatic water) this follows immediately from the law of continuity: no influx occurs at the face of the well if the well is stopped.

For semi-confined ground water this inflow is not exactly zero, but equal to the amount of water that whould seep through the part of the semi-permeable layer overlying the well. Of course this amount is very small.

Example Two wells in an infinite aquifer with semi-confined ground water

We assume the axes of the wells to have horizontal co-ordinates $(0, -a)$ and $(0, +a)$ and we assume the radius of the wells to be r_W (fig. 6.1.2.–1).

Let the total discharge of each of the wells be Q_W and the thickness, permeability and leakage factor H, k and λ respectively.

According to 5.1.2.1. (formula (7)) the potential field of the well in $(0, a)$ in the absence of the other one is given by (for $r_W \ll \lambda$)

$$\varphi_1(x, y) = \frac{Q_W}{2\pi kH} \cdot K_0\left(\frac{r_1}{\lambda}\right) \tag{1}$$

in which

$$r_1 = \sqrt{x^2 + (y - a)^2} \tag{2}$$

Also the potential field of the well in $(0, -a)$ in the absence of the other well is

$$\varphi_2(x, y) = \frac{Q_W}{2\pi kH} K_0\left(\frac{r_2}{\lambda}\right) \tag{3}$$

in which

$$r_2 = \sqrt{x^2 + (y + a)^2}. \tag{4}$$

The total potential field is found by superposition to be

$$\varphi(x, y) = \frac{Q_W}{2\pi kH} \left\{ K_0\left(\frac{r_1}{\lambda}\right) + K_0\left(\frac{r_2}{\lambda}\right)\right\}. \tag{5}$$

For this it follows that the potential along the faces of the wells can be very closely approximated as

$$\varphi_W = \frac{Q_W}{2\pi kH}\left\{\ln \frac{1.123\lambda}{r_W} + K_0\left(\frac{2a}{\lambda}\right)\right\}$$

(if $r_W \ll a$ and $r_W \ll \lambda$, see 5.1.2.1. formula (8)), since, if the point (x, y) is situated for instance on the face of the well in $(0, a)$, the distance r_2 from $(x\ y)$, to the axis of the other well varies very little, and practically equals $2a$ (the distance between the axes of the wells).

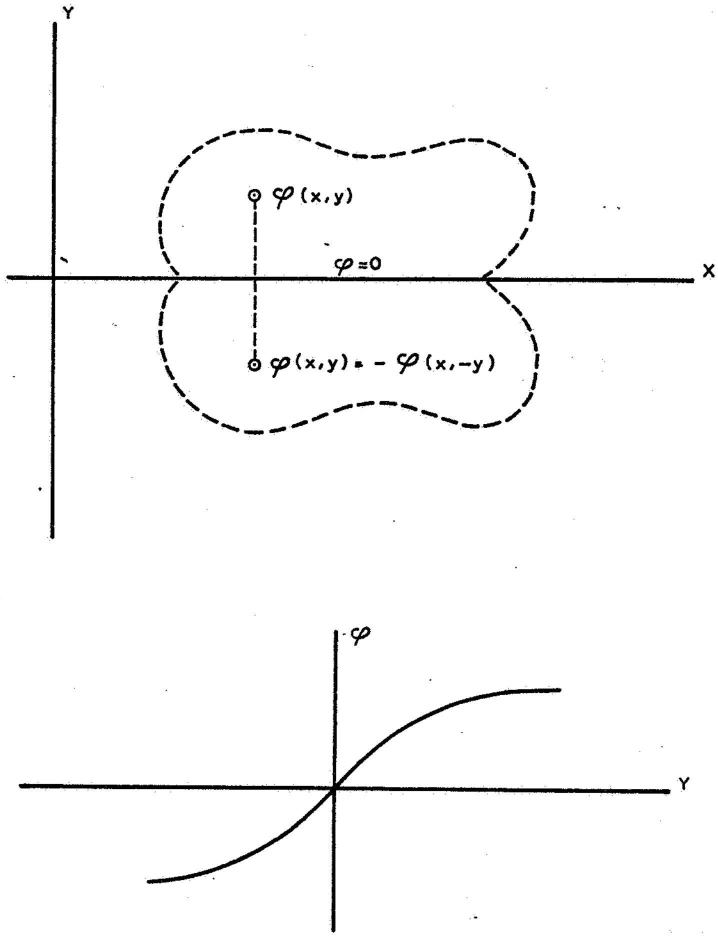


Fig. 6.1.3.–1

Remark

In formula (5) the potential φ is given as the sum of two potentials φ_1 and φ_2, the first of which is given in polar co-ordinates with (0, a) as origin, and the second in polar co-ordinates with (0, $-a$) as origin. In order to verify that φ really satisfies the differential equation (3b) of 6.1.1., we might for instance replace r_1 and r_2 by the expressions (2) and (4) respectively thereby obtaining φ as a function of x and y only. However, this is unncessary.

The functions φ_1 and φ_2 respectively satisfy the differential equation, as may be verified by writing for instance them out in polar co-ordinates with (0, a) and (0, $-a$) as origin. The sum of φ_1 and φ_2 also satisfies the differential equation.

6.1.3. THE IMAGE METHOD

If one or more of the boundaries of the aquifer are vertical planes, the image method can often be used. This method is founded on the following principles.

Let $\varphi(x, y)$ for $y > 0$, be a solution of the equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} - \frac{\varphi}{\lambda^2} = 0\,, \tag{1}$$

(in which λ may also be infinite, viz. for fully confined or phreatic ground water), and for $y = 0$ satisfy the boundary condition

$$\varphi(x, 0) = 0. \tag{2}$$

Then the function $\varphi(x, y)$, hitherto only known in the half-plane $y > 0$, can be continued in the other half-plane $y < 0$ by defining it in that area as

$$\varphi(x, y) = -\varphi(x, -y) \tag{3}$$

(if y is negative, $-y$ is positive, consequently the right-hand member of (3) is known).

According to (3) the function $\varphi(x, y)$ extended in this way is what is called an *odd-function* of y (see fig. 6.1.3.–1). For that reason the function $\varphi(x, y)$ in the half-plane $y < 0$ is called an odd image of the function $\varphi(x, y)$ in the plane $y > 0$.

According to (3) the derivates of the function (defined in this way for $y < 0$) are

$$\begin{aligned} \frac{\partial\varphi(x,y)}{\partial x} &= -\frac{\partial\varphi(x,-y)}{\partial x}\,, & \frac{\partial\varphi(x,y)}{\partial y} &= +\frac{\partial\varphi(x,-y)}{\partial y}\,, \\ \frac{\partial^2\varphi(x,y)}{\partial x^2} &= -\frac{\partial^2\varphi(x,y)}{\partial x^2}\,, & \frac{\partial^2\varphi(x,y)}{\partial y^2} &= -\frac{\partial^2\varphi(x,-y)}{\partial y^2}\,. \end{aligned} \tag{4}$$

It follows in the first place that the function $\varphi(x, y)$ satisfies the differential equation

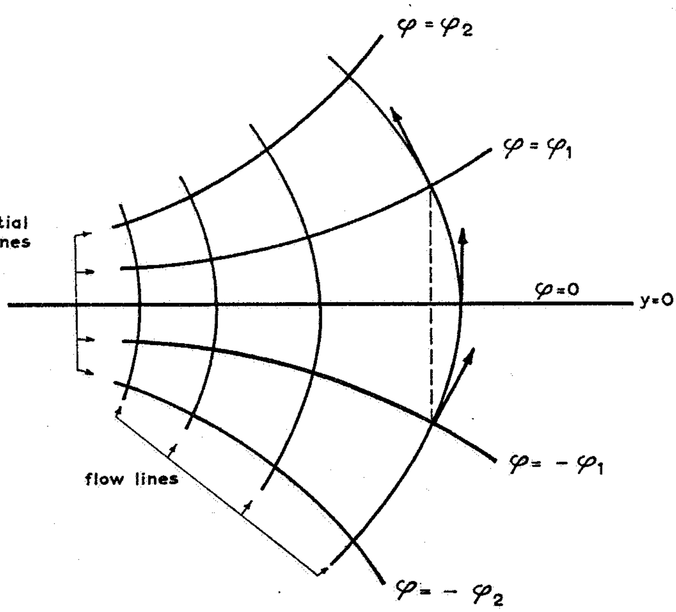


Fig. **6.1.3.–2**

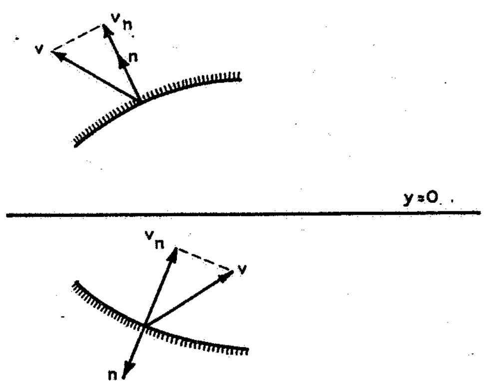


Fig. **6.1.3.–3**

(1) for $y < 0$ too, and that $\varphi, \frac{\partial \varphi}{\partial x}$ and $\frac{\partial \varphi}{\partial y}$ are continuous if $y = 0$ (this holds good for φ and $\frac{\partial \varphi}{\partial x}$ because they are zero). Moreover it is obvious that equation (3), which defines $\varphi(x, y)$ for $y < 0$, also holds good for $y > 0$. For if we substitute $-y'$ for y in (3), we get

$$\varphi(x, y') = -\varphi(x, y'),$$

which is valid if $y' > 0$.

The same thing holds good with regard to formulas (4).

Finally, on combining (4) with Darcy's Law we get

$$v_x(x, y) = -v_x(x, -y)$$

and (5)

$$v_y(x, -y) = v_y(x, -y),$$

which are valid for positive and negative values of y. So the x-component of the velocity changes its sign, the y-component does not. Consequently the flow patterns in both areas are also images of each other (since $v_x' = 0$ if $y = 0$, the flow lines intersect the x-axis at right angles, their course showing no refraction, see fig. 6.1.3.–2).

If the potential φ in the area $y > 0$ satisfies certain boundary conditions (wells, boundaries of the aquifer), it follows from (3) and (4) that the extended function in the area $y < 0$ will satisfy analogous boundary conditions. For instance, a circular well of radius r_w, discharge Q_w, water level φ_w with its centre at x_0, y_0 will correspond to a circular well of radius r_w, discharge $-Q_w$, water level $-\varphi_w$ with its centre at $x_0, -y_0$. Owing to the change of sign of the velocity in the image area, the discharge also changes its sign. The well need not be a circular one; in general the velocity component v_n normal to a boundary plane changes its sign (see fig. 6.1.3.–3).

Let $\varphi(x, y)$ be the potential of the ground water, which is zero for $y = 0$ and which for $y > 0$ satisfies the differential equation (1) and certain boundary conditions. By introducing images of these boundary conditions one can determine the potential distribution in the image area. The boundary condition for $y = 0$ is satisfied automatically [1]) and need not be taken into consideration.

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[1]) This always holds good, because there is only one solution to each of the boundary value problems arising here. Indeed, in principle, the potential that satisfies all boundary conditions in both the area and the image area may be obtained by extending from the original area the potential that satisfies the boundary condition at $y = 0$. However, since the potential is determined unequivocally within the doubled area it must be identical with the one in the half area (though it may have been obtained in a different way), consequently it must also satisfy the boundary condition at $y = 0$.

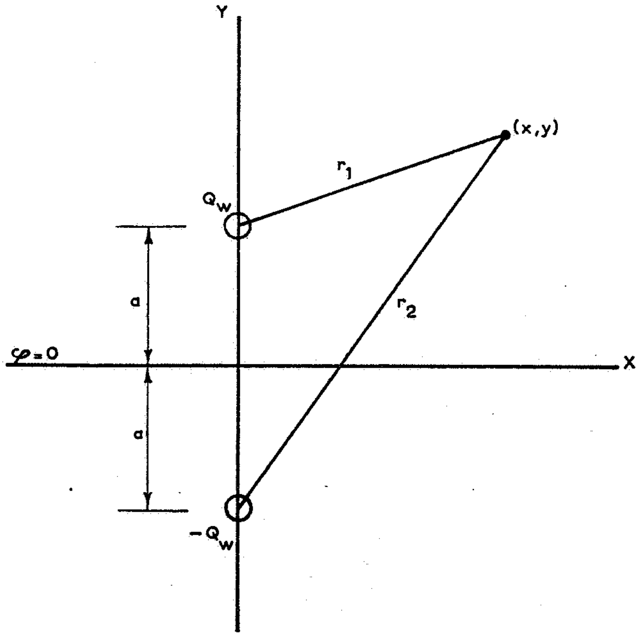


Fig. **6.1.3.–4**

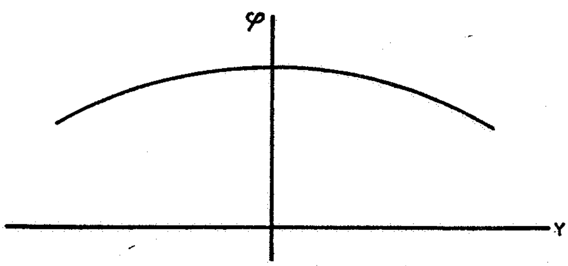


Fig. **6.1.3.–5**

Example Well in a leaky aquifer, bounded by a straight canal (infinite half-plane)

Place the x-axis along the canal, the potential of the canal being zero. Place the axis of the well at the point $(0, a)$. Let the radius of the well be r_W, the total discharge of the well Q_W and the formation constants kH and λ. At infinity the potential is supposed to be zero.

Using the image method, the potential distribution may be found by placing an image well of a total discharge $-Q_W$ at the point $(0, -a)$ (see fig. 6.1.3.-4). By means of formula (7) in 5.1.2.1. we find (superposition method; see 6.1.2.) that

$$\varphi(x, y) = \frac{Q_W}{2\pi kH}\left[K_0\left(\frac{r_1}{\lambda}\right) - K_0\left(\frac{r_2}{\lambda}\right)\right], \tag{6}$$

$$r_1 = \sqrt{x^2 + (y-a)^2}, \qquad r_2 = \sqrt{x^2 + (y+a)^2}.$$

For $y = 0$, r_1 equals r_2 therefore it is obvious that the boundary condition at $y = 0$ will be satisfied. It is evident that

$$\varphi(x, -y) = -\varphi(x, y).$$

Of course the potential is not constant along the face of the wells, but as $r_W \ll 2a$, the deviation will be so small as to be negligible.

Besides the odd images dealt with above there are even images. The latter may be used when dealing with vertical boundaries where $\frac{\partial \varphi}{\partial n} = 0$ (i.e. $v_n = 0$; impermeable barrier).

Placing the x-axis along this boundary, $\varphi(x, y)$ becomes an *even-function* of y if

$$\varphi(x, y) = \varphi(x, -y) \tag{7}$$

(see fig. 6.1.3.–5).

Now obviously

$$\frac{\partial \varphi(x,y)}{\partial x} = \frac{\partial \varphi(x,-y)}{\partial x} \quad \text{and} \quad \frac{\partial \varphi(x,y)}{\partial y} = -\frac{\partial \varphi(x,-y)}{\partial y}. \tag{8}$$

Since $\frac{\partial \varphi}{\partial y} = 0$ if $y = 0$, not only φ and $\frac{\partial \varphi}{\partial x}$ are continuous for $y = 0$, but also $\frac{\partial \varphi}{\partial y}$.

The equations (8) show that the velocities in the image area have the same sign as those in the real area (see fig. 6.1.3.–6). The discharge of an image well also has the same sign as that of the corresponding real well.

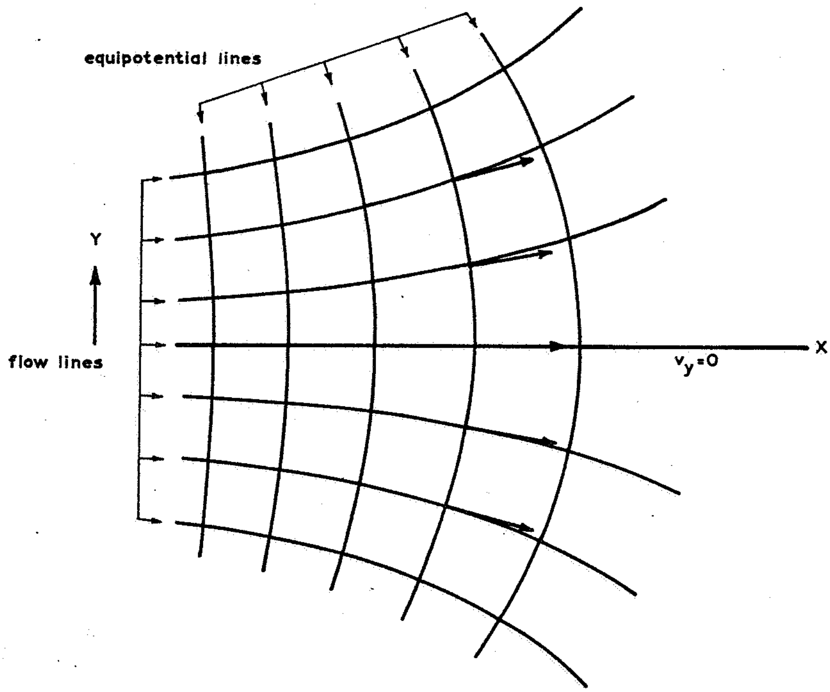


Fig. **6.1.3.–6**

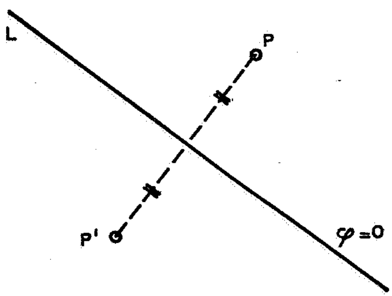


Fig. **6.1.3.–7**

Example Well in a leaky aquifer, bounded by a vertical impermeable barrier

If the canal in the former example is replaced by a vertical impermeable barrier, the potential distribution may be found by the introduction of even images. Obviously, an image well of discharge Q_W must be placed at the point $(0, -a)$. This problem has been dealt with in the example in 6.1.2. It is quite clear that the potential is an even function of y and that $v_y = 0$ if $y = 0$.

Remark 1

Of course we can also use the image method when the vertical boundary of the aquifer does not coincide with the x-axis (see fig. 6.1.3.–7 in which $\varphi = 0$ along the boundary L and therefore the introduction of odd images gives us $\varphi(P) = -\varphi(P')$). The proof is the same as it was for the other cases, because it will always be possible to reduce such a problem to the one dealt with by means of a proper co-ordinate transformation.

Remark 2

When we are dealing with fully confined water or with phreatic water, the potential along the vertical boundary of the aquifer need not be zero; the image method may be used also when along the boundary L the potential φ is a constant $(=\varphi_0)$. It is then known that $\varphi' = \varphi - \varphi_0$ also satisfies the differential equation (1) (this is not true in a leaky aquifer) and satisfies the boundary condition $\varphi' = 0$ along the straight line L.

Using the image method we then get

$$\varphi'(P') = -\varphi'(P)$$

or

$$\varphi(P') = -\varphi(P) + 2\varphi_0.$$

Remark 3

When we are dealing with confined or phreatic water, we may also introduce images when the vertical boundary of the aquifer is a circle. An example of this can be found in 6.2.2.2.

6.2. Well in an aquifer with various boundaries

The case of a well in an aquifer that is either infinite or has a circular boundary concentric with the well has been dealt with in chapter 5. Other cases will now be considered.

As a rule the image method and the superposition method will be used, the general principles of which are explained in 6.1.2. and 6.1.3. Once the flow pattern has been found, it is usually easy to check whether the boundary conditions have been satisfied.

6.2.1. Infinite half-plane with a fixed potential along its boundary (flow from an infinite line source)

6.2.1.1. *Confined ground water*

Let the well be situated at the point $(0, a)$ and let the aquifer $(y > 0)$ be bounded by a canal along the axis $y = 0$ having potential $\varphi = 0$. According to the theory underlying the image method the real flow pattern is similar to the one produced by wells at the points $(0, a)$ and $(0, -a)$ having discharges of Q and $-Q_W$ respectively, (see fig. 6.1.3.–4).

With the solution given in 5.1.1. we get for the potential produced by the well at the point $(0, a)$, the other well being absent,

$$\varphi_1(x, y) = -\frac{Q_W}{2\pi kH} \ln r_1 + C_1 ,$$

in which $r_1 = \sqrt{x^2 + (y - a)^2}$ and C_1 is a constant that has not yet been determined.

In an analogous manner we find that the potential caused by the well at the point $(0, -a)$, the other well being absent, is expressed by

$$\varphi_2(x, y) = \frac{Q_W}{2\pi kH} \ln r_2 + C_2 ,$$

in which $r_2 = \sqrt{x^2 + (y + a)^2}$.

Superposition gives the flow pattern of the two wells together, viz.

$$\varphi(x, y) = \frac{Q_W}{2\pi kH} \ln \frac{r_2}{r_1} + C .$$

The constant C is determined by the condition that $\varphi = 0$ if $y = 0$ (that is: $r_1 = r_2$). Then we see that

$$\varphi(x, y) = \frac{Q_W}{4\pi kH} \ln \frac{x^2 + (y + a)^2}{x^2 + (y - a)^2} . \tag{1}$$

It is easy to understand that at infinity the potential φ approaches zero. Introducing $x = R \cos \Theta$ and $y = R \sin \Theta$ (polar co-ordinates with regard to the axis of the well) equation (1) becomes (for $R \gg a$)

$$\varphi(R, \Theta) = \frac{Q_W}{4\pi kH} \ln \frac{R^2 + 2\,a\,r \sin\Theta + a^2}{R^2 - 2\,a\,R \sin\Theta + a^2}$$

$$= \frac{Q_W}{4\pi kH} \ln \frac{1 + \frac{2a}{R} \sin\Theta + \ldots.}{1 - \frac{2a}{R} \sin\Theta + \ldots.}$$

$$= \frac{Q_W}{4\pi kH} \left[\frac{4a}{R} \sin\Theta + \ldots.\right],$$

in which higher powers of $\frac{a}{R}$ have been neglected. The right-hand member of the equation approaches zero as R approaches infinity.

All the water discharged by the well comes from the canal. The amount of canal water entering the aquifer between $-x_0$ and $+x_0$ is

$$H\int_{-x_0}^{x_0} v_y(x,0)\,dx = -kH\int_{-x_0}^{x_0} \left.\frac{\partial\varphi}{\partial y}\right|_{y=0} dx = -\frac{Q_W}{4\pi}\int_{-x_0}^{x_0} \left[\frac{2(y+a)}{x^2+(y+a)^2} - \frac{2(y-a)}{x^2+(y-a)^2}\right]_{y=0} dx =$$

$$= -\frac{Q_W}{\pi}\int_{-x_0}^{x_0} \frac{a}{x^2+a^2}\,dx = -\frac{2Q_W}{\pi} \arctan\frac{x_0}{a} =$$

$$= -\frac{2Q_W}{\pi}\left\{\frac{\pi}{2} \arctan\frac{a}{x_0}\right\} = -Q_W\left\{1 - \frac{2}{\pi}\arctan\frac{a}{x_0}\right\}.$$

For $x > \frac{40a}{\pi}$ is $\frac{2}{\pi} \arctan\frac{a}{x_0} < \frac{2}{\pi} \arctan\left(\frac{\pi}{2} 0.05\right) \approx 0.05$.

Consequently, about 95% of the water discharged by the well enters the aquifer within the interval $-\frac{40a}{\pi} < x < \frac{40a}{\pi}$ (if Q_W is negative).

The formulas for the equipotential lines are easy to find for the present case, for according to (1) we see that

$$\varphi(x, y) = \frac{Q_W}{2\pi kH} \ln\frac{1}{\alpha},$$

in which

$$\frac{1}{\alpha_2} = \frac{x^2 + (y+a)^2}{x^2 + (y-a)^2}. \qquad (2)$$

Hence

$$x^2 + y^2 + a^2 - \frac{1+\alpha^2}{1-\alpha^2}\, 2\,a\,y = 0$$

or

$$x^2 + \left(y - \frac{1+\alpha^2}{1-\alpha^2}a\right)^2 = \frac{4\alpha^2}{(1-\alpha^2)^2}a^2 . \qquad (3)$$

Equation (3) respresents a circle of radius ρ, with its centre at $x = 0$, $y = b$, in which

$$b = \frac{1+\alpha^2}{1-\alpha^2}a , \qquad (4)$$

$$\rho = \frac{2\alpha}{|1-\alpha^2|}a = \sqrt{b^2 - a^2} \qquad (5)$$

From (4) we get $\alpha^2 = \frac{b-a}{b+a}$, consequently on the circle of radius $\rho = \sqrt{b^2 - a^2}$ with its centre at (0, b) we have

$$\varphi(x, y) = \frac{Q_W}{4\pi kH} \ln \frac{b+a}{b-a} . \qquad (6)$$

Summarizing, we see that the equipotential lines are circles of radii $\rho = \sqrt{b^2 - a^2}$ [1]) having their centres at the point (0, b) in which $b > a$. The potentials on these circles are given by equation (6) [2]).

Obviously, the face of the well (the circle $x^2 + (y - a)^2 = r^2$) is not an equipotential line [3]). Substituting $r \cos \Theta$ for x and $a + r \sin \Theta$ for y (polar co-ordinates with regard to the axis of the well), one finds from (1) (if r is small relative to a) that

$$\begin{aligned}\varphi(x, y) &= \frac{Q_W}{4\pi kH} \ln \frac{4a^2 + 4a\, r \sin \Theta + r^2}{r^2} = \\ &= \frac{Q_W}{2\pi kH}\left[\ln \frac{2a}{r} + \frac{1}{2} \ln \left(1 + \frac{r}{a} \sin \Theta + \frac{r^2}{4a^2}\right)\right] = \\ &= \frac{Q_W}{2\pi kH}\left[\ln \frac{2a}{r} + \frac{r}{2a} \sin \Theta + \ldots\ldots\right] . \qquad (7)\end{aligned}$$

Since in practice, however, the radius of the well is very small relative to a, the potential along the well face has an almost constant value (independent of Θ) expressed as

$$\varphi_W = \frac{Q_W}{2\pi kH} \ln \frac{2a}{r_W} \qquad (8)$$

1) Expressed more mathematically: the equipotential lines are a series of circles with the circles $x^2 + (y \pm a)^2 = 0$ as point-circles, as is immediately apparent from (2).

2) Equation (1) shows that the potential in the centre of such a circle can be expressed as

$$\varphi(0, b) = \frac{Q_W}{2\pi kH} \ln \frac{b+a}{b-a}$$

which is double the value of the potential at the circumference.

3) Because a circular equipotential line with its centre at (0, a) would have a radius $\rho = 0$.

6.2.1.1.

Remark 1

Placing point sinks at the points $(0, \pm\sqrt{a^2 - r_W^2})$ one finds that

$$\varphi(x, y) = \frac{Q_W}{4\pi kH} \ln \frac{x^2 + (y + \sqrt{a^2 - r_W^2})^2}{x^2 + (y - \sqrt{a^2 - r_W^2})^2}, \tag{9}$$

from which the potential on the circle $x^2 + (y - a)^2 = r_W^2$ can be found, viz.

$$\varphi(x, y) = \varphi_W = \tfrac{1}{2}\varphi(0, a) = \frac{Q_W}{4\pi kH} \ln \frac{a + \sqrt{a^2 - r_W^2}}{a - \sqrt{a^2 - r_W^2}}$$

or

$$\varphi_W = \frac{Q_W}{2\pi kH} \ln \frac{a + \sqrt{a^2 - r_W^2}}{r_W}. \tag{10}$$

The face of the well in this case is an exactly equipotential plane. As r_W is very small relative to a, the difference between (9) and (1), or between (10) and (8) is of no consequence whatever.

Remark 2

Of course, formula (7) also applies to values of r other than the radius of the well, consequently it gives the potential distribution in the vicinity of the well. When $r < 0.22a$, the second member is less than 5% of the first member, so that formula (8) – the formula for a well in the centre of a circular island of radius $2a$ and with potential $\varphi = 0$ along the boundary - may be used as a good approximation in this area.

6.2.1.2. *Semi-confined ground water*

This case has been dealt with in the example in 6.1.3. We find that

$$\varphi(x, y) = \frac{Q_W}{2\pi kH}\left[K_0\left(\frac{r_1}{\lambda}\right) - K_0\left(\frac{r_2}{\lambda}\right)\right], \tag{1}$$

in which

$r_1 = \sqrt{x^2 + (y - a)^2}$ and $r_2 = \sqrt{x^2 + (y + a)^2}$.

When $a \gg \lambda$, the second term in the right-hand member of equation (1) is of little significance for all values of $y > 0$, so formula (1) is practically identical with the formula for a well in an infinite semi-confined aquifer.

When $a \ll \lambda$, the formula for a confined aquifer (as a first approximation) will apply to the vicinity of the well (formula (1) in 6.2.1.1.). For when $a \ll \lambda$, both r_1 and r_2 are small relative to λ in the vicinity of the well, so that we may use the formulas in the Appendix for both K_0-functions. We find that

$$\varphi(x, y) = \frac{Q_W}{2\pi kH}\left[\left\{1 + \frac{r_1^2}{4\lambda^2} + \ldots\ldots\right\} \ln\frac{r_2}{r_1} - \frac{r_2^2 - r_1^2}{4\lambda^2}\left(\ln\frac{1.123}{r^2} + 1\right) + ..\right] \approx$$

$$\approx \frac{Q_W}{2\pi kH} \ln\frac{r_2}{r_1}.$$

The quantity of water that flows from the boundary towards the well is

$$Q_b = -Q_W\, e^{-\frac{a}{\lambda}}. \tag{2}$$

(Q_b, an afflux into the aquifer, is negative when Q_W is positive).

For we see from (1) that

$$v_y = -k\frac{\partial\varphi}{\partial y} = -\frac{Q_W}{2\pi H}\left[\frac{y-a}{\lambda r_1}K'_0\left(\frac{r_1}{\lambda}\right) - \frac{y+a}{\lambda r_2}K'_0\left(\frac{r_2}{\lambda}\right)\right],$$

consequently

$$v_y(x, 0) = \frac{Q_W}{\pi H}\frac{a}{\lambda\sqrt{x^2+a^2}}K'_0\left(\frac{\sqrt{x^2+a^2}}{\lambda}\right),$$

hence

$$Q_b = H\int_{-\infty}^{+\infty} v_y(x, 0)\, dx = \frac{2Q_W}{\pi}\int_0^{\infty}\frac{a}{\lambda\sqrt{x^2+a^2}}K'_0\left(\frac{\sqrt{x^2+a^2}}{\lambda}\right) dx,$$

or

$$Q_b = \frac{2Q_W}{\pi}\frac{d\int_0^{\infty} K_0\left(\frac{\sqrt{x^2+a^2}}{\lambda}\right) dx}{da}.$$

Using the corresponding formula in the Appendix we find that

$$Q_b = Q_W\frac{d\left(\lambda\, e^{-\frac{a}{\lambda}}\right)}{da} = -Q_W\, e^{-\frac{a}{\lambda}}.$$

From this result it appears once again, that if $a \gg \lambda$, the recharging of the aquifer is almost entirely a matter of vertical percolation through the covering layer; the existence of the boundary canal is not very important. If, however, $a \ll \lambda$, most of the recharging water comes from the canal.

6.2.1.2.

6.2.1.3. *Phreatic ground water*

If the same situation is considered as mentioned in 6.2.1.1. but with phreatic water, the solution may be obtained direct from that given in 6.2.1.1. By virtue of the statement made at the end of 6.1.1. and if the water level at the boundary is H, the following relation for the level $h\ (x, y)$ of the phreatic surface can be found:

$$\tfrac{1}{2}\{h(x, y)\}^2 = \tfrac{1}{2}H^2 + \frac{Q_W}{2\pi k} \ln \frac{r_2}{r_1}.$$

If $h = H + z$, this equation is reduced to (see (3) in 5.2.1.)

$$z\left(1 + \frac{z}{2H}\right) = \frac{Q_W}{2\pi kH} \ln \frac{r_2}{r_1},$$

from which it follows that approximately

$$z = \frac{Q_W}{2\pi kH} \ln \frac{r_2}{r_1} \text{ at some distance from the well.}$$

The water level h_W in the well can be expressed by the following formula (good approximation):

$$\tfrac{1}{2}h_W^2 = \tfrac{1}{2}H^2 + \frac{Q_W}{2\pi k} \ln \frac{2a}{r_W}$$

(this indicates the water level in the well, not the level of the phreatic surface at the well face; see 5.3.1.2.).

6.2.2. CIRCULAR AREA WITH A FIXED POTENTIAL ALONG ITS BOUNDARY AND A WELL OUT OF CENTRE

6.2.2.1. *Confined ground water*

Suppose the aquifer is bounded by a circular canal with radius R, a potential $\varphi = 0$ and its centre coinciding with the origin of the co-ordinate axes. The well is located at the point with co-ordinates $(0, b)$ (see fig. 6.2.2.1.–1).

Since it was shown (see 6.2.1.) that the equipotentential lines of the potential field of two point sources with opposite discharges are circles, it is obvious to introduce an image point source with discharge $-Q_W$ at a point $(0, b')$, b' being an unknown distance larger than R (method of images applied to a circular area).

The potential procuded by the action of both sources together, can now be expressed as

$$\varphi(x, y) = \frac{Q_W}{2\pi kH} \ln \alpha \frac{r_2}{r_1},$$

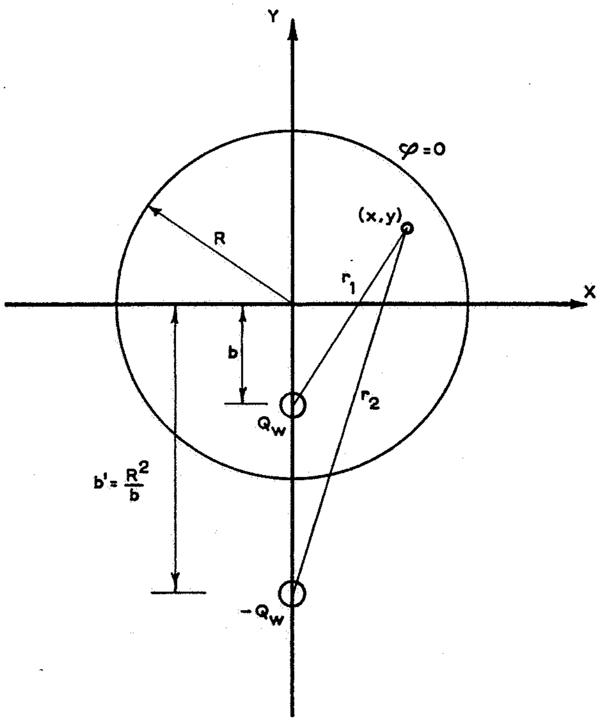


Fig. **6.2.2.1.–1**

in which $r_1 = \sqrt{x_2 + (y+b)^2}$, $r_2 = \sqrt{x^2 + (y+b')^2}$, α being a positive constant yet to be determined, ensuring that along the circle $x^2 + y^2 = R^2$ the potential not only remains constant but also equals 0. This condition is satisfied

$$\text{if } r_1^2 = \alpha^2 r_2^2 \text{ for } x = R\cos\Theta \text{ and } y = R\sin\Theta,$$

$$\text{or } R^2 + 2Rb\sin\Theta + b^2 = \alpha^2(R^2 + 2Rb'\sin\Theta + b'^2).$$

As this equation must hold good for all values of Θ between 0 and 2π, it follows that

$$R^2 + b^2 = \alpha^2(R^2 + b'^2),\ b = \alpha^2 b'.$$

Elimination of α^2 gives

$$b'(R^2 + b^2) = b(R^2 + b'^2), \text{ or } (b' - b)(R^2 - bb') = 0.$$

As $b' > R > b$, it is seen that

$$b' = \frac{R^2}{b}, \text{ so } \alpha = \frac{b}{R}. \tag{1}$$

Therefore the potential will be

$$\varphi(x, y) = \frac{Q_W}{2\pi kH} \ln \frac{b}{R}\frac{r_2}{r_1}. \tag{2}$$

It can be seen that (2) will give for the potential at the well face (with a slight error, if $r_W \ll b' - b$)

$$\varphi_W = \frac{Q_W}{2\pi kH} \ln \frac{b}{R}\frac{b' - b}{r_W} =$$

$$= \frac{Q_W}{2\pi kH} \ln \frac{R^2 - b^2}{Rr_W} =$$

$$= \frac{Q_W}{2\pi kH} \ln \frac{R(1 - b^2/R^2)}{r_W} = \tag{3a}$$

$$= \frac{Q_W}{2\pi kH} \ln \frac{2a(1 - a/2R)}{r_W}, \tag{3b}$$

in which $a = R - b$ (shortest distance from the axis of the well to the boundary).

If $b \ll R$ (circular island with a well somewhat out of centre) an approximate solution can be found as follows. Since in this case $b' \ll R$, it is true for all points inside the circle that $x^2 + y^2 < R^2$;

$$r_2 = \sqrt{b'^2 + 2b'y + x^2 + y^2} = b'\sqrt{1 + \frac{2y}{b'} + \frac{x^2 + y^2}{b'^2}} = b'\left(1 + \frac{y}{b'} + \ldots\right).$$

6.2.2.1.

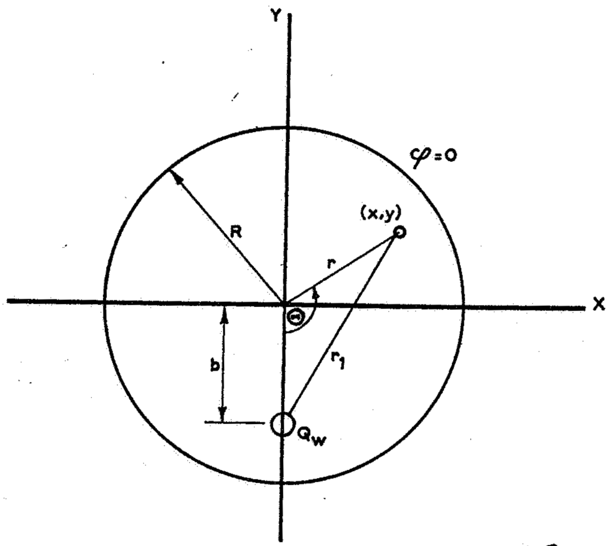


Fig. **6.2.2.2.–1**

Therefore

$$\varphi(x, y) = \frac{Q_W}{2\pi kH} \ln \left\{ \frac{bb'}{Rr_1} \left(1 + \frac{y}{b'} + \ldots \right) \right\} = \frac{Q_W}{2\pi kH} \left\{ \ln \frac{R}{r_1} + \frac{y}{b'} + \ldots \right\}.$$

Here a correction is obtained to the formula in 5.1.1., for which the well was situated in the centre of the island.

6.2.2.2. *Semi-confined ground water*

Let the radius of the circular boundary with potential $\varphi = 0$ be R and the co-ordinates of the axis of the well with regard to the centre of the area be $(0, -b)$ (see fig. 6.2.2.2.–1). It is evident that the potential can be expressed as

$$\varphi(x, y) = \frac{Q_W}{2\pi kH} K_0\left(\frac{r_1}{\lambda}\right) + \varphi_1(x, y), \tag{1}$$

in which $r_1 = \sqrt{x^2 + (y + b)^2}$.

In this formula φ_1 must satisfy the differential equation and the boundary condition

$$\varphi_1(x, y) = -\frac{Q_W}{2\pi kH} K_0\left(\frac{r_1}{\lambda}\right)$$

for $x^2 + y^2 = R^2$. Moreover, $\varphi_1(x, y)$ must be continuous and limited within the circle as opposed to the first term in (1)).

If polar co-ordinates r and Θ with regard to the origin are used (it is advisable to take $\Theta = 0$ along the negative y-axis) the differential equation for $\varphi_1(r, \Theta)$ is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \varphi_1}{\partial \Theta^2} - \frac{1}{\lambda^2}\varphi_1 = 0, \tag{2}$$

and the boundary condition is

$$\varphi_1(R, \Theta) = -\frac{Q_W}{2\pi kH} K_0\left(\frac{1}{\lambda}\sqrt{R^2 + b^2 - 2Rb \cos \Theta}\right). \tag{3}$$

It can be shown quite easily that the following functions satisfy the differential equations (2):

$$F_n(r, \Theta) = \{A_n I_n(r) + B_n K_n(r)\} \times \{C_n \cos n\Theta + D_n \sin n\Theta\},$$

A_n, B_n, C_n and D_n being constants and n any number.

The condition that $F_n(r, \Theta + 2\pi) = F_n(r, \Theta)$ is only satisfied if n is a whole number, whereas the condition that F_n be limited for $r = 0$ gives $B_n = 0$. Furthermore, as $\varphi_1(r, \Theta)$ must of necessity be an even function of Θ, it may be said that

$$\varphi_1(r, \Theta) = \sum_{n=0}^{\infty} A_n I_n\left(\frac{r}{\lambda}\right) \cos n\Theta . \tag{4}$$

Now if $b < R$ it follows that

$$K_0\left(\frac{1}{\lambda}\sqrt{R^2 + b^2 - 2bR \cos \Theta}\right) = \sum_{n=0}^{\infty} \varepsilon_n K_n\left(\frac{R}{\lambda}\right) I_n\left(\frac{b}{\lambda}\right) \cos n\Theta , \tag{5}$$

in which $\varepsilon_0 = 1, \varepsilon_n = 2$ $(n = 1, 2, \ldots)$.

Equations (3), (4) and (5) give

$$\sum_{n=0}^{\infty} A_n I_n\left(\frac{R}{\lambda}\right) \cos n\Theta = -\frac{Q_w}{2\pi kH} \sum_{n=0}^{\infty} \varepsilon_n K_n\left(\frac{R}{\lambda}\right) I_n\left(\frac{b}{\lambda}\right) \cos n\Theta , \tag{6}$$

so

$$A_n = -\frac{Q_w}{2\pi kH} \varepsilon_n K_n\left(\frac{R}{\lambda}\right) \frac{I_n\left(\frac{b}{\lambda}\right)}{I_n\left(\frac{R}{\lambda}\right)} ,$$

so that

$$\varphi(r, \Theta) = \frac{Q_w}{2\pi kH}\left[K_0\left(\frac{r_1}{\lambda}\right) - \sum_{n=0}^{\infty} \varepsilon_n K_n\left(\frac{R}{\lambda}\right) \frac{I_n\left(\frac{b}{\lambda}\right)}{I_n\left(\frac{R}{\lambda}\right)} I_n\left(\frac{r}{\lambda}\right) \cos n\Theta\right], \tag{7}$$

in which $r_1 = \sqrt{r^2 + b^2 - 2br \cos \Theta}$.

Since $I_n(0) = 0$ if $n \geqslant 1$, it is quite clear that for $b = 0$ the formula (4) in 5.1.2.2. will emerge from equation (6).

The part Q_b of the total discharge which is derived from the canal is computed as follows

$$Q_b = -H\int_0^{2\pi} v_r(R, \Theta) R d\Theta = k\,H\,R \int_0^{2\pi} \left(\frac{\partial\varphi}{\partial r}\right)_{r=R} d\Theta . \tag{8}$$

Substituting r for R in (5) it is seen from (5) and (6) that for $r > b$

$$\varphi(r, \Theta) = \frac{Q_w}{2\pi kH} \sum_{n=0}^{\infty} \varepsilon_n \frac{I_n\left(\frac{b}{\lambda}\right)}{I_n\left(\frac{R}{\lambda}\right)} \left[K_n\left(\frac{r}{\lambda}\right) I_n\left(\frac{R}{\lambda}\right) - K_n\left(\frac{R}{\lambda}\right) I_n\left(\frac{r}{\lambda}\right)\right] \cos n\Theta ,$$

hence

$$\left.\frac{\partial \varphi}{\partial r}\right|_{r=R} = \frac{Q_W}{2\pi kH} \sum_{n=0}^{\infty} \varepsilon_n \frac{I_n\left(\frac{b}{\lambda}\right)}{I_n\left(\frac{R}{\lambda}\right)} \cos n\Theta \times \left[K_n'\left(\frac{R}{\lambda}\right) I_n\left(\frac{R}{\lambda}\right) - K_n\left(\frac{R}{\lambda}\right) I_n'\left(\frac{R}{\lambda}\right)\right] =$$

$$= -\frac{Q_W}{2\pi kH} \frac{1}{R} \sum_{n=0}^{\infty} \varepsilon_n \frac{I_n\left(\frac{b}{\lambda}\right)}{I_n\left(\frac{R}{\lambda}\right)} \cos n\Theta, \qquad (9)$$

because

$$K_n'(u)\, I_n(u) - K_n(u)\, I_n'(u) = -\frac{1}{u} \text{ (see Appendix).}$$

From (8) and (9) it follows that

$$Q_b = -\frac{Q_W}{2\pi} \sum_{n=0}^{\infty} \varepsilon_n \frac{I_n\left(\frac{b}{\lambda}\right)}{I_n\left(\frac{R}{\lambda}\right)} \int_0^{2\pi} \cos n\Theta \, d\Theta = -Q_W \frac{I_0\left(\frac{b}{\lambda}\right)}{I_0\left(\frac{R}{\lambda}\right)}. \qquad (10)$$

(Q_b is an injection and therefore negative if Q_W is taken positive).

If $b = 0$, formula (10) changes into formula (6) in 5.1.2.2.

Remark

If b and R are both large with regard to λ and if use is made of the elaborations in the Appendix it is seen that

$$Q_b = -Q_W \, e^{-(R-b)/\lambda} \sqrt{1 + \frac{R-b}{b}}$$

(see formula (2) in 6.2.1.2., which results from the above equation if R and b approach infinity while $a = R - b$ remains unchanged).

6.2.3. Infinite quadrant

If the well is situated in an aquifer bounded by two canals with potential $\varphi = 0$ intersecting at right angles, the potential function can be obtained by the use of a set of images. If the positive x- and y-axis are made to coincide with the canals and the well is designated by the co-ordinates a and b, image wells with discharge $-Q_W$ are situated at the points (a, b) and $(a, -b)$ and one image well with discharge $+Q_W$ at the point $(-a, -b)$ (see fig. 6.2.3.–1). From the figure it is evident that the potential

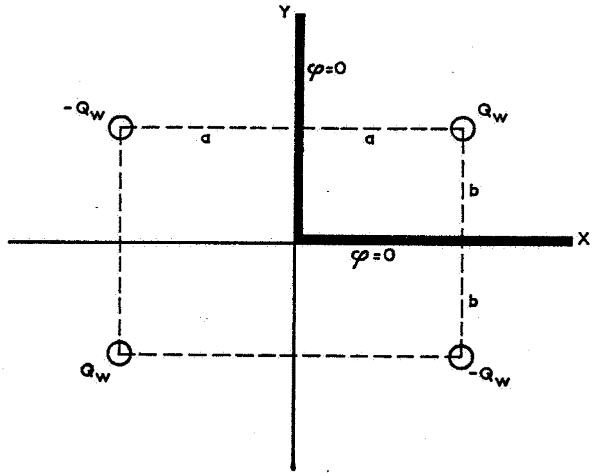


Fig. **6.2.3.–1**

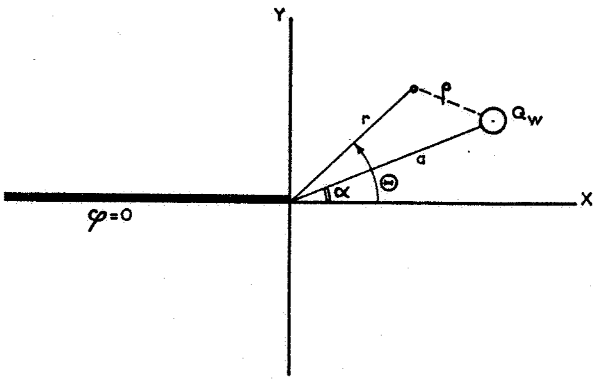
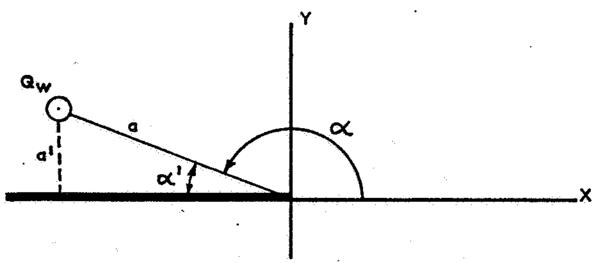


Fig. **6.2.4.–1**



6.2.4.-2.

resulting from the simultaneous actions of all the wells will be an odd function both of x and of y, and so must be 0 along the co-ordinate axes. This applies to confined and semi-confined ground water alike.

For instance, for confined water the potential is expressed by

$$\varphi(x, y) = \frac{Q_W}{2\pi kH} \ln \frac{\sqrt{(x+a)^2 + (y-b)^2}\ \sqrt{(x-a)^2 + (y+b)^2}}{\sqrt{(x-a)^2 + (y-b)^2}\ \sqrt{(x+a)^2 + (y+b)^2}}.$$

Hence the potential on the well face is

$$\varphi_W = \frac{Q_W}{2\pi kH} \ln \frac{2ab}{r_W\sqrt{a^2+b^2}}.$$

If the boundary $x = 0$ is not a canal but an impermeable barrier ($v_x = 0$) "even" reflection with respect to this boundary must be applied. Then an image well with a discharge $+Q_W$ is located at $(-a, b)$ and an image well with a discharge $-Q_W$ at $(-a, -b)$.

6.2.4. Half-infinite line source (confined ground water)

Suppose a fully penetrating canal with potential $\varphi = 0$ is situated along the full length of the negative x-axis, thus acting as a half-infinite line source (see fig. 6.2.4.–1). The well is situated at the point indicated by the polar co-ordinates $r = a$, $\Theta = \alpha$ $(-\pi < \alpha < \pi)$.

In this case the potential is expressed by

$$\varphi(r, \Theta) = \frac{Q_W}{4\pi kH} \ln \frac{r + 2\sqrt{ra}\cos((\Theta+\alpha)/2) + a}{r - 2\sqrt{ra}\cos((\Theta-\alpha)/2) + a}. \tag{1}$$

This formula can be arrived at with the aid of the method of conformal mapping. On obtaining the formula it is easy to show that $\varphi(r, \Theta)$ satisfies the differential equation (3b*) in 6.1.1. if $\lambda^2 = \infty$, and that $\varphi = 0$ if $\Theta = \pm\pi$. In order to investigate the behaviour of φ near the well, the following considerations should be borne in mind. If the distrance from the point (r, Θ) to the point (a, α) is called ρ, the cosine rule will give

$$r^2 - 2ar\cos(\Theta - \alpha) + a^2 = \rho^2, \quad \text{or}$$

$$r^2 - 2ar\left(2\cos^2\frac{\Theta-\alpha}{2} - 1\right) + a^2 = \rho^2, \quad \text{or}$$

$$(r+a)^2 - 4\,ar\cos^2\frac{\Theta-\alpha}{2} = \rho^2, \quad \text{or}$$

$$\left(r - 2\sqrt{ar}\cos\frac{\Theta-\alpha}{2} + a\right)\left(r + 2\sqrt{ar}\cos\frac{\Theta-\alpha}{2} + a\right) = \rho^2.$$

So (1) can by expressed by

$$\varphi(r, \Theta) = \frac{Q_W}{4\pi kH} \ln \frac{\left(r + 2\sqrt{ar}\cos\frac{\Theta-\alpha}{2} + a\right)\left(r + 2\sqrt{ar}\cos\frac{\Theta+\alpha}{2} + a\right)}{\rho^2}.$$

Close to the well r is about equal to a and Θ is about equal to α. If ρ is small with regard to a it is seen that approximately

$$\varphi = \frac{Q_W}{4\pi kH} \ln \frac{4a.\,2a.\,(1+\cos\alpha)}{\rho^2}, \quad \varphi = \frac{Q_W}{2\pi kH} \ln \frac{4a\cos(\alpha/2)}{\rho}.$$

The potential on the well face ($\rho = r_W$) is expressed as

$$\varphi_W = \frac{Q_W}{2\pi kH} \ln \frac{4a\cos(\alpha/2)}{r_W}.$$

<u>Remark</u>

If α approaches π and if $\alpha' = \pi - \alpha$ and $a' = a \sin\alpha = a$ sind α' (see fig. 6.2.4.–2) it follows that

$$\varphi_W = \frac{Q_W}{2\pi kH} \ln \frac{4a'}{r_W}\frac{\sin(\alpha'/2)}{\sin\alpha'} = \frac{Q_W}{2\pi kH} \ln \frac{2a'}{r_W\cos(\alpha'/2)} =$$

$$= \frac{Q_W}{2\pi kH}\left\{\ln\frac{2a'}{r_W} - \ln\cos\frac{\alpha'}{2}\right\} = \frac{Q_W}{2\pi kH}\left\{\ln\frac{2a'}{r_W} + \frac{\alpha'^2}{8} + \ldots\right\},$$

since

$$\cos\frac{\alpha'}{2} = 1 - \frac{\alpha'^2}{8} + \ldots \text{ and } \ln\left(1 - \frac{\alpha'^2}{8} + \ldots\right) = -\frac{\alpha'^2}{8} + \ldots$$

If the correction term $\frac{\alpha'^2}{8}$ is also omitted the formula is reduced to that for the flow from an infinite canal into a well. This means that if the distance between the well and the the canal is small in proportion to that between the well and the end of the canal, the fact that the canal is half-infinite instead of infinite is of little consequence.

6.2.5. INFINITE STRIP

6.2.5.1. *Confined ground water*

Suppose the aquifer is bounded by parallel canals along $x = 0$ and $x = b$ (see fig. 6.2.5.1.–1), the well being in the point $(a, 0)$ $(0 < a < b)$.

To meet the boundary conditions for $x = 0$ an imaginary well discharging $-Q_W$ can be located at the point $(-a, 0)$. To meet the boundary conditions for $x = b$ not only the (real) well at $(a, 0)$ has to be reflected, but also the imaginary well in $(-a, 0)$, that is to say imaginary wells discharging $-Q_W$ and $+Q_W$ respectively have to be located at the points $(2b - a, 0)$ and $(2b + a, 0)$. As a result of these new wells, however, the boundary conditions are no longer valid for $x = 0$. This can be met by placing imaginary wells discharging $+Q_W$ and $-Q_W$ respectively at $(-2b + a, 0)$ and $(-2b - a, 0)$. Now, however, the boundary conditions for $x = b$ are not valid etc.

In this way wells discharging $+Q_W$ are obtained at the points $y = 0$, $x = a$, $a \pm 2b$, $a \pm 4b, \ldots.$ and wells discharging $-Q_W$ at the points $y = 0$, $x = -a$, $-a \pm 2b$, $-a \pm 4b, \ldots.$

The total potential of these wells is calculated by superposition

$$\varphi(x,y) = \frac{Q_W}{2\pi kH} \sum_{n=-\infty}^{\infty} \ln \frac{\sqrt{(x+a-2nb)^2 + y^2}}{\sqrt{(x-a-2nb)^2 + y^2}}. \tag{1}$$

It will be clear that this formula is not very practical for carrying out calculations. Formula (1) can be reduced, but it is simpler to use the results, which have been derived for infinite well series. The total potential of a series of wells each discharging Q_W at the points $y = 0$, $x = 0, \pm b, \pm 2b, \ldots.$ has been worked out as follows.

$$\varphi(x, y) = -\frac{Q_W}{2\pi kH} \left[\frac{\pi |y|}{b} + \tfrac{1}{2} \ln \left(1\text{-}2e^{-\frac{2\pi |y|}{b}} \cos \frac{2\pi x}{b} + e^{-\frac{4\pi |y|}{b}} \right) \right] + C_0. \tag{2}$$

In this formula C_0 is an indefinite constant.

In the above-mentioned case there are two series of wells discharging $+Q_W$ and $-Q_W$ respectively. In applicating formula (2) (in which b has to be substituted by $2b$ and x by $x - a$ and $x + a$ respectively) the potential at that point will be found to be

$$\varphi(x, y) = -\frac{Q_W}{2\pi kH} \left[\frac{\pi |y|}{2b} + \tfrac{1}{2} \ln \left(1\text{-}2e^{-\frac{\pi |y|}{b}} \cos \frac{\pi (x-a)}{b} + e^{-\frac{2\pi |y|}{b}} \right) \right] +$$

$$+ \frac{Q_W}{2\pi kH} \left[\frac{\pi |y|}{2b} + \tfrac{1}{2} \ln \left(1\text{-}2e^{-\frac{\pi |y|}{b}} \cos \frac{\pi (x+a)}{b} + e^{-\frac{2\pi |y|}{b}} \right) \right] + C_1 =$$

$$= \frac{Q_W}{2\pi kH} \ln \frac{1\text{-}2e^{-\frac{\pi |y|}{b}} \cos \frac{\pi(x+a)}{b} + e^{-\frac{2\pi |y|}{b}}}{1\text{-}2e^{-\frac{\pi |y|}{b}} \cos \frac{\pi(x-a)}{b} + e^{-\frac{2\pi |y|}{b}}}. \tag{3}$$

Here the constant C_1 has to be taken as zero, because in this case $\varphi = 0$ for $x = 0$ and $x = b$.

6.2.5.1.

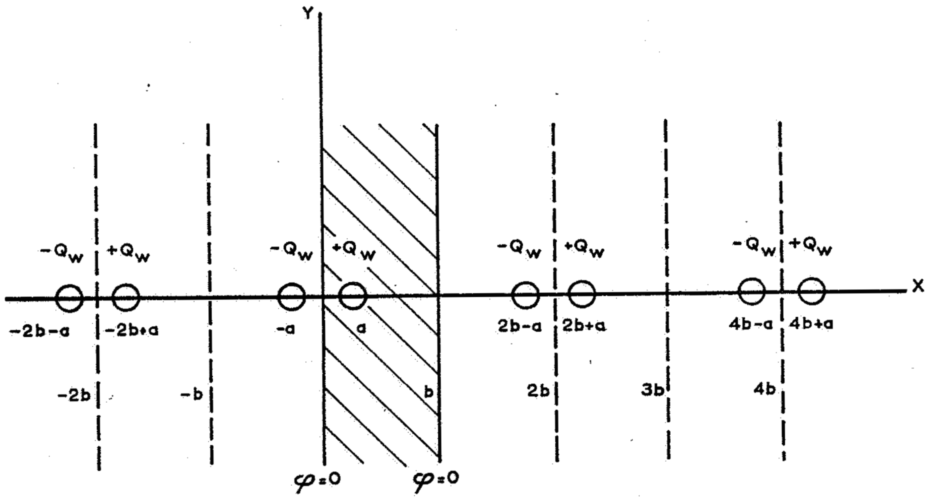

Y
X
-Qw
+Qw
-Qw
+Qw
-Qw
+Qw
-Qw
+Qw
-2b-a
-2b+a
-a
a
2b-a
2b+a
4b-a
4b+a
-2b
-b
b
2b
3b
4b
φ=0
φ=0

Fig. 6.2.5.1.–1

The following formula, which gives a very close approximation, can be used to determine the potential on the face of the well

$$\varphi_W = \frac{Q_W}{2\pi kH} \ln \frac{2b \sin(\pi a/b)}{\pi r_W}. \tag{4}$$

This formula is derived as follows: If $|x - a|$ and $|y|$ are small with regard to b the following formula, which is a very close approximation, will apply.

$$1\text{–}2e^{-\frac{\pi|y|}{b}} \cos \frac{\pi(x-a)}{b} + e^{-\frac{2\pi|y|}{b}} =$$

$$= 1\text{–}2\left(1 - \frac{\pi|y|}{b} + \frac{\pi^2 y^2}{2b^2} + \ldots\right)\left(1 - \frac{\pi^2(x-a)^2}{2b^2} + \ldots\right) +$$

$$+ 1 - \frac{2\pi|y|}{b} + \frac{2\pi^2 y^2}{b^2} + \ldots = \frac{\pi^2}{b^2}[(x-a)^2 + y^2] + \ldots = \left(\frac{\pi r_W}{b}\right)^2 + \ldots$$

In the numerator of the logarithm in formula (3) a may be substituted for x and 0 for y, so that we obtain

$$2\left(1 - \cos\frac{2\pi a}{b}\right) = 4\sin^2\frac{\pi a}{b}.$$

From this relation formula (4) is derived.

Further calculations may be carried out with the aid of formula (3), to determine what proportion of the total discharge is derived from the canal at $x = 0$. We then find.

$$Q_{x=0} = H\int_{-\infty}^{+\infty} v_x(0, y)dy = -Q_w\left(1 - \frac{a}{b}\right).$$

Similarly

$$Q_{x=b} = -H\int_{-\infty}^{+\infty} v_x(b, y)dy = -Q_W\frac{a}{b}. \tag{5}$$

Finally it should be noted that the potential can also be expressed in the form of a Fourier series (as a result of the introduction of images the potential $\varphi(x, y)$ has become a periodic function of x with a period $2b$; moreover, since φ is an odd-function of x, a series of the type $\sum_{m=1}^{\infty} f_m(y) \sin\frac{m\pi x}{b}$ can be expected).

6.2.5.1.

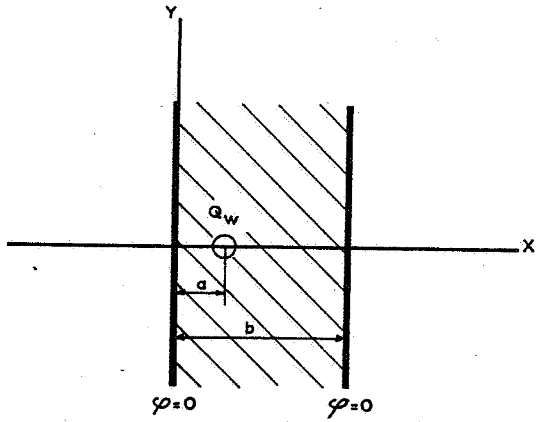


Fig. 6.2.5.2.–1

With the Fourier series the potential of an infinite well series is found to be

$$\varphi(x, y) = \frac{Q_W}{2\pi kH} \sum_{m=1}^{\infty} \frac{1}{m} e^{-\frac{\pi m |y|}{b}} \left[\cos \frac{\pi m(x-a)}{b} - \cos \frac{\pi m(x+a)}{b}\right] =$$

$$= \frac{Q_W}{2\pi kH} \sum_{m=1}^{\infty} \frac{2}{m} \sin \frac{\pi ma}{b} \cdot \sin \frac{\pi mx}{b} \cdot e^{-\frac{\pi m |y|}{b}} \qquad (6)$$

The behaviour of φ for higher values of $|y|$ is clear from this formula. The first term of the series diminishes slowest. But as $e^{-\pi} = 0.044$, the potential for $|y| = b$ is already rather low and for $|y| > 2b$ it is practically zero.

6.2.5.2. *Semi-confined ground water*

The situation is the same as in 6.2.5.1. (see fig. 6.2.5.2.–1). With the aid of the image method the potential is found to be

$$\varphi(x, y) = \frac{Q_W}{2\pi kH} \sum_{n=-\infty}^{\infty} \left[K_0(\lambda^{-1}\sqrt{(x-a-2nb)^2 + y^2}) - K_0(\lambda^{-1}\sqrt{(x+a-2nb)^2 + y^2})\right] \cdot \qquad (1)$$

If b is of the same order as or higher than λ this expression is well suited to the calculation of the potential at a certain point, since but few terms materially affect the outcome. If, however, $b \ll \lambda$ (as will often be the case) it is more advantageous to determine φ by means of a Fourier series. This series can be derived direct from the one determined for the case of an infinite well series. If, in this formula b is replaced by $2b$ and x by $x - a$ and $x + a$ respectively, we get

$$\varphi(x,y) = \frac{Q_W}{2\pi kH} \sum_{m=1}^{\infty} \frac{1}{\sqrt{m^2+\alpha^2}} e^{-\frac{\pi |y| \sqrt{m^2+\alpha^2}}{b}} \left[\cos \frac{\pi m(x-a)}{b} + \cos \frac{\pi m(x+a)}{b}\right] =$$

$$= \frac{Q_W}{2\pi kH} \sum_{m=1}^{\infty} \frac{2}{\sqrt{m^2+\alpha^2}} \sin \frac{\pi ma}{b} \sin \frac{\pi mx}{b} e^{-\frac{\pi |y|}{b}\sqrt{m^2+\alpha^2}}, \qquad (2)$$

in which $\alpha = \frac{b}{\pi\lambda}$. (3)

If α is small (e.g. $\alpha < 0.1$) the potential is practically the same as that for confined water. The proportion of the total well discharge derived from the canal at $x = 0$ is

$$Q_{x=0} = -Q_W \frac{\sinh((b-a)/\lambda)}{\sinh(b/a)} \qquad (4a)$$

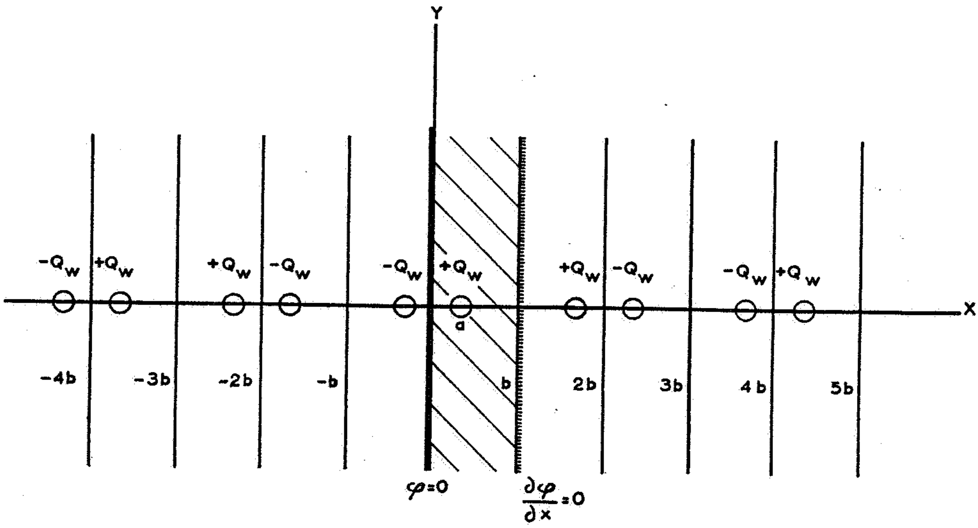


Fig. 6.2.5.3.–1

This proves true because the series of wells obtained after repeated introduction of images can be grouped in the following manner (see fig. 6.2.5.1.–1):

1°. a series of wells each discharing $+Q_W$ at $y = 0, x = a, a + 2b, a + 4b, \ldots$. and the images thereof with a discharge of $-Q_W$ at $y = 0$, $x = -a, -a - 2b, -a - 4b, \ldots$.

2°. a series of wells each discharging $-Q_W$ at $y = 0$, $x = -a + 2b, -a + 4b, \ldots$. and their images each with a discharge of $+Q_W$ at $y = 0$, $x = a-2b, a-4b, \ldots$.

Formula (2) in 6.2.1.2. can now be applied to each pair consisting of a well and its image. The discharge across the line $x = 0$ (in the direction of the positive x-axis)we then find to be

$$Q_{x=0} = -Q_W\left[\sum_{n=0}^{\infty} e^{-\frac{a+2nb}{\lambda}} - \sum_{n=1}^{\infty} e^{-\frac{-a+2nb}{\lambda}}\right] =$$

$$= -Q_W \frac{e^{-a/\lambda} - e^{-(2b-a)/\lambda}}{1 - e^{-2b/\lambda}} = -Q_W \frac{e^{(b-a)/\lambda} - e^{-(b-a)/\lambda}}{e^{b/\lambda} - e^{-b/\lambda}} =$$

$$= -Q_W \frac{\sinh((b-a/\lambda)}{\sinh\ (b/\lambda)}.$$

Similarly the amount of water flowing across the line $y = b$ (in the direction of the negative x-axis) is found to be

$$Q_{x=b} = -Q_W \frac{\sinh(a/\lambda)}{\sinh(b/\lambda)} \tag{4b}$$

Consequently, the leakage through the semi-permeable layer is

$$Q_z = -Q_W\left[1 - \frac{\sinh(a/\lambda) + \sinh((b-a)/\lambda)}{\sinh(b/\lambda)}\right] =$$

$$= -Q_W\left[1 - \frac{\cosh((b-2a)/2\lambda)}{\cosh(b/2\lambda)}\right] = -Q_W \cdot \frac{2\sinh(a/2\lambda)\cdot\sinh((b-a)/2\lambda)}{\cosh(b/2\lambda)} \tag{5}$$

If b (and therefore also a) is small with regard to λ, Q_z will, of course, be very small with regard to Q_W. Then formulas (4a) and (4b) approach formula (5) in 6.2.5.1.

6.2.5.3. *Infinite strip between canal and impervious barrier*

Supposing the aquifer is bounded by a canal with potential $\varphi = 0$ along $x = 0$ and an impervious barrier along $x = b$ (this can also be an approximation of a case in which the profile for $x > b$ has a much lower value of kH).

The well is found at the point $(a, 0)$ (with $0 < a < b$). Since $v_x = 0$ or $\frac{\partial \varphi}{\partial x} = 0$ along $x = b$, the reflection with regard to this line must of necessity be "even" (see 6.1.3.). As can be seen from figure 6.2.5.3.–1, the potential field can be calculated by superposition of the potential fields of wells with a discharge of $+Q_W$ at $y = 0$, $x = a + 4nb$ and at $y = 0$, $x = -a + (4n + 2)b$ and of wells with a discharge of $-Q_W$ at $y = 0$, $x = -a + 4nb$ and at $y = 0$, $x = a + (4n + 2)b$. The index n is given the successive values of $0, \pm 1, \pm 2 \ldots$

From this and with the aid of the formulas for an infinite well series we find for confined water

$$\varphi(x, y) = \frac{Q_W}{4\pi kH} \times$$

$$\times \ln \left[\frac{1 - 2e^{-\frac{\pi|y|}{2b}} \cos \frac{\pi(a+a)}{2b} + e^{-\frac{\pi|y|}{b}}}{1 - 2e^{-\frac{\pi|y|}{2b}} \cos \frac{\pi(x-a)}{2b} + e^{-\frac{\pi|y|}{b}}} \cdot \frac{1 + 2e^{-\frac{\pi|y|}{2b}} \cos \frac{\pi(x-a)}{2b} + e^{-\frac{\pi|y|}{b}}}{1 + 2e^{-\frac{\pi|y|}{2b}} \cos \frac{\pi(x+a)}{2b} + e^{-\frac{\pi|y|}{b}}} \right] \cdot \quad (1)$$

From this it follows that the potential on the face of the well is

$$\varphi_W = \frac{Q_W}{2\pi kH} \ln \left(\frac{4b}{\pi r_W} \tan \frac{\pi a}{2b} \right) \cdot \quad (2)$$

Fourier series for $\varphi(x, y)$ can also be drawn up again:

$$\varphi(x, y) = \frac{Q_W}{2\pi kH} \sum_{m=0}^{\infty} \frac{4}{2m+1} \sin \frac{(2m+1)\pi a}{2b} \sin \frac{(2m+1)\pi x}{2b} e^{-\frac{(2m+1)\pi|y|}{b}} \cdot \quad (3)$$

6.2.6. THE POTENTIAL IN THE VICINITY OF A WELL

The condition of permanent flow towards a well in confined water of phreatic water is only reasonably closely approached after a finite period, if at finite distances from the well the aquifer is in communication with open water from which the ground water in the aquifer can be replenished. Then the potential in the vicinity of the well with a discharge of Q_W can always be written (in polar co-ordinates with regard to the well axis) as follows:

$$\varphi(r, \Theta) = \frac{Q_W}{2\pi kH} \ln \frac{R_{eq}}{r} + \varphi_c(r, \Theta), \quad (1)$$

in which the "correction" $\varphi_c(r, \Theta)$ satisfies the potential equation and will also be small for $r = 0$. The quantity R_{eq}, which has the dimension of length, can always be chosen in such a manner that $\varphi_c(0, \Theta) = 0$.

The value of R_{eq} determined in this way depends only on the shape of the area and on the situation of the well (not on Q_w or kH), provided there is only one well in the area and provided the potential along the boundary is zero. If the area is bounded by a circular canal with radius R, concentric with the well, then $R_{eq} = R$ and $\varphi_c = 0$ (see 5.1.1.). For a more complicated area the quantity R_{eq} can therefore be called the *equivalent radius.*

In the above-mentioned cases the following values for R_{eq} were found:

Circular area with eccentric well (6.2.2.)

$$R_{eq} = R\,(1 - b^2/R^2) = 2a\,(1 - a/2R) \text{ (see fig. 6.2.6.–1).}$$

Infinite half-plane (6.2.1.1.)

$$R_{eq} = 2a. \text{ (see fig. 6.2.6.–2).}$$

Infinite quadrant (with canals as boundaries, 6.2.3.)

$$R_{eq} = \frac{2a}{\sqrt{1+(a^2/b^2)}} = \frac{2ab}{\sqrt{a^2+b^2}} \text{ (see fig. 6.2.6.–3).}$$

Infinite quadrant (with one canal and one impervious barrier, 6.2.3.)

$$R_{eq} = 2a\sqrt{1+(a^2/b^2)} = \frac{2a}{b}\sqrt{a^2+b^2} \text{ (see fig. 6.2.6.–4).}$$

Infinite strip (between canals, 6.2.5.1.)

$$R_{eq} = \frac{2b}{\pi} \sin\frac{\pi a}{b} \text{ (see fig. 6.2.6.–5 and fig. 6.2.6.–5a).}$$

Infinite strip (between canal and impervious barrier, 6.2.5.3.)

$$R_{eq} = \frac{4b}{\pi} \tan\frac{\pi a}{2b} \text{ (see fig. 6.2.6.–6 and fig. 6.2.6.–6a).}$$

Half infinite canal (6.2.4.)

$$R_{eq} = 4a\cos(\alpha/2) \text{ (see fig. 6.2.6.–7) and}$$

$$R_{eq} = \frac{2a'}{\cos(\alpha'/2)} \text{ (see fig. 6.2.6.–7a).}$$

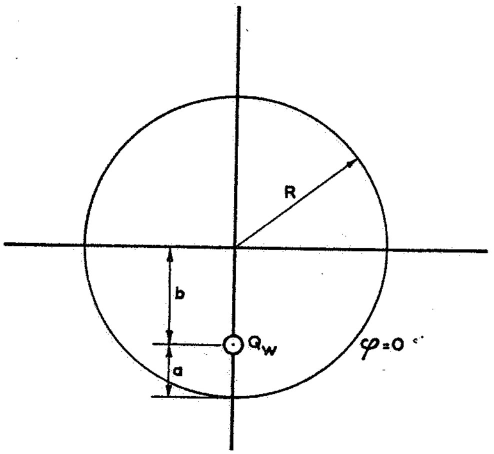


Fig. **6.2.6.–1**

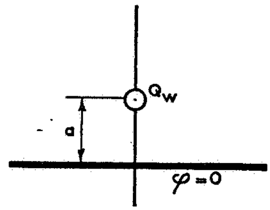


Fig. **6.2.6.–2**

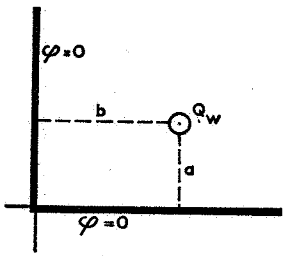


Fig. **6.2.6.–3**

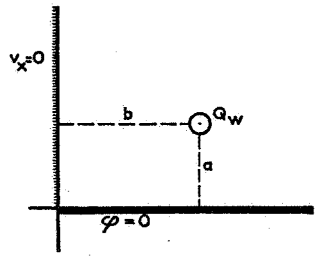


Fig. **6.2.6.–4**

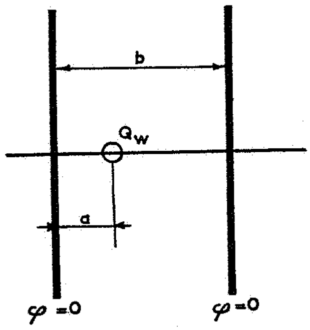


Fig. **6.2.6.–5**

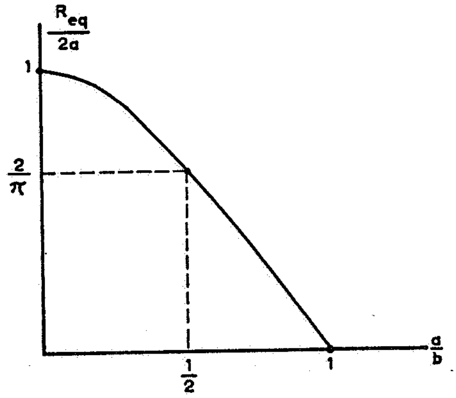


Fig. **6.2.6.–5a**

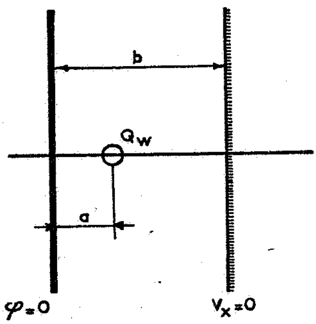


Fig. **6.2.6.–6**

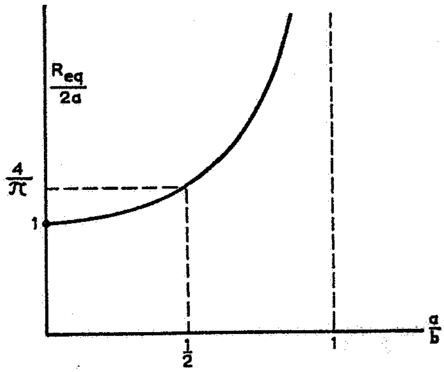


Fig. **6.2.6.–6a**

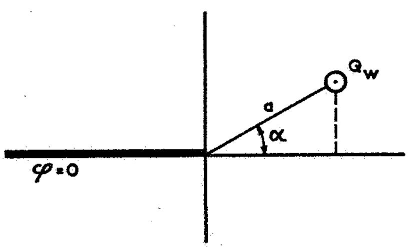


Fig. **6.2.6.–7**

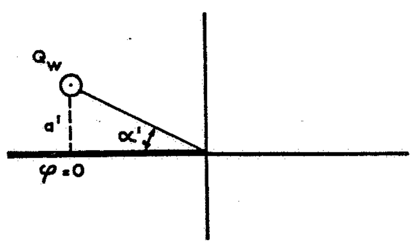


Fig. **6.2.6.–7a**

These examples would point to the existence of the following rules:

1°. In general R_{eq} is somewhat less than twice the shortest distance a from the well to the boundary of the area.

2°. Exceptions to this rule are:

   a. cases in which the replenishment obviously comes from different directions (circular area with slightly eccentric well: $R_{eq} \sim a$; well midway between two canals: $R_{eq} = \frac{4a}{\pi}$);

   b. areas with re-entering angles; R_{eq} is higher here, up to about $4a$ (the half infinite canal is an extreme example, for $\alpha = 0$, $R_{eq} = 4a$);

   c. areas of which parts of the boundary are composed of impervious barriers; here R_{eq} is larger than $2a$; if the well is stituated near the impervious barrier, R_{eq} becomes very large (infinite quadrant and infinite strip).

Within a certain distance of the well $\varphi_c(r, \Theta)$ is low and in that case a good approximation is

$$\varphi = \frac{Q_w}{2\pi kH} \ln \frac{R_{eq}}{r}. \tag{2}$$

This formula will always be the first to be tried when analyzing pumping-tests, or to determine the capacity of pumps. Of course, no definite indication can be given of the area for which formula (2) is a good approximation, but from some examples it is evident that the deviation is usually lower than 5%, as long as $r < 0.1$ or $0.05\ R_{eq}$.

As $\ln (R_{eq}/r)$ is high for $r \ll R_{eq}$, formula (2) is, of course, not very sensitive to a somewhat incorrect value of R_{eq}. For instance, the discrepancy in $\ln (R_{eq}/r)$ for $r < 0.05\ R_{eq}$ is less than 7% for a discrepancy of 20% in R_{eq}.

Therefore, if the boundary of the area has an irregular shape it is often possible to estimate a value of R_{eq} so that formula (2) gives reasonably reliable results by applying the above-mentioned rules.

In the case of semi-confined water a permanent flow can also exist without a flow across the boundary, as here we are concerned with a flow through the semi-permeable layer. The following formula obtains in the vicinity of a well in an infinite aquifer (see 5.1.2.1.).

$$\varphi = \frac{Q_w}{2\pi kH} \ln \frac{1.123\,\lambda}{r} \tag{3}$$

(with a discrepancy lower than 5% as long as $r < 0.33\ \lambda$). Here $1.123\ \lambda$ could be called the equivalent radius. If there is a flow across boundaries at a finite distance, two extreme cases can immediately be distinguished. If the shortest distance a from the well to the boundary is great with regard to λ (e.g. $a > 4\lambda$) the potential behaves in much the same way as that of a well in an infinite aquifer and the flow takes place almost completely through the semi-permeable layer. In the vicinity of the well formula (3) holds good. If, on the other hand, $a \ll \lambda$ (for instance $a < 0.1\,\lambda$) the potential in the vicinity of the well is almost equal to that of confined water (formula (2)). The flow in that case mainly takes place across the boundary. In intermediate cases the behaviour of the potential is, of course, more complicated. Near the well the potential can be written again as in formula (2), but the equivalent radius R_{eq} has a value depending both on the configuration of the area and on λ.

6.3. Multiple-well systems

The problem of ground-water flow to more than one well can be solved by means of the superposition method (see 6.1.2. in which a simple example has already been dealt with). A condition to be fulfilled in that case is that the distances between the individual wells and the distances between the wells and the boundary of the aquifer are large with respect to the diameter of the wells.

If in a certain area there are N wells with yields $Q_1, Q_2, \ldots Q_N$, the total potential $\varphi(x, y)$ at a certain point (x, y) is found by adding up the potentials $\varphi_1(x, y), \varphi_2(x, y), \ldots.$ $\varphi_N(x, y)$ which appear at that point if there is only one well or if the other wells have been stopped.

The potential of the n^{th} well may be expressed as

$$\varphi_n(x, y) = \frac{Q_n}{2\pi kH} f_n(x, y),$$

in which $f_n(x, y)$ is a dimensionless function, depending only on the location of the point (x, y), on the location of the n^{th} well and on the shape and (in the case of semi-confined water) on the leakage factor of the aquifer.

The total potential, therefore, becomes

$$\varphi(x, y) = \sum_{n=1}^{N} \varphi_n(x, y) = \sum_{n=1}^{N} \frac{Q_n}{2\pi kH} f_n(x, y). \tag{1}$$

If all the yields Q_n are known, $\varphi(x, y)$ is thereby completely determined. However, it is a different matter if the yields are not known, but the potentials on the well faces are known. Then before formula (1) can be used the yields must be computed from the potentials given.

For the potential φ_{mn} on the well face of the m^{th} well with the centre (x_m, y_m) resulting from the action of the n^{th} well, and, if $m \neq n$, it is seen that a very good

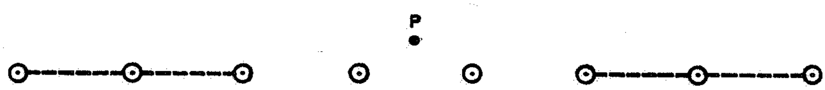


Fig. 6.3.–1

approximation is

$$\varphi_{mn} = \frac{Q_n}{2\pi kH} f_{nm},$$

in which $f_{nm} = f_n(x_m, y_m)$.

The potential φ_{mm} of the m^{th} well resulting from its own influence may be expressed in the same way, viz.,

$$\varphi_{mm} = \frac{Q_m}{2\pi kH} f_{mm}.$$

φ_{mm} is therefore the quantity which in the foregoing has always been denoted by φ_w [1]).

Therefore the total potential on the well face of the m^{th} well may be expressed as

$$\varphi_m = \sum_{n=1}^{N} \varphi_{mn} = \sum_{n=1}^{N} \frac{Q_n}{2\pi kH} f_{nm}, \qquad (2)$$

$$m = 1, 2, \ldots N.$$

If the potentials φ_m are given, it is possible to solve the yields Q_n with N equations of system (2) with N unknows and these values can then be substituted in formula (1).

To solve system (2) it is often possible to make sure of the fact that the coefficients f_{nn} are much larger than the coefficients f_{nm} with $m \neq n$. This property offers an opportunity of using iteration. Then instead of equation (2) the following equation should be used

$$\frac{Q_m}{2\pi kH} = \frac{\varphi_m}{f_{mm}} - \sum_{n=1}^{N}{}' \frac{Q_n}{2\pi kH} \cdot \frac{f_{nm}}{f_{mm}}, \qquad (2a)$$

$$m = 1, 2, \ldots N$$

in which the dash above the Σ means that the term $n = m$ is omitted.

Since $f_{nm} \ll f_{mm}$ when $n \neq m$ it follows from equation (2*a*) that

$$\frac{Q_m}{2\pi kH} \approx \frac{\varphi_m}{f_{mm}}.$$

If this approximation is substituted in the second term of (2*a*), the first approximation becomes

$$\frac{Q_m}{2\pi kH} = \frac{\varphi_m}{f_{mm}} - \sum_{n=1}^{N}{}' \frac{f_{nm}}{f_{mm}} \cdot \frac{\varphi_n}{f_{nn}} \qquad (3)$$

Etc. etc.

---

1) For a well having a radius $r_w^{(m)}$ in completely confined water it is seen that

$$f_{mm} = \ln \frac{R_{eq}^{(m)}}{r_w^{(m)}} \quad \text{(see fig. 6.2.6.)}$$

in which $R_{eq}^{(m)}$ is the equivalent radius of the m^{th} well.

Remark

If all the potentials φ_m are equal to φ_w, equation (3) may also be written as

$$Q_m = \frac{2\pi kH\varphi_w}{f_{mm}} \left\{1 - \sum_{n=1}^{N}{}' \frac{f_{nm}}{f_{nn}}\right\}. \tag{3a}$$

Examples

1°. Two wells in semi-confined ground water (see 6.1.2.).

2°. Flow from an infinite canal into three wells in a fully confined aquifer (see fig. 6.3.–1).

In this case

$$f_{11} = f_{22} = f_{33} = \ln \frac{2a}{r_w}$$

$$f_{12} = f_{21} = f_{23} = f_{32} = \ln \frac{\sqrt{b^2 + 4a^2}}{b} = \tfrac{1}{2} \ln \left(1 + \frac{4a^2}{b^2}\right)$$

$$f_{13} = f_{31} = \ln \frac{\sqrt{4b^2 + 4a^2}}{2b} = \tfrac{1}{2} \ln \left(1 + \frac{a^2}{b^2}\right)$$

(compare 6.2.1.1.).

a. If the discharge of each well is equal, the total potential becomes

$$\varphi(x, y) = \frac{Q_w}{2\pi kH} \ln \frac{\sqrt{(x+b)^2 + (y+a)^2}\,\sqrt{x^2 + (y+a)^2}\,\sqrt{(x-b)^2 + (y+a)^2}}{\sqrt{(x+b)^2 + (y-a)^2}\,\sqrt{x^2 + (y-a)^2}\,\sqrt{(x-b)^2 + (y-a)^2}}$$

and the potentials on the well faces can be expressed as

$$\varphi_1 = \varphi_3 = \frac{Q_w}{2\pi kH} \ln \frac{2a}{r_w} \left\{1 + \frac{\ln\left(1 + \frac{4a^2}{b^2}\right)}{2 \ln \frac{2a}{r_w}} + \frac{\ln\left(1 + \frac{a^2}{b^2}\right)}{2 \ln \frac{2a}{r_w}}\right\},$$

$$\varphi_2 = \frac{Q_w}{2\pi kH} \ln \frac{2a}{r_w} \left\{1 + \frac{\ln\left(1 + \frac{4a^2}{b^2}\right)}{\ln \frac{2a}{r_w}}\right\}.$$

The potentials on the well faces are therefore larger than if only one well were present and the potential on the face of the middle well is the largest, due to its being screened by the outer wells.

6.3.

If, for instance, $a = b$ and $r_W = 0.002a$, it can be seen that

$$\frac{\ln\left(1+\frac{4a^2}{b^2}\right)}{2\ln\frac{2a}{r_W}} = 0.116\,, \qquad \frac{\ln\left(1+\frac{a^2}{b^2}\right)}{2\ln\frac{2a}{r_W}} = 0.051\,.$$

Therefore

$$\varphi_1 = \varphi_3 = 1.167\,\frac{Q_W}{2\pi kH}\ln\frac{2a}{r_W}$$

$$\varphi_2 = 1.232\,\frac{Q_W}{2\pi kH}\ln\frac{2a}{r_W}\,.$$

b. If all the potentials on the well faces are equal ($\varphi_1 = \varphi_2 = \varphi_3 = \varphi_W$) the equations according to (2) $\left(\text{after division by } \frac{f_{mm}}{2\pi kH}\right)$ can be written:

$$Q_1 + Q_2\frac{\ln\left(1+\frac{4a^2}{b^2}\right)}{2\ln\frac{2a}{r_W}} + Q_3\frac{\ln\left(1+\frac{a^2}{b^2}\right)}{2\ln\frac{2a}{r_W}} = \frac{2\pi kH}{\ln\frac{2a}{r_W}}\varphi_W$$

$$Q_1\frac{\ln\left(1+\frac{4a^2}{b^2}\right)}{2\ln\frac{2a}{r_W}} + Q_2 + Q_3\frac{\ln\left(1+\frac{4a^2}{b^2}\right)}{2\ln\frac{2a}{r_W}} = \frac{2\pi kH}{\ln\frac{2a}{r_W}}\varphi_W$$

$$Q_1\frac{\ln\left(1+\frac{a^2}{b^2}\right)}{2\ln\frac{2a}{r_W}} + Q_2\frac{\ln\left(1+\frac{4a^2}{b^2}\right)}{2\ln\frac{2a}{r_W}} + Q_3 = \frac{2\pi kH}{\ln\frac{2a}{r_W}}\varphi_W\,.$$

Consequently if $a = b$ and $r_W = 0.002a$ it is seen that

$$Q_1 = Q_3 = 0.863\,\frac{2\pi kH\varphi_W}{\ln\frac{2a}{r_W}}\,, \qquad Q_2 = 0.800\,\frac{2\pi kH\varphi_W}{\ln\frac{2a}{r_W}}\,. \tag{4}$$

Of course the discharge of the middle well is the lowest (because of screening). It is also clear that the total discharge is not three times but only 2.46 times as large as the discharge of one well with the same potential at the well face. Of course, this factor becomes smaller as the quotient b/a diminishes, because as it does so the wells influence each other more and more.

If the discharge of each well is calculated with the aid of the iteration method outlined, the first approximation (equation (3*a*)) will give as factors in formulas (4) 0.833, and 0.768, respectively. This approximation is satisfactory. In the second approximation one finds 0.868 and 0.807, respectively. So here the error is smaller than 1%.

Now the total potential at the point (x, y) is

$$\varphi(x, y) = \frac{Q_1}{2\pi kH} \ln \frac{\sqrt{(x+b)^2+(y+a)^2}}{\sqrt{(x+b)^2+(y-a)^2}} + \frac{Q_2}{2\pi kH} \ln \frac{\sqrt{x^2+(y+a)^2}}{\sqrt{x^2+(y-a)^2}} + \frac{Q_3}{2\pi kH} \ln \frac{\sqrt{(x-b)^2+(y+a)^2}}{\sqrt{(x-b)^2+(y-a)^2}}$$

in which the calculated values of Q_1, Q_2 and Q_3 must be substituted.

6.3.

7. DETERMINATION OF THE VALUES OF THE FORMATION CONSTANTS BY MEANS OF PUMPING TESTS

7.1. PUMPING TESTS

With the formulas of the preceding chapters the drawdown of the potential in the vicinity of a number of pumped wells (e.g. in the well field of a water company or for the drainage of a foundation pit) can be easily computed if the type of the geo-hydrological profile and the values of the formation constants are known. These constants are the vertical resistance to water passage through the semi-pervious layers and the horizontal transmissibility of the water-bearing aquifers. Generally speaking these data are only partly known or quite unknown. Even the structure of the geo-hydrological profile is mostly not known.

The geo-hydrological profile can be ascertained by drilling a borehole; often several borings are needed to get a clear concept of the structural features such as the extent of a confining layer and the thickness of a water-bearing stratum at several places, etc.

The values of the formation constants can be computed by means of a pumping test in which the drawdown of the potential at various distances from the constantly discharging well is determined.

A pumping test is simple in principle. One well is needed for pumping and a number for observation purposes. The latter must be located at various distances from the pumped well in at least two, preferably in four directions. The initial ground-water potential in the observation wells and if possible also in the pumped well are measured, whereupon pumping is started. The discharge should be constant and as high as possible. The water levels should be observed at regular intervals during the test until equilibrium is attained; then the potentials are measured again in all wells. A third measurement must be carried out some time after the withdrawal of water has stopped. The difference between the potential head at the end of the pumping period and that prior to or after pumping is the drawdown of the potential due to the withdrawal of water.

In practice, however, a pumping test is not very simple. The figures obtained for the drawdown of the potential are only correct if no other factors have affected the potentials during the pumping test.

Examples of such factors are: a change in the original ground-water table, modifications due to neighbouring withdrawals of ground water, alterations in barometric pressure, tidal fluctuations, etc.

Such effects are very difficult to control. In princple it can be done by sinking additional observation wells outside the area of the pumped well affected and to

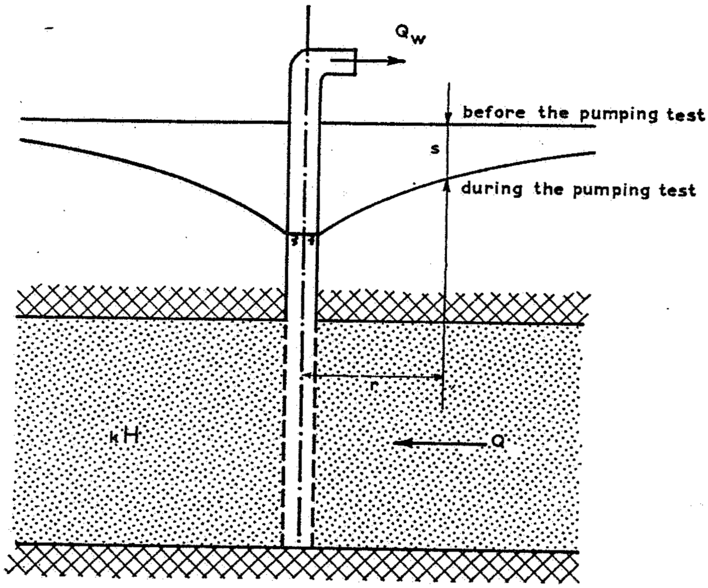


Fig. 7.2.-1

superimpose on the results of the pumping test the changes in potential recorded in these observation wells. The additional observation wells, however, are generally rather far from the observation wells the drawdown of which has to be corrected. So the correction itself cannot be quite exact. The effect of this correction and the error in it can be relatively reduced by pushing the yield of the pumping test up as high as possible; this is of paramount importance. The greater expenses entailed is more than offset by the more accurate results obtained. Other errors may creep in due to the fact that it is often extremely difficult to ascertain the moment at which a state of equilibrium is reached during and after pumping. In cases of completely and of partially confined ground water a state of equilibrium is reached quickly, i.e., after some hours or days. In the case of phreatic water on the other hand one has to wait longer, sometimes such a long time that analysis of the results of the pumping test based on the state of equilibrium becomes impossible. Patience and prolonged observation of the drawdown of the potentials at regular intervals are the only remedies. Errors which cannot be eliminated by more powerful pumping or by waiting longer are those due to variations in the values of the formation constants in several directions and at various distances. Often the correct interpretation of the results of such a pumping test is very difficult.

Though it may often be imperfect, a pumping test will provide the relation between the distance to the pumped well and the magnitude of the drawdown of the potential caused by pumping at a certain rate. If several pumping tests have to be analysed it is desirable to reduce the observed drawdown of the potential to a standard rate, e.g. 250 m³/day, in order to achieve uniformity.

7.2. Analysis of a pumping test in confined ground water (see 5.1.1.)

The formula for the drawdown of the potential when pumping a well in completely confined ground water in the geo-hydrological profile given in figure 7.2.–1 is

$$s = \frac{Q_W}{2\pi kH} \ln r + C.$$

If s is plotted on a linear scale and r on a logarithmic scale a straight line is obtained. The slope of this line is a measure of the transmissibility kH.

7.3. Analysis of a pumping test in semi-confined ground water (see 5.1.2.1.)

The following formula applies to the drawdown of a well in a semi-confined aquifer as represented in figure 7.3.–1.

$$s = \frac{Q_W}{2\pi kH} K_0\left(\frac{r}{\lambda}\right).$$

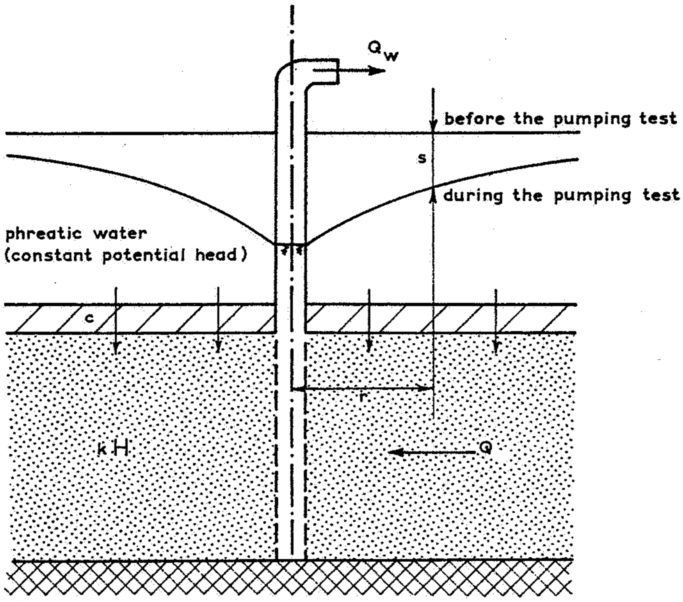


Fig. 7.3.–1

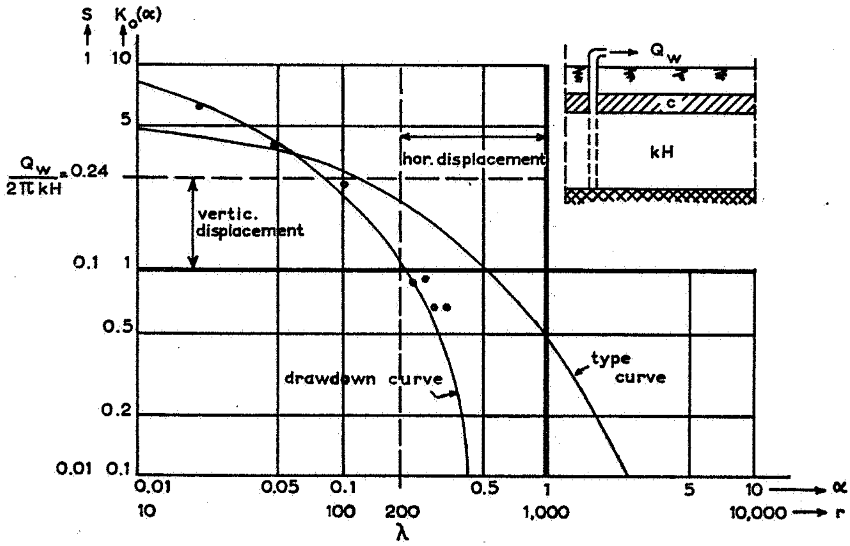


Fig. 7.3.–2

The observed drawdown of the potential and the distance r from the well are plotted on double logarithmic paper. The graph of the function $K_0(\alpha)$ is drawn on the same scale but on tracing paper. This graph is superimposed on the first one and is then shifted without rotating it till it coincides as nearly as possible with the points plotted. The vertical displacement of the axis stands for the quotient $\frac{Q_W}{2\pi kH}$, from which the transmissibility kH can be derived, and the horizontal displacement stands for the leakage factor λ, i.e. $\sqrt{kHc}$, from which the resistance c can be calculated (fig. 7.3.-2). There are two points to which attention should be drawn.

First of all it should be realised that the well is usually a partially penetrating one; consequently the observed drawdowns of the potential at short distances from the well are greater than in the case of a fully penetrating well (for which the formula holds good); these drawdown figures, therefore, may not be used for the analysis proper. They can, however, be used as a check for the analysis, after they have been adjusted to eliminate the effect of partial penetration. Such adjustments to the drawdown figures can be made with the aid of the formulas given in 5.4.

In the second place it should be noted that the relative accuracy of the observations decreases as the distances increase; so less value should be attached to observations made at greater distance.

The analysis of the results of a pumping-test carried out 2 km south of Hengelo in 1942 is a good example. This test was part of investigations aimed at finding a suitable area for an additional well field for the municipal waterworks of Hengelo and was carried out under the supervision of the Rijksinstituut voor Drinkwatervoorziening (Government Institute for Water Supply). As appeared from the results of the borings carried out for the construction of the pumping and observation wells, the subsoil down to about 10 m consists of fine silty sand with layers of peat and loam below it. Underneath them lies a sand stratum 10 tot 15 m thick, the deepest layers of it being coarse-grained. The upper part of the next formation consists of fine silty sand, the lower part is loam.

There is phreatic water in the upper water-bearing sand layer, separated by the peat and loam layers at a depth of 10 m from the second water-bearing sand stratum with semi-confined ground water. During the 24 hour pumping test 11 m^3 p.h. was withdrawn from the latter stratum. No perceptible drop in the phreatic level took place during that period.

Application of the above-mentioned method of analysis gives the following result (see fig. 7.3.-2)

$$\frac{Q_W}{2\pi kH} = 0.24 \text{ m} \quad \text{or} \quad kH = 174 \text{ m}^2/\text{day} ;$$
$$\lambda = 200 \text{ m} \quad \text{or} \quad c = 230 \text{ days} .$$

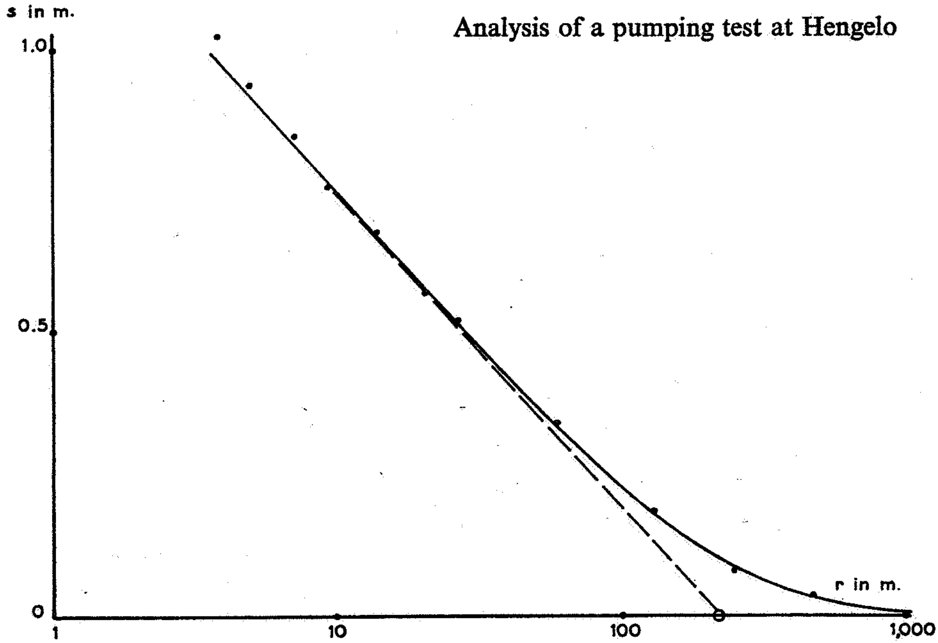

s in m.
1.0
0.5
0
Analysis of a pumping test at Hengelo
r in m.
1
10
100
1,000

Fig. 7.3.-3

If the leakage factor λ is large or if the distance r is small and consequently $\frac{r}{\lambda}$ is small, we have

$$K_0\left(\frac{r}{\lambda}\right) \approx \ln \frac{1.123}{r} \lambda \cdot$$

Then the drawdown of the potential can be expressed as

$$s = \frac{Q_W}{2\pi kH} \ln \frac{1.123}{r} \lambda \cdot$$

In the vicinity of the well the relation between s on a linear scale and r on a logarithmic scale plots as a straight line. The slope of this line indicates the transmissibility kH. The point of intersection of this line and the axis $s = o$ indicates directly the value of λ and c (fig. 7.3.–3).

7.4. ANALYSIS OF A PUMPING TEST IN AN AQUIFER BETWEEN TWO SEMI-PERVIOUS LAYERS (see 5.1.3.)

The following formulas hold good for a fully penetrating well in the profile of the strata given in fig. 7.4.–1.

$$s_1(r) = \frac{Q_W}{2\pi k_1 H_1} \frac{1}{\lambda_1^2 - \lambda_2^2} \left[(\alpha_2\lambda_1^2 - 1)\lambda_2^2 K_0\left(\frac{r}{\lambda_1}\right) - (\alpha_2\lambda_2^2 - 1)\lambda_1^2 K_0\left(\frac{r}{\lambda_2}\right)\right],$$

and

$$s_2(r) = \frac{Q_W}{2\pi k_1 H_1} \frac{\alpha_2 \lambda_1^2 \lambda_2^2}{\lambda_1^2 - \lambda_2^2} \left[K_0\left(\frac{r}{\lambda_1}\right) - K_0\left(\frac{r}{\lambda_2}\right)\right] \cdot$$

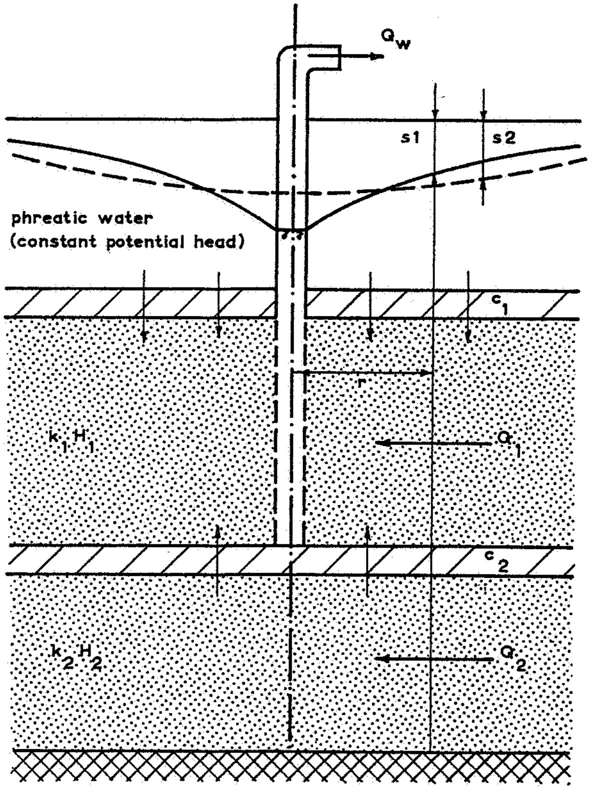
These intricate formulas with 4 unknown quantities are not easy to analyse and do not always produce an unequivocal result.
How the results of pumping tests are analysed depends on the data available. These data will almost always include the drawdown of the potential in the stratum tapped by the well.

Since $K_0(x) \approx \ln \frac{1.123}{x}$ for small values of x (see Appendix) the formula for s_1 can be reduced to

$$s_1(r) = \frac{Q_W}{2\pi k_1 H_1} \frac{2.3026}{\lambda_1^2 - \lambda_2^2} [(\alpha_2\lambda_1^2 - 1)\lambda_2^2 \log \lambda_1 - (\alpha_2\lambda_2^2 - 1)\lambda_1^2 \log \lambda^2] +$$

$$+ \frac{Q_W}{2\pi k_1 H_1} 0.11601 - \frac{Q_W}{2\pi k_1 H_1} 2.30261 \log r$$

or $s_1(r) = B - A \log r$.



Qw
s1
s2
phreatic water
(constant potential head)
c1
r
k1 H1
Q1
c2
k2 H2
Q2

Fig. **7.4.-1**

The graph of this formula is a straight line if it is plotted on semi-logarithmic paper. The slope of this line is related to $k_1 H_1$ and its position on the graph to $k_1 H_1$, $k_2 H_2$, c_1 and c_2. This linear relation gives a good approximation only for short distances from the pumped well. The straight line should therefore be drawn as an asymptote to the curve drawn through the points plotted, taking into account the effect of the partial penetration of the well due to the short distances r used in this analysis. When, however, after some trials k_1 has been found the effect of partial penetration can easily be computed with the formulas given in 5.4.

The analysis of the results cf a pumping test carried out at Velsen in 1937 is a good example. This test done at the south side of the Noordzeekanaal (North Sea canal) formed part of a geo-hydrological investigation for the construction of a tunnel under the canal. The investigation was carried out by the Rijksinstituut voor Drinkwater-voorziening. At the site of the pumping test the subsoil consists of a series of hardly permeable layers down to about 16 m below ordnance datum N.A.P., its lower peat layers in particular offering great resistance to ground-water flow, then a water-bearing sandy stratum down to about N.A.P.—40 m, then a relatively thin layer of clay and underneath that another water-bearing stratum extending to a great depth and resting on a very thick impermeable base.

During the test water was withdrawn from the sandy stratum between N.A.P.—1616 and —40 m via a partially penetrating well. The ground-water level in the layers above N.A.P.—16 m did not change during pumping.

The drawdown of the potential caused by pumping at a reduced standard yield of 250 m³/day has been plotted in figure 7.4.–2. Measured on the well face the drawdown was 0.924 m. Since the effect of partial penetration amounts to 0.29 m for a screen radius of 0.15 m the drawdown of the potential of a fully penetrating well would have been 0.63 m measured on the well face.

The points plotted in figure 7.4.–2 give the tangent

$$s_1 (r) = 0.480 - 0.191 \log r.$$

From the value $A = 0.191$ it follows that

$$k_1 H_1 = \frac{250 \times 2.3026}{2 \times 0.191} = 480 \text{ m}^2/\text{day}$$

and from the value $B = 0.480$ a relation between the three remaining geo-hydrological constants can be established. This relation is given in the first table of page 165; it has been composed by means of the nomograph in figure 7.4.–3 (pages 168 and 169) by determining the value of c_2 for each combination of c_1 and $k_2 H_2$ selected.

Analysis of a pumping test at Velsen

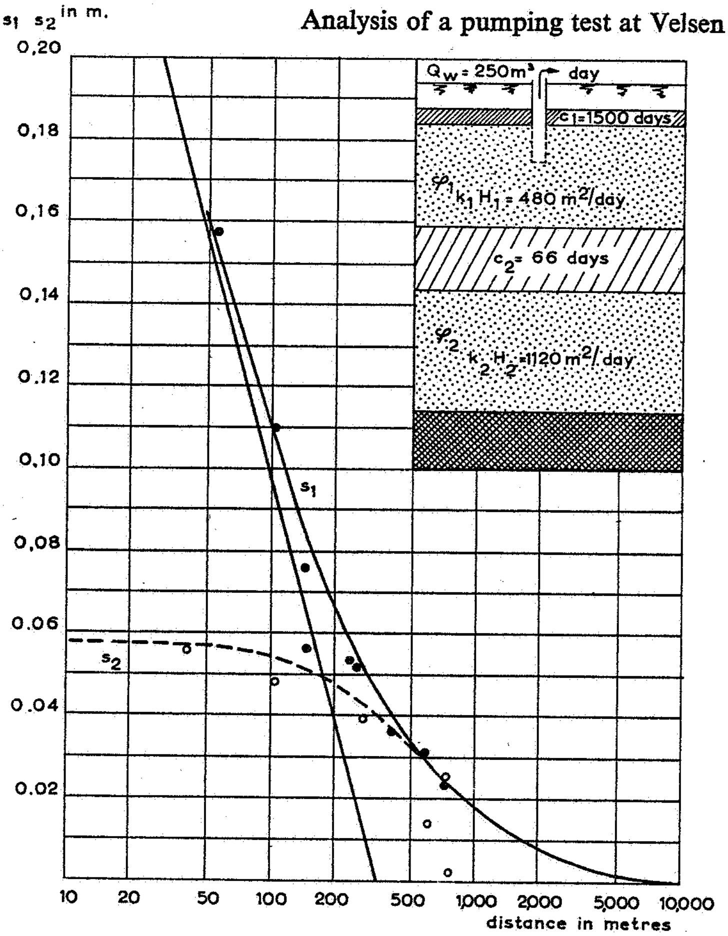


Fig. 7.4.–2

k_2H_2	$c_1 = 1{,}000$	2,000	5,000	10,000 days
1,000 *m²/day*	$c_2 =$ 76	51	31	22
2,000 „	117	91	68	56
3,000 „	138	113	91	79
4,000 „	150	129	104	94 days

If observations of the drawdown $s_2\,(r)$ in the stratum underneath the second confining layer are available then, according to

$$K_0(x) \approx \ln \frac{1.123}{x}, \text{ for small values of } x,\ s_2 \text{ can be reduced to}$$

$$s_2(r) = \frac{Q_W}{2\pi k_1 H_1} \times \frac{\alpha_2 \lambda_1^2 \lambda_2^2}{\lambda_1^2 - \lambda_2^2} \times 2.3026 \log \frac{\lambda_1}{\lambda_2}$$

holding good only for small values of r or $s_2\,(r) =$ constant. The pumping test at Velsen led to $s_2\,(o) = 0.058$ m (see fig. 7.4.–2). With this value and the nomograph in figure 7.4.–3 the following relation between k_2H_2, c_1 and c_2 can be established.

k_2H_2	$c_1 = 1{,}000$	2,000	5,000	10,000 days
1,000 m²/day	$c_2 =$ 60	120	301	603
2,000 „	5	9	23	45
3,000 „	0	1	2	3
4,000 „	0	0	0	0 days

Both relations given in the foregoing tables have 2 degrees of freedom. Combining these data gives the table below, which still leaves 1 degree of freedom between the 3 formation constants k_2H_2, c_1 and c_2.

$c_1 =$	1,000	2,000	5,000	10.000	days
$c_2 =$	72	62	57	54	days
$k_2H_2 =$	930	1240	1660	1930	m²/days

Finally the fourth relation between the 4 formation constants can be derived from the grade line of the potentials at a greater distance from the pumped well. Mathematical formulation of this relation is impossible. Therefore technicians of the Municipal Waterworks of Amsterdam have drawn standard curves which give s_1 and s_2 as a function of the distance to the pumped well for values of $c_1 = 1{,}000 - 2{,}000 -$

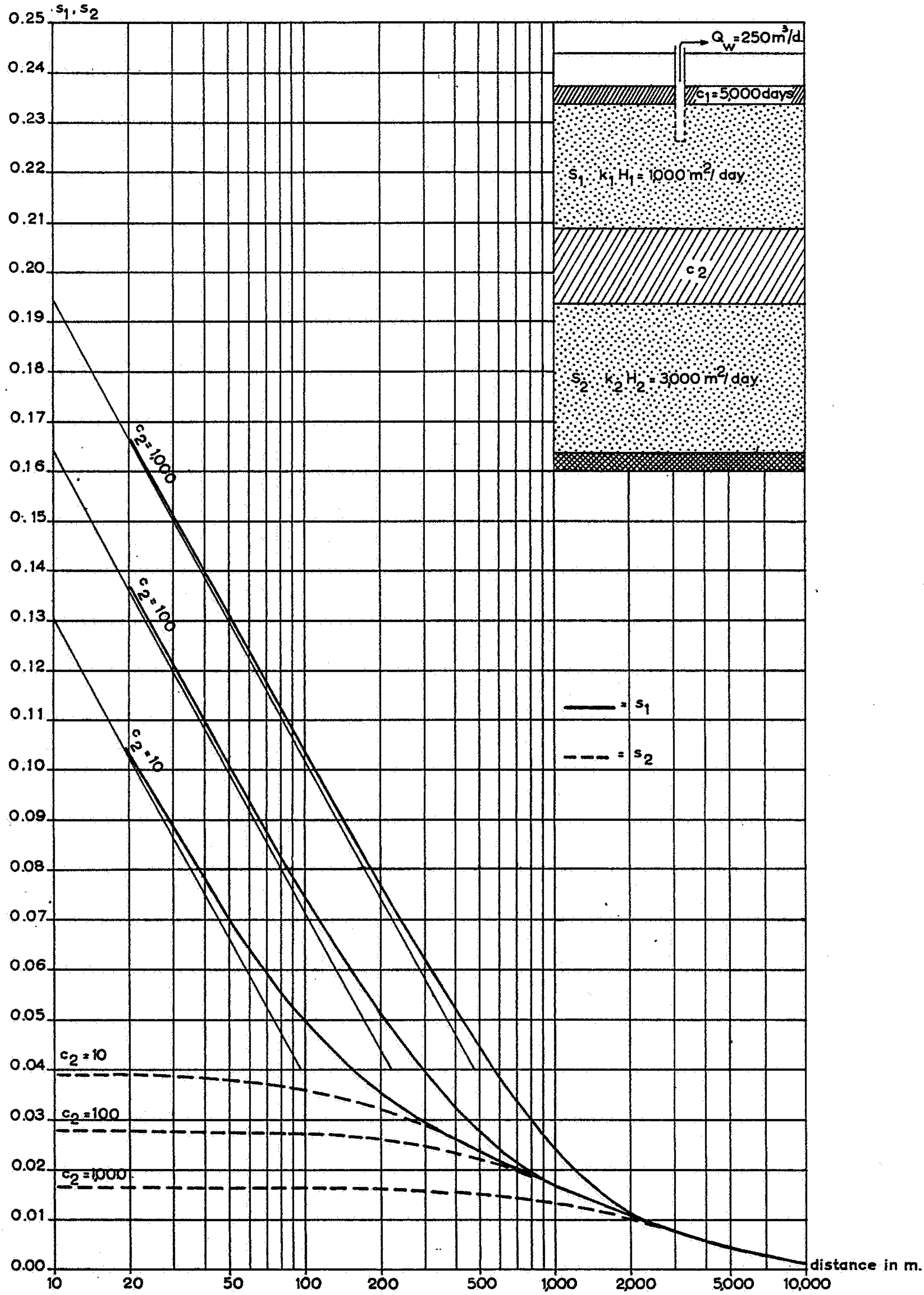


Fig. 7.4.–4 Drawdown curves for a well in an aquifer between two semi-permeable layers

5,000 and 10,000 days, $c_2 = 10$ — 100 and 1,000 days, $k_1H_1 = 500$ — 1,000 and 1,500 m²/day and $\Sigma\, kH = 3{,}000$ — 4,000 — 5,000 and 6,000 m²/day. Figure 7.4.–4 gives an example of such a set of curves. With

$$\begin{aligned} c_1 &= 1{,}500 \text{ days} \\ k_1H_1 &= 480 \text{ m}^2/\text{day} \\ c_2 &= 66 \text{ days} \\ k_2H_2 &= 1{,}120 \text{ m}^2/\text{day} \end{aligned}$$

the grade line of the potentials and the standard curves match as nearly as possible. This result is reasonably accurate for the values obtained for k_1H_1, c_2 and k_2H_2. For the value of c_1 the result is less accurate.

7.5. ANALYSIS OF A PUMPING TEST IN PHREATIC GROUND WATER (see 5.2.2.)

When the well is situated at a great distance from open water it takes so long time, fore a state of equilibrium to be reached that an analysis based on the formulas of steady flow is impossible (see 7.1.1.). The determination of the transmissibility must therefore be done by means of the formulas for nonsteady flow. If the location of the well is regarded as the centre of a circular island, the state of equilibrium can be reasonably well approximated by

$$s = \frac{-Q_W}{2\pi kH} \ln \frac{R}{r}$$

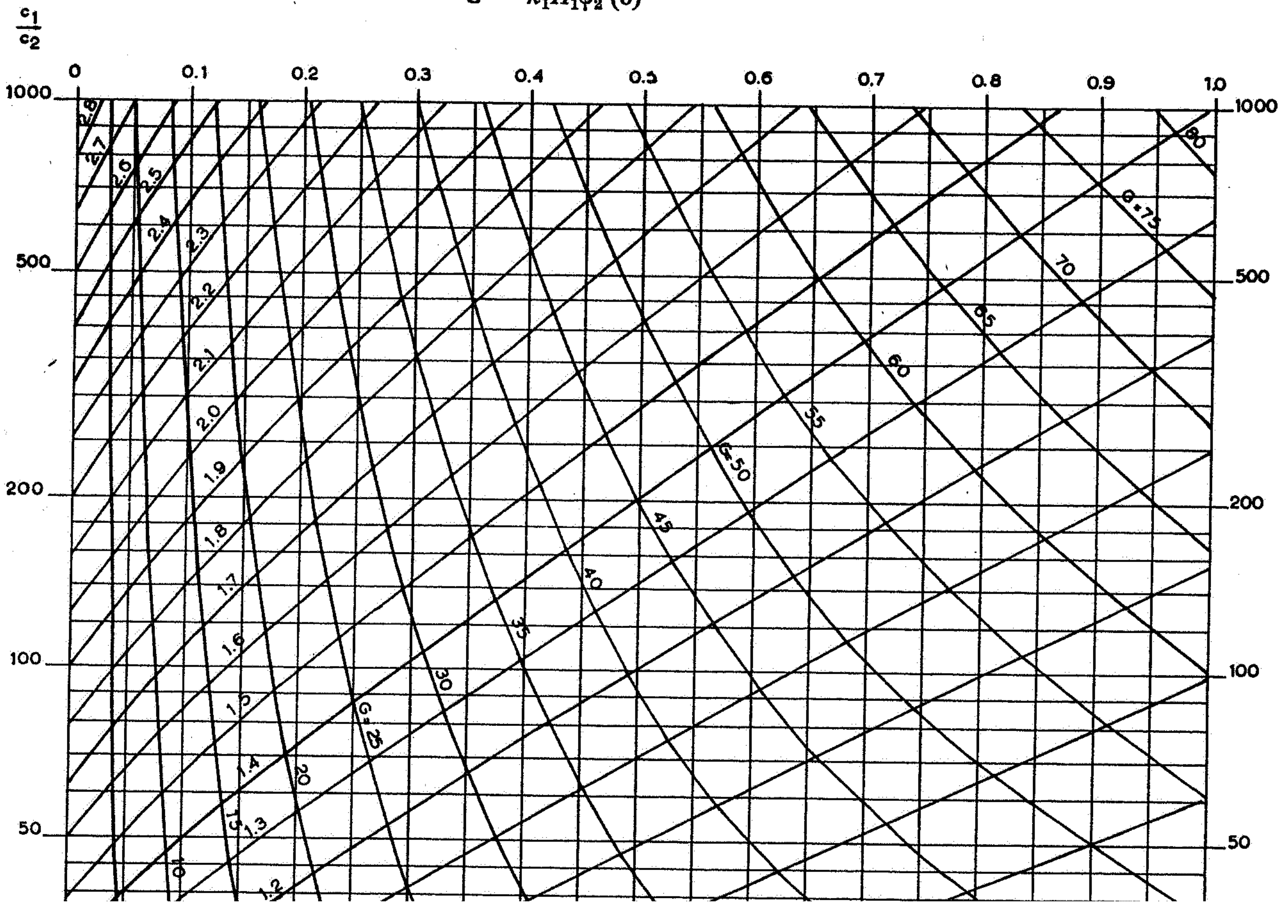
which is a straight line when plotted on semi-logarithmic paper. With s on the linear scale and r on the logarithmic scale the slope of this line represents the transmissibility of the soil.

Formulas used:

$$s_1(r) \sim B - A \log_{10} r$$

$$F = -\frac{Bk_1H_1}{45{,}81} + \log_{10}(k_1H_1c_1)$$

$$G = k_1H_1\varphi_2(0)$$



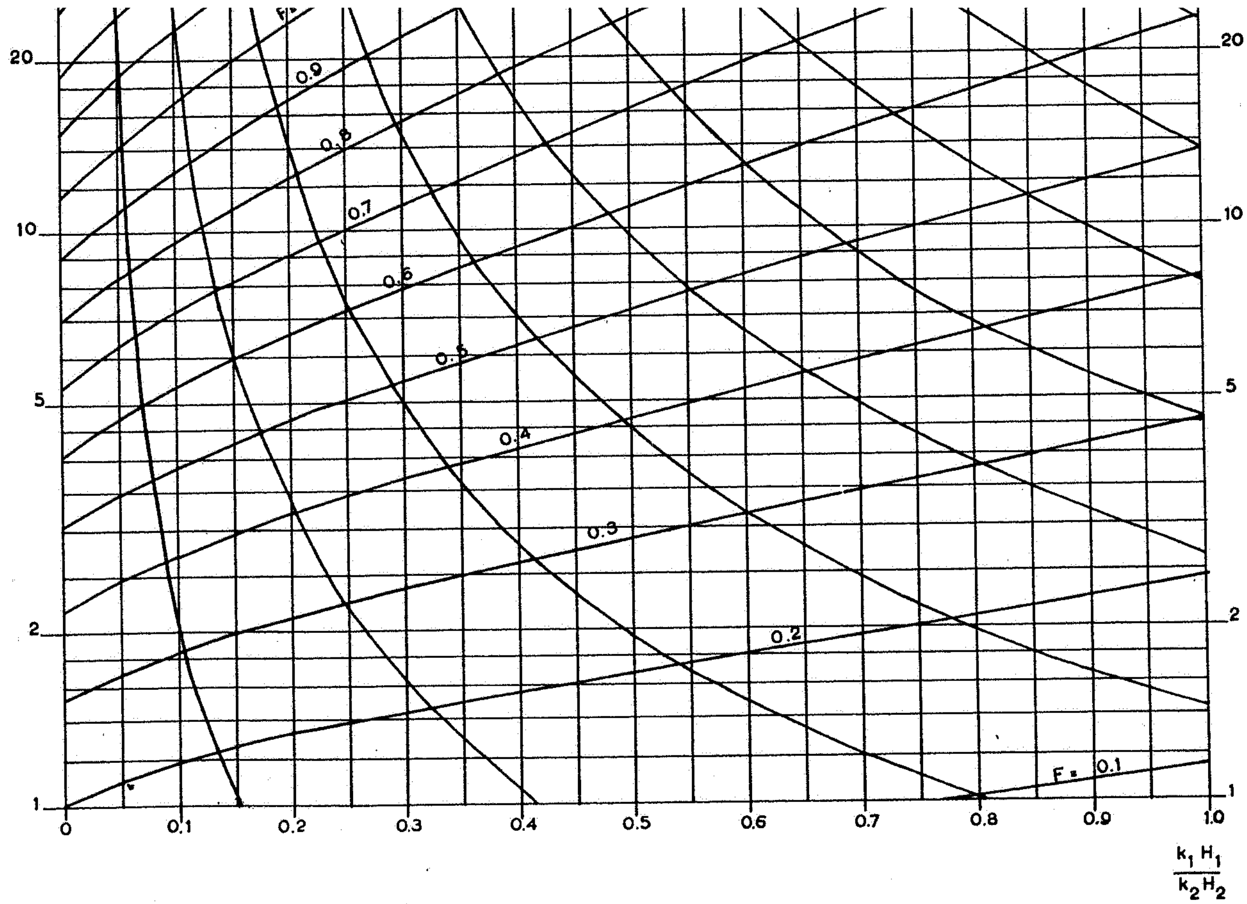


Fig. **7.4.–3** Pumping test of a well in an aquifer between two semi-pervious layers. Relation between c_1, c_2, k_1H_1 and k_2H_2.

8. APPENDIX

The *Bessel equation* of the n^{th} order ($n = 0, 1, \ldots.$)

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0 \qquad (1)$$

has two independent solutions, viz. the functions $J_n(x)$ and $Y_n(x)$ (Bessel function and Neumann function of the n^{th} order resp.), defined by the equations

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{x}{2}\right)^{n+2k}$$

$$Y_n(x) = -\frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{2}{x}\right)^{n-2k} + \frac{2}{\pi}\left(\ln\frac{x}{2} + \gamma\right) J_n(x) +$$

$$-\frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(1 + \tfrac{1}{2} + \ldots + \frac{1}{k} + 1 + \tfrac{1}{2} + \ldots + \frac{1}{k+n}\right) \left(\frac{x}{2}\right)^{n+2k}$$

(there is no first series for $n = 0$).

γ is the Euler constant, the value of which is 0.5772 . . .

N.B. It should be noted that

$$\gamma + \ln\frac{x}{2} = -\ln\frac{1.123\ldots}{x}.$$

The *Hankel functions* of the first and second kind and of the n^{th} order are defined by the equations

$$H_n^{(1)}(x) = J_n(x) + i\,Y_n(x), \text{ and}$$
$$H_n^{(2)}(x) = J_n(x) - i\,Y_n(x), \text{ respectively.}$$

These functions also constitute independent solutions of equation (1).

In hydrological applications formula (1) is usually replaced by the *modified Bessel equation* of the n^{th} order

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(1 + \frac{n^2}{x^2}\right) y = 0.$$

On substituting ix for x it appears that $J_n(ix)$ and $Y_n(ix)$ are independent solutions of this equation.

As $x^{-n} J_n(x)$ is an even function of x, $J_n(ix)$ is real for real x and for even n, and it is purely imaginary for odd n. Therefore the function

$$I_n(x) = i^{-n} J_n(ix)$$

(modified Bessel function of the n^{th} order) is, for each n, real for real x.

The function $Y_n(ix)$ for positive real x is neither zero nor purely imaginary. As $\ln\left(\frac{ix}{2}\right) = \frac{\pi i}{2} + \ln\left(\frac{x}{2}\right)$, for even n the imaginary part of $Y_n(ix)$ equals $J_n(ix)$. Consequently, for even n the function $Y_n(ix) - i\,J_n(ix) = -\,iH_n^{(1)}(ix)$ is real for positive real x. In a similar way it appears that this function is purely imaginary if n is odd. Therefore, as a second solution of the modified Bessel equation we could choose the function $i^{n+(1)}\,H_n^{(1)}(ix)$, which is real for positive real x. It is more usual, however, to choose for this the function

$$K_n(x) = \frac{\pi}{2} i^{n+1} H_n^{(1)}(ix) = \frac{\pi}{2} i^{n+1} [J_n(ix) + i\,Y_n(ix)]$$

(modified Hankel function of the n^{th} order).

We see that

$$I_n(x) = \sum_{k=0}^{\infty} \frac{1}{k!\,(k+n)!}\left(\frac{x}{2}\right)^{n+2k},$$

$$K_n(x) = \tfrac{1}{2}\sum_{k=0}^{n-1} (-1)^k \frac{(n-k-1)!}{k!}\left(\frac{2}{x}\right)^{n-2k} - (-1)^n\left(\ln\frac{x}{2} + \gamma\right) I_n(x) +$$

$$+ \tfrac{1}{2}(-1)^n \sum_{k=0}^{\infty} \frac{1}{k!\,(k+n)!}\left(1 + \tfrac{1}{2} + \ldots + \frac{1}{k} + 1 + \tfrac{1}{2} + \ldots + \frac{1}{k+n}\right)\left(\frac{x}{2}\right)^{2k+n}$$

(there is no first series for $n = 0$).

For $n = 0$, or 1 we have

$$I_0(x) = 1 + \left(\frac{x}{2}\right)^2 + \frac{1}{2!2!}\left(\frac{x}{2}\right)^4 + \ldots,$$

$$I_1(x) = \frac{x}{2}\left\{1 + \frac{1}{1!2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^4 + \ldots\right\},$$

$$K_0(x) = I_0(x)\ln\frac{1.123..}{x} + \left(\frac{x}{2}\right)^2 + \frac{1+\frac{1}{2}}{2!2!}\left(\frac{x}{2}\right)^4 + \frac{1+\frac{1}{2}+\frac{1}{3}}{3!3!}\left(\frac{x}{2}\right)^6 + \ldots,$$

$$K_1(x) = \frac{1}{x} - I_1(x)\ln\frac{1.123..}{x} - \tfrac{1}{2}\left\{\frac{x}{2} + \frac{1+1+\frac{1}{2}}{1!2!}\left(\frac{x}{2}\right)^3 + \frac{1+\frac{1}{2}+1+\frac{1}{2}+\frac{1}{3}}{2!3!}\left(\frac{x}{2}\right)^5 + \ldots\right\}.$$

For small x we have

$$K_0(x) \sim \ln\frac{1.123}{x}$$

with an error of less than 1‰ for $x < 0.06$,
1% for $x < 0.18$,
5% for $x < 0.33$,

and

$$K_1(x) \sim \frac{1}{x}$$

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with an error of less than 1‰ for $x < 0.02$,
1% for $x < 0.08$,
5% for $x < 0.21$.

Formulas for the differential

$$I_n'(x) = \frac{n}{x} I_n(x) + I_{n+1}(x) =$$
$$= -\frac{n}{x} I_n(x) + I_{n-1}(x) \cdot$$
$$K_n'(x) = \frac{n}{x} K_n(x) - K_{n+1}(x)$$
$$= -\frac{n}{x} K_n(x) - K_{n-1}(x)$$

Therefore specifically $I_0'(x) = I_1(x)$, $K_0'(x) = -K_1(x)$. The functions $I_n(x)$ and $K_n(x)$ for positive x are positive and monotonously rising or falling, respectively.

Wronski determinant

$$\begin{vmatrix} I_n(x) & K_n(x) \\ I_n'(x) & K_n'(x) \end{vmatrix} = I_n(x) K_n'(x) - I_n'(x) K_n(x) = -\frac{1}{x}.$$

Asymptotic series

$$I_n(x) \sim \frac{1}{\sqrt{2\pi x}} e^{x} \left\{1 - \frac{4n^2 - 1^2}{1!8x} + \frac{(4n^2 - 1^2)(4n^2 - 3^2)}{2!(8x)^2} - \dots\right\},$$

$$K_n(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \left\{1 + \frac{4n^2 - 1^2}{1!8x} + \frac{(4n^2 - 1^2)(4n^2 - 3^2)}{2!(8x)^2} + \dots\right\}.$$

These series are divergent but for large x they give very good approximations if they are cut short after a certain number of terms. When cutting short the error is of the order of the first term to be omitted, therefore the proportional error for increasing x will approach zero, provided that a fixed number of terms is being considered.

Integral relations

We have

$$\int_0^\infty \frac{e^{-b\sqrt{a^2+t^2}}}{\sqrt{a^2 + t^2}} \cos t\, u\, dt = K_0(a\sqrt{b^2 + u^2}) \quad (a > 0, b \geqslant 0).$$

For $b = 0$ it follows (if $u \neq 0$, $a > 0$) that

$$\int_0^\infty \frac{1}{\sqrt{a^2 + t^2}} \cos t\, u\, dt = K_0(a\,|\,u\,|).$$

We also have

$$\int_0^\infty K_0(a\sqrt{b^2 + u^2}) \cos t\, u\, du = \frac{\pi}{2} \cdot \frac{e^{-b\sqrt{a^2+t^2}}}{\sqrt{a^2 + t^2}}, \qquad a > 0, b \geqslant 0.$$

For $t = 0$ it follows that

$$\int_0^\infty K_0(a\sqrt{b^2 + u^2})\; du = \frac{\pi}{2a}\, e^{-ab}, \quad a > 0, b \geqslant 0.$$

Hence by differentiation to b

$$\int \frac{ab}{\sqrt{b^2 + u^2}}\, K_0'(a\sqrt{b^2 + u^2})\, du = -\frac{\pi}{2}\, e^{-ab}.$$

Addition formula

For $0 \leqslant u < v$ we have

$$K_0(\sqrt{u^2 + v^2 - 2uv\cos\Theta}) = \sum_{n=0}^{\infty} \varepsilon_n\, i_n(u)\, K_n(v) \cos n\,\Theta\,,$$

in which $\varepsilon_0 = 1$, $\varepsilon_n = 2$ for $n \geqslant 1$.

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x	$K_0(x)$	$K_1(x)$	$I_0(x)$	$I_1(x)$	x	$K_0(x)$	$K_1(x)$	$I_0(x)$	$I_1(x)$
0,00	∞	∞	1,00000	0,00000	0,50	0,92442	1,65644	1,06348	0,25789
0,01	4,72124	99,97389	1,00003	0,00500	0,51	0,90806	1,61489	1,06609	0,26338
0,02	4,02846	49,95472	1,00010	0,01000	0,52	0,89212	1,57492	1,06875	0,26889
0,03	3,62353	33,27149	1,00023	0,01500	0,53	0,87656	1,53645	1,07147	0,27441
0,04	3,33654	24,92329	1,00040	0,02000	0,54	0,86138	1,49938	1,07424	0,27996
0,05	3,11423	19,90967	1,00063	0,02501	0,55	0,84657	1,46366	1,07707	0,28553
0,06	2,93288	16,56373	1,00090	0,03001	0,56	0,83210	1,42921	1,07995	0,29112
0,07	2,77982	14,17100	1,00123	0,03502	0,57	0,81798	1,39596	1,08289	0,29673
0,08	2,64749	12,37421	1,00160	0,04003	0,58	0,80418	1,36385	1,08588	0,30237
0,09	2,53102	10,97486	1,00203	0,04505	0,59	0,79070	1,33282	1,08894	0,30802
0,10	2,42707	9,85384	1,00250	0,05006	0,60	0,77752	1,30283	1,09205	0,31370
0,11	2,33327	8,93534	1,00303	0,05508	0,61	0,76464	1,27383	1,09521	0,31941
0,12	2,24786	8,16878	1,00360	0,06011	0,62	0,75204	1,24576	1,09843	0,32514
0,13	2,16950	7,51919	1,00423	0,06514	0,63	0,73972	1,21859	1,10171	0,33089
0,14	2,09717	6,96154	1,00491	0,07017	0,64	0,72767	1,19227	1,10505	0,33667
0,15	2,03003	6,47750	1,00563	0,07521	0,65	0,71587	1,16676	1,10845	0,34247
0,16	1,96742	6,05330	1,00641	0,08026	0,66	0,70433	1,14204	1,11190	0,34830
0,17	1,90880	5,67842	1,00724	0,08531	0,67	0,69303	1,11806	1,11541	0,35415
0,18	1,85371	5,34467	1,00812	0,09036	0,68	0,68197	1,09479	1,11898	0,36003
0,19	1,80179	5,04558	1,00905	0,09543	0,69	0,67113	1,07221	1,12261	0,36594
0,20	1,75270	4,77597	1,01003	0,10050	0,70	0,66052	1,05028	1,12630	0,37188
0,21	1,70619	4,53167	1,01106	0,10558	0,71	0,65012	1,02898	1,13005	0,37784
0,22	1,66200	4,30923	1,01214	0,11067	0,72	0,63994	1,00829	1,13386	0,38384
0,23	1,61994	4,10582	1,01327	0,11576	0,73	0,62996	0,98817	1,13773	0,38986
0,24	1,57983	3,91908	1,01445	0,12087	0,74	0,62017	0,96861	1,14166	0,39591
0,25	1,54151	3,74703	1,01569	0,12598	0,75	0,61058	0,94958	1,14565	0,40199
0,26	1,50484	3,58797	1,01697	0,13110	0,76	0,60118	0,93107	1,14970	0,40810
0,27	1,46971	3,44049	1,01831	0,13623	0,77	0,59196	0,91305	1,15381	0,41425
0,28	1,43600	3,30335	1,01970	0,14138	0,78	0,58292	0,89551	1,15798	0,42042
0,29	1,40361	3,17549	1,02114	0,14653	0,79	0,57405	0,87842	1,16222	0,42663
0,30	1,37246	3,05599	1,02263	0,15169	0,80	0,56535	0,86178	1,16651	0,43286
0,31	1,34247	2,94406	1,02417	0,15687	0,81	0,55681	0,84557	1,17087	0,43914
0,32	1,31356	2,83898	1,02576	0,16206	0,82	0,54843	0,82976	1,17530	0,44544
0,33	1,28567	2,74016	1,02741	0,16726	0,83	0,54021	0,81435	1,17978	0,45178
0,34	1,25873	2,64703	1,02911	0,17247	0,84	0,53215	0,79933	1,18433	0,45815
0,35	1,23271	2,55912	1,03086	0,17769	0,85	0,52423	0,78468	1,18895	0,46456
0,36	1,20754	2,47601	1,03266	0,18293	0,86	0,51645	0,77038	1,19362	0,47100
0,37	1,18317	2,39730	1,03452	0,18818	0,87	0,50882	0,75643	1,19837	0,47748
0,38	1,15958	2,32265	1,03643	0,19345	0,88	0,50132	0,74281	1,20317	0,48399
0,39	1,13671	2,25176	1,03839	1,19873	0,89	0,49396	0,72952	1,20805	0,49054
0,40	1,11453	2,18435	1,04040	0,20403	0,90	0,48673	0,71653	1,21299	0,49713
0,41	1,09301	2,12018	1,04247	0,20934	0,91	0,47963	0,70385	1,21799	0,50375
0,42	1,07212	2,05900	1,04459	0,21466	0,92	0,47265	0,69147	1,22306	0,51041
0,43	1,05182	2,00062	1,04676	0,22001	0,93	0,46580	0,67937	1,22820	0,51712
0,44	1,03209	1,94485	1,04899	0,22537	0,94	0,45906	0,66754	1,23340	0,52386
0,45	1,01291	1,89152	1,05127	0,23074	0,95	0,45245	0,65598	1,23868	0,53064
0,46	0,99426	1,84048	1,05360	0,23614	0,96	0,44594	0,64468	1,24402	0,53746
0,47	0,97610	1,79157	1,05599	0,24155	0,97	0,43955	0,63363	1,24942	0,54432
0,48	0,95842	1,74467	1,05843	0,24698	0,98	0,43327	0,62282	1,25490	0,55123
0,49	0,94120	1,69967	1,06093	0,25243	0,99	0,42710	0,61225	1,26045	0,55817
0,50	0,92442	1,65644	1,06348	0,25789	1,00	0,42102	0,60191	1,26607	0,56516

x	$K_0(x)$	$K_1(x)$	$I_0(x)$	$I_1(x)$	x	$K_0(x)$	$K_1(x)$	$I_0(x)$	$I_1(x)$
1,00	0,42102	0,60191	1,26607	0,56516	1,50	0,21381	0,27739	1,64672	0,98167
1,01	0,41506	0,59179	1,27175	0,57219	1,51	0,21105	0,27343	1,65659	0,99163
1,02	0,40919	0,58189	1,27751	0,57926	1,52	0,20834	0,26954	1,66656	0,00166
1,03	0,40342	0,57219	1,28334	0,58638	1,53	0,20566	0,26572	1,67662	1,01178
1,04	0,39774	0,56270	1,28924	0,59354	1,54	0,20302	0,26196	1,68679	1,02197
1,05	0,39216	0,55341	1,29521	0,60075	1,55	0,20042	0,25826	1,69706	1,03224
1,06	0,38667	0,54432	1,30125	0,60801	1,56	0,19786	0,25462	1,70744	1,04259
1,07	0,38128	0,53541	1,30737	0,61531	1,57	0,19533	0,25104	1,71791	1,05302
1,08	0,37597	0,52668	1,31356	0,62265	1,58	0,19284	0,24751	1,72850	1,06354
1,09	0,37074	0,51814	1,31982	0,63005	1,59	0,19038	0,24404	1,73919	1,07413
1,10	0,36560	0,50976	1,32616	0,63749	1,60	0,18795	0,24063	1,74998	1,08481
1,11	0,36055	0,50155	1,33257	0,64498	1,61	0,18557	0,23728	1,76088	1,09557
1,12	0,35557	0,49351	1,33906	0,65252	1,62	0,18321	0,23397	1,77189	1,10642
1,13	0,35068	0,48563	1,34562	0,66011	1,63	0,18089	0,23072	1,78301	1,11735
1,14	0,34586	0,47790	1,35226	0,66775	1,64	0,17859	0,22753	1,79424	1,12837
1,15	0,34112	0,47033	1,35898	0,67544	1,65	0,17633	0,22438	1,80558	1,13948
1,16	0,33645	0,46290	1,36577	0,68318	1,66	0,17411	0,22128	1,81703	1,15067
1,17	0,33186	0,45561	1,37264	0,69098	1,67	0,17191	0,21823	1,82859	1,16195
1,18	0,32734	0,44847	1,37959	0,69882	1,68	0,16974	0,21523	1,84027	1,17333
1,19	0,32289	0,44146	1,38662	0,70672	1,69	0,16760	0,21227	1,85206	1,18479
1,20	0,31851	0,43459	1,39373	0,71468	1,70	0,16550	0,20936	1,86396	1,19635
1,21	0,31420	0,42785	1,40091	0,72269	1,71	0,16342	0,20650	1,87599	1,20800
1,22	0,30995	0,42124	1,40818	0,73075	1,72	0,16137	0,20368	1,88813	1,21974
1,23	0,30577	0,41474	1,41553	0,73887	1,73	0,15934	0,20090	1,90038	1,23158
1,24	0,30166	0,40838	1,42296	0,74705	1,74	0,15735	0,19817	1,91276	1,24351
1,25	0,29760	0,40212	1,43047	0,75528	1,75	0,15538	0,19548	1,92525	1,25554
1,26	0,29361	0,39599	1,43806	0,76357	1,76	0,15344	0,19283	1,93787	1,26766
1,27	0,28968	0,38997	1,44574	0,77192	1,77	0,15152	0,19022	1,95061	1,27989
1,28	0,28581	0,38405	1,45350	0,78033	1,78	0,14963	0,18765	1,96347	1,29221
1,29	0,28200	0,37825	1,46135	0,78880	1,79	0,14777	0,18512	1,97645	1,30464
1,30	0,27825	0,37255	1,46928	0,79733	1,80	0,14593	0,18262	1,98956	1,31717
1,31	0,27455	0,36695	1,47729	0,80592	1,81	0,14412	0,18017	2,00279	1,32980
1,32	0,27091	0,36145	1,48540	0,81457	1,82	0,14233	0,17775	2,01616	1,34253
1,33	0,26732	0,35605	1,49359	0,82329	1,83	0,14056	0,17537	2,02965	1,35537
1,34	0,26379	0,35075	1,50186	0,83206	1,84	0,13882	0,17302	2,04326	1,36831
1,35	0,26031	0,34554	1,51023	0,84090	1,85	0,13710	0,17071	2,05701	1,38136
1,36	0,25688	0,34043	1,51868	0,84981	1,86	0,13541	0,16843	2,07089	1,39452
1,37	0,25350	0,33540	1,52722	0,85878	1,87	0,13373	0,16619	2,08490	1,40778
1,38	0,25017	0,33046	1,53586	0,86782	1,88	0,13208	0,16398	2,09905	1,42116
1,39	0,24689	0,32561	1,54458	0,87692	1,89	0,13045	0,16180	2,11333	1,43465
1,40	0,24365	0,32084	1,55340	0,88609	1,90	0,12885	0,15966	2,12774	1,44824
1,41	0,24047	0,31615	1,56230	0,89533	1,91	1,12726	0,15755	2,14229	1,46196
1,42	0,23733	0,31154	1,57130	0,90464	1,92	0,12569	0,15547	2,15698	1,47578
1,43	0,23423	0,30701	1,58040	0,91402	1,93	0,12415	0,15341	2,17181	1,48972
1,44	0,23119	0,30256	1,58958	0,92346	1,94	0,12263	0,15139	2,18677	1,50378
1,45	0,22819	0,29819	1,59886	0,93298	1,95	0,12112	0,14940	2,20188	1,51796
1,46	0,22523	0,29389	1,60824	0,94257	1,96	0,11964	0,14744	2,21713	1,53225
1,47	0,22231	0,28966	1,61772	0,95223	1,97	0,11817	0,14550	2,23253	1,54666
1,48	0,21943	0,28550	1,62729	0,96197	1,98	0,11673	0,14360	2,24807	1,56120
1,49	0,21660	0,28141	1,63696	0,97178	1,99	0,11530	0,14172	2,26375	1,57586
1,50	0,21381	0,27739	1,64672	0,98167	2,00	0,11389	0,13987	2,27959	1,59064

x	$K_0(x)$	$K_1(x)$	$I_0(x)$	$I_1(x)$	x	$K_0(x)$	$K_1(x)$	$I_0(x)$	$I_1(x)$
2,00	0,11389	0,13987	2,27959	1,59064	2,50	0,06235	0,07389	3,28984	2,51672
2,01	0,11250	0,13804	2,29557	1,60554	2,51	0,06161	0,07298	3,31512	2,53965
2,02	0,11113	0,13624	2,31170	1,62057	2,52	0,06089	0,07208	3,34063	2,56278
2,03	0,10978	0,13447	2,32798	1,63573	2,53	0,06017	0,07119	3,36638	2,58612
2,04	0,10844	0,13272	2,34441	1,65102	2,54	0,05946	0,07031	3,39236	2,60967
2,05	0,10712	0,13100	2,36100	1,66643	2,55	0,05877	0,06945	3,41857	2,63342
2,06	0,10582	0,12930	2,37774	1,68198	2,56	0,05808	0,06859	3,44503	2,65739
2,07	0,10454	0,12763	2,39464	1,69766	2,57	0,05739	0,06775	3,47172	2,68156
2,08	0,10327	0,12598	2,41169	1,71347	2,58	0,05672	0,06692	3,49866	2,70595
2,09	0,10202	0,12435	2,42891	1,72942	2,59	0,05606	0,06609	3,52584	2,73056
2,10	0,10078	0,12275	2,44628	1,74550	2,60	0,05540	0,06528	3,55327	2,75538
2,11	0,09956	0,12117	2,46382	1,76172	2,61	0,05475	0,06448	3,58095	2,78043
2,12	0,09836	0,11961	2,48152	1,77808	2,62	0,05411	0,06369	3,60888	2,80570
2,13	0,09717	0,11807	2,49938	1,79458	2,63	0,05348	0,06292	3,63706	2,83119
2,14	0,09600	0,11655	2,51741	1,81122	2,64	0,05285	0,06215	3,66550	2,85691
2,15	0,09484	0,11506	2,53561	1,82800	2,65	0,05223	0,06139	3,69420	2,88286
2,16	0,09370	0,11359	2,55397	1,84492	2,66	0,05162	0,06064	3,72316	2,90904
2,17	0,09257	0,11213	2,57250	1,86199	2,67	0,05102	0,05990	3,75238	2,93545
2,18	0,09145	0,11070	2,59121	1,87921	2,68	0,05042	0,05917	3,78187	2,96210
2,19	0,09035	0,10929	2,61009	1,89658	2,69	0,04984	0,05845	3,81163	2,98898
2,20	0,08927	0,10790	2,62914	1,82800	2,70	0,04926	0,05774	3,84165	3,01611
2,21	0,08820	0,10652	2,64837	1,93176	2,71	0,04868	0,05704	3,87195	3,04347
2,22	0,08714	0,10517	2,66778	1,94958	2,72	0,04811	0,05634	3,90252	3,07109
2,23	0,08609	0,10383	2,68736	1,96755	2,73	0,04755	0,05566	3,93337	3,09894
2,24	0,08506	0,10252	2,70713	1,98568	2,74	0,04700	0,05498	3,96450	3,12705
2,25	0,08404	0,10122	2,72708	2,00397	2,75	0,04645	0,05432	3,99591	3,15541
2,26	0,08304	0,09993	2,74721	2,02241	2,76	0,04592	0,05366	4,02761	3,18402
2,27	0,08204	0,09867	2,76753	2,04101	2,77	0,04538	0,05301	4,05959	3,21289
2,28	0,08106	0,09742	2,78803	2,05978	2,78	0,04485	0,05237	4,09187	3,24202
2,29	0,08010	0,09620	2,80872	2,07871	2,79	0,04433	0,05174	4,12444	3,27140
2,30	0,07914	0,09498	2,82961	2,09780	2,80	0,04382	0,05111	4,15730	3,30106
2,31	0,07820	0,09379	2,85068	2,11706	2,81	0,04331	0,05050	4,19046	3,33097
2,32	0,07726	0,09261	2,87195	2,13648	2,82	0,04281	0,04989	4,22392	3,36116
2,33	0,07634	0,09144	2,89341	2,15608	2,83	0,04231	0,04929	4,25768	3,39161
2,34	0,07544	0,09029	2,91507	2,17585	2,84	0,04182	0,04869	4,29175	3,42234
2,35	0,07454	0,08916	2,93693	2,19578	2,85	0,04134	0,04811	4,32613	3,45335
2,36	0,07365	0,08804	2,95899	2,21590	2,86	0,04086	0,04753	4,36082	3,48463
2,37	0,07278	0,08694	2,98125	2,23619	2,87	0,04039	0,04696	4,39582	3,51620
2,38	0,07191	0,08586	3,00371	2,25665	2,88	0,03992	0,04639	4,43114	3,54805
2,39	0,07106	0,08478	3,02638	2,27730	2,89	0,03946	0,04584	4,46678	3,58018
2,40	0,07022	0,08372	3,04926	2,29812	2,90	0,03901	0,04529	4,50275	3,61261
2,41	0,06939	0,08268	3,07234	2,31913	2,91	0,03856	0,04474	4,53904	3,64532
2,42	0,06856	0,08165	3,09564	2,34033	2,92	0,03811	0,04421	4,57566	3,67834
2,43	0,06775	0,08063	3,11915	2,36170	2,93	0,03767	0,04368	4,61261	3,71164
2,44	0,06695	0,07963	3,14287	2,38327	2,94	0,03724	0,04316	4,64989	3,74525
2,45	0,06616	0,07864	3,16682	2,40503	2,95	0,03681	0,04264	4,68751	3,77916
2,46	0,06538	0,07767	3,19098	2,42698	2,96	0,03638	0,04213	4,72547	3,81338
2,47	0,06461	0,07670	3,21536	2,44912	2,97	0,03597	0,04163	4,76378	3,84791
2,48	0,06384	0,07575	3,23996	2,47145	2,98	0,03555	0,04113	4,80243	3,88275
2,49	0,06309	0,07482	3,26479	2,49398	2,99	0,03514	0,04064	4,84144	3,91790
2,50	0,06235	0,07389	3,28984	2,51672	3,00	0,03474	0,04016	4,88079	3,95337

x	$K_0(x)$	$K_1(x)$	$I_0(x)$	$I_1(x)$	x	$K_0(x)$	$K_1(x)$	$I_0(x)$	$I_1(x)$
3,00	0,03474	0,04016	4,88079	3,95337	3,50	0,01960	0,02224	7,37820	6,20583
3,01	0,03434	0,03968	4,92050	3,98916	3,51	0,01938	0,02198	7,44054	6,26214
3,02	0,03395	0,03921	4,96058	4,02528	3,52	0,01916	0,02173	7,50345	6,31897
3,03	0,03356	0,03874	5,00101	4,06172	3,53	0,01894	0,02147	7,56692	6,37631
3,04	0,03317	0,03828	5,04181	4,09849	3,54	0,01873	0,02123	7,63098	6,43418
3,05	0,03279	0,03782	5,08298	4,13559	3,55	0,01852	0,02098	7,69561	6,49258
3,06	0,03241	0,03738	5,12453	4,17303	3,56	0,01831	0,02074	7,76083	6,55152
3,07	0,03204	0,03693	5,16644	4,21081	3,57	0,01810	0,02050	7,82664	6,61099
3,08	0,03168	0,03649	5,20874	4,24893	3,58	0,01790	0,02026	7,89305	6,67101
3,09	0,03131	0,03606	5,25142	4,28739	3,59	0,01770	0,02003	7,96006	6,73159
3,10	0,03095	0,03563	5,29449	4,32621	3,60	0,01750	0,01979	8,02768	6,79271
3,11	0,03060	0,03521	5,33795	4,36537	3,61	0,01730	0,01957	8,09592	6,85440
3,12	0,03025	0,03480	5,38180	4,40489	3,62	0,01711	0,01934	8,16477	6,91666
3,13	0,02990	0,03438	5,42605	4,44477	3,63	0,01692	0,01912	8,23425	6,97949
3,14	0,02956	0,03398	5,47070	4,48501	3,64	0,01673	0,01890	8,30437	7,04289
3,15	0,02922	0,03358	5,51575	4,52562	3,65	0,01654	0,01868	8,37511	7,10688
3,16	0,02889	0,03318	5,56121	4,56660	3,66	0,01635	0,01846	8,44651	7,17145
3,17	0,02856	0,03279	5,60708	4,60794	3,67	0,01617	0,01825	8,51855	7,23662
3,18	0,02824	0,03240	5,65337	4,64967	3,68	0,01599	0,01804	8,59124	7,30239
3,19	0,02791	0,03202	5,70008	4,69177	3,69	0,01581	0,01783	8,66460	7,36876
3,20	0,02759	0,03164	5,74721	4,73425	3,70	0,01563	0,01763	8,73862	7,43575
3,21	0,02728	0,03127	5,79476	4,77113	3,71	0,01546	0,01743	8,81331	7,50334
3,22	0,02697	0,03090	5,84275	4,82039	3,72	0,01528	0,01722	8,88869	7,57156
3,23	0,02666	0,03054	5,89117	4,86404	3,73	0,01511	0,01703	8,96475	7,64041
3,24	0,02636	0,03018	5,94003	4,90809	3,74	0,01494	0,01683	9,04150	7,70989
3,25	0,02606	0,02983	5,98934	4,95255	3,75	0,01477	0,01664	9,11895	7,78002
3,26	0,02576	0,02948	6,03909	4,99740	3,76	0,01461	0,01645	9,19710	7,85078
3,27	0,02547	0,02913	6,08929	5,04267	3,77	0,01445	0,01626	9,27596	7,92220
3,28	0,02518	0,02879	6,13994	5,08835	3,78	0,01428	0,01607	9,35555	7,99428
3,29	0,02489	0,02845	6,19105	5,13444	3,79	0,01412	0,01589	9,43585	8,06701
3,30	0,02461	0,02812	6,24263	5,18096	3,80	0,01397	0,01571	9,51689	8,14042
3,31	0,02433	0,02779	6,29467	5,22790	3,81	0,01381	0,01553	9,59866	8,21451
3,32	0,02405	0,02746	6,34719	5,27527	3,82	0,01366	0,01535	9,68118	8,28928
3,33	0,02378	0,02714	6,40018	5,32306	3,83	0,01350	0,01517	9,76445	8,36474
3,34	0,02351	0,02682	6,45365	5,37130	3,84	0,01335	0,01500	9,84848	8,44089
3,35	0,02325	0,02651	6,50761	5,41998	3,85	0,01320	0,01483	9,93327	8,51775
3,36	0,02298	0,02620	6,56205	5,46910	3,86	0,01306	0,01466	10,01883	8,59531
3,37	0,02272	0,02589	6,61699	5,51866	3,87	0,01291	0,01449	10,10518	8,67359
3,38	0,02246	0,02559	6,67243	5,56868	3,88	0,01277	0,01432	10,19231	8,75259
3,39	0,02221	0,02529	6,72837	5,61916	3,89	0,01262	0,01416	10,28023	8,83232
3,40	0,02196	0,02500	6,78481	5,67010	3,90	0,01248	0,01400	10,36896	8,91279
3,41	0,02171	0,02471	6,84177	5,72151	3,91	0,01234	0,01384	10,45849	8,99400
3,42	0,02146	0,02442	6,89924	5,77338	3,92	0,01221	0,01368	10,54884	9,07595
3,43	0,02122	0,02414	6,95724	5,82573	3,93	0,01207	0,01353	10,64001	9,15867
3,44	0,02098	0,02385	7,01576	5,87856	3,94	0,01194	0,01337	10,73202	9,24215
3,45	0,02074	0,02358	7,07481	5,93187	3,95	0,01180	0,01322	10,82486	9,32640
3,46	0,02051	0,02330	7,13440	5,98567	3,96	0,01167	0,01307	10,91855	9,41143
3,47	0,02028	0,02303	7,19453	6,03996	3,97	0,01154	0,01292	11,01309	9,49724
3,48	0,02005	0,02276	7,25520	6,09475	3,98	0,01141	0,01277	11,10849	9,58384
3,49	0,01982	0,02250	7,31642	6,15004	3,99	0,01129	0,01263	11,20477	9,67125
3,50	0,01960	0.02224	7,37820	6,20583	4,00	0,01116	0,01248	11,30192	9,75947

x	$K_0(x)$	$K_1(x)$	$I_0(x)$	$I_1(x)$	x	$K_0(x)$	$K_1(x)_1$	$I_1(x)$	$I_1(x)$
4,00	0,01116	0,01248	11,30192	9,75947	4,50	0,00640	0,00708	17,48117	15,38922
4,01	0,01104	0,01234	11,39996	9,84849	4,51	0,00633	0,00700	17,63577	15,53049
4,02	0,01091	0,01220	11,49889	9,93835	4,52	0,00626	0,00692	17,79179	15,67307
4,03	0,01079	0,01206	11,59873	10,02903	4,53	0,00619	0,00684	17,94924	15,81698
4,04	0,01067	0,01193	11,69948	10,12055	4,54	0,00612	0,00677	18,10813	15,96223
4,05	0,01055	0,01179	11,80114	10,21292	4,55	0,00606	0,00669	18,26848	16,10883
4,06	0,01044	0,01166	11,90374	10,30614	4,56	0,00599	0,00662	18,43031	16,25679
4,07	0,01032	0,01152	12,00727	10,40023	4,57	0,00592	0,00654	18,59362	16,40614
4,08	0,01021	0,01139	12,11175	10,49519	4,58	0,00586	0,00647	18,75844	16,55687
4,09	0,01009	0,01126	12,21718	10,59102	4,59	0,00579	0,00640	18,92477	16,70901
4,10	0,00998	0,01114	12,32357	10,68774	4,60	0,00573	0,00633	19,09262	16,86256
4,11	0,00987	0,01101	12,43093	10,78536	4,61	0,00567	0,00625	19,26202	17,01755
4,12	0,00976	0,01089	12,53928	10,88388	4,62	0,00561	0,00618	19,43298	17,17398
4,13	0,00965	0,01076	12,64862	10,98331	4,63	0,00554	0,00612	19,60551	17,33187
4,14	0,00954	0,01064	12,75895	11,08367	4,64	0,00548	0,00605	19,77962	17,49123
4,15	0,00944	0,01052	12,87029	11,18495	4,65	0,00542	0,00598	19,95534	17,65207
4,16	0,00933	0,01040	12,98265	11,28717	4,66	0,00536	0,00591	20,13267	17,81442
4,17	0,00923	0,01028	13,09604	11,39034	4,67	0,00530	0,00585	20,31163	17,97827
4,18	0,00913	0,01017	13,21046	11,49447	4,68	0,00525	0,00578	20,49224	18,14366
4,19	0,00903	0,01005	13,32593	11,59966	4,69	0,00519	0,00572	20,67451	18,31059
4,20	0,00893	0,00994	13,44246	11,70562	4,70	0,00513	0,00565	20,85846	18,47907
4,21	0,00883	0,00983	13,56005	11,81267	4,71	0,00508	0,00559	21,04410	18,64913
4,22	0,00873	0,00971	13,67871	11,92071	4,72	0,00502	0,00553	21,23144	18,82077
4,23	0,00863	0,00961	13,79846	12,02975	4,73	0,00497	0,00547	21,42052	18,99401
4,24	0,00854	0,00950	13,91931	12,13980	4,74	0,00491	0,00541	21,61133	19,16887
4,25	0,00844	0,00939	14,04126	12,25087	4,75	0,00486	0,00535	21,80390	19,34536
4,26	0,00835	0,00928	14,16433	12,36298	4,76	0,00480	0,00529	21,99824	19,52350
4,27	0,00826	0,00918	14,28853	12,47612	4,77	0,00475	0,00523	22,19437	19,70330
4,28	0,00817	0,00908	14,41386	12,59032	4,78	0,00470	0,00517	22,39231	19,88478
4,29	0,00808	0,00897	14,54034	12,70557	4,79	0,00465	0,00511	22,59208	20,06795
4,30	0,00799	0,00887	14,66797	12,82189	4,80	0,00460	0,00506	22,79368	20,25283
4,31	0,00790	0,00877	14,79678	12,93930	4,81	0,00455	0,00500	22,99714	20,43944
4,32	0,00781	0,00867	14,92676	13,05779	4,82	0,00450	0,00494	23,20247	20,62780
4,33	0,00773	0,00858	15,05794	13,17738	4,83	0,00445	0,00489	23,40970	20,81791
4,34	0,00764	0,00848	15,19031	13,29809	4,84	0,00440	0,00483	23,61884	21,00979
4,35	0,00756	0,00838	15,32390	13,41991	4,85	0,00435	0,00478	23,82990	21,20347
4,36	0,00747	0,00829	15,45872	13,54287	4,86	0,00430	0,00473	24,04291	21,39896
4,37	0,00739	0,00820	15,59476	13,66696	4,87	0,00426	0,00468	24,25789	21,59627
4,38	0,00731	0,00810	15,73206	13,79222	4,88	0,00421	0,00462	24,47484	21,79543
4,39	0,00723	0,00801	15,87061	13,91863	4,89	0,00416	0,00457	24,69380	21,99645
4,40	0,00715	0,00792	16,01044	14,04622	4,90	0,00412	0,00452	24,91478	22,19935
4,41	0,00707	0,00783	16,15154	14,17500	4,91	0,00407	0,00447	25,13779	22,40414
4,42	0,00699	0,00775	16,29394	14,30497	4,92	0,00403	0,00442	25,36287	22,61085
4,43	0,00692	0,00766	16,43764	14,43615	4,93	0,00399	0,00437	25,59002	22,81949
4,44	0,00684	0,00757	16,58267	14,56855	4,94	0,00394	0,00432	25,81926	23,03008
4,45	0,00676	0,00749	16,72902	14,70218	4,95	0,00390	0,00428	26,05063	23,24264
4,46	0,00669	0,00740	16,87671	14,83706	4,96	0,00386	0,00423	26,28412	23,45719
4,47	0,00662	0,00732	17,02576	14,97319	4,97	0,00381	0,00418	26,51978	23,67375
4,48	0,00654	0,00724	17,17618	15,11058	4,98	0,00377	0,00414	26,75761	23,89233
4,49	0,00647	0,00716	17,32798	15,24926	4,99	0,00373	0,00409	26,99763	24,11295
4,50	0,00640	0,00708	17,48117	15,38922	5,00	0,00369	0,00404	27,23987	24,33564

x	$K_0(x)$	$K_1(x)$	$I_0(x)$	$I_1(x)$
5,0	0,003691	0,004045	27,239872	24,335642
5,1	0,003308	0,003619	29,788855	26,680436
5,2	0,002966	0,003239	32,583593	29,254310
5,3	0,002659	0,002900	35,648105	32,079892
5,4	0,002385	0,002597	39,008788	35,182059
5,5	0,002139	0,002326	42,694645	38,588165
5,6	0,001918	0,002083	46,737551	42,328288
5,7	0,001721	0,001866	51,172536	46,435504
5,8	0,001544	0,001673	56,038097	50,946185
5,9	0,001386	0,001499	61,376550	55,900332
6,0	0,001244	0,001344	67,234407	61,341937
6,1	0,001117	0,001205	73,662794	67,319385
6,2	0,001003	0,001081	80,717913	73,885894
6,3	0,000900	0,000969	88,461553	81,100002
6,4	0,000908	0,000869	96,961640	89,026097
6,5	0,000726	0,000780	106,292858	97,735011
6,6	0,000652	0,000700	116,537324	107,304661
6,7	0,000586	0,000628	127,785330	117,820769
6,8	0,000526	0,000564	140,136160	129,377639
6,9	0,000473	0,000506	153,698996	142,079028
7,0	0,000425	0,000454	168,593909	156,039093
7,1	0,000382	0,000408	184,952944	171,383438
7,2	0,000343	0,000366	202,921330	188,250271
7,3	0,000308	0,000329	222,658800	206,791670
7,4	0,000277	0,000295	244,341043	227,174982
7,5	0,000249	0,000265	268,161312	249,584365
7,6	0,000224	0,000238	294,332184	274,222480
7,7	0,000201	0,000214	323,087508	301,312360
7,8	0,000181	0,000192	354,684536	331,099464
7,9	0,000163	0,000173	389,406283	363,853944
8,0	0,000146	0,000155	427,564116	399,873137
8,1	0,000132	0,000140	469,500607	439,484309
8,2	0,000118	0,000126	515,592677	483,047683
8,3	0,000107	0,000113	566,255056	530,959766
8,4	0,000096	0,000101	621,944087	583,657020
8,5	0,000086	0,000091	683,161927	641,619903
8,6	0,000078	0,000082	750,461160	705,377315
8,7	0,000070	0,000074	824,449884	775,511507
8,8	0,000063	0,000066	905,797315	852,663473
8,9	0,000057	0,000060	995,239948	937,538901
9,0	0,000051	0,000054	1093,588355	1030,914723
9,1	0,000046	0,000048	1201,734657	1133,646332
9,2	0,000041	0,000043	1320,660768	1246,675533
9,3	0,000037	0,000039	1451,447466	1371,039295
9,4	0,000033	0,000035	1595,284378	1507,879402
9,5	0,000030	0,000032	1753,480991	1658,453078
9,6	0,000027	0,000028	1927,478769	1824,144695
9,7	0,000024	0,000026	2118,864504	2006,478672
9,8	0,000022	0,000023	2329,385016	2207,133683
9,9	0,000020	0,000021	2560,963353	2427,958313
10,0	0,000018	0,000019	2815,716629	2670,988304