The above equations for flow and continuity are valid for steady flow. When investigating a particular flow problem, we can only determine its solution if we know what happens at the boundaries of the flow region. Hence, the boundary conditions should be properly defined. They may include statements on the hydraulic head, or on the inflow and outflow conditions at the boundary, or that a boundary may be a streamline, and so on.

7.7 Boundary Conditions

From theory, we know that partial differential equations like Laplace's have an infinite number of solutions. So how do we choose the one solution that applies to a given problem? Boundary conditions in problems of groundwater flow describe the specific conditions that are to be imposed at the boundaries of the flow region. These boundaries are not necessarily impervious layers or walls confining the groundwater. Rather, they are geometrical surfaces where, at all points, we know either the flow velocity of the groundwater, or an equipotential line, or a given function of both. Some characteristic boundary conditions will be briefly discussed in the following sections.

7.7.1 Impervious Layers

The boundary of an impervious layer can be regarded as a streamline because there is no flow across it. The flow velocity component normal to such a boundary therefore vanishes, and we have Ψ = constant and $d\Psi/ds = 0$. In practice, a layer is considered impervious if its hydraulic conductivity is very low compared with the hydraulic conductivity of adjacent layers.

7.7.2 Planes of Symmetry

Planes of symmetry are illustrated in Figure 7.20 by the two lines marked A-B (running vertically through the drain axis) and the boundary line marked C-D (running parallel to A-B, but midway between the drains). Because of the symmetry of the system, the

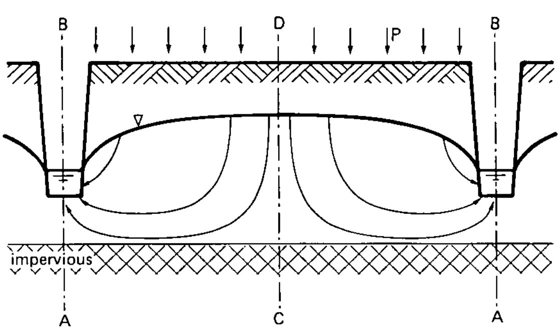

Figure 7.20 Boundary conditions for steady flow to drainage canals

pattern of equipotential lines and streamlines on one side of the 'boundary' mirrors the pattern on the other side. Hence, any horizontal flow velocity component immediately adjacent to that 'boundary' must be matched on the other side by a component in the opposite direction. The net flow across the boundary must therefore be zero, and the plane of symmetry, like an impervious layer, is a streamline of the system.

7.7.3 Free Water Surface

The free water surface is defined as the surface where water pressure equals atmospheric pressure. It is assumed that the free water surface limits the groundwater flow region, i.e. no groundwater flow occurs above this surface. This assumption is untrue in most instances of flow through soils, but it is useful when we are analyzing flow through a layer whose capillary fringe is thin in comparison with its thickness, D.

In a free water surface, the pressure component of the total head, $p/\rho g$, is zero; hence the hydraulic head at the water's surface is equal to the elevation component of this surface at a given point: $h = z$.

If there is no percolation of water towards the free water surface, the flow velocity component normal to that surface is zero and the free water surface then represents a streamline.

If there is percolation, however, the vertical recharge, R, determines the value of the streamlines. In Figure 7.21, a rainfall intensity is assumed, of which R recharges the groundwater body. The free water surface is neither an equipotential line nor a streamline, and the starting points of streamlines are at regular distances from each other.

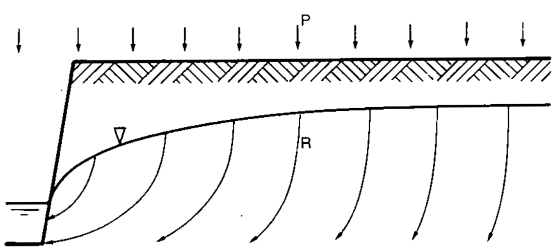

Figure 7.21 Boundary conditions for a free water surface

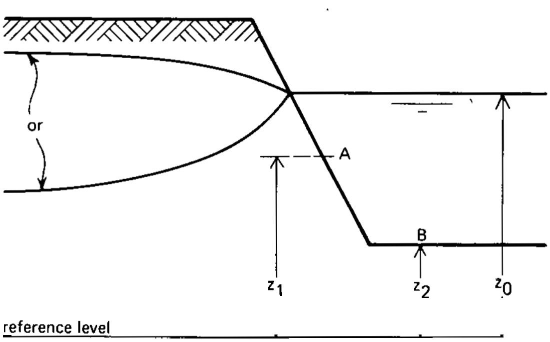

Figure 7.22 Boundary conditions for water at rest or for slowly-moving water

7.7.4 Boundary Conditions for Water at Rest or for Slowly-Moving Water

These boundaries are found along the bottom and side slopes of canals and reservoirs, and where upward groundwater flow meets downward percolating water. The hydrostatic pressure along the bottom or side of a canal, i.e. the pressure due to a certain level of water above the bottom or side of a canal, is expressed by (Figure 7.22)

$$p = \rho g (z_o - z) \tag{7.60}$$

where

z = the elevation of a given point above the reference level (m)
z_o = the elevation of the water level in the canal (m)

It then follows that

$$h = z_o = \frac{p}{\rho g} + z \tag{7.61}$$

Now the right-hand expression represents the potential or hydraulic head, so the potential head at each point along the canal is equal to the height of the water level, z_o, in the canal. In Figure 7.22, we have

Point	Elevation head	$h = z + p/\rho g$	Pressure $(p/\rho g)$
A	z_1	z_o	$z_o - z_1$
B	z_2	z_o	$z_o - z_2$

7.7.5 Seepage Surface

At all points in the soil above the watertable, the pressure head is negative, while below the watertable, it is generally positive. An exception occurs if the watertable intersects the surface of the soil, as shown in Figure 7.23. In this case, a seepage surface occurs. A seepage surface is defined as the boundary where water leaves the soil mass and then continues to flow in a thin film along its surface. Seepage surfaces can also occur on the downstream face of dams.

Along a seepage surface, the pressure $p = 0$ (atmospheric pressure). Hence the

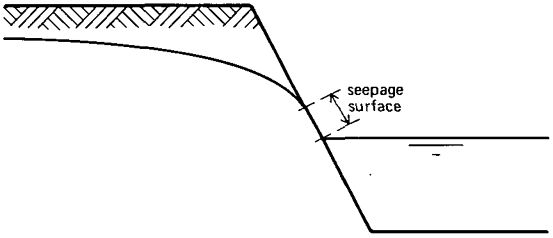

Figure 7.23 Boundary conditions for a seepage surface

hydraulic head at any point on the seepage surface is equal to the elevation at that point, or $h = z$. A seepage surface is not a streamline because, in the interior of the soils mass, the component of the flow-velocity vector perpendicular to the boundary is not zero.

7.8 The Dupuit-Forchheimer Theory

7.8.1 The Dupuit-Forchheimer Assumptions

Groundwater-flow patterns bounded by the watertable (known as unconfined flow patterns) are difficult to calculate. Obtaining a mathematically exact solution with the Laplace equation is complex because the non-linear free water surface is both the boundary condition of, and the solution to, the drainage problem. As a result, complex calculations do not always give better answers than a simplified method because our knowledge of this boundary condition is imprecise, the soil is heterogeneous, and the groundwater recharge from rainfall or irrigation is not uniform.

A method first developed by Dupuit in 1863, and improved by Forchheimer (1930), gives good solutions to problems of flow to parallel canals and to pumped wells. In addition to assuming that:

- The flow pattern is steady;
- Darcy's equation is applicable.

Dupuit also assumed that:

- In a vertical section, MN, of the aquifer, all velocity vectors are horizontal and equal to $v = -K(dy/dx)$ (see Figure 7.24);
- The hydraulic gradient between two infinitely adjacent sections, MN and M′N′, exactly equals

$$s = \frac{dy}{MM'} = \sin\theta$$

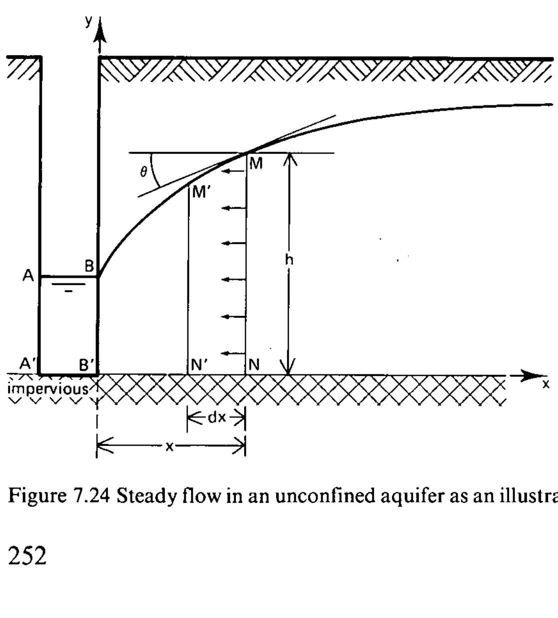

Figure 7.24 Steady flow in an unconfined aquifer as an illustration of Dupuit's assumptions

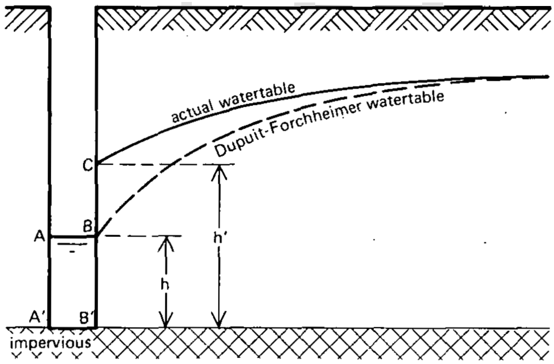

Figure 7.25 The watertable near a seepage surface cannot be explained by Dupuit's assumptions

which can be written as

$$s = \frac{dy}{dx} = \text{tg}\ \theta \qquad (7.62)$$

In part, Dupuit's assumptions are contradictory, and the last assumption is valid only if θ remains small. Calculations that incorporate these assumptions will therefore indicate a lower watertable in the vicinity of pumped wells and when a seepage surface (illustrated as BC in Figure 7.25) can be expected. We can easily see that a seepage surface will occur if the water surface AB approaches A′B′.

Nevertheless, with Dupuit's assumptions, we can solve a variety of groundwater-flow problems with satisfactory accuracy. Forchheimer used the assumptions to develop a general equation for the free water surface. He applied the continuity equation to a vertical column of water, which, in a flow region, is bounded above by the phreatic surface and below by an impervious layer, and whose height is h (see Figure 7.26).

If we assume that the surface of the impervious layer is horizontal, i.e. that it coincides with the plane delineated by the horizontal coordinates x and y, the horizontal components of the flow velocity are

$$v_x = K \frac{\partial h}{\partial x} \quad \text{and} \quad v_y = -K \frac{\partial h}{\partial y} \qquad (7.63)$$

If q_x is the flow rate in the x-direction, then the water entering through the left face of the column is the product of the area, h × dy, and the velocity, v_x. Thus

$$q_x = -K\left(h \frac{\partial h}{\partial x}\right) dy \qquad (7.64)$$

As water moves from the left-hand to the right-hand face of the column, we see that the flow rate changes by $\partial q_x/\partial x$. When the water leaves the right-hand face of the column, q_x has changed to

$$q_{(x+dx)} \text{ i.e. to } q_x + \frac{\partial q_x}{\partial x} dx$$

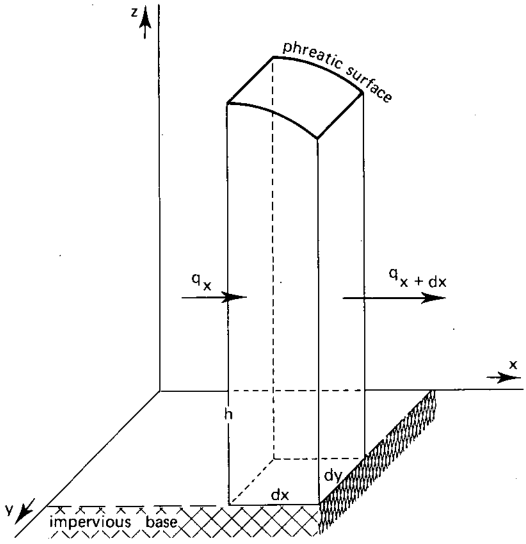

Figure 7.26 Approximate horizontal flow in a fluid space column as an assumption for deriving Forchheimer's linearized continuity equation

The difference between outflow and inflow per unit time in the x-direction is

$$q_{(x+dx)} - q_x = \frac{\partial q_x}{\partial x} dx = -K \frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right) dx\, dy \tag{7.65}$$

Similarly, the change in flow in the y-direction is

$$\frac{\partial q_y}{\partial y} dy = -K \frac{\partial}{\partial y}\left(h \frac{\partial h}{\partial y}\right) dx\, dy \tag{7.66}$$

If we assume steady flow, the continuity equation requires that the sum of the changes adds up to zero. Hence the sum of the right-hand expressions of Equations 7.65 and 7.66 equals

$$-K\left[\frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(h \frac{\partial h}{\partial y}\right)\right] dx\, dy = 0 \tag{7.67}$$

Equation 7.67 can also be written as

$$\frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(h \frac{\partial h}{\partial y}\right) = 0 \tag{7.68}$$

or

$$\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} = 0 \tag{7.69}$$

Equations 7.67 to 7.69 are alternative forms of the Forchheimer equation for steady flow.

7.8.2 Steady Flow above an Impervious Horizontal Boundary

Let us regard the xz-plane in Figure 7.27 as the plane of flow in which h_1 and h_2 are the known elevations of two points of the steady watertable. For this flow pattern, Equation 7.68 reduces to

$$\frac{d}{dx}\left(h\frac{dh}{dx}\right) = 0$$

which, after integration, yields the equation of the parabola

$$h^2 = C_1x + C_2 \qquad (7.70)$$

The integration constants, C_1 and C_2, can be solved by applying the boundary conditions $x = 0$, $h = h_1$, and $x = L$, $h = h_2$. Substituting the values of C_1 and C_2 into Equation 7.70, we obtain the expression for the elevation h at any intermediate point

$$h = \sqrt{h_1^2 - (h_1^2 - h_2^2)\frac{x}{L}} \qquad (7.71)$$

According to Darcy, the discharge per unit width through any vertical section between Points 1 and 2 in Figure 7.27 is

$$q = -K\,h\frac{dh}{dx} \qquad (7.72)$$

which, after integration and substitution of the boundary conditions, yields

$$q = \frac{K}{2L}(h_1^2 - h_2^2) \qquad (7.73)$$

Equation 7.73 is called Dupuit's formula.

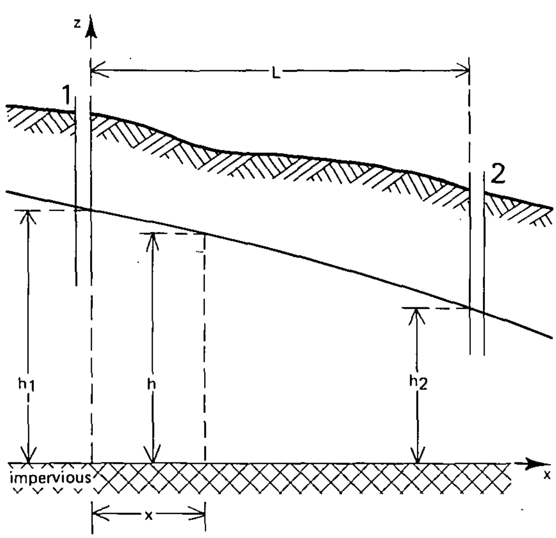

Figure 7.27 Steady flow above an impervious horizontal boundary

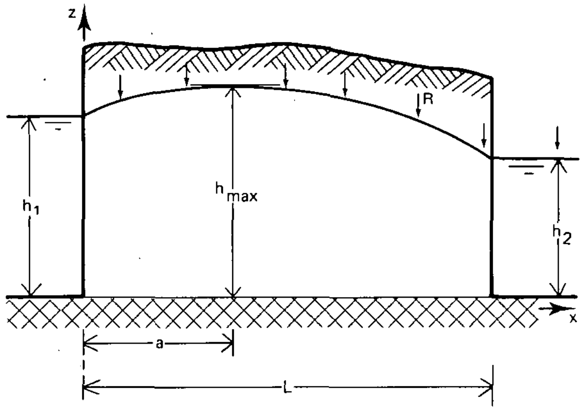

Figure 7.28 Watertable subject to recharge at a rate of R per unit area

7.8.3 Watertable subject to Recharge or Capillary Rise

We assume that the watertable in Figure 7.28 has a uniform rate of flow, R, per unit area ($R > 0$ if there is recharge, and $R < 0$ if there is capillary rise). For two-dimensional flow, the right-hand expression in Equation 7.67 will equal Rdxdy. Hence

$$-K\left[\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)\right]dxdy = Rdxdy \tag{7.74}$$

For flow in the xz-plane shown in Figure 7.28, this equation reduces to

$$K\frac{d}{dx}\left(h\frac{dh}{dx}\right) + R = 0$$

Upon integration, it becomes

$$Kh^2 + Rx^2 = C_1x + C_2 \tag{7.76}$$

If there is recharge ($R > 0$), Equation 7.76 is an ellipse; if there is capillary rise from the groundwater profile ($R < 0$), it is a hyperbola.

Limiting ourselves to recharges, we can derive several useful approximate relationships. If we substitute the boundary conditions, $x = 0$, $h = h_1$ and $x = L$, $h = h_2$, into Equation 7.76, we obtain the general equation for the watertable

$$h = \sqrt{h_1{}^2 - \frac{(h_1{}^2 - h_2{}^2)x}{L} + \frac{R}{K}(L - x)x} \tag{7.77}$$

If $R = 0$, this equation gives the approximate groundwater profile for flow through a dam or a dike. It then equals Equation 7.71.

When the water levels in the (drainage) canals shown in Figure 7.28 are equal ($h_1 = h_2 = h_0$), the maximum value of h is reached, because of symmetry, at

$a = x = L/2$. After substituting these conditions into Equation 7.77, we obtain (see also Chapter 8, Section 8.2.1))

$$h_{max} = \sqrt{h_0^2 + \frac{RL^2}{4K}}$$

or

$$L = \sqrt{\frac{4K}{R}(h_{max}^2 - h_0^2)} \tag{7.78}$$

Then, to determine the flow rate through a vertical section in Figure 7.28, we substitute Equation 7.72 into Equation 7.75, which yields

$$\frac{dq_x}{dx} = R$$

After integrating and substituting the boundary condition ($x = 0, q_x = q_l$), we obtain

$$q_x = Rx + q_l \tag{7.79}$$

Substituting Equation 7.72 into Equation 7.79 and integrating the result with the boundary conditions of Equation 7.77 gives

$$q_l = \frac{K}{2L}(h_1^2 - h_2^2) - \frac{RL}{2} \tag{7.80}$$

which, when substituted back into Equation 7.79, finally gives

$$q_x = \frac{K}{2L}(h_1^2 - h_2^2) - R\left(\frac{L}{2} - x\right) \tag{7.81}$$

It should be noted here that, if $R = 0$, this equation will be similar to Equation 7.73.

The distance $x = a$ (see Figure 7.28), for which the elevation of the groundwater profile is at its maximum (h_{max}), can be found from
Equation 7.81 by substituting $q_x = 0$. Hence

$$a = \frac{L}{2} - \frac{K}{2RL}(h_1^2 - h_2^2) \tag{7.82}$$

7.8.4 Steady Flow towards a Well

As a last example, the flow towards a fully penetrating well will be analyzed (Figure 7.29). A homogeneous and isotropic aquifer is assumed, bounded below by a horizontal impervious layer. While being pumped, such a well receives water over the full thickness of the saturated aquifer because the length of the well screen equals the saturated thickness of the aquifer.

The initial watertable is horizontal, but attains a curved shape after pumping is started. Water is then flowing from all directions towards the well (radial flow). It is further assumed that there is no vertical recharge, and that the groundwater flow towards the well is in a steady state, i.e. the hydraulic heads along the perimeter of

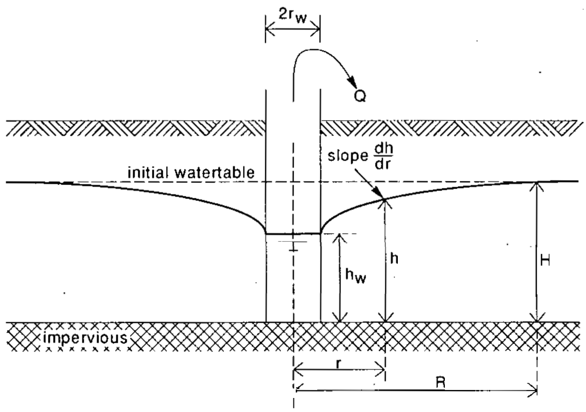

Figure 7.29 Horizontal radial flow towards a pumped well that fully penetrates an unconfined aquifer. No recharge from rainfall.

any circle concentric with the well are constant (radial symmetry). Flow through any cylinder at a distance from the centre of the well can be found by applying the continuity equation and the equation of Darcy. We hereby assume that the hydraulic gradient in this cylinder equals the slope of the watertable at the circle of this cylinder, dh/dr. Substituting this gradient, and the area of flow, $A = 2\pi rh$, into the Darcy equation yields

$$Q = 2\,\pi\, rh\, K \frac{dh}{dr} \tag{7.83}$$

where Q is the steady radial well flow and K is the hydraulic conductivity of the aquifer. On integration between the limits $h = h_w$ at $r = r_w$ and $h = H$ at $r = r_e$ Equation 7.83 yields

$$H^2 - h_w^{\,2} = \frac{Q}{\pi k} \ln \frac{r_e}{r_w} \tag{7.84}$$

where

r_w = the well radius (m)
r_e = the radius of influence of the well (m)

After being rearranged, this yields the Dupuit equation

$$Q = \frac{\pi\, K\, (H^2 - h_w^{\,2})}{\ln \frac{r_e}{r_w}} \tag{7.85}$$

We can obtain a specific solution to this equation by substituting a pair of values of h and r measured in two observation wells at different distances from the centre of the pumped well. For $r = r_1$ with $h = h_1$ and for $r = r_2$ with $h = h_2$, Equation 7.85 then reads

$$Q = \pi\, K \frac{h_2^{\,2} - h_1^{\,2}}{\ln \frac{r_2}{r_1}} \tag{7.86}$$

If the head loss is small compared with the saturated thickness of the aquifer, D, we can approximate $h_2 + h_1 = 2D$. Equation 7.86 then becomes

$$Q = 2\pi KD \frac{h_2 - h_1}{\ln \frac{r_2}{r_1}} \tag{7.87}$$

Because of the Dupuit-Forchheimer assumptions listed in Section 7.8.1, this equation cannot accurately describe the drawdown curve near the well. For distances farther from the well, however, the equation can be used without appreciable errors (see also Section 10.4.4).

7.9 The Relaxation Method

The relaxation method is a numerical way of calculating an approximate solution to the Laplace equation (Equation 7.59) for two-dimensional flow. It is based on the replacement of differential coefficients by finite difference expressions.

Let us now assume that we know the groundwater levels, h_1, h_2, h_3, h_4, at four points (Figure 7.30), and that we want to estimate the level, h_0.

Studying the watertable in the x-direction, we can assign an arbitrary value to the h_0-level and connect h_1, h_0, and h_3 as shown.

The physical meaning of the first differential coefficient of a function is the slope of that function (watertable) at a given point. Hence

$$\left(\frac{\partial h}{\partial x}\right)_{3\to 0} \approx \frac{h_0 - h_3}{\Delta x} \tag{7.88}$$

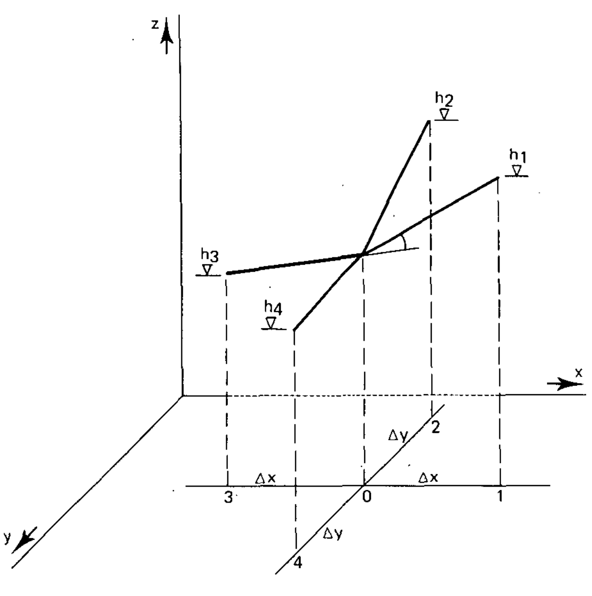

Figure 7.30 Estimating the h_0-level

and also

$$\left(\frac{\partial h}{\partial x}\right)_{0\to 1} \approx \frac{h_1 - h_0}{\Delta x} \tag{7.89}$$

The physical meaning of the second differential coefficient of a function is the rate of change in the slope of that function (describing the watertable) at a given point. Thus, for distance 0.5Δx around point 0, we can write

$$\left(\frac{\partial^2 h}{\partial x^2}\right)_0 = \frac{\left(\frac{\partial h}{\partial x}\right)_{0\to 1} - \left(\frac{\partial h}{\partial x}\right)_{3\to 0}}{\Delta x} \tag{7.90}$$

Substituting Equations 7.88 and 7.89 into this equation yields

$$\left(\frac{\partial^2 h}{\partial x^2}\right)_0 = \frac{h_1 + h_3 - 2h_0}{\Delta x^2} \tag{7.91}$$

A similar procedure for levels h_2, h_0, and h_4 in the y-direction yields

$$\left(\frac{\partial^2 h}{\partial y^2}\right)_0 = \frac{h_2 + h_4 - 2h_0}{\Delta y^2} \tag{7.92}$$

Substituting Equations 7.91 and 7.92 into the Laplace equation (Equation 7.59) yields

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{h_1 + h_3 - 2h_0}{\Delta x^2} + \frac{h_2 + h_4 - 2h_0}{\Delta y^2} = 0 \tag{7.93}$$

If a grid is used to study the watertable elevation where the distance $\Delta x = \Delta y$, Equation 7.93 reduces to

$$h_0 = \frac{h_1 + h_2 + h_3 + h_4}{4}$$

To illustrate the use of the relaxation method, let us consider Figure 7.31, where there are twelve known groundwater levels at the boundary of a grid. To draw a family of equipotential lines as accurately as possible (watertable-contour map), we assign initially-estimated levels to the four central grid points.

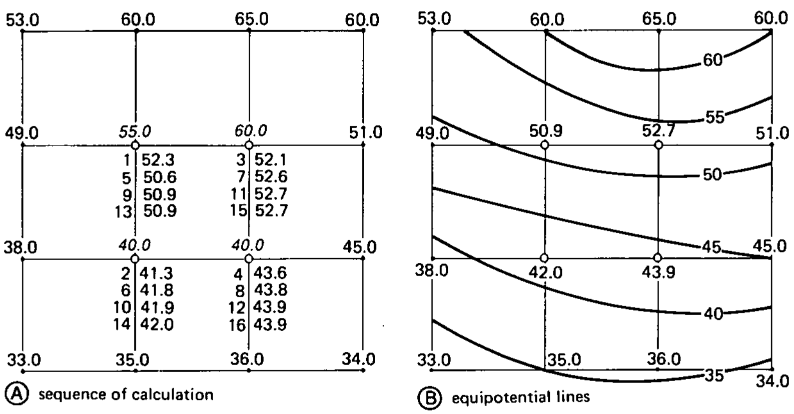

Figure 7.31 Illustration of the relaxation method

Subsequently, we use Equation 7.94 to improve the first estimate.

Hence

1→ (60.0 + 60.0 + 49.0 + 40.0)/4 = 52.3
2→ (40.0 + 52.3 + 38.0 + 35.0)/4 = 41.3
(Note that the level 55.0 is not used.)
3→ (51.0 + 65.0 + 52.3 + 40.0)/4 = 52.1
4→ (45.0 + 52.1 + 41.3 + 36.0)/4 = 43.6, and so on.

As soon as the difference between the subsequent estimates becomes sufficiently small, we stop our calculations and use the final estimate to draw the equipotential lines (Figure 7.31B).

Working out these calculations on paper is, of course, laborious, but, fortunately nowadays, we can use a computer.

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