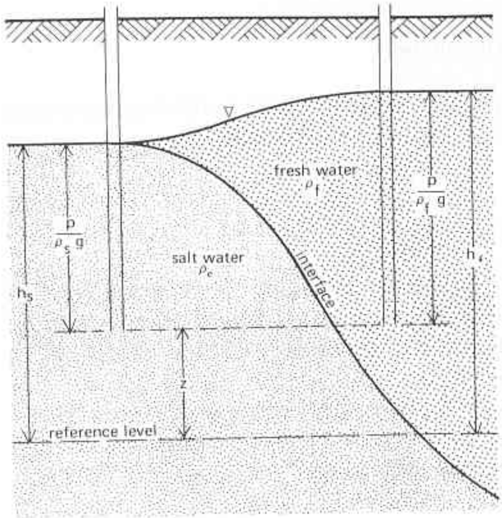

Figure 7.6 Hydraulic heads in bodies of fresh water and salt water

and the saltwater head, h_s, as

$$h_s = z + \frac{p}{\rho_s g}$$

Where ρ_f is the mass density of fresh water and ρ_s is the mass density of salt water, we obtain, after eliminating p/g,

$$h_f = \frac{h_s \rho_s}{\rho_f} - z\,\frac{\rho_s - \rho_f}{\rho_f} \tag{7.17}$$

If the reference level coincides with the bottom of the piezometer, i.e. if z = 0, the corresponding fresh-water head can be expressed as

$$h_f = h_s\,\frac{\rho_s}{\rho_f} \tag{7.18}$$

If, for example, the hydraulic head in salt water is 30 m above the reference level that is assumed to coincide with the bottom of the piezometer, and the mass density of the groundwater is 1025 kg/m^3, then the length of a column of fresh water of the same weight is

$$h_f = 30\,\frac{1025}{1000} = 30.75 \text{ m}$$

7.4 Darcy's Equation

7.4.1 General Formulation

The fundamental equation describing the flow of groundwater through soil was derived by Darcy (1856). He performed his experiments using an instrument like the one shown in Figure 7.7.

Darcy observed that the volume of water flowing through a sand column per unit of time was proportional to the column's cross-sectional area and its difference in head ($h_1 - h_2 = \Delta h$), and inversely proportional to the length of the sand column. This relation can be written as

$$Q = K \frac{\Delta h}{L} A \qquad (7.19)$$

where

- Q = the rate of flow through the column (m^3/s)
- Δh = the head loss (m)
- L = the length of the column (m)
- A = the cross-sectional area of the column (m^2)
- K = a proportionality coefficient, called hydraulic conductivity (m/s)

In this context, it should be noted that $Q/A = v$ is not the actual velocity at which a particle of water flows through the pores of the soil. It is a fictitious velocity that is better referred to as the 'discharge per unit area', or 'apparent velocity'. For design purposes, the discharge per unit area is more important than the actual velocity, v_m, at which water moves through the pores ($v_m > v_{apparent}$).

Nevertheless, the ratio between the apparent velocity and the actual velocity, is directly related to the value of K in Equation 7.19. The v_m-value can be calculated as a function of Q/A and porosity. The porosity of a sample of sand, or any other porous material, is the ratio of the volume of voids, in the sample, V_v, to the total volume of the sample, V.

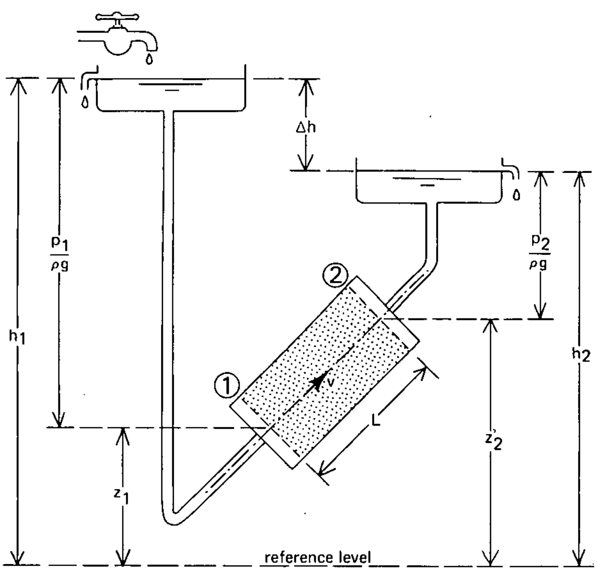

Figure 7.7 Pressure distribution and head loss in flow through a sand column

Porosity was defined in Chapter 3 as

$$\varepsilon = \frac{V_v}{V} \tag{7.20}$$

If a plane section is drawn through a random grain formation, it will intersect the grains and the voids between them. The net area occupied by the voids, a, will then stand in the same proportion to the cross-sectional area, in the sample, A, as the ratio V_V/V does. Therefore

$$\frac{a}{A} = \frac{V_v}{V} = \varepsilon \tag{7.21}$$

and also

$$v_m = \frac{Q}{\varepsilon A} = \frac{v}{\varepsilon} \tag{7.22}$$

In alluvial soils, the porosity, ε, varies from about 0.2 to 0.55 (Chapter 3).

7.4.2 The K-Value in Darcy's Equation

The proportionality coefficient, K, in Equation 7.19 represents the 'discharge per unit area' at unit hydraulic gradient. K depends mainly on the porosity of the material and on the size and shape of the material's grains. To a lesser extent, K depends on the grain-size distribution and the temperature of the water.

The influence of grain size on the velocity at which groundwater flows through pores can best be explained by laminar flow in pipes. This is exceptional because, in nearly all problems of practical hydraulic engineering, the flow is turbulent. The flow of water through a porous medium is possibly the only laminar-flow problem that will confront an irrigation and drainage engineer.

In 1843, Poiseuille published his well-known equation to describe laminar flow in pipes

$$v_p = \alpha\, s^u \tag{7.23}$$

where

- v_p = laminar flow velocity in the pipe (m/s)
- α = coefficient (m/s)
- s = hydraulic gradient (–)
- u = an exponent that approximates unity (–)

The coefficient, α, in Equation 7.23 can be derived from theoretical considerations. For a circular pipe, for example, Equation 7.23 becomes

$$v_p = \frac{A\rho g}{8\pi\eta}\, s \tag{7.24}$$

Because the cross-sectional area of the pipe, A, equals $\pi d^2/4$, where d is the diameter of the pipe, Equation 7.24 can be written as

$$v_p = \frac{d^2\rho g}{32\,\eta}\, s \tag{7.25}$$

This equation was developed for a straight pipe, but it can also serve for the flow of water through porous material (see Equations 7.19 and 7.22). Equation 7.25 can thus be rewritten as

$$v_p = \frac{Q}{\varepsilon A} = \frac{d^2 \rho g}{32\,\eta}\frac{\Delta h}{L} \tag{7.26}$$

where d is the mean diameter of the pores between the grains. If we compare Equations 7.19 and 7.26, we find that

$$K = \frac{d^2 \rho g \varepsilon}{32\,\eta} \tag{7.27}$$

The influence of porosity, ε, on the hydraulic conductivity, K, is clearly shown in this equation.

We saw in Table 7.1 that both the mass density of water, ρ, and its dynamic viscosity, η, are influenced by the temperature of the water. In practice, the relation between mass density and temperature is ignored, and the value of the mass density is taken as a constant, 1000 kg/m^3. Nevertheless, as is obvious from Table 7.1, it is not always possible to ignore the influence of temperature on viscosity, and thus on the K-value. We can determine the hydraulic conductivity, K, at a temperature of x °C if we substitute the value of K measured at y °C into the following equation

$$K_{x^\circ} = K_{y^\circ}\frac{\eta_{y^\circ}}{\eta_{x^\circ}} \tag{7.28}$$

For example, if the hydraulic conductivity of a soil measured in the laboratory at 20 °C is found to be 2 m/d, while the groundwater temperature is 10 °C, then

$$K_{10^\circ} = K_{20^\circ}\frac{\eta_{20^\circ}}{\eta_{10^\circ}} = 2 \times \frac{1.01 \times 10^{-3}}{1.31 \times 10^{-3}} = 1.5 \text{ m/d}$$

Figure 7.8A shows a sample of coarse sand, and Figure 7.8B a sample of fine sand. Let the line AB be the path of a water particle flowing through both samples. It is clear that the flow path of the water particle is wider in the coarse sand than in the fine sand. We can thus draw the general conclusion that the hydraulic conductivity of a coarse material is greater than that of a fine one (see Equation 7.27).

Up until now, however, we have tacitly assumed that, in any type of soil, all the grains are of about the same diameter. This would be true for sand that has been carefully sieved, but natural soils usually consist of grains of many different sizes.

The influence of grain-size distribution on hydraulic conductivity is illustrated in Figure 7.9, which shows the flow path of a water particle through a mixture of the fine and coarse sands shown in Figure 7.8A and B. Following the flow path AB, we can see that the mean diameter of the pores in the mixture is determined by the smaller grains, and is only slightly affected by the larger grains. But, the larger grains partially block the passages that were present in the fine sand. Thus, in the mixture, the water particle is forced to travel a longer path to pass around the larger grains. In other words, while the mean diameter of the pores in the fine sand is almost the same as that in the mixture, the porosity of the mixture is less than the porosity of the fine sand. And, although the average size of the grains in the mixture is larger than that in the fine sand, the hydraulic conductivity of the mixture will be less than that of the fine sand.

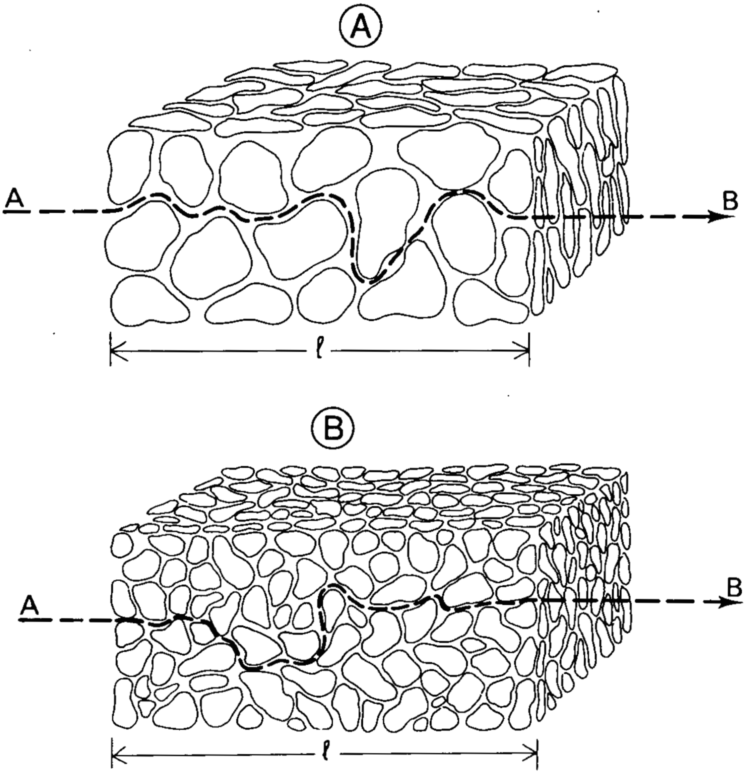

Figure 7.8 Flow path through two samples of sand (Leliavsky 1965)

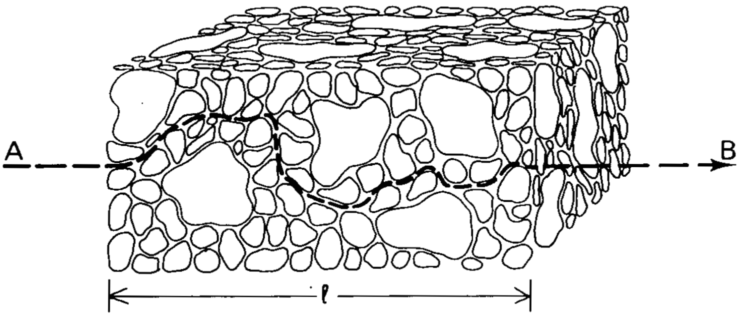

Figure 7.9 Flow path through a mixed sand sample (Leliavsky 1965)

So far, we have considered soils to be isotropic. Isotropy is the condition in which all significant properties are independent of direction. This means that the hydraulic conductivity is the same for any direction of flow. In reality, however, soils are seldom isotropic; instead, they are made up of layers with different hydraulic conductivities. Figure 7.10 presents a simple example of flow through a naturally-deposited, stratified soil. It is clear that a water particle following either flow path AB or CD will meet greater resistance than a water particle following paths EF or GH. The actual flow

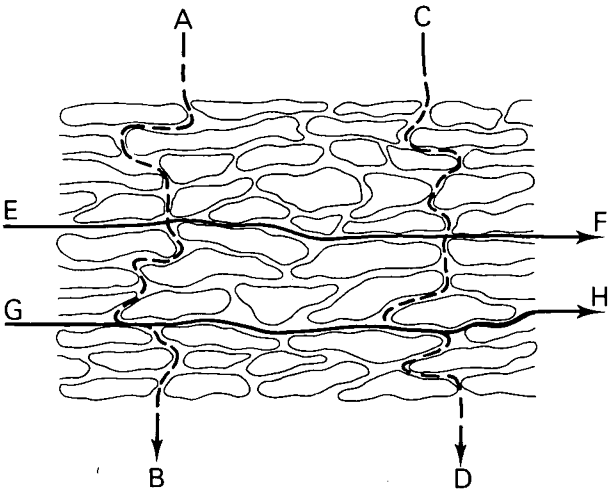

Figure 7.10 Flow through a naturally deposited, stratified soil (after Leliavsky 1955)

Table 7.2 K-values in m/d for granular materials (Davis 1969)

Clay soils (surface)	0.01	-	0.5
Deep clay beds	10^{-8}	-	10^{-2}
Loam soils (surface)	0.1	-	1
Fine sand	1	-	5
Medium sand	5	-	20
Coarse sand	20	-	100
Gravel	100	-	1000
Sand and gravel mixtures	5	-	100
Clay, sand, and gravel mixtures	0.001	-	0.1

paths of AB and CD are longer than those of EF and GH. This means that the hydraulic conductivity of this soil is greater for horizontal flow than for vertical flow.

The difference between horizontal and vertical K-values can only be determined by testing undisturbed samples or by conducting tests in situ (Chapter 12).

The orders of magnitude of K-values for granular materials are listed in Table 7.2.

7.4.3 Validity of Darcy's Equation

Like most empirical equations, Darcy's equation can be applied only within certain limits. Darcy's equation is valid only if the flow is laminar. As a criterion for laminar flow, we use the Reynolds number, Re, which is defined as (see also Chapter 19)

$$Re = \frac{v d_{50} \rho}{\eta} \tag{7.29}$$

where

v = discharge per unit area or the apparent velocity (m/s)

d_{50} = mean diameter of soil grains (m)
ρ = mass density (kg/m^3)
η = dynamic viscosity (kg/m s)

These four parameters thus determine the extent to which Darcy's equation can be applied. For a drainage engineer, it will suffice to accept the validity of Darcy's equation if the Reynolds number is equal to or less than unity (see also Muskat 1946). Hence

$$Re = \frac{v d_{50} \rho}{\eta} \leq 1 \tag{7.30}$$

If we substitute the known values of ρ and η for water at 10°C into Equation 7.30, we arrive at

$$v d_{50} \leq 1.3 \times 10^{-6} \text{ m}^2/\text{s} \tag{7.31}$$

In natural conditions of groundwater flow, it is unlikely that this limit of application will be exceeded. It may occur, however, if d_{50} is large (coarse gravel), if the hydraulic gradient is steep (close to pumped wells), or if most of the groundwater flows through cavities in (calcareous) soils.

7.5 Some Applications of Darcy's Equation

7.5.1 Horizontal Flow through Layered Soil

Consider Figure 7.11, where water flows from an irrigation canal to parallel drains. In the cross-section, water flows in a horizontal direction through three layers, each of which has a different hydraulic conductivity (K_1, K_2, and K_3) and a different thickness (D_1, D_2, and D_3). If we assume that there is no flow across the boundaries between the individual layers, then the hydraulic gradient, $s = (h_1 - h_2)/L = \Delta h/L$, applies to the flow through each layer.

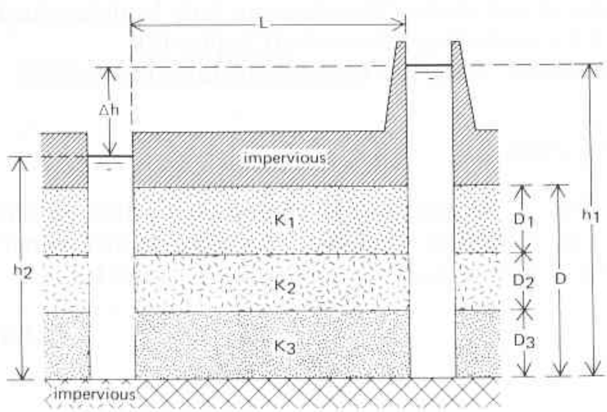

Figure 7.11 Horizontal flow through a layered soil

The flow rate per unit length of canal (q_1, q_2, and q_3) can be expressed by

$$q_1 = K_1 D_1 s$$
$$q_2 = K_2 D_2 s$$
$$q_3 = K_3 D_3 s$$

and the total flow through the cross-section by

$$q = q_1 + q_2 + q_3 = (K_1 D_1 + K_2 D_2 + K_3 D_3) s \tag{7.32}$$

or

$$q = \sum_{m=1}^{n} (K_m D_m) s$$

The product, KD, is the transmissivity of a soil layer in which the flow is horizontal. Layers with a high KD-value may thus contribute more to horizontal flow.

7.5.2 Vertical Flow through Layered Soils

Figure 7.12 is a cross-section of an irrigated field (basin with ponded water), where water is flowing vertically downward through a soil profile made up of layers of different thicknesses and different hydraulic conductivities. If we assume that the soil is saturated and no water can escape laterally, the discharge per unit area, i.e. the apparent velocity, is the same for each layer. Hence

$$v = K_1 \frac{h_1 - h_2}{D_1} \quad \text{or} \quad v \frac{D_1}{K_1} = h_1 - h_2$$

$$v = K_2 \frac{h_2 - h_3}{D_2} \quad \text{or} \quad v \frac{D_2}{K_2} = h_2 - h_3$$

$$v = K_3 \frac{h_3 - h_4}{D_3} \quad \text{or} \quad v \frac{D_3}{K_3} = h_3 - h_4$$

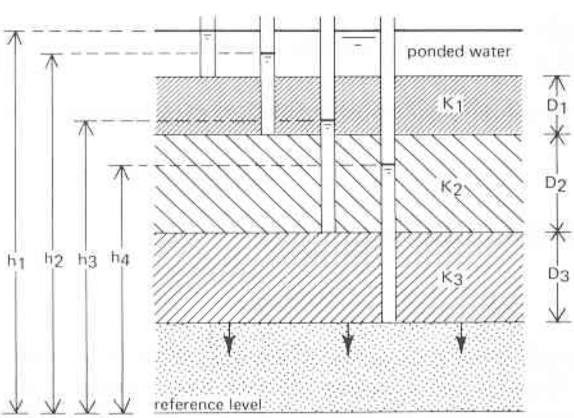

Figure 7.12 Vertical downward flow through layered soil

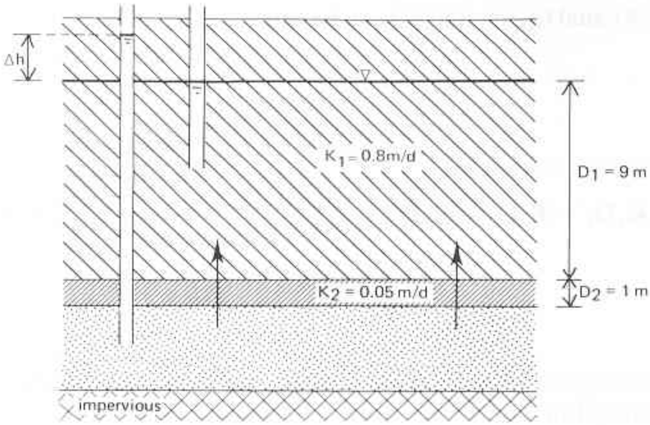

Figure 7.13 Vertical upward flow through two clay layers with different hydraulic conductivities and different thicknesses

Since $(h_1 - h_2) + (h_2 - h_3) + (h_3 - h_4) = h_1 - h_4 = \Delta h$, adding the equations yields

$$\frac{Q}{A} = v = \frac{\Delta h}{\frac{D_1}{K_1} + \frac{D_2}{K_2} + \frac{D_3}{K_3}} \tag{7.33}$$

As an example, let us consider Figure 7.13. It depicts an upper clay layer in which the watertable is assumed to remain stable because of factors such as drainage or evaporation. The saturated thickness of the clay layer is $D_1 = 9$ m; its hydraulic conductivity for vertical flow is $K_1 = 0.8$ m/d. Below this layer is a second clay layer; it is 1 m thick, and its hydraulic conductivity for vertical flow is $K_2 = 0.05$ m/d. The second clay layer rests on a sand layer that contains groundwater whose hydraulic head lies above the watertable in the upper clay layer ($\Delta h = 0.05$ m). The head difference causes a vertical upward flow from the sand layer through the overlying clay layers. According to Equation 7.33, the rate of this upward flow per unit area is

$$\frac{Q}{A} = \frac{0.05}{9/0.8 + 1/0.05} = \frac{0.05}{11.25 + 20} = \frac{0.05}{31.25} = 0.0016 \text{ m/d}$$

This shows that the second layer, with a high K/D-ratio, influences groundwater flow more than the thick first layer.

7.6 Streamlines and Equipotential Lines

7.6.1 Streamlines

In reality, all flow is three-dimensional. In irrigation and drainage engineering, however, we can regard groundwater flow as being essentially two-dimensional. This means that, in the direction normal to the plane of Figure 7.14, the thickness of the aquifer in the analysis equals unity. Such a two-dimensional flow problem is shown

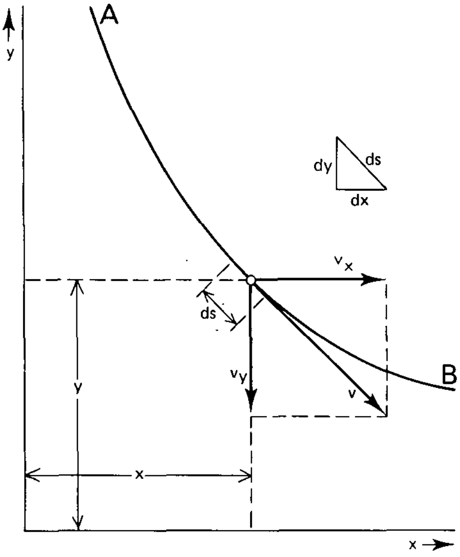

Figure 7.14 Two-dimensional path of a water particle

in Figure 7.14, where AB is the path of a moving particle of water. In this figure, v represents the velocity vector, which is of necessity tangential to the path AB at the point marked. It follows that

$$\frac{v_y}{v_x} = \frac{dy}{dx} \tag{7.34}$$

or

$$-v_y\,dx + v_x\,dy = 0 \tag{7.35}$$

This, generally stated, is the equation that describes the streamline AB at time t_1. If we assume that $v_x = f(x,y)$ and that $v_y = -g(x,y)$, where $f(x,y)$ and $-g(x,y)$ are two functions of the coordinates x and y, then, because of the continuity equation (Equation 7.9), the left-hand part of Equation 7.35 will always be a total differential of a certain function $\Psi(x,y)$. The equation representing a streamline can thus be written as

$$-v_y\,dx + v_x\,dy = \frac{\delta\Psi}{\delta x}dx + \frac{\delta\Psi}{\delta y}dy = d\Psi = 0 \tag{7.36}$$

so that

$$v_x = +\frac{\delta\Psi}{\delta y} \tag{7.37}$$

and

$$v_y = -\frac{\delta\Psi}{\delta x} \tag{7.38}$$

It follows from Equation 7.36 that the function $\Psi(x,y)$ is a constant for every streamline. This function is called the 'stream function'. We can therefore draw a number of streamlines, e.g. $(AB)_1$, $(AB)_2$, and so on, each corresponding to a different value of the same stream function Ψ, e.g. Ψ_1, Ψ_2, and so on (Figure 7.15).

It is also useful to know the discharge, Δq, which flows between these streamlines. This can be calculated by adding all the elementary discharges, $dq = v_s dn$, passing through a section, ab. Expressed as an equation, this reads

$$\Delta q = \int_a^b dq = \int_a^b v_s\, dn \tag{7.39}$$

For the s,n-coordinates, Equations 7.37 and 7.38 can be rewritten as

$$v_s = + \frac{d\Psi}{dn} \tag{7.40}$$

and

$$v_n = 0 \tag{7.41}$$

If we substitute Equation 7.40 into Equation 7.39 and adapt the limits of integration, we find that

$$\Delta q = \int_{\Psi_1}^{\Psi_2} \frac{d\Psi}{dn}\, dn = \Psi_2 - \Psi_1 = -\Delta\Psi \tag{7.42}$$

This is an important result; it shows that the difference between two values of the stream function equals the discharge passing between the two corresponding streamlines. Thus, once the streamlines have been drawn, they show not only the flow direction, but also the relative magnitude of the velocity along the channel between the two streamlines. In other words, because of continuity, the velocity at any point in the stream channel varies in inverse proportion to the spacing of the streamline near that point.

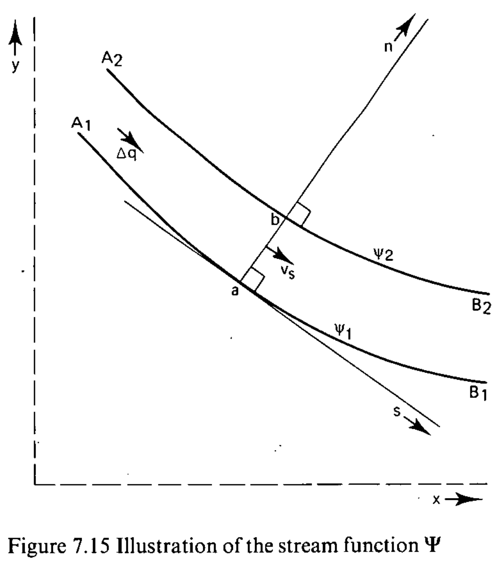

Figure 7.15 Illustration of the stream function Ψ

7.6.2 Equipotential Lines

To express two-dimensional flow in the xy-plane (Figure 7.16), we must rewrite Darcy's equation as

$$v_x = -K \frac{\delta h}{\delta x} \quad \text{and} \quad v_y = -K \frac{\delta h}{\delta y} \qquad (7.43)$$

Obviously, the rate of change of $K \times h = K(z + p/\rho g)$ determines the flow velocity. The product Kh can be replaced by Φ, i.e. the 'velocity potential'. Hence, we can rewrite Equation 7.43 as

$$v_x = -\frac{\delta \Phi}{\delta x} \quad \text{and} \quad v_y = -\frac{\delta \Phi}{\delta y} \qquad (7.44)$$

For lines with a constant value of $\Phi(x,y)$, i.e. lines that connect points with the same velocity potential, the total differential equals zero. Hence

$$\frac{\delta \Phi}{\delta x} dx + \frac{\delta \Phi}{\delta y} dy = d\Phi = 0 \qquad (7.45)$$

If we substitute v_x and v_y as given in Equation 7.44 into Equation 7.45, we see that

$$v_x\, dx + v_y\, dy = 0 \qquad (7.46)$$

This equation describes a line with a constant value, Φ_1, of Φ, i.e. an equipotential line.

Following the streamline's tangent, i.e. following the s-direction, we can rewrite Equation 7.44 as

$$v_s = -\frac{d\Phi}{ds} \quad \text{and} \quad v_n = 0 \qquad (7.47)$$

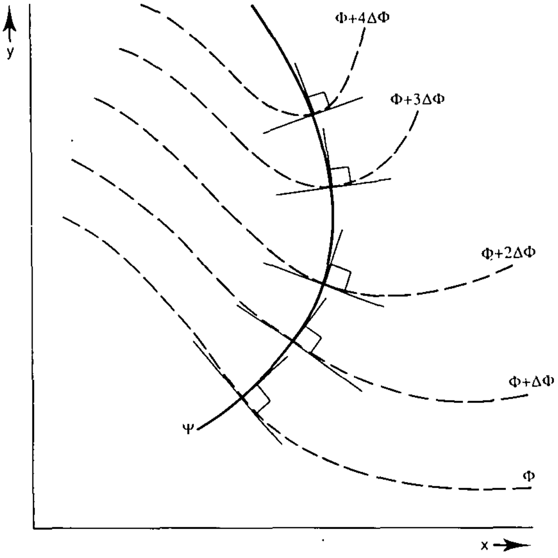

Figure 7.16 Illustration of equipotential lines

Thus, if we draw a set of curves, for each one of which Φ is constant, and if we choose these curves in such a way that the interval between them is $\Delta\Phi$, we obtain a diagram like the one in Figure 7.16. This could, for example, be part of a watertable-contour map.

The equipotential lines describe a surface whose slope, $d\Phi/ds$, is a measure of the velocity, v_s; so, the closer the equipotentials, the greater the velocity.

7.6.3 Flow-Net Diagrams

If we reconsider both the equation that describes a streamline (Equation 7.35) and the equation that describes an equipotential line (Equation 7.46), we can see that for the streamline

$$\frac{dy}{dx} = \frac{v_y}{v_x}$$

and for the equipotential line

$$\frac{dy}{dx} = -\frac{v_x}{v_y}$$

The product of their slopes equals −1. Streamlines always intersect equipotential lines at right angles, as we saw in Figure 7.16. Flow patterns can therefore be shown by a family of streamlines that intersects a family of equipotential lines at right angles. An example is shown in Figure 7.17.

It has been explicitly stated that streamlines and equipotential lines are directly

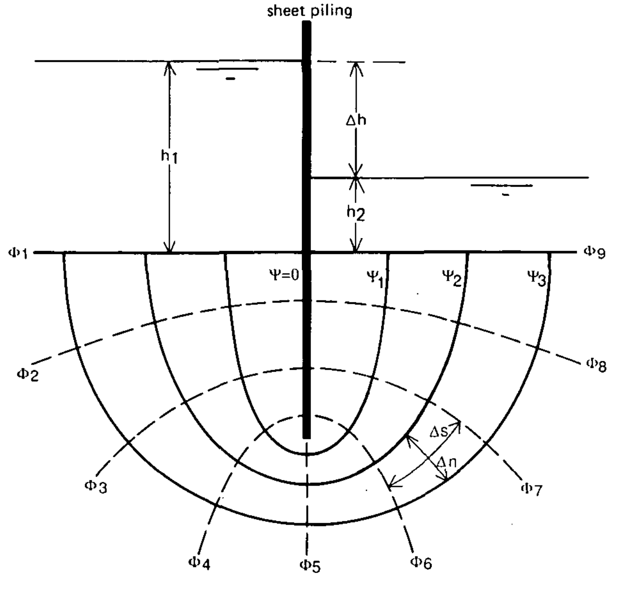

Figure 7.17 Flow-net diagram illustrating two-dimensional seepage under sheet piling in a homogeneous soil

related to each other. In fact, if a family of streamlines is established for a given flow pattern, there will then be only one corresponding family of equipotential lines, and vice versa. This can be seen if we rewrite Equations 7.40 and 7.47 as

$$d\Psi = v_s dn \quad \text{or} \quad \Delta\Psi = v_s \Delta n \tag{7.48}$$

and

$$-d\Phi = v_s ds \quad \text{or} \quad -\Delta\Phi = v_s \Delta s \tag{7.49}$$

If the flow net is constructed in such a way that $\Delta n = \Delta s$, it will consist of 'approximate squares', and

$$\Delta\Psi = -\Delta\Phi \tag{7.50}$$

A flow-net diagram of approximate squares is more convenient to draw and is easier to check for accuracy than a flow net of approximate rectangles. Harr (1962) recommends the following procedure for drawing a flow net:

1 Draw the boundaries of the flow region to scale so that all equipotential lines and streamlines that are drawn can be terminated on these boundaries (Figure 7.18A);
2 Sketch lightly three or four streamlines, keeping in mind that they are only a few of the infinite number of curves that must provide a smooth transition between the boundary streamlines. As an aid in the spacing of lines, note that the distance between adjacent streamlines increases in the direction of the larger radius of curvature (Figure 7.18B);
3 Sketch the equipotential lines, bearing in mind that they must intersect all streamlines, including the boundary streamlines, at right angles, and that the enclosed figures must be approximate squares (Figure 7.18B);
4 Adjust the locations of the streamlines and the equipotential lines to satisfy the requirements of Step 3 (Figures 7.18C and D). This is a trial-and-error process with the amount of correction being dependent upon the position of the initial streamlines. The speed with which a successful flow net can be drawn is highly contingent on the experience and judgement of the individual.

In the judgement of the designer, the flow nets shown in Figure 7.18B and C might be 'equally good'. In Figure 7.18D, the two flow nets have been superposed and it appears that the two 'equally good' flow nets do not coincide. The designer of a flow net can improve his design if he realizes that all sides of an approximate square must have a tangent point with a circle drawn within the square (Leliavsky 1955). In addition to the above four rules, it is recommended to:

5 Fill the area of Figure 7.18A with auxiliary circles as shown in Figure 7.18E;
6 Draw curved lines through the points of contact of the circles (Figure 7.18E) and omit the circles to obtain the flow net diagram (Figure 7.18F);
7 As a final check on the accuracy of the flow net, draw the diagonals of the squares. These should also form curves that intersect each other at right angles (Figure 7.18G).

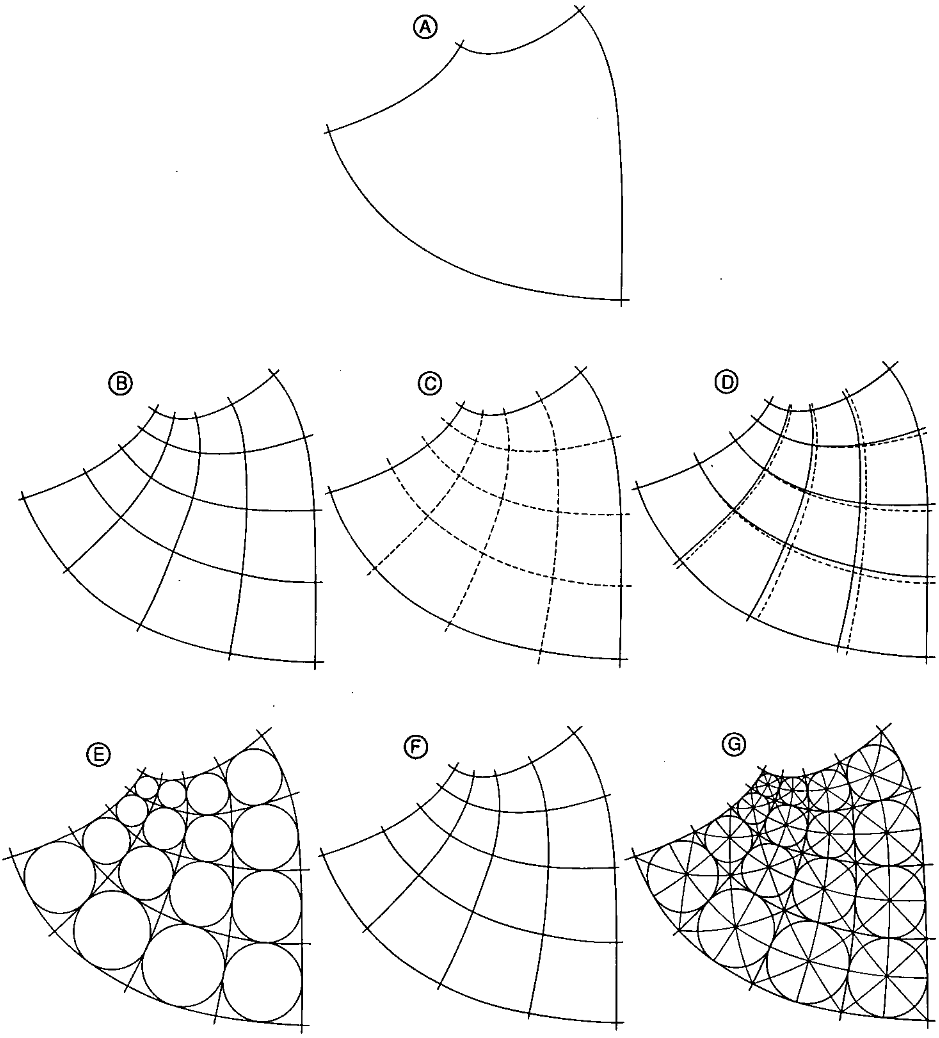

Figure 7.18 Methods of drawing a flow-net diagram (after Leliavsky 1955)

7.6.4 Refraction of Streamlines

When streamlines in a soil layer with a given hydraulic conductivity, K_1, cross the interface into a layer with a different hydraulic conductivity, K_2, their flow paths are refracted in the same way as light is refracted (Figure 7.19).

Let us consider two water particles that are following separate streamlines, Ψ_a and Ψ_b. They arrive simultaneously at Points A and C_1. In the time it takes the second particle to flow from C_1 to B, the first particle has flowed at a different velocity to C_2; it has crossed the interface. If we draw the equipotential lines, Φ_1 and Φ_2, we see that the flow paths of the streamlines have been refracted.

In Figure 7.19, we consider the discharge Δq that flows between two streamlines,

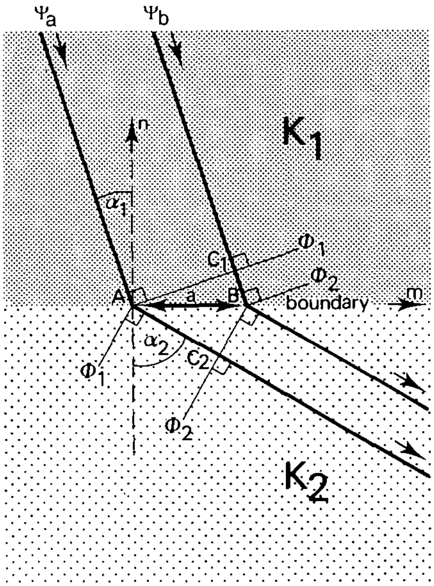

Figure 7.19 Refraction of streamlines

Ψ_a and Ψ_b. From the continuity equation (Equation 7.11), we know that this discharge must be the same in both layers, even though their K-values are different. Hence

$$\Delta q = K_1 s_1 \, a \cos \alpha_1 = K_2 s_2 \, a \cos \alpha_2 \tag{7.51}$$

where

α_1 = angle of entry
α_2 = angle of refraction
a = the unit area between Points A and B in Figure 7.19
s_1, s_2 = the respective values of the hydraulic gradient in the two layers, being

$$s_1 = \frac{\Phi_1 - \Phi_2}{a \sin \alpha_1} \tag{7.52}$$

$$s_2 = \frac{\Phi_1 - \Phi_2}{a \sin \alpha_2} \tag{7.53}$$

As can be seen in Figure 7.19, Φ_1 and Φ_2 are two different equipotential lines. Substituting Equations 7.52 and 7.53 into Equation 7.51, we see that

$$K_1 \frac{\Phi_1 - \Phi_2}{a \sin \alpha_1} a \cos \alpha_1 = K_2 \frac{\Phi_1 - \Phi_2}{a \sin \alpha_2} a \cos \alpha_2$$

or

$$\frac{K_1}{K_2} = \frac{\tan \alpha_1}{\tan \alpha_2} \tag{7.54}$$

If $\alpha_1 = 0$, then $\alpha_2 = 0$ as well. This was tacitly assumed in Section 7.5.2. And, Equation 7.54 shows that if $K_2 \gg K_1$, then α_2 is very large compared with $\tan \alpha_1$. Such a situation is found where a clay layer (K_1) covers a drained sand layer (K_2). In the clay layer, the flow is almost vertical; in the sand layer, it is almost horizontal. This supports the general assumption that, in a semi-confined aquifer, groundwater flow in the sand can be regarded as horizontal, and in the covering clay layer as vertical.

7.6.5 The Laplace Equation

For two-dimensional flow to occur, Equations 7.37, 7.38, and 7.44 dictate that

$$-\frac{\delta\Phi}{\delta x} = \frac{\delta\Psi}{\delta y} \tag{7.55}$$

and

$$\frac{\delta\Psi}{\delta x} = \frac{\delta\Phi}{\delta y} \tag{7.56}$$

These two important conditions are called the Cauchy-Rieman equations. They are necessary, but in themselves are insufficient to calculate two-dimensional flow; the existence and continuity of all partial derivatives of $\Phi(x,y)$ and $\Psi(x,y)$ must be verified as well. We must therefore be able to differentiate Equation 7.55 with respect to x and Equation 7.56 with respect to y. Adding the results, we find that

$$\frac{\delta^2\Phi}{\delta x^2} + \frac{\delta^2\Phi}{\delta y^2} = 0 \tag{7.57}$$

This equation can also be obtained by substituting Equation 7.44 directly into Equation 7.9. The continuity equation for two-dimensional flow would then read

$$\frac{\delta v_x}{\delta x} + \frac{\delta v_y}{\delta y} = 0 \tag{7.58}$$

Equation 7.57 is the well-known Laplace equation for two-dimensional flow. For homogeneous and isotropic soils (hence for $K_x = K_y = K =$ constant), the Laplace equation is often written with h instead of $\Phi = Kh$. Under these conditions, Equation 7.57 reduces to

$$\frac{\delta^2 h}{\delta x^2} + \frac{\delta^2 h}{\delta y^2} = 0 \tag{7.59}$$

Laplace's equation is also written as

$$\nabla^2 h = 0$$

where the symbol ∇, called 'del', is used to denote the differential operator

$$\frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z}$$

and ∇^2, called 'del squared', is used for

$$\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}$$

which is called the Laplacean operator.