2.1 Introduction to dynamic simulation

Th. J. Ferrari

2.1.1 *Introduction*

In the preceding chapter the definitions of concepts related to the state variable approach for simulation of living systems have been given. The method of construction of models according to this approach is introduced in this and the two following sections (Section 2.2 and 2.3). Particular attention is given in this section to the system dynamics of the most simple unit of a system. Such a unit consists of a number of elements and may contain a feedback loop. With a number of such units, larger systems with a closed structure can be described, with which the behaviour of the larger systems can be analyzed. In applying system dynamics for simulation of living systems, one does not need to have much knowledge of the mathematics of integration. In the computation of state variables, we use in fact often only the elementary arithmetical operations of adding, subtracting, multiplying and dividing.

How the knowledge of a system, i.e. of the factors on which the rates of a process depend and of the relationships between these factors, can be translated into a relational diagram of this system, according to certain conventions, will be shown in Subsection 2.1.2. Such relational diagrams are not necessary, but they do form an useful help to gain a better insight into the mutual relationships and enable surveying of the relevant factors. With these diagrams, the definition can be facilitated of the rate equations (the differential equations) to compute the rate variables and of the state equations (the integral equations) to compute the state variables (Subsections 2.1.3 and 2.1.4). Although integration in simple systems can often be done analytically, the numeric solution will be emphasized here (Subsection 2.1.5). This solution is based on a repetitive computation of changes occurring during successive, small time-steps. It will be shown that the analytical solution for models of systems of plant growth is impossible to use in practice, even for relatively simple systems. The differential equation is an important element of the description of the feedback phenomenon (Subsection 2.1.6). The time coefficient of a system or a process is the subject of Subsection 2.1.7; it can be used to characterize delays and mathematical, dispersion (Subsection 2.1.8), included in many models.

2.1.2 *Relational diagrams*

Relational diagrams are not necessary, but they have several advantages, so many people find them useful for building and elaborating more abstract models. At the start of research, a relational diagram summarizes the most important elements and relationships and helps the researcher to maintain an overall picture. Especially when problems are complex, it simplifies the definition of rate and state equations. It makes also the content and characteristics of a model easily accessible to others. Finally, a relational diagram improves the comprehensibility of a model so that consequence of different concepts of the structure of a system on its behaviour, and the significance of certain structures (loops) for the behaviour stand out more clearly.

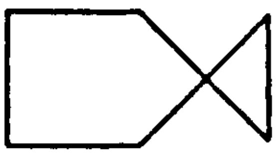
An example of a simple relational diagram is shown in Figure 2 of Section 1.2. Its representation is based on a number of conventions summarized in Figure 9.

Figure 9. Basic elements of relational diagrams. Abbreviated names of variables represented by these elements are usually written in or next to them. Note that driving variables are often underlined or placed between parentheses. Intermediate variables are often characterized by a circle.

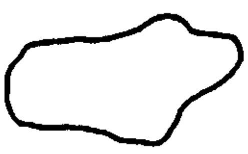
A state variable, or integral of the flow; final result of what has happened

Flow and direction of an action by which an amount, or state variable, is changed; if necessary, different sorts of lines can be used to distinguish between various states but no broken lines.

Flow and direction of information.

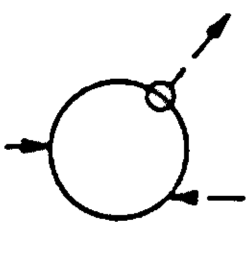


Valve in a flow, that indicates that a decision takes place here; the lines of incoming information indicate upon which factors the decision depends.



Source and sink of quantities in whose content one is not interested. This symbol is often omitted.

A constant or parameter.



Auxiliary or intermediate variable in the flow of material or of information.

The relations between elements in a diagram are indicated in a qualitative way only. Quantification of relations occurs during the definition of rate and state equations, and this is to be discussed in the following subsection. Sometimes during the research it appears that a factor supposed to be constant is variable after all (see Subsection 1.4.3). Then it has to be replaced by something else, perhaps a table, an auxiliary equation or a connection with an integral. One sometimes indicates with a + or − sign whether a loop concerns a positive or negative feedback.

Special emphasis must be given to the flows. The flow of material or energy is presented by solid arrows and the flow of information by dotted arrows. The solid lines connect state variables. Information is only transmitted and usually not processed, thus information is given, directly or indirectly, only to the decision functions and never to state variables. The use of information does not affect the information source itself. The flow of information can be delayed, and as such be a part of a process itself.

2.1.3 *Rate and integral*

The rate by which the numerical value of a state variable changes is expressed in the dimension: amount per time. Depending on the nature of the state, this amount can relate to different quantities, such as weight, length, number and even rate. The rate itself may be constant for a certain period; it may also change without a clear pattern (at random) or according to certain rules. These so-called decision rules must then be converted into differential equations. Note that decision does not have a human connotation: the differential equation describing a chemical reaction can be considered as a decision rule. Neither does it yield only a 'yes' or 'no': it can have any value.

It has great advantages to illustrate with simple examples a discussion on nature and function of a differential equation and on the integration associated with this equation. The principles used hereby are essentially the same as for more complex phenomena. The simplest case is the solution of a differential equation to describe the constant speed or rate of change in position of a vehicle. Plotted against time in hours (h) on a graph, this speed (km h^{-1}) is shown as a straight line parallel to the time axis (Figure 10). What is the result of this speed after a certain period? In other words, what is the distance covered? This question can be answered easily. The speed is multiplied by the length of time or period concerned and the distance covered is obtained as a result. One has now integrated the differential equation $ds/dt = c$, in which s is the distance (km), t the time and c a constant with the dimension km h^{-1}. The differential quotient or derivative ds/dt is a notation for the rate during an infinitely small time interval dt; another notation is $\dot{s}$.

In a graph this integration is the same as the computation of the area delimited by the time axis, by the line parallel to this time axis at a value c of the rate ordinate and by the both lines, parallel to the vertical rate axis, at two

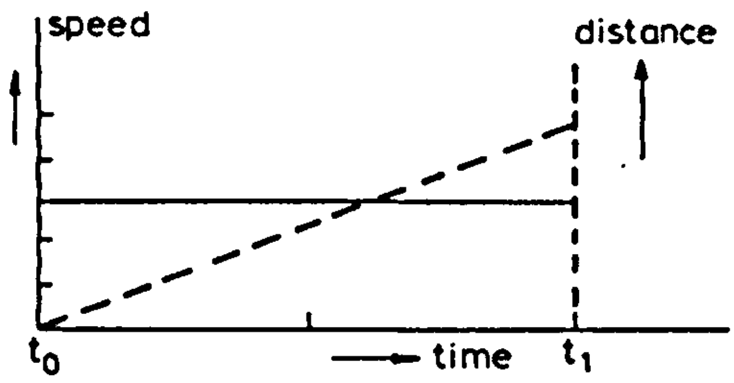


Figure 10. Speed in km h^{-1} (solid line) and distance covered in km (broken line) as function of time in h. The area delimited by the speed line, the two lines at the moments t_0 and t_1 and the time axis equals the distance covered after $t_1 - t_0$ time units.

points of time indicating the period. The result of the integration, *s*, is plotted as a function of time and given in Figure 10 by the dashed line through the origin with slope *c*. In this graph the slope of a straight line or the tangent to a curve represents the speed or rate at a certain moment.

Exercise 1
In a graph, the distance in meters on the y axis is plotted against the time in seconds on the x axis; the result is a straight line.
a. What does the slope of the line represent?
b. What is its dimension?
c. What can be said about it if the line is parallel to the x axis?

Such a procedure is always performed by integration. The right side of the differential equation is mostly more complex. The mathematician usually tries to integrate such differential equations mathematically by introducing boundary conditions or constraints in the model. This analytical, or mathematical, solution often gives a better understanding of the behaviour of the system than the numerical solution (computation of surfaces, for example) but can be applied less easily to practical situations. The necessary constraints can also be unacceptable in view of the purpose of the modelling effort.

2.1.4 Differential and finite difference equations

One starts mostly with simple differential equations. But with an increase in knowledge, there is the tendency to make the differential equations more complex so that they can no longer be integrated analytically. One might expect an equation that cannot be solved to be of little practical value. However, the numerical solution of such equations is often not very difficult (see Subsection 2.1.5).

Both methods can be illustrated with the help of some examples. Exercise 2 refers to the problem of filling a tank with water. The constructor has regulated the rate of filling by a valve with a time switch. From this one can derive that the rate as function of time is represented by the differential equation $dw/dt = -(1.2/30) \cdot t + 1.2$ up till time $t=30$, in which w is the number of litres and t the time in seconds; dw/dt is the rate of change with which the water is flowing into the tank; the rate is zero after $t=30$.

An analytical integration of this differential equation gives the amount of water w in the tank as a function of time according to $w = -1.2/60 \cdot t^2 + 1.2 \cdot t$. With this equation it is now possible to compute the inflow of water for each period between the points $t = 0$ and $t = 30$.

Exercise 2

Plot the rate of water flowing into a tank in litres per second on the y axis against the time in seconds on the x axis using the differential equation given above. The result is a straight line descending with a rate of 1.2 litres per second at point $t = 0$ and with a rate equal to zero after 30 seconds.

a. What is the amount of water in the tank after 30 seconds if there was no water in the tank at point $t = 0$?

b. Calculate the amount of water in the tank as a function of time between $t = 0$ and $t = 30$.

c. Check the results of the calculations based on the differential equation using the integral given.

We stress that the rate of inflow is not dependent on the amount of water already present in the tank in this example. However rates generally depend on states in the system (Subsection 1.1.3). For instance, an ecologist may think, on biological grounds, that the number of animals in an area increases by a certain percentage every year. Now the rate of increase, expressed in numbers of animals per year, would be determined by the number of animals already present and consequently would not be constant in successive years. Under such conditions, the following differential equation holds: $dy/dt = c \cdot y$, in which y is the number of animals at a certain moment and c the annual relative growth rate. The rate is a function of the number of animals y; in a graph this is represented by a straight line through the origin. This differential equation can be integrated analytically and produces the well-known exponential growth curve $y_t = y_0 \cdot e^{ct}$, in which t is the time in the chosen units and e the base of the natural or Napierian logarithms. The subscripts of y represent time; consequently, y_t and y_0 are the amounts of moment t and at the start of the calculation, respectively.

Exercise 3
Sketch the graph of the exponential growth curve with an annual relative growth rate of 0.03 over a period of 50 years, starting with a herd of 10^4 animals. Draw one graph on a linear scale and one with $\ln y$ instead of y on the ordinate.

2.1.5 Numerical integration

Until now we have investigated the influence of a certain rate equation on the state variable by analytically integrating this equation. The numeric value of a state variabale could be represented as a function of time by performing this integration for different time spans. In this way we can compute its behaviour. It is also possible to calculate the evolution of the value of the state variable by computing the changes during a number of successive short periods. The rate during such a short period can be supposed to be constant. One starts with a certain initial state y_0. By using the rate equation concerned, one can calculate the absolute rate during the next time interval or time step Δt and the subsequent change in state during this time interval. The new state again causes a new rate which holds for the next interval Δt, and so on.

The following example explains this procedure. Suppose that the rate at which an amount of water w is flowing into the tank through an adjustable valve, is given by the differential equation $\mathrm{d}w/\mathrm{d}t = 1/4 \cdot (16 - w)$. Suppose also that there is no water in the tank at $t = 0$; thus $w_0 = 0$. The rate by which water is flowing at that moment into the tank equals: $1/4 \cdot (16 - 0) = 4\ \mathrm{l\ s^{-1}}$. If we take the length of the time interval Δt equal to 2 s, then 8 l water will have flowed into the tank after 2 s, and w becomes 8 l. During the following time interval of 2 s, the rate is than: $1/4 \cdot (16 - 8) = 2\ \mathrm{l\ s^{-1}}$. Therefore, during this time step 4 l is flowing into the tank, so that the total quantity of water in the tank equals to 8 + 4 = 12 l. The calculation proceeds as follows:

time (s)	inflow during the interval (l)	amount of water in tank (l)	difference from the maximum of 16 l (l)	rate of inflow ($\mathrm{l\ s^{-1}}$)
0 } Δt		0	16	4
2	8	8	8	2
4				
6				
8				
10				
12				

Exercise 4

Complete the calculation and plot the amounts of water in the tank against time.

a. What do you notice?

b. When is the rate of inflow zero?

c. What happens if 8 is substituted for 4 in the fraction 1/4?

d. Suggest a name for this fraction, and what is its dimension?

In filling out such a scheme one has performed a numerical integration. Analytical integration may also be applied here. Integrating the differential equation $dw/dt = 1/4 \cdot (16 - w)$ gives the equation $w_t = 16 - (16 - 0) \cdot e^{-t/4}$. Figure 11 shows the state variable w as a function of time.

Exercise 5

Plot the results of the calculation of Exercise 4 in Figure 11.

a. Are the results of this calculation of the amount of water in the tank underestimated or overestimated compared with those of the analytical solution?

b. How do you explain this difference and in which way could it be corrected? (see also Subsection 1.1.3).

c. Repeat the calculations from t_0 onwards with $\Delta t = 1$ s.

With the calculation just discussed a numerical integration is performed in basically the same way as with a computer. The researcher converts continuous differential equations into finite difference equations, or rate equations, with a difference quotient $\Delta y/\Delta t$; with the aid of these finite-difference equations the new states or amounts are computed.

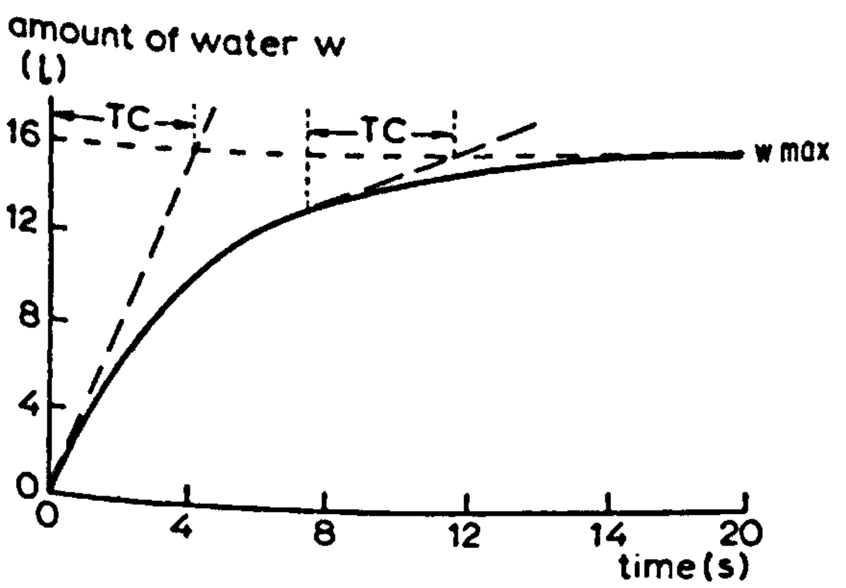


Figure 11. The amount of water w as function of time. The curve is the integral of the differential equation $dw/dt = 1/4 \cdot (16 - w)$ in which w is the amount of water at moment t. For explanation of TC, see Subsection 2.1.7.

The state equations describe how the changes are effected and form the integral. Their basic form is always:

$$\text{state}_{t+\Delta t} = \text{state}_t + \Delta t \cdot rate_t$$

(see Subsection 1.1.3). This computation is repeated many times by resetting $t+\Delta t$ to t after integration. Initial values of the state variables, and not of rate variables, define the situation at the onset of simulation. The numerical integration method of Euler (the rectangular method, see Section 2.3) is the most straightforward and works mathematically just as given here. More sophisticated integration methods that compensate considerably for inaccuracies inherent to numeric integration also exist (Exercise 5, Section 2.2 and 2.3). The use of simulation languages facilitates the expression of rate and state equations in a form that can be processed by a computer. In the formulation of the ultimate program used by the computer, the time subscripts can usually be dropped.

It is worthwhile to consider the implications of this formulation of numerical integration. State variables are updated after each time interval: they obtain then a new value by adding to them the rate of change multiplied with the duration of the time interval. Numerical integration requires that changes in the state variables and in all other elements of a system are small during a time interval. If this is not the case, the duration of the integration time interval is too long and should be reduced. Obviously, the rates of change are unaffected by this adaptation: they may be quite large, but then the corresponding time interval should be very short.

Proper numerical integration requires the computation of all rates of change before the integration starts. Simulation languages may take care of this (see Subsection 2.2.4) so that the modeller is not bothered by it. When using other computer languages the programmer should take care that this requirement is met.

Rate equations and the state equations can be extended in various ways. Rates of different, parallel processes can be included in a state equation. A rate equation may contain every combination of state variables and constants required by the problem. Furthermore, the number of rate equations and state equations can be increased as specified by the content of the problem. When the equations are formulated, the following points have to be taken into account. The time interval Δt is only found in state equations. A rate does not depend directly on another rate (Section 1.1) and because in reality a rate can be determined only indirectly through changes of state variables, rate equations contain only states, other variables and constants. The states are altered only by rates. The dimension of an element in the equation does not determine by itself whether it is a rate variable or not.

2.1.6 Feedback loops

Study of the behaviour of man-made control and servo-mechanisms has shown that the structure of a system may be more significant for its behaviour than the individual elements are. An important structure is the closed loop, or feedback loop, in which the state of an element or variable determines the degree of action or flow, which subsequently changes this state. This process takes place in a continuously circulating loop. There are two kinds of feedback loops.

In a positive feedback system, the action enhances the state, and vice versa, so that the action becomes greater and greater until a limitation within the system is encountered. An example is the exponential growth according to $y_t = y_0 \cdot e^{ct}$ with as underlying differential or growth-rate equation $dy/dt = c \cdot y$. This is the model of, for example, the growth of capital at a fixed interest per year, and of the growth of algae in a lake with a constant relative or intrinsic growth rate $(dy/dt)/y$. The absolute increase per time unit is determined by the amounts already present, so that the increase in amounts is enormous until it becomes restricted by internal limitations of the system. A positive feedback loop produces, as it were, a departure from some reference, neutral condition or goal, which is often that of zero activity. Such an equilibrium state in a positive feedback loop is often called an 'unstable' equilibrium.

Unlike the positive feedback, the negative feedback loop tends to return the system to an equilibrium situation; a departure from this equilibrium produces an action to return the value of the state variable to this equilibrium level. An example is a mechanism for the automatic filling of a tank with water up to a certain level. The tank may then be filled according to the equation $w_t = w_{max} - (w_{max} - w_0) \cdot e^{-ct}$, obtained by integration of the differential equation $dw/dt = c \cdot (w_{max} - w)$. The term w_0 is the amount of water in the tank at the beginning of the calcultation, w_{max} is the maximum level in the tank, which in this case is also the equilibrium value.

Exercise 6

a. Trace how the negative feedback loop in the last example works with the help of the differential equation.

b. In a certain crop, leaves are formed and die simultaneously. They are formed at a rate of 50 kg ha^{-1} d^{-1}. The rate of dying of the leaves is described by $dy/dt = -c \cdot y$, in which y is the amount of leaves and c the relative death rate, which equals 0.03 g g^{-1} d^{-1}. Does this system contain a positive or a negative feedback loop?

c. What is the rate equation and what is the equilibrium of the system?

Because a feedback system has a closed boundary, its behaviour must be accounted for by the structure only: it arises from the properties inside the system. Although factors outside the system do influence it, they are not essential for the pattern of behaviour. State variables and decision functions are parts of

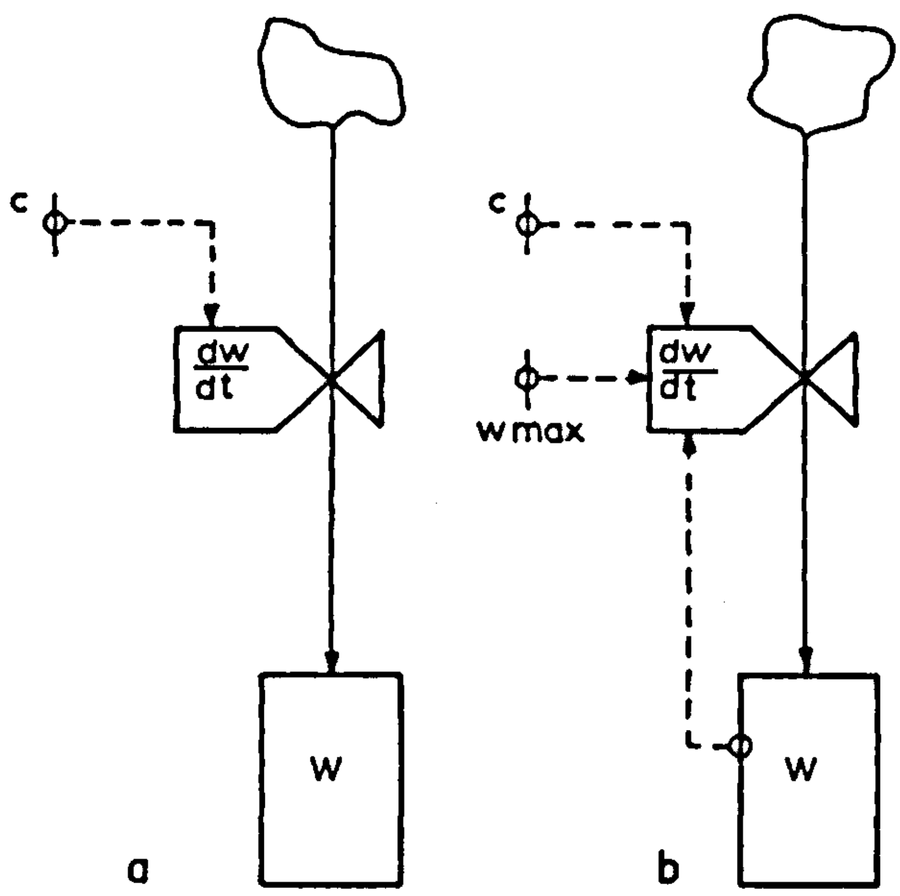


Figure 12. Relational diagram of systems of filling a tank with water. a. Without feedback loop and without maximum level. b. With feedback loop and a maximum level.

this feedback loop and are connected by an information chain or flow. The components of a decision function are the equilibrium situation, the state as observed by the decision function, the discrepancy between this state and the equilibrium and finally the necessary action resulting from this discrepancy.

The importance of the feedback structure and how it works can be illustrated best by two comparable examples, one with and one without a feedback structure. Both examples refer to the filling of a tank with water through an adjustable valve and are worked out in Figure 12 by relational diagrams. In Figure 12a there is no connection between the water-level in the tank and the aperture of the valve; the valve is not affected by the water-level: it has an aperture that does not change. In Figure 12b, however, the valve is affected by the water-level in the tank: through the float in the tank and the level between float and valve, the water-level determines the position of the valve and thus the aperture and the flow rate or decision. The builder of this system fitted a valve whose aperture closes increasingly with a rising level of water, until the flow is cut of when the water level in the tank has attained its maximum. The system reacts instantaneously to information; it is striving towards an equilibrium, namely the maximum level. Consequently, this system contains a negative feedback loop.

In the first example (Figure 12a) the rate of flow is constant and independent of the water-level, and the relevant differential equation is $dw/dt = c$. By integration, one can derive that the amount of water w at every moment can be calculated from $w_t = c \cdot t$. In the example with the feedback structure, the aperture of the valve on which the rate of flow depends, is a function of the amount of water and therefore not constant. How is this function derived? It is reasonable to suppose that the rate of flow is a constant fraction of the difference between the maximum amount w_{max} and the instantaneous level w. The lower the water in the tank, the faster the flow. The differential equation now becomes $dw/dt = c \cdot (w_{max} - w)$, from which after integration the amount of water in the tank can be calculated as function of time according to $w_t = w_{max} - (w_{max} - w_0) \cdot e^{-ct}$; the parameter c is a constant and w_0 represents as usual the initial value. The rate of flow is zero at the moment that the water-level in the tank has reached the maximum. This situation is called a static equilibrium because the rate of flow at that level has become zero and will be zero again some time after disturbance of the water-level. (This state must be distinguished from a dynamic equilibrium, in which the total amount present does not change, but where the rates are not equal to zero).

2.1.7 Time coefficient

Many processes are described by characteristic rates. For the exponential growth, the relative growth rate *(rgr)*, is such a characteristic rate. In the equation in which the rate is given as a function of the amount, $dy/dt = rgr \cdot y$, the left side presents an amount per time unit, and the right side an amount and the relative growth rate. The dimensions on both sides of the equal sign should equal each other. This means that the *rgr* has the dimension T^{-1}. In comparing processes in systems it is customary, especially in the technical sciences, not to use this characteristic rate but its inverse: the time coefficient. This coefficient is important for the behaviour of the system and has often a characteristic name. It is discussed briefly in the Subsections 1.1.4 and 1.4.4, and we will return to it later on. In the equation of the exponential growth, the relative growth rate equals the inverse of the time coefficient (TC). The equations become then $dy/dt = (1/TC) \cdot y$ and $y_t = y_0 \cdot e^{t/TC}$, respectively.

Exercise 7

a. Calculate the time coefficients when annual relative growth rates are 1.50, 0.25, 0.05, 0.02 and 0.001.
b. What is the value of y_t after one year, starting with the initial value $y_0 = 100$?
c. Compare the percentage of the total increase with the relative growth rate.

A warning for mistakes may appropriate here. The time coefficient is calculated correctly in Exercise 7. But when the growth percentage, given to indicate the increase in amount after one year (e.g. an increase from 100 to 125 corresponds with a growth percentage of 25% per year) is used to calculate the TC, an incorrect result is obtained. In Exercise 7 it is shown that this growth percentage of 25% per year is not identical with a *rgr* of 0.25. The relative growth rate is in fact less: after one year, y_t becomes $y_t = 125 = 100 + e^{rgr \cdot 1}$, hence $rgr = \ln 1.25 = 0.223\ yr^{-1}$; the TC equals 4.48 yr instead of $1/0.25 = 4$. Especially with large relative growth rates this difference between the percentage of increase and the relative growth rate can be considerable. The cause of the difference is the feedback in the exponential increase or decrease. The relative growth rate is therefore smaller than the percentage of increase after a unit of time for an exponential increase, and the relative rate of decrease is larger than the percentage of decrease after a unit of time for an exponential decrease.

The time coefficient TC appears to be an important element to characterize the behaviour of a system. It determines mainly the reaction rate and indirectly the behaviour of the system. To demonstrate this we confine ourselves to its influence in the most simple feedback systems: the exponential growth curve and the system by which a tank is filled automatically with water. In such feedback systems with only one state variable, TC is the time needed to bring the system into equilibrium if the rate of change were constant. This applies to any point of the exponential growth curve, as is illustrated by the extension of the tangent until it intercepts the equilibrium line (Figure 11).

Exercise 8

a. Prove this statement by using the rate equation of the system for automatic filling of a tank with water. Verify that this holds indeed for every point of the integrated function!

b. This is also correct for a positive feedback system, but the formulation is different. Why?

The significance of the time coefficient is generally recognized, as is indicated by well known names of this coefficient and of related concepts in various sciences: the time constant, transmission time in control-system theory, the doubling time, the average total residence time, the delay time, the extinction time and the relaxation time. In population biology and in many crop growth models the inverse of the time coefficient, the relative growth rate $(dy/dt)/y$, is mostly used.

Exercise 9

For exponential growth the doubling time, defined as the time needed to double the amount, equals 0.7 • TC and is therefore smaller than the time coefficient.

a. How can the factor 0.7 be derived?

b. What could be a definition of half-life or half-value time? The half-life period equals 0.7 • TC.

The relaxation time, often used in physics, is the time needed to decrease the tate to 1/e-th, or 0.37th, part of the original value. It is the time coefficient of he exponential return to the original state and can be used as a measure of the peed with which a system is absorbing disturbances. Suppose that a population of N animals decreases by death according to the exponential death curve $y_t =$ '$_0 \cdot \mathrm{e}^{-t/\mathrm{TC}}$, and that there is no increase by birth or by migration. In this case the ime coefficient equals the average residence time. The average residence time of nimals among other living beings is called the average life-time of the animals.

Finally, the time coefficient is important for the length of the time interval Δt or numerical integration. We have already seen that the drawbacks of numeri-al solutions can be overcome partly by using small time intervals (see Subsec-ion 2.3.6). However, smaller intervals require extra computer time and cost noney. Therefore, the tendency to increase the length of the time interval does aise the question how far this enlargement may proceed without invalidating he prediction. As a rule the length of the interval should not be greater than ne-fifth to one-quarter of the smallest time coefficient of the system (see Sub-ections 1.1.4 and 1.4.4). If the interval in the simulation procedure is too large, he behaviour of the simulation model will have nothing to do with reality. For nstance, if the time interval of integration of the simple system of the automatic illing of a tank is taken to be 2 • TC, oscillations will occur.

'.1.8 *Exponential delays and dispersion*

Closely related to the concept of the average residence time is that of delay ime. This concept involves a change of place, amount or form that is not realized mmediately. A transformation of raw material into a product requires time for nanufacture, just as the transfer of oil from the mining area through the pipe-ine to the client takes time. The transmission of information takes time, ex-ressed as an information delay.

Oil pumped into a pipe takes some time to arrive at its destination, but once here, it is all there. This type of delay is called a pipeline delay. Exponential de-ays are another type of delay: not all material arrives at the same time, but ome early, some late and most in between.

Suppose for instance that a group of seeds have been wetted at the same mo-nent. They do not germinate immediately, but after a delay. The average ger-nination time (or residence time or delay time) may be 10 days. Only a few eeds will take exactly 10 days to germinate because there are differences in the ate of germination between individual seeds. Hence the germination dates how a dispersion according to a certain frequency-distribution curve. These ispersions are met in all kinds of problems. Some examples are: the difference n biological-response time to signals or to manipulation, and the difference in hysiological development of biological subjects; responses to changes of rates lso show similar patterns.

The phenomena delay and dispersion as they occur in reality can be described

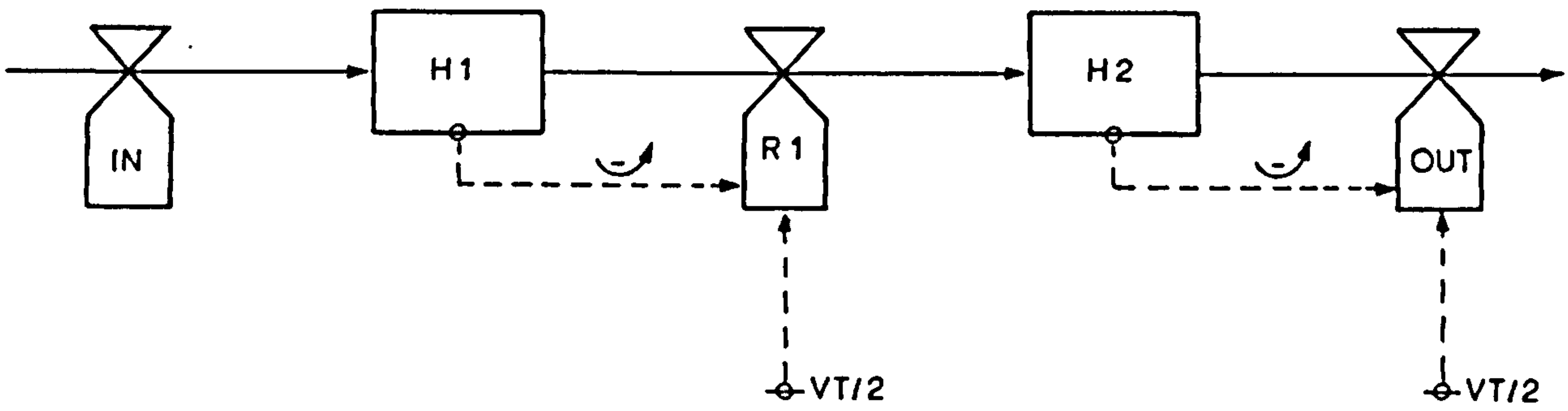


Figure 13. Relational diagram of an exponential delay of the second order of a rate. The input IN is changed; this change is effectuated in the output in a delayed and transformed way.

well with help of certain structures of system dynamics. It appears that distribution curves, representing the dispersion, can be obtained by using a cascade of successive integrations. An example is the relational diagram Figure 13, which represents a second order exponential delay of a rate (two integrals H1 and H2, between the rates IN and OUT) with a total delay time of VT. The corresponding state and rate equations are:

$$H1_t = H1_{t-1} + \Delta t \cdot (IN_{t-1} - R1_{t-1}) \text{ and } R1_t = H1_t/(VT/2),$$
$$H2_t = H2_{t-1} + \Delta t \cdot (R1_{t-1} - OUT_{t-1}) \text{ and } OUT_t = H2_t/(VT/2).$$

Figure 14 presents the rates R1 and OUT that result from a stepwise INput. The response of R1 to IN represents a first order exponential delay. The average delay time of a first order exponential delay equals the time coefficient of the filling process of H1 by IN.

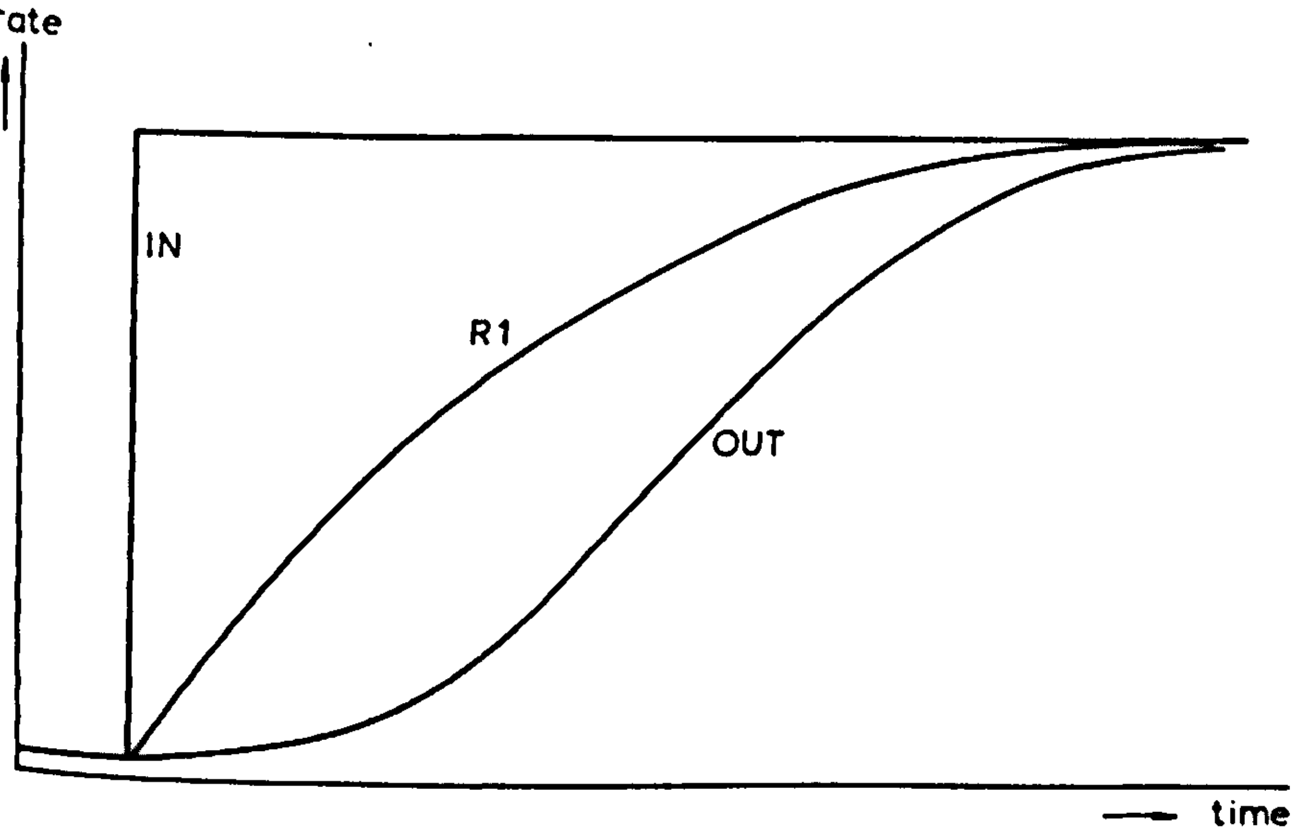
Exercise 10

a. Why has VT/2 been taken as time coefficient in both rate equations?

b. What are the values of H1 and H2, assuming that the steady state is reached for a constant inflow rate IN?

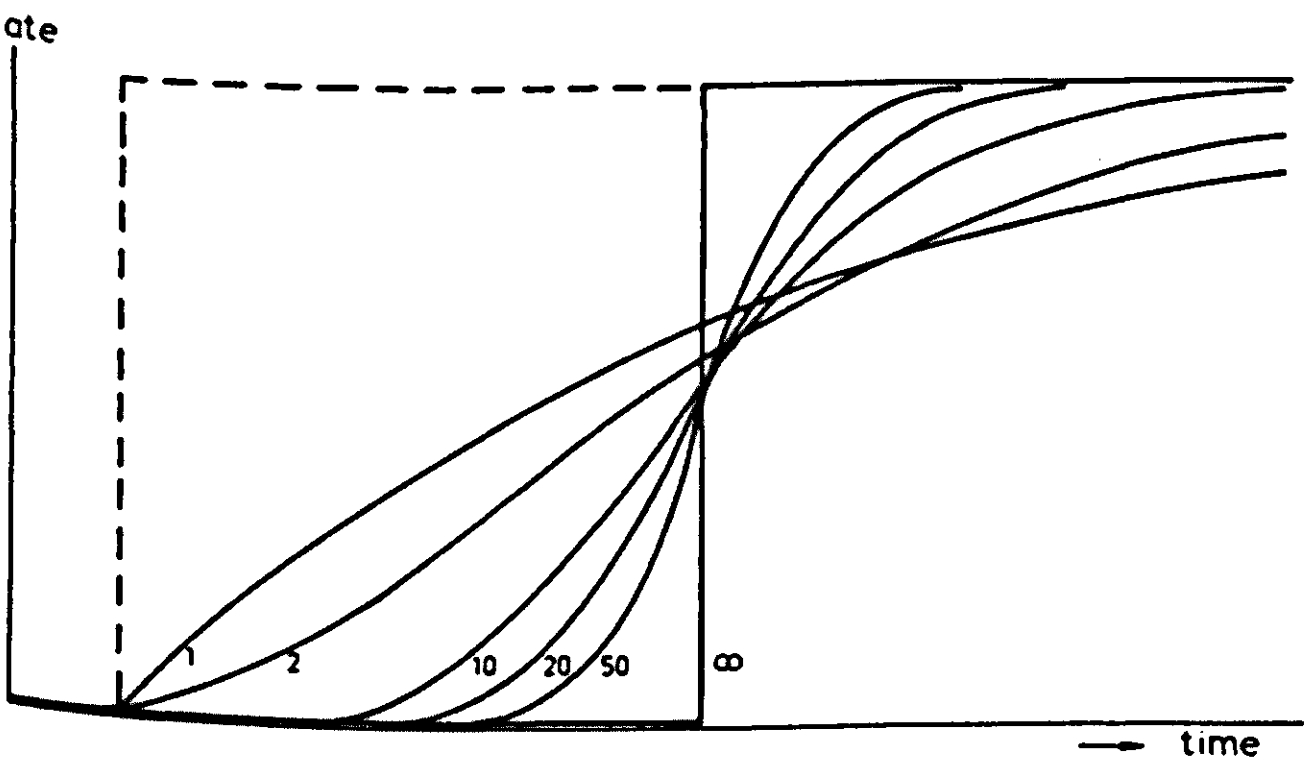
c. In a lake district, water flows from one reservoir to the other. The outflow from each reservoir is proportional with the content. We consider two similar lakes in succession. In a steady state the rate of inflow (IN) is 100 m^3 per week; the total delay time (VT) is 8 weeks. Outflow from the last lake can be represented by the second order exponential delay.
Assume that the inflow rate (IN) in the first reservoir (H1) is doubled suddenly. Calculate the time course of R1 and OUT (in m^3 per week) at intervals of 2 weeks, for a period of 3 months. Draw a figure of R1 and OUT against time and compare with Figure 14.

d. What equilibrium state is reached at the end?



'igure 14. The responses of the rates R1 and OUT to a stepwise changing input (IN) for he model shown in Figure 13.

Delays of higher order than the second order can be formulated in a similar vay. It appears that cascades of successive first-order delays yield dispersion urves that are distinct from those of Figure 14. Some of these curves are brought ogether into Figure 15 where the simulated output, resulting from a sudden and ermanent change in the input, is given. The greater the order of the delay, the teeper the distribution curve and the narrower the distribution. The relation-hip between order and standard deviation is formulated by the expression N = VT/s)^2, in which N represents the order or the number of integrations, VT otal delay time and s the standard deviation in time units. With a delay of in-inite order the deviation disappears and a 'pipeline' effect is obtained. This omputed dispersion can be used in model studies to simulate the dispersion ound in nature. This technique can be applied in many fields.



igure 15. Some characteristic patterns of responses of the output rates on sudden and ermanent changes in the input rate (-----). Delays of the 1st, 2nd, 10th, 20th, 50th and finite order are shown.