Non-stationary flow solution for water levels in open channels for TOXSWA

Alterra report 2166
ISSN 1566-7197

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Commissioned by the Ministry of Economic Affairs, Agriculture and Innovation
Project code: BO-12-07-002-ALT-6

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Alterra report 2166

Alterra, part of Wageningen UR
Wageningen, 2011

Abstract

Opheusden, J.H.J. van, J. Molenaar, W.H.J. Beltman and P.I. Adriaanse, 2011. *Non-stationary flow solution for water levels in open channels for TOXSWA.* Wageningen, Alterra, Alterra report 2166. 42 pp.; 18 fig.; 2 ref.

We study non-stationary flow in open discharge channels. A model is derived from basic principles, conservation of mass and momentum, which is solved numerically for the cross sectional area and discharge as a function of time and position along the channel. The model describes the effect of external inflow from fields adjacent to the channel. Several scenarios are calculated, both for very slowly, and more rapidly flowing water courses

Keywords: non-stationary flow, open channel flow, surface water, TOXSWA

ISSN 1566-7197

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Alterra report 2166
Wageningen, March 2011

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Summary

This report describes model results for non-stationary flow in straight discharge channels with a fixed prismatic cross section and a fixed bottom slope. A model is derived for gradually varying flow from first principles, that is conservation of mass (the water conservation equation) and momentum. Using balance equations for these quantities over a short sample stretch of the channel, two partial differential equations are derived, one for the discharge, the second for the cross sectional area, both as a function of time and position along the channel.

The obtained equations are largely identical to the standard de Saint Venant equations for gradually varied flow. Important is that also the effect of an external flow is considered, describing the drainage of runoff from fields adjacent to the channel. The friction with the channel walls is modelled using the Chézy-Manning relation for the friction slope. Viscous effects are neglected. In an earlier study we investigated the steady state for this model. Given the external flow as a function of position, the equations for the discharge and cross sectional area in the steady state are solved numerically. The only restriction is that the principle of gradually varying flow must be maintained, the external flow density must not change too rapidly along the channel.

An estimate for the relaxation time towards the steady state, based on the hydraulic retention time, indicates that for channels with slow flow (low Froude numbers) the rate of change of the external flow in practical situations exceeds that of the equilibration process. Hence a quasi steady state approximation to the process may not provide a good description, and a fully dynamical description is needed. The present model provides such a description.

The model needs the specification of two independent boundary conditions and an initial condition. In the two different scenarios investigated these are given by a weir at the downstream end of the channel, and a fixed discharge at the upstream end. The weir relation specifies the water level, and hence the cross sectional area immediately in front of the weir, given the discharge over the weir. As initial condition the steady state without external flow is considered. A numerical scheme with a Forward-Central method as a predictor and a Crank-Nicolson correction step is used to calculate the transition to a steady state with fixed external flow over the central part of the channel.

For very slowly moving water courses without external flow, it appears the water level is almost horizontal with respect to a fixed datum, such as sea level. The water level in the water course in such cases can be found in good approximation by analytical solution of the differential equation, it increases linearly to match the decrease of the bottom of the channel along the slope.

Preface

The TOXSWA model (TOXic substances in Surface WAters) simulates the fate of plant protection products in small surface waters. Under EU Council Directive 91/414 the exposure concentrations of plant protection products in surface water need to be assessed to evaluate their risk to aquatic organisms. In the current assessment procedure, the exposure concentrations in water and in sediment are calculated with FOCUS_TOXSWA for the EU Surface Water Scenarios. In FOCUS_TOXSWA versions TOXSWA simulates watercourses with transient flow conditions, assuming a constant water depth along the length of the watercourse. This approach was improved by dropping the assumption of a constant water depth in the watercourse and elaboration of the exact integration of the water conservation equation and discharge relations, resulting in a stationary flow solution. However, for relatively slow flow the pseudo-stationary approach appears not suited to describe the system behaviour under slowly changing external inflow. Therefore in this study solutions for non-stationary flow have been developed. These solutions will be implemented in TOXSWA to improve the simulation of its hydrology.

1 Introduction

We study the effect of a varying external flow to the water flow in a discharge channel. This flow is described by a large number of variables, such as the discharge, the slope and shape of the channel, possible curves, roughness of the bed, possible vegetation, and external discharges from neighbouring fields draining into the channel. Moreover most of these parameters can vary in time and position along the channel. To simplify matters we will study straight discharge channels, and assume several variables to be constant along the channel, specifically the bottom slope, the shape, and the friction parameter.

In an earlier study (Van Opheusden *e.a.*) we have investigated the same problem for constant external flow, and calculated the stationary (*i.e.* time independent) flow solution as a function of position along the channel for the given external flow. It was estimated in that study that the rate at which this stationary state is reached after a system change ranges from one hour to about a day, depending on the scenario investigated. If the changes in the system are slow compared to this time scale, for instance for seasonal changes in vegetation, or weekly changes in external flow, the stationary solution can appropriately be interpreted as a quasi-stationary approximation, in which the transient behaviour is neglected. In practice changes in external flow, however, occur at a daily rate, or even faster, and the stationary state may never be reached. In order to investigate the effect of rapid changes in the system, we here study non-stationary flow in straight discharge channels with a fixed prismatic cross section, that is both as a function of time and position along the channel.

2 Model description

We model open channel flow in a straight channel with a constant bottom slope. We use a variable *x* to identify the position along the channel, and *t* to describe the time. The discharge as a function of time and position then is denoted as $Q(x,t)$. The discharge and cross sectional area $A(x,t)$ are the two main variables we study. The water level $h(x,t)$ and the average flow velocity $v(x,t)$ are directly related to these two variables. We will derive and specify four relations between these four basic system variables that will allow us to calculate the water level for a given scenario. Additionally we will need the values of two of the variables at any given point along the channel, for any given time, and their initial values along the full channel, to fully specify the solution.

With those four relations and the additional so called boundary and initial conditions, we can calculate the water level profile as a function of time and position. In this section we derive the relations, mention the additional assumptions, and in the next section we will use them to investigate several scenarios.

2.1 Prismatic channel

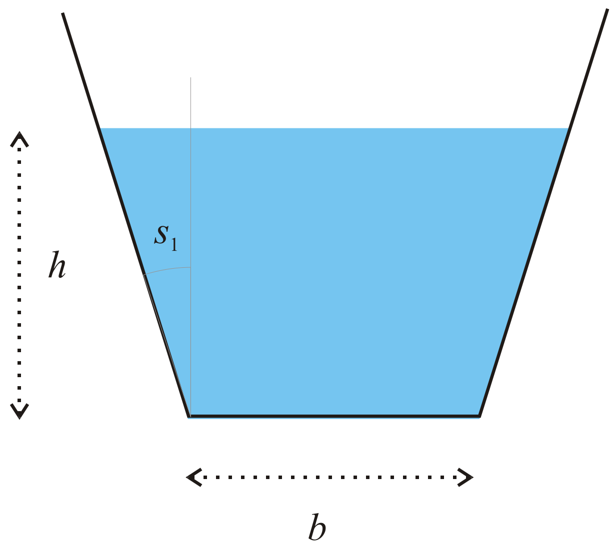


Figure 1
Wetted cross sectional area of the channel. The parameter s_1 gives the slope of the bank, h is the water level and b is the bottom width. The slope and bottom width are assumed to be constant along the channel.

We consider so called prismatic channels, whose cross section has a trapezoidal shape (Figure 1). The width at the bottom is *b*, and the slope of the banks is determined by a parameter s_1. For a rectangular channel we have $s_1 = 0$, for banks with a slope of 45 degrees $s_1 = 1$. The wetted cross sectional area *A* of the channel is given by the formula

$$A(x,t) = h(x,t)\,(b + s_1 h(x,t))\,, \tag{1}$$

where $h(x,t)$ is the water level in the channel. The bottom width b and the inclination s_1 of the banks are constant along the channel. The water level, and hence the cross sectional area of the flow, can vary with time t and position x, as indicated in the equation. See appendix 1 for a list of all variables used in this report. Formula (1) expresses the cross section $A(x,t)$ in the water level $h(x,t)$.

2.2 Average velocity

The second relation is

$$Q(x,t) = A(x,t)\, v(x,t)\,, \tag{2}$$

expressing the discharge Q as the product of the cross sectional area and the average velocity

$$v(x,t) = \frac{\int u(x,y,z,t)\, dA}{A(x,t)}\,, \tag{3}$$

with $u(x,y,z,t)$ the local velocity at any point of the cross section. In general we do not know u, so we only use the average velocity v. Note that we assume the average velocity to be perpendicular to the cross section. As long as we have a channel with a small bottom slope and a water level that changes only gradually, this approximation will be quite reasonable, but if the model leads to large changes in water level over short distances, we must be especially careful.

2.3 Water conservation equation

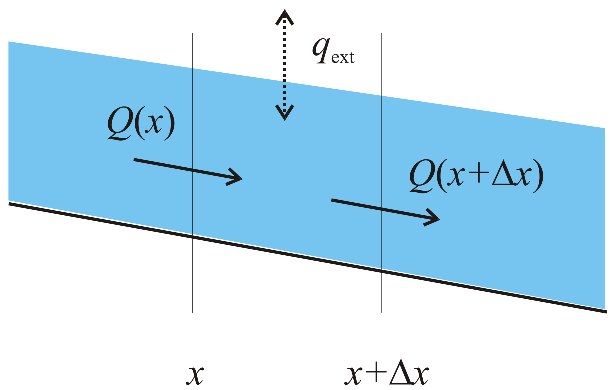


Figure 2
Volume flows for a sample section of the channel. Differences in discharge along the channel and external inflow result in changes in water level.

A third relation between the system variables is given by the continuity equation, in this context also termed the water conservation equation. The general physics law of conservation of mass, because of the incompressibility of the water, translates into a conservation of volume. If we consider a section of the channel between points x and $x+\Delta x$ (Figure 2) the total change in volume in that section must be the result of a net inflow or outflow:

$$change \;=\; inflow \;-\; outflow\,.$$

The discharge can vary with position, so the inflow on the left not necessarily matches the outflow on the right, and there also may be an external flow. If we consider the total change in volume between the end points of the channel section over a time interval Δt we find

$$\Delta V = Q(x,t)\cdot\Delta t - Q(x+\Delta x,t)\cdot\Delta t + q_{\text{ext}}(x,t)\cdot\Delta x\cdot\Delta t\,, \tag{4}$$

where the external discharge density q_{ext} is taken per unit length along the channel, and a positive value stands for water being drained into the channel (note that other authors, *e.g.* Jain, use the opposite convention). Moreover, since the distance between the two points is fixed, a change in volume must give a change in wetted cross sectional area:

$$\Delta V = \Delta A\cdot\Delta x\,. \tag{5}$$

Combination of these two equations gives

$$\Delta A\cdot\Delta x = Q(x,t)\cdot\Delta t - Q(x+\Delta x,t)\cdot\Delta t + q_{\text{ext}}(x,t)\cdot\Delta x\cdot\Delta t\,, \tag{6}$$

which when divided by Δx and Δt yields

$$\frac{\Delta A}{\Delta t} = \frac{Q(x,t)-Q(x+\Delta x,t)}{\Delta x} + q_{\text{ext}}(x,t)\,. \tag{7}$$

In the limit of $\Delta t \to 0$ and $\Delta x \to 0$ the difference quotients become derivatives

$$\frac{\partial A(x,t)}{\partial t} = -\frac{\partial Q(x,t)}{\partial x} + q_{\text{ext}}(x,t)\,. \tag{8}$$

This is the familiar continuity equation, which relates the rate of change of the cross sectional area at a given point to the gradient of the discharge and the external discharge density at that same point.

2.4 Momentum conservation equation

As stated, we have four variables $Q(x,t)$, $A(x,t)$, $h(x,t)$ and $v(x,t)$, that specify the flow situation along the channel. With equations (1) and (2) we can express the water level and the average velocity in the cross sectional area and the discharge, eliminating the former from the equations. We need a fourth relation, next to relations (1), (2) and (8) we already have discussed, to fully describe the problem and calculate the discharge and wetted cross section.

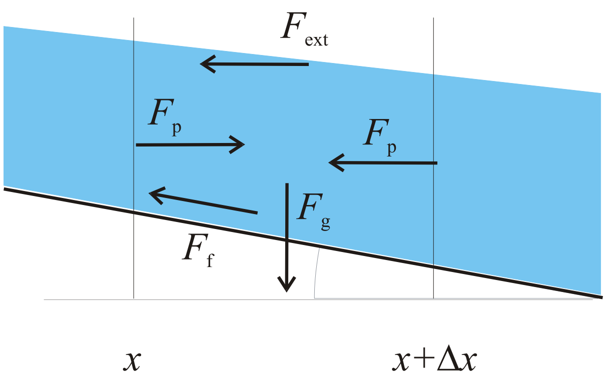


Figure 3
Forces on a sample section of the channel. Gravity, friction, hydrostatic pressure differences and a possible external force give an acceleration to the sample section.

This fourth relation follows from the principle of momentum conservation, as it follows from Newton's second law of motion. We have chosen for momentum conservation instead of energy conservation, since it is easier to account for the effect of the external inflow if we have no specific information on how the process occurs. If there is more specific information about how actually the external inflow occurs, energy conservation more easily allows for incorporation of that information. We will discuss the issue in detail to make clear the assumptions we make. Again we consider a short sample section of the channel between points *x* and $x+\Delta x$ (Figure 3). We consider all forces on the volume element.

The force on the sample section (Figure 3) contains four distinguishable terms, gravity F_g, hydrostatic pressure (which also derives from gravity), wall friction and an external force due to the external flow of water. Newton's law states

$$F_{\mathrm{g}} + F_{\mathrm{p}} - F_{\mathrm{f}} + F_{\mathrm{ext}} = \frac{dp}{dt}, \tag{9}$$

where $p = mv$ is the momentum, with *m* the mass of the total amount of water in the section. The gravitational force is *mg*, with *g* the acceleration of gravity. The component in the direction of the flow is $F_{\mathrm{g}} = mgS$. Note that we assume the bottom slope *S* to be small, and use $\cos(S) = 0$ and $\sin(S) = S$ without further specification. The force exerted by the hydrostatic pressure is given by integrating the pressure over both end surfaces

$$\int \rho gh \, dA = \tfrac{1}{2} \rho ghA, \tag{10}$$

which is relatively simple because of the prismatic shape of the channel. The net force is the difference between the pressure forces on the end surfaces

$$F_{\mathrm{p}} = \tfrac{1}{2} \rho gh(x,t) A(x,t) - \tfrac{1}{2} \rho gh(x+\Delta x,t) A(x+\Delta x,t) = -\tfrac{1}{2} \rho g \Delta x \frac{\partial (hA)}{\partial x}. \tag{11}$$

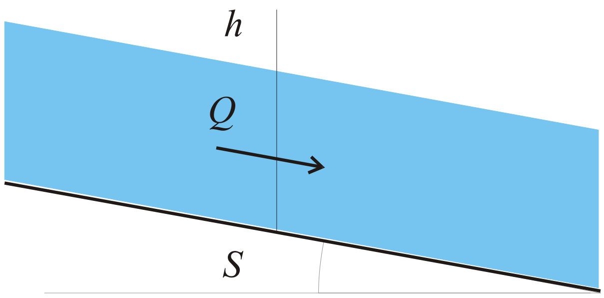


Figure 4
In uniform flow conditions all variables are constant along the length of the channel. Friction forces are exactly balanced by other forces. As shown in the earlier report about the stationary state, it can only be maintained along a part of the upstream reach of the channel. Uniform flow within the framework of this model provides an unstable stationary state.

The friction force requires quite some assumptions. The Chézy-Manning relation defines the friction in a channel with a fixed water level *h* and a constant discharge *Q*, so-called uniform flow conditions (Figure 4). The discharge in this particular case is given by the empirical Manning relation

$$Q = k_{\mathrm{M}} A R^{2/3} S^{1/2}, \tag{12}$$

with k_{M} the Manning coefficient or roughness factor of the bottom and walls of the channel. We will take this value to be constant along the length of the channel. The variable R is the so called hydraulic radius of the channel, which is defined by the shape of the channel

$$R = \frac{A}{P}, \tag{13}$$

with P the total length over which the water in the cross section is in contact with the walls and the bottom, the so called wetted perimeter. For a trapezoidal channel the wetted perimeter is expressed in the basic variables through

$$P = b + 2h\sqrt{s_1^2 + 1}. \tag{14}$$

Uniform flow is a very specific type of flow. Since in uniform flow all parameters are constant and there is no external flow, we have

$$F_{\mathrm{g}} - F_{\mathrm{f}} = mgS - F_{\mathrm{f}} = 0. \tag{15}$$

The particular value of the slope for which uniform flow applies is called the friction slope, and we identify it as

$$S = S_{\mathrm{f}} = \frac{F_{\mathrm{f}}}{mg}. \tag{16}$$

The Chézy-Manning relation now defines the friction slope in terms of Q and A as

$$S_{\mathrm{f}} = \frac{Q^2}{k_{\mathrm{M}}^2 A^2 R^{4/3}}. \tag{17}$$

We assume that the Manning relation also applies for non-uniform flow, and use it to calculate the frictional losses as a function of the system parameters.

Finally we take the external force to be zero. Whether that is a correct assumption depends very much on how the external water is added to the flow, in most cases this will give some negative force, impeding the flow, which means that we are overestimating the acceleration of the water in the case of nonzero external flow. The three other forces together give

$$F_{\mathrm{tot}} = mgS - mgS_{\mathrm{f}} - \tfrac{1}{2}\rho g\Delta x \frac{\partial(hA)}{\partial x} = \rho A\Delta x g(S - S_{\mathrm{f}}) - \tfrac{1}{2}\rho g\Delta x \frac{\partial(hA)}{\partial x}, \tag{18}$$

where $m = \rho A \Delta x$ is the mass of the section.

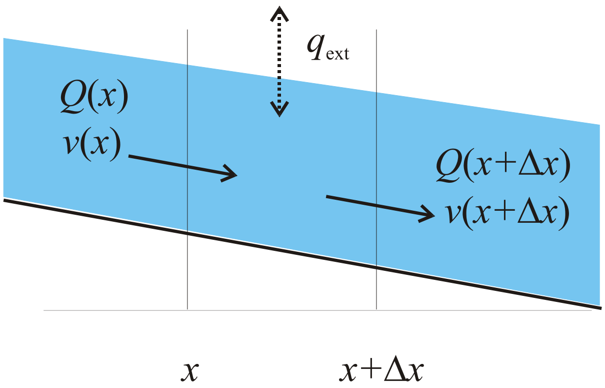


Figure 5
Momentum change due to flow. When a fixed sample section of the channel is monitored, the momentum of the water inside that section changes as water flows in from the upstream part of the channel, while the water currently present moves downstream and out of the sample section.

The forces are the rates at which momentum is produced (gain) or deleted (loss), inside the sample volume. The momentum inside the sample section of the channel between the fixed points *x* and *x+*Δ*x* also changes due to the flow itself. The total balance equation has the form

$$change \ = \ inflow \ - \ outflow \ + \ gain \ - \ losses \ .$$

The total momentum flux is given by integrating the momentum flux density over the cross section

$$\int \rho u \cdot u \, dA = \rho \cdot \int u^2 \, dA = \rho \beta Q v , \tag{19}$$

where

$$\beta \equiv \frac{\int u^2 \, dA}{v^2 A} = \frac{\int u^2 \, dA}{vQ} . \tag{20}$$

The momentum correction factor β, defined here, indicates that the average of the velocity squared is not the same as the square of the average velocity. Since we do not have detailed information about the local velocity *u*, we again need to be satisfied with only the average velocity, as defined in (2). Further we assume that the external flow does not carry any momentum, water is added from the surrounding fields at zero velocity. Over a time interval Δ*t* the change in momentum is

$$\Delta p = \rho\beta(x,t)Q(x,t)v(x,t)\Delta t - \rho\beta(x+\Delta x,t)Q(x+\Delta x,t)v(x+\Delta x,t)\Delta t + F_{\text{tot}}\Delta t . \tag{21}$$

Assuming β to be constant in time and along the channel, this can be approximated to first order as

$$\Delta p = -\rho\beta\Delta t \frac{\partial}{\partial x}\big(Q(x,t)v(x,t)\big)\Delta x + F_{\text{tot}}\Delta t . \tag{22}$$

Dividing by Δ*t* and taking the limit $\Delta t \to 0$ we obtain

$$\frac{\partial p}{\partial t} = -\rho\beta \frac{\partial (Qv)}{\partial x}\Delta x + F_{\text{tot}} . \tag{23}$$

Substituting $p = mv = \rho A v \Delta x = \rho Q \Delta x$, inserting equation (18) for the force, and dividing both sides of the relation by $\rho \Delta x$ we find

$$\frac{\partial Q(x,t)}{\partial t} = gA(x,t)(S - S_{\mathrm{f}}(x,t)) - \tfrac{1}{2} g \frac{\partial (h(x,t)A(x,t))}{\partial x} - \beta \frac{\partial (Q(x,t)v(x,t))}{\partial x}. \qquad (24)$$

Together with the continuity equation

$$\frac{\partial A(x,t)}{\partial t} = -\frac{\partial Q(x,t)}{\partial x} + q_{\mathrm{ext}}(x,t),$$

these are the dynamic equations that describe the problem. In the stationary case both time derivatives are zero, and the equations reduce to the stationary equations we derived earlier, with the exception that here we have a momentum correction factor β instead of the α obtained for the stationary case, which was derived from energy conservation. For a rectangular channel of width *b* the equations can be written as

$$\begin{cases} \dfrac{\partial A}{\partial t} + \dfrac{\partial Q}{\partial x} = q_{\mathrm{ext}} \\ \dfrac{\partial Q}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{gA^2}{2b} + \dfrac{\beta Q^2}{A}\right) = gA(S - S_{\mathrm{f}}) \end{cases}, \qquad (25)$$

where we have dropped the explicit time and position dependence of the variables. Apart from the external flow density and the momentum correction coefficient these are the familiar de Saint Venant equations for unsteady flow as formulated in 1871. They can be written in a mathematically more compact form as

$$\frac{\partial}{\partial t}\begin{pmatrix} A \\ Q \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ gh - v^2 & 2v \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} A \\ Q \end{pmatrix} = \begin{pmatrix} q_{\mathrm{ext}} \\ gA(S - S_{\mathrm{f}}) \end{pmatrix}, \qquad (26)$$

(where we have taken $\beta = 1$ for simplicity). The eigenvalues of the velocity matrix give the local velocities at which disturbances propagate

$$v_{\pm} = v \pm \sqrt{gh}, \qquad (27)$$

where *v* is the familiar flow velocity and the second term, called the celerity, is the propagation rate of gravity waves in shallow water. Their ratio is the Froude number

$$Fr = \frac{v}{\sqrt{gh}}. \qquad (28)$$

For a small Froude number, when the celerity exceeds the flow velocity, disturbances can travel upstream.

To make this latter statement more explicit, we consider a disturbance δA and δQ of the stationary state

$$\begin{pmatrix} A(x,t) \\ Q(x,t) \end{pmatrix} = \begin{pmatrix} A_s(x,t) \\ Q_s(x,t) \end{pmatrix} + \begin{pmatrix} \delta A(x,t) \\ \delta Q(x,t) \end{pmatrix}. \qquad (29)$$

The stationary state satisfies

$$\begin{pmatrix} 0 & 1 \\ gh - v^2 & 2v \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} A_s \\ Q_s \end{pmatrix} \equiv J \frac{\partial}{\partial x}\begin{pmatrix} A_s \\ Q_s \end{pmatrix} = \begin{pmatrix} q_{\mathrm{ext}} \\ gA_s(S - S_{\mathrm{f}}) \end{pmatrix}. \qquad (30)$$

Using equation (26) this implies for the disturbance

$$\frac{\partial}{\partial t}\begin{pmatrix} \delta A \\ \delta Q \end{pmatrix} + J \frac{\partial}{\partial x}\begin{pmatrix} \delta A \\ \delta Q \end{pmatrix} = \begin{pmatrix} 0 \\ g\delta A(S - S_{\mathrm{f}}) \end{pmatrix}. \qquad (31)$$

If we take the partial derivative with respect to time and position respectively, we find

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix}\delta A\\ \delta Q\end{pmatrix}+J\frac{\partial^2}{\partial t\partial x}\begin{pmatrix}\delta A\\ \delta Q\end{pmatrix}=\begin{pmatrix}0\\ g\delta A_t(S-S_\mathrm{f})\end{pmatrix}\quad\text{and}$$
$$\frac{\partial^2}{\partial x\partial t}\begin{pmatrix}\delta A\\ \delta Q\end{pmatrix}+J\frac{\partial^2}{\partial x^2}\begin{pmatrix}\delta A\\ \delta Q\end{pmatrix}+J_x\frac{\partial}{\partial x}\begin{pmatrix}\delta A\\ \delta Q\end{pmatrix}=\begin{pmatrix}0\\ g\delta A_x(S-S_\mathrm{f})\end{pmatrix}. \tag{32}$$

Elimination of the cross-derivative yields

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix}\delta A\\ \delta Q\end{pmatrix}-J^2\frac{\partial^2}{\partial x^2}\begin{pmatrix}\delta A\\ \delta Q\end{pmatrix}-JJ_x\frac{\partial}{\partial x}\begin{pmatrix}\delta A\\ \delta Q\end{pmatrix}=g(S-S_\mathrm{f})\begin{pmatrix}\delta A_x\\ \delta A_t+2v\delta A_x\end{pmatrix}. \tag{33}$$

The disturbances apparently satisfy the wave equation (the left hand side of this equation) with a position dependent wave velocity given by the eigenvalues of the matrix *J* and a dissipation given by the right hand side of equation 33. Note that for uniform flow the friction slope is exactly equal to the bottom slope, so there is no dissipation to first order. This corroborates earlier observations that the uniform flow state is actually unstable.

3 Numerical solution of the equations

We have two first order partial differential equations. To complete this to a full initial value problem we need two boundary conditions, and an initial condition. The initial condition will specify the cross sectional area and the discharge at time zero along the channel, as boundary conditions we use a fixed discharge at the beginning of the channel section, while we assume the presence of a weir with specified weir characteristics at the downstream end. As initial condition we take a steady state solution with the external flow set to zero. In our test runs we use a channel with a length of 500 m, with an external inflow only over the centre 100 m, the position of the drained field. The solution method for the stationary state has been discussed in detail in an earlier report. For the non-stationary state the external flow is kept at a specified non-zero level, and the relaxation to equilibrium is monitored.

The partial differential equations are solved numerically. First we discretise the position *x* along the channel

$$x_k = x_0 + k\Delta x, \;\; k = 1, 2, .., K \;,$$

where we take $x_0 = 0$ at the beginning of the channel section. The time discretisation is

$$t_n = t_0 + n\Delta t, \;\; n = 1, 2, .., N \;,$$

where we take $t_0 = 0$ at the beginning of the process. The system is integrated with a predictor-corrector method with a Forward-Central predictor step, and a Crank-Nicolson correction step. The method is also referred to in literature as Heun's method, or modified Euler method (although there are several methods by that latter name). In the Forward-Central step the derivatives with respect to time are approximated with a first order forward differential quotient

$$\frac{\partial A(x_k,t_n)}{\partial t} \cong \frac{A(x_k,t_n+\Delta t) - A(x_k,t_n)}{\Delta t} = \frac{A(x_k,t_{n+1}) - A(x_k,t_n)}{\Delta t}, \tag{34}$$

and similarly for the discharge. The derivatives with respect to position are approximated by a central difference quotient

$$\frac{\partial A(x_k,t_n)}{\partial x} \cong \frac{A(x_k+\Delta x,t_n) - A(x_k-\Delta x,t_n)}{2\Delta x} = \frac{A(x_{k+1},t_n) - A(x_{k-1},t_n)}{2\Delta x}. \tag{35}$$

At the boundaries a quotient using three end points is applied, with the same order of accuracy as the central difference. The advantage of the Forward-Central (FC) method is that it leads to an explicit algorithm even for non-linear equations, as we have here. The FC result is used as an approximation for the values at t_{n+1} that are needed for the Crank-Nicolson (CN) method. When we use the short-hand notation

$$A_{k,n} = A(x_k,t_n),$$

the two equations can be written as an explicit prescription to calculate the variables

$$\begin{aligned} A_{k,n+1} &= A_{k,n} + \Delta t\left[-\frac{Q_{k+1,n} - Q_{k-1,n}}{2\Delta x} + q_{k,n}\right] \\ Q_{k,n+1} &= Q_{k,n} + \Delta t\left[gA_{k,n}(S - S_{\mathrm{f}}) - \frac{g}{2}\frac{A_{k+1,n}h_{k+1,n} - A_{k-1,n}h_{k-1,n}}{2\Delta x} - \beta\frac{Q_{k+1,n}v_{k+1,n} - Q_{k-1,n}v_{k-1,n}}{2\Delta x}\right] \end{aligned} \tag{36}$$

The friction slope is also a (complicated) function of the system variables, and hence in general a function of position (and time).

In the present system the smallest time scale is given by the time it takes the disturbances to spread over the interval between two grid points. In order to be able to correctly describe the rapid fluctuations in the system we need a time step small enough to catch this behaviour

$$\Delta t < \frac{\Delta x}{v + \sqrt{gh}} . \tag{37}$$

The larger time scale is that of the approach of the perturbed system to equilibrium, the hydraulic retention time

$$\tau = \frac{L}{v} . \tag{38}$$

The ratio between the time scales

$$\frac{L}{\Delta x} \frac{v + \sqrt{gh}}{v} = \frac{L}{\Delta x} \frac{1 + Fr}{Fr} \tag{39}$$

can be a large number when the Froude number is small, the first factor is simply the detail with which the flow needs to be known. For slow flows, with small Froude numbers, many integration steps are needed to both describe the rapid changes and the slow relaxation towards the stationary state.

During testing it turned out that a time step just slightly below the requirement of equation 37 still gave rise to instabilities, so we needed to choose somewhat smaller steps still.

4 Results

In all the test runs we have used a rectangular channel. For a non-rectangular channel it is little bit more complicated to calculate the water level from the given cross sectional area (see Appendix 2), if necessary it could be included without difficulty.

The model describes a channel of 500m long, with a weir at the downstream end, a fixed discharge at the beginning upstream, and a drained part at the central 100m. We have studied two scenarios, starting from a stationary state without drainage to one with drainage, one for a channel with a fairly steep slope and a relatively high discharge (the stream scenario), and one for a much less steep channel with a significantly smaller discharge (the ditch scenario). First the stationary state is determined by numerically solving the equations as described in the earlier study. These are the same equations as equation 25 above. The numerical model is unstable for larger time steps. Even in the stationary state small numerical rounding errors cause artificial fluctuations that eventually grow and lead to unphysical results. Similarly the actual physical disturbances caused by changing the drainage level and the pursuing external flow in the central part of the channel cause short wavelength fluctuations that grow in time. For time steps sufficiently smaller than the smallest time scale in the discretized system we did obtain stable results. For our two tests we needed to go to time steps as small as 0.1 and 0.25 s, while the time scale as given by equation 37 was of the order of a second.

The weir at the downstream end of the channel is the same in both scenarios. The weir relation

$$A = b\left(h_{\mathrm{w}} + (Q / C_{\mathrm{w}} w_{\mathrm{w}})^{2/3}\right), \tag{40}$$

with w_w and h_w the width and height of the weir respectively, and C_w the weir constant, is used to express the cross sectional area A immediately in front of the weir in terms of the local value of the discharge. The values we used were a weir height h_w = 0.5 m, a weir width w_w = 0.5 m, and a weir constant C_w = 1.7 in the appropriate units (1/ $m^{1/2}$ s).

4.1 Stream scenario

The stream scenario describes a rectangular channel with a slope $S = 0.002$, a channel bottom width $b = 1$ m, an inflowing discharge $Q = 0.15$ m^3/s, a Manning coefficient $k_M = 11$ (in appropriate units, see Appendix 1) and an external flow of $q_{ext} = 0.0001$ m^2/s (8.64 m^3/m/day). The equations for the discharge and cross sectional area were integrated numerically with the FC-predictor and CN-corrector method described above, with a time step $\Delta t = 0.25$ s, for a spatial grid distance $\Delta x = 5$ m. A typical water level in this scenario is 0.75 m (Figure 9), and a typical flow velocity 0.2 m/s (Figure 8), so the time scale as given by equation 37 is 1.7 s. The system relaxes to its new equilibrium state in about 4500 s (see Figure 6). In real time the calculation completed in under a second.

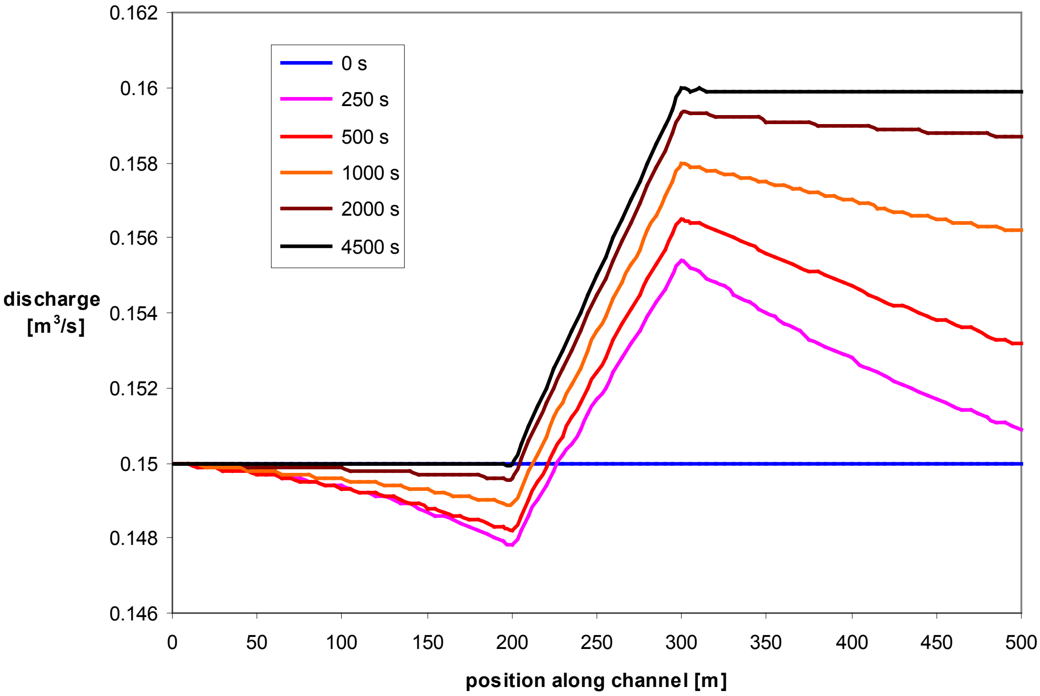


Figure 6
The discharge in the stream scenario as a function of position for various times. In the initial stationary state, without drainage, the discharge is constant. Once the external flow is switched on, the discharge profile gradually develops to the stationary state, constant to the inflow level upstream of the drained field, linearly increasing along the field, and constant again in the downstream reach. Note that the discharge initially decreases upstream.

The celerity in the stream scenario is about 2.7 m/s, the velocity is about 0.2 m/s, so the Froude number is about 0.07, still relatively slow flow. The results for the discharge show that the upstream part of the channel is less affected than the downstream part during the equilibration process, though there is a temporary effect. The result for the cross sectional shows that for this variable the upstream and downstream part are influenced much more equally. Also in the final stationary state the drained part does not stand out as clearly as it does for the discharge. The curves for the average velocity show mainly the same salient behaviour as that for the discharge. The actual change in water level, as indicated by Figure 9, is small, as could have been expected for the relative sizes of the additional discharge caused by the external flow and the initial discharge in the stream.

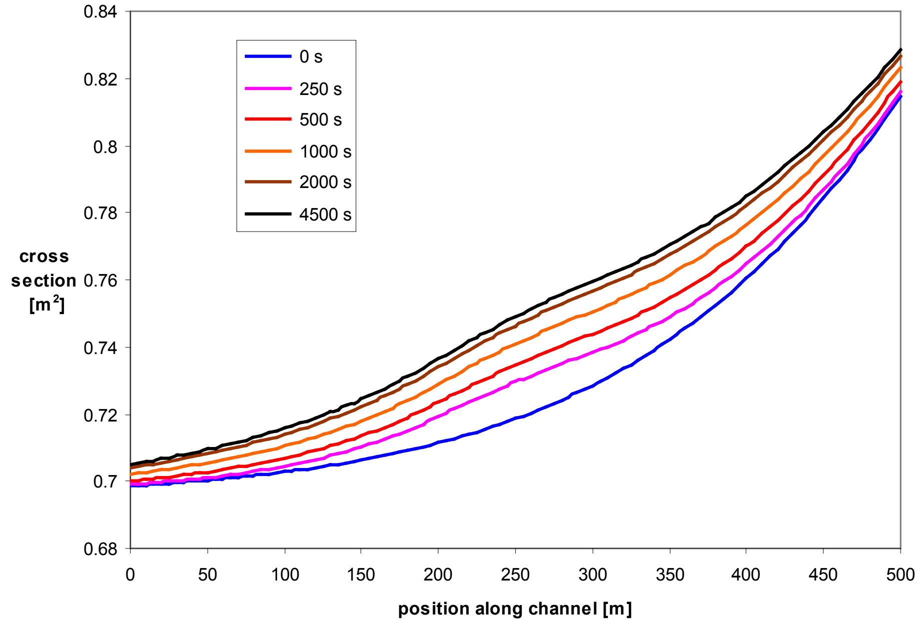


Figure 7
The cross sectional area in the stream scenario as a function of position for various times. The initial stationary state, without drainage, gradually develops to the stationary state with drainage over a period of about 4500 s. The changes extend over the full length of the channel, the upstream changes are not significantly smaller than the downstream ones.

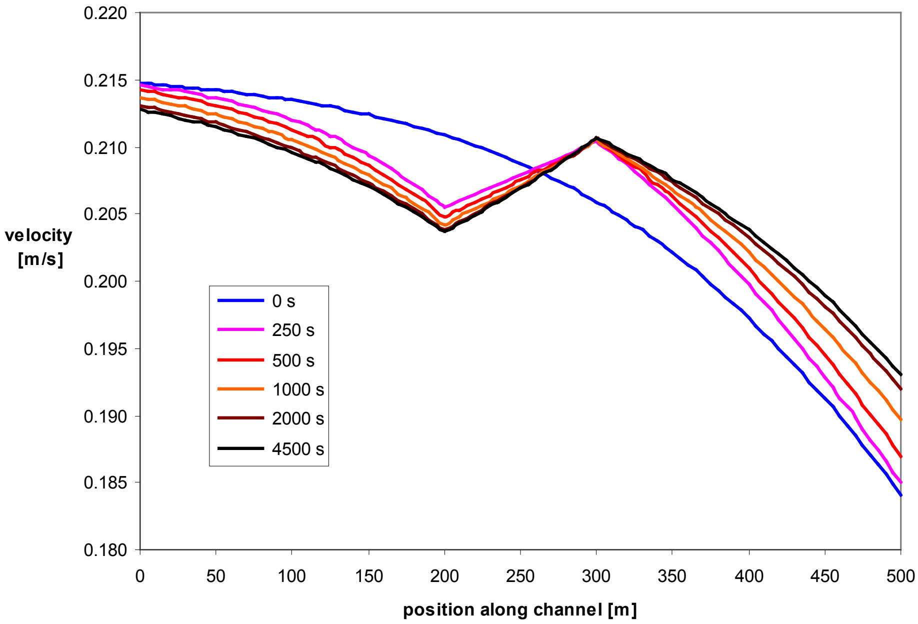


Figure 8
The average flow velocity in the stream scenario as a function of position for various times. When the external flow is switched on the velocity along the drained field rapidly assumes the linearly increasing profile of the new stationary state, it takes longer before also the upstream and downstream profile adapt to the new situation.

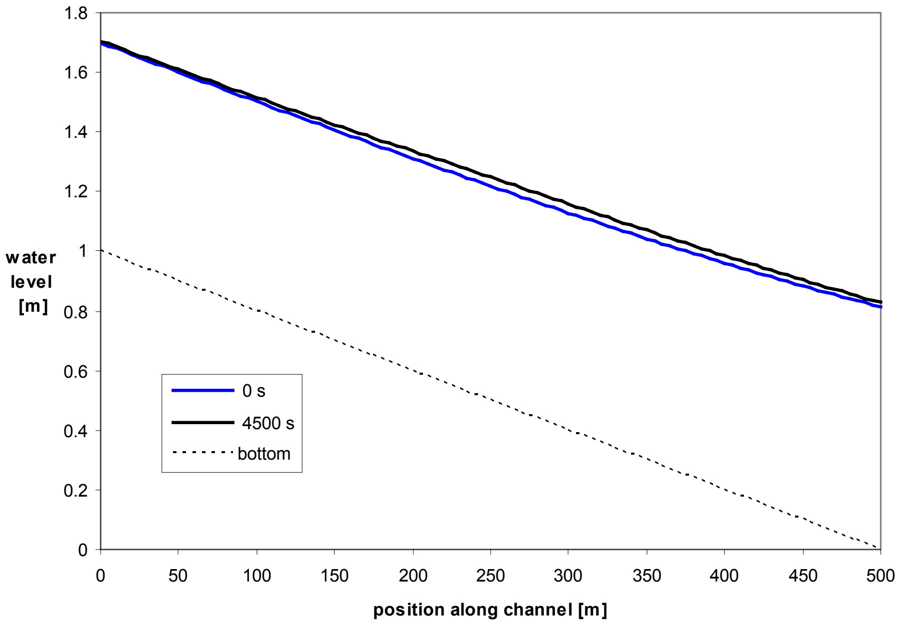


Figure 9
The water level with respect to a fixed horizontal datum, the weir bottom, as a function of position along the drained field for the initial stationary state, and as it has developed to the new stationary state, with external inflow due to the drainage. Also the channel bottom is given. In both cases the flow looks almost uniform over this short interval, the two previous figures show the actual variation with position.

4.2 Ditch scenario

The ditch scenario describes a rectangular channel with a slope $S = 0.0001$, a channel bottom width $b = 1$ m, an inflowing discharge $Q = 0.006$ m^3/s, a Manning coefficient $k_M = 25$ (in appropriate units) and an external flow of $q_{ext} = 10^{-6}$ m^2/s (0.0864 m^3/m/day). The equations for the discharge and cross sectional area were integrated with a time step $\Delta t = 0.1$ s, for a spatial grid distance $\Delta x = 5$ m. A typical water level in this scenario is 0.5 m, and a typical flow velocity 0.001 m/s, so the time scale as given by equation 37 is 2.2 s. The system relaxes to its new equilibrium state in about 20.000 s, the actual calculation is performed in a few seconds.

With a celerity of 2.2 m/s, a flow velocity of 0.0013 m/s, and an ensuing Froude number of 0.0006, it is clear that we may expect the flow of the water to have little effect on the waves caused by switching on the external flow. Figure 10 shows two almost identical waves travelling in the upstream and downstream direction. The upstream wave is reflected by the upstream boundary, then travels downstream along the channel.

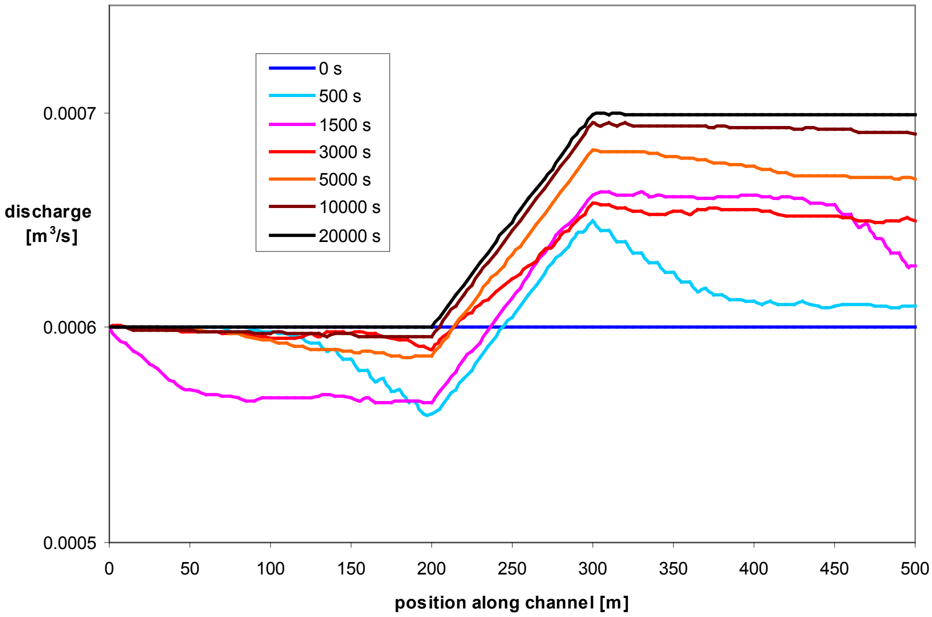


Figure 10
The discharge in the ditch scenario as a function of position for various times. In the initial stationary state, without drainage, the discharge is constant. Once the external flow is switched on, the discharge rapidly develops a linearly increasing profile along the field, while the upstream and downstream part are influence almost the same. Perturbation waves are emitted in both directions, leading to decrease and increase in discharge. The upstream wave is reflected at the inflow point, while the downstream wave leads to an increase at the weir. Only after the reflected upstream wave has travelled the full length of the channel the new stationary state develops.

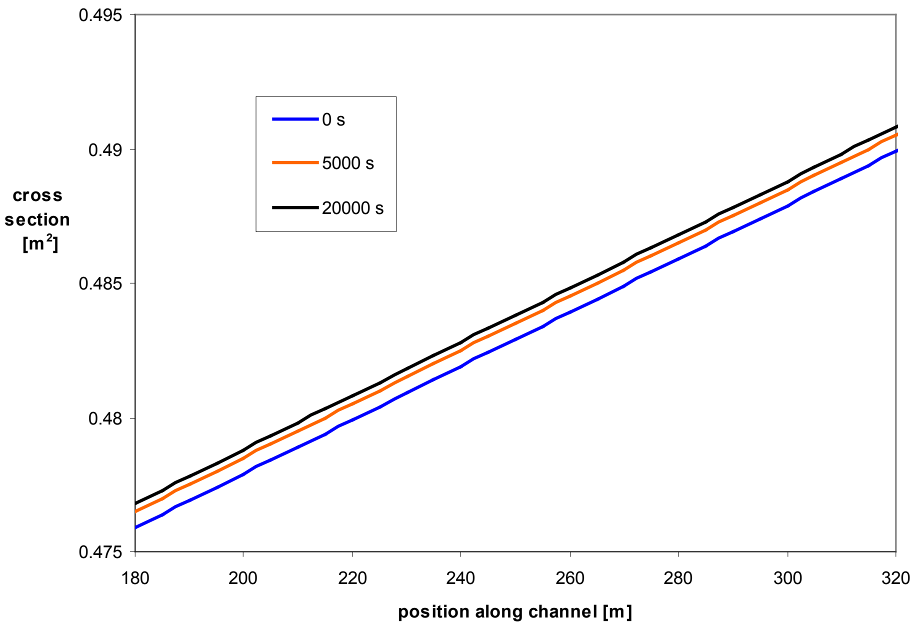


Figure 11
The cross sectional area in the ditch scenario as a function of position for three different times. The initial stationary state, without drainage, gradually develops to the stationary state with drainage over a period of about 20000 s. Only the part along the drained field is shown, otherwise the three curves would fully overlap; changes are very small indeed.

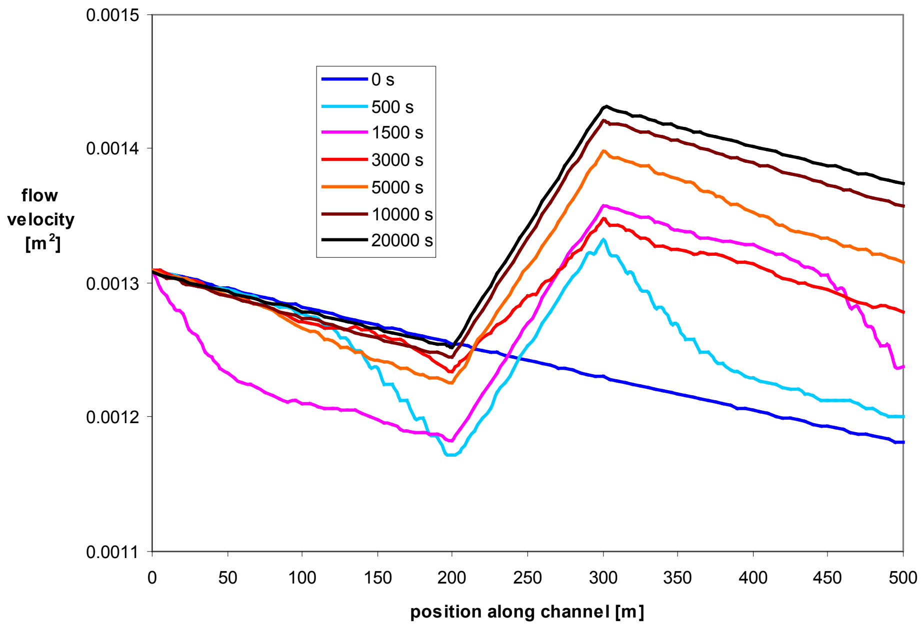


Figure 12
The average flow velocity in the ditch scenario as a function of position for various times. As the cross sectional area is almost the same linear function of position for all times, and the average velocity equals the discharge divided by the cross sectional area, the velocity profiles essentially show the same behaviour as the discharge.

4.3 Simple ditch scenario

The simple ditch scenario decribes a channel without an upstream reach, the ditch begins at position zero right next to the drained field, with zero discharge. The length of the ditch is L = 300 m; the slope S = 0.0001, bottom width: b = 1 m, k_M = 25 m$^{1/3}$/s, and β = 1.2 are identical to the ditch scenario discussed above. The cross section of the ditch is rectangular. The external flow is q_{ext} = 10^{-6} m^2/s over the full length of the ditch. At the end there is a weir with the same specifications as in the previous scenarios (h_w = 0.5 m, w_w = 0.5, C_w = 1.7 m$^{-1/2}$/s m). The calculation start at t = 0 when the external flow is changed from zero to its scenario value.

The results indicate the behaviour of this system is rather simple. The cross section increases linearly at all times. The data show small low amplitude waves travelling rapidly across the surface, which dissipate once the system reaches the new stationary state. The increase of the cross section, and hence the water level, is almost exactly compensated by the decrease in bottom level, the water level as measured with respect to a horizontal datum is almost constant along the full length of the ditch. The observed increase in water level at the weir end after one hour may be a spurious result, related to the low discharge level (which may even become negative due to surface waves). Such negative discharge levels are also observed at the dead end of the ditch, at x = 0. The overall picture for the discharge is a linear increase over the full length of the ditch, with a small levelling off near the weir.

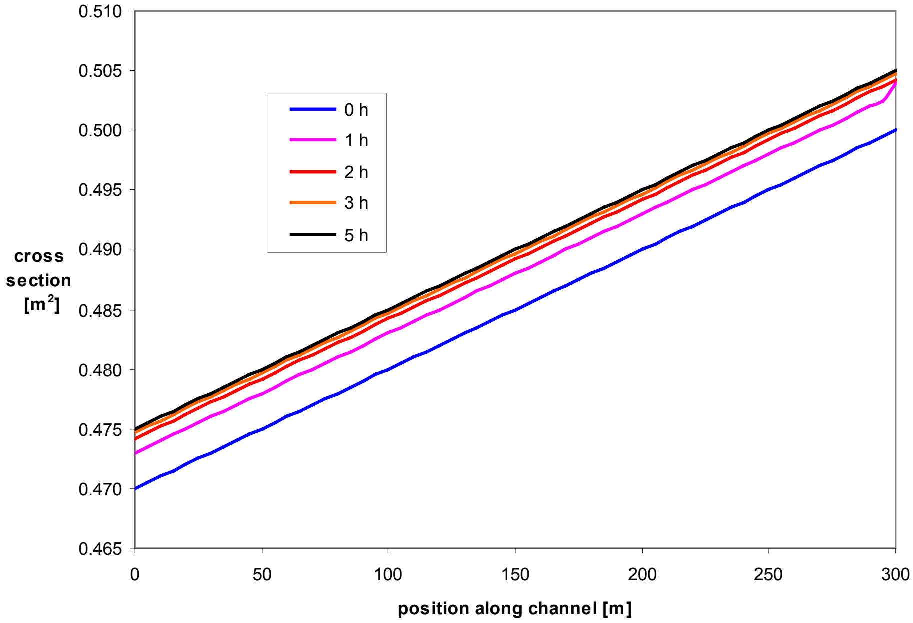


Figure 13
Cross sectional area in the ditch as a function of position along the channel for various times, when the external flow is increased from zero to 10^{-6} m^2/s at time zero. It takes about 5 hours for the model system to fully relax to its stationary state. After one hour about one half of the effect is reached. The water level in the ditch as measured with respect to a horizontal datum is almost fully level during the full process, the 3 cm decrease in bottom level over the length of the ditch exactly matches the increase in cross section.

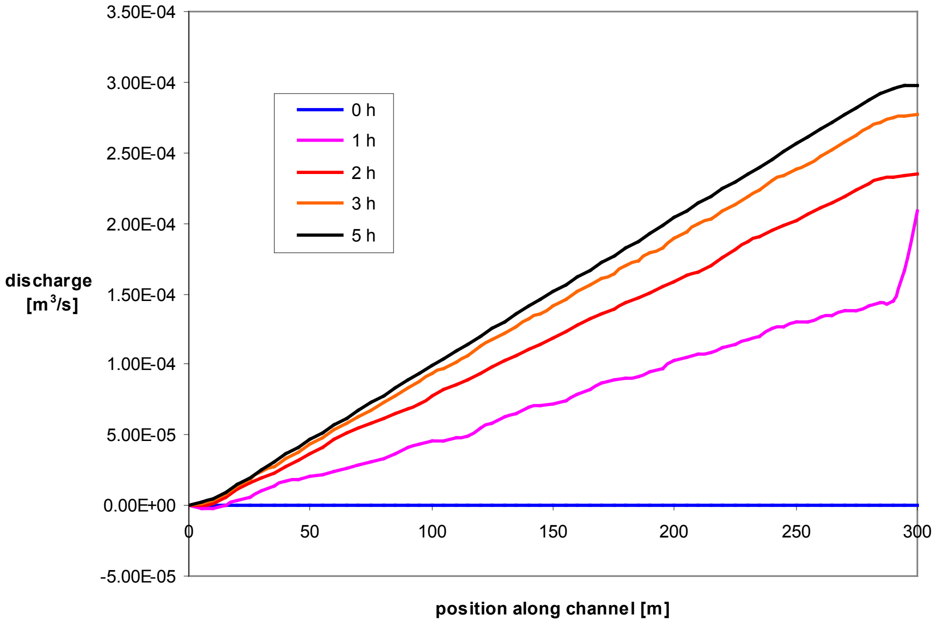


Figure 14
Discharge in the ditch as a function of position along the channel for various times. At all times the discharge is a linearly increasing function of position. Equilibrium is obtained when the discharge at the weir matches the total inflow of 3.10^{-4} m^3/s.

The results also indicate a much simpler model can be used for this system. We ignore the small surface waves, and consider only the overall water level in the ditch as measured with respect to a fixed datum, being the bottom of the weir. This water level is assumed to be constant along the full length of the ditch, but is still a function of time. We have a non-stationary model in which the full channel is modelled as a single compartment. When the water level is lower than the height of the weir, it increases because of the external inflow at a rate

$$\frac{d}{dt}h(t) = h'(t) = \frac{q_{\text{ext}}(t)}{b + 2s_1 h(t)} . \tag{41}$$

Note that for a rectangular channel (s_1 = 0) the denominator is just b. The effect of a base flow Q_B at the beginning of the channel (or anywhere for that matter) is given by

$$h'(t) = \frac{Q_B(t)}{L(b + 2s_1 h(t))} , \tag{42}$$

with L the length of the ditch. When the water level at the weir is higher than the height of the weir, the weir relation can be used to find the total discharge in the weir crest

$$Q(t) = C_w w_w h_{\text{crest}}^{3/2}(t) = C_w w_w (h(t) - h_w)^{3/2} . \tag{43}$$

This leads to a decrease in water level. Assuming again the water level is uniform, this leads to a balance equation

$$h'(t) = \frac{q_{\text{ext}}(t)}{b + 2s_1 h(t)} + \frac{Q_B(t)}{L(b + 2s_1 h(t))} - C_w w_w \frac{(h(t) - h_w)^{3/2}}{L(b + 2s_1 h(t))} . \tag{44}$$

This differential equation is most easily solved numerically with a standard forward Euler approximation. We used an Excel sheet to perform that calculation. Even for the simple case of a rectangular channel a formal analytical solution only returns a function $t(h)$, which must be inverted numerically to find the water level at a given time (see Appendix 3).

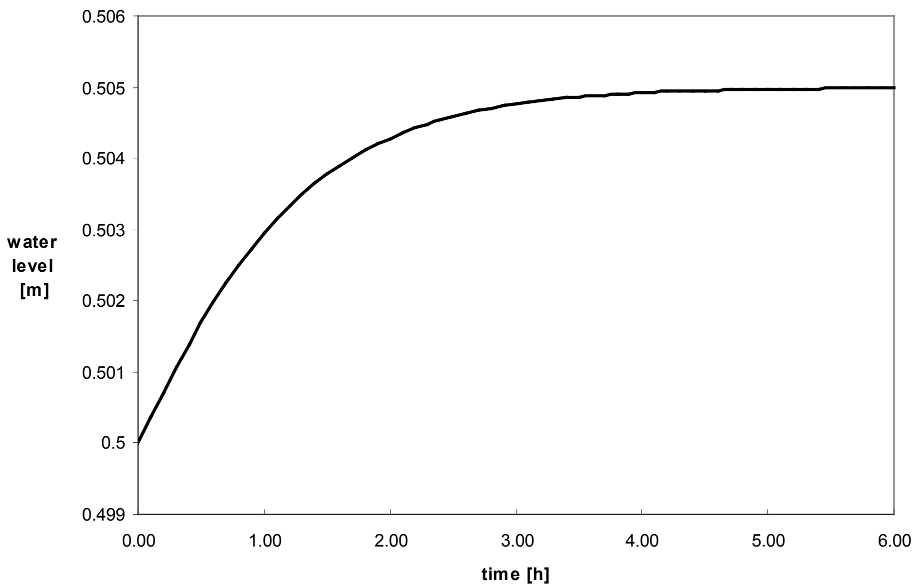


Figure 15
Water level with respect to the bottom of the weir at the end of the channel as a function of time. The external inflow is constant, the base flow is zero. The results agree with the much more detailed calculation above, after about five hours the water level reaches its stationary value, 5 mm above the weir. After one hour the increase in level is about 3 mm.

The results (Figure 15) indicate that the water level increases by 5 mm to the stationary state with a relaxation time of about 1.5 hours. Since this is a non-linear equation, this relaxation rate depends on the water level. The stationary level (at a constant external inflow and base flow) can be found directly from equation 44. The discharge at the weir can be found from the water level using the weir relation equation 43. It has the same characteristics as the water level itself. Note that the discharge along the channel is a linear function of position at all times, increasing from zero at the dead end of the ditch to the value as reported in Figure 16.

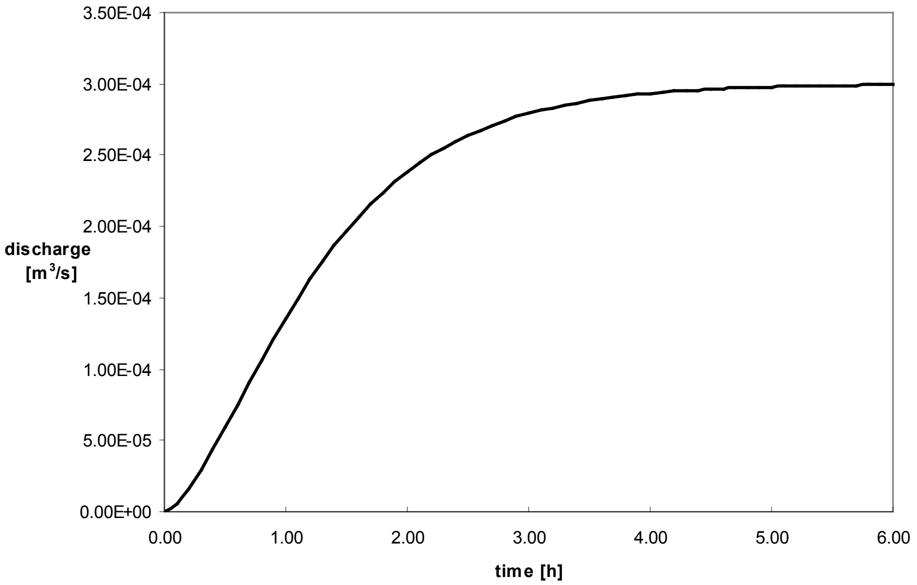


Figure 16
The discharge, as given by the weir relation, as a function of time.

4.4 Wet winter scenario

The simplified model is also applied to a ditch of L = 500 m and bottom width b = 2.16 m; the slope S = 0.0001, k_M = 25 $m^{1/3}$/s, and β = 1.2 are again identical to the ditch scenarios discussed above. The cross section of the ditch is prismatic, with bank slope parameter s_1 = 1 (a 45° angle). The external flow is constant over the full length of the ditch, and is changing daily according to the data in Figure 17. At the end there is a weir with specifications h_w = 0.23 m, w_w = 0.5 m, C_w = 1.7 $m^{-1/2}$/s m. The base flow is zero. The calculation starts at t = 0 when the external flow is changed from zero to its first scenario value (which happens to be zero as well).

The water level in the ditch reaches its equilibrium value in a few hours to slightly more than a day, depending on inflow levels. In the present calculation a time step of one hour was used, without less detailed data about the inflow level smaller time steps do not give more reliable results. For larger steps the calculation becomes unstable at the higher inflow levels.

As the relaxation time in almost all cases is considerably less than the data interval for the external inflow, one might use just the stationary value for the water level, as can be deduced from equation 43 by setting h'(t)=0.

$$h_s(t) = h_w + \left(\frac{Q_B(t) + q_{ext}(t)L}{C_w w_w} \right)^{2/3} . \tag{45}$$

The result is identical to the one obtained in Figure 18, within the resolution of that graph. Only when the results are inspected on a timescale of hours the approach to the stationary state is observed.

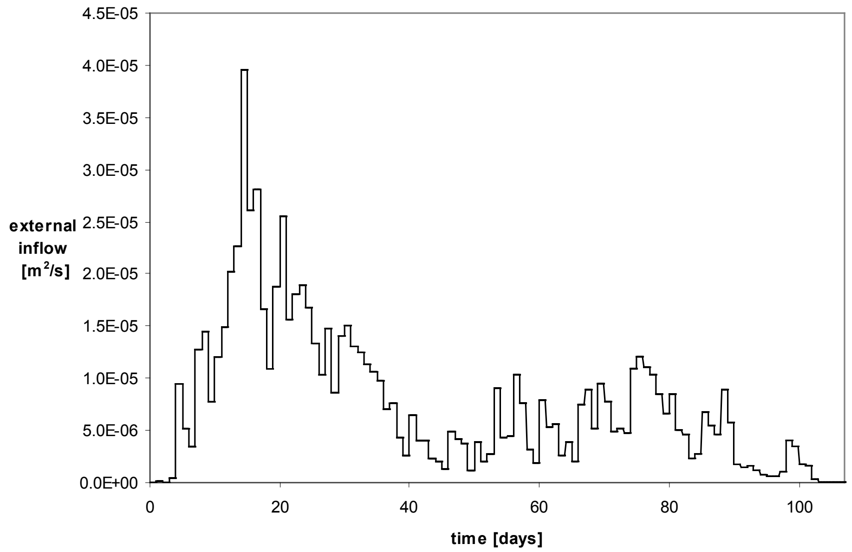


Figure 17

External inflow between October 20th 1998 and February 4th 1999 in a ditch. The drainage data refer to the calculated drainage over that period and are constant over a day. This was a relatively wet winter.

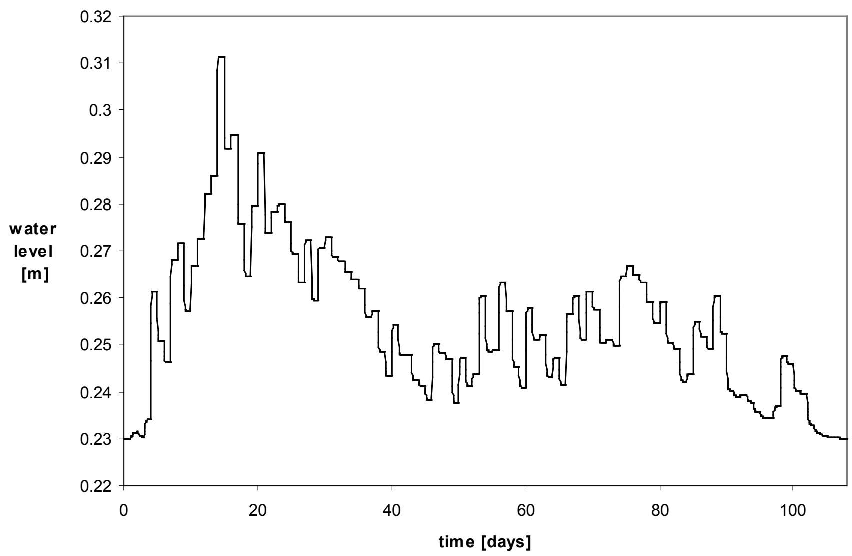


Figure 18

Water level in the ditch during the 107 day period. Values were calculated using the simplified model. Detailed calculations with the full model indicate that for this system the simplified model is adequate.

5 Conclusions and recommendations

The results from the first two scenarios show that for slow flow conditions the full watercourse is influenced by disturbances at any point or reach. Moreover the time needed for the effect to spread over the full course is proportional to the full length of the channel, which can be considerably longer than the part right adjacent to the drained fields. The concept of a local disturbance, both in time and position, is a false one.

The implications for practical application are profound. When the effect of adding runoff water from an drained field to an adjacent watercourse is studied, not only that part of the watercourse, but also the full upstream and downstream reaches should be modelled properly to determine the effect on the flow. The presence of a weir, as used at the downstream end in the scenarios considered, does provide an effective barrier for disturbances. If such a weir is actually present it will absorb any waves propagating downstream, no reflected waves in the upstream direction have been observed in the scenarios studied here. The boundary condition used at the upstream end of the channel, that of a fixed discharge, does lead to such reflections, leading to an increase in relaxation time of the full system. In fact that effect is to be expected, if downstream disturbance travel upstream, eventually these will be carried downstream again before the full system equilibrates.
Recommendation: When calculating the effect of external flow to a water course, the upstream and downstream part of the channel should be modelled to comply with the actual system, using long artificial upstream and downstream reaches will give qualitatively incorrect results.

This investigation studies only the effect of additional water to flow in a channel. Details on how precisely the water is added have not been taken into consideration. The energy balance, as used in derivation of the stationary model, provides more handles to include specific external inflow properties than the momentum balance used here. There is no a priori expectation on what would be the correct way of modelling the inflow in more detail, it depends largely on how exactly it occurs in practice, as a thin sheet of water flowing in along the banks, or in a number of small outlet pipes from a drainage system dropping the water from some elevation. In most scenarios the total external discharge was small compared to the native discharge in the channel, and in general so was the size of the effect on the channel flow. If the total external discharge exceeds the one in the upstream channel, details of how the water flows into the channel may be important.
Conclusion: When more details are known about the actual inflow of water, and the total external discharge is considerable, these details can be included most easily in the present model if an energy balance is developed.

More important may be that this investigation did not study the effect of possible pollutants in the external discharge. The mass of the water carried with that discharge in the model mixes fully with that of the water in the channel. The waves as mentioned are waves in the full channel flow, they do not necessarily pertain to the externally added mass, again that depends on how actually the runoff is added to the flow. The spreading of contaminants will to a large extent depend on the dispersion properties of the polluting materials in water, but it is not a priori clear that a simple one dimensional advection-dispersion equation, using the results in this study for the advection profiles, would provide a good description. It would be a proper starting point for modelling this phenomenon though.
Conclusion: To study the propagation of pollutants in the channel, an advection-dispersion equation based on the advection model in this study is a good starting point.

6 Literature

Opheusden, J.H.J van, J. Molenaar, W.H.J. Beltman and P.I. Adriaanse, 2010. *Stationary flow solution for water levels in open channels*. Alterra report 2084, Alterra, Wageningen, The Netherlands.

Jain, S.C., 2001. *Open Channel Flow*, Wiley, NY, 2001.

Appendix 1 List of variables

A	wetted cross sectional area	L^2
b	bottom width of channel	L
C_w	weir constant	$1 / L^{1/2} / T$
F_f	friction loss from channel bed	$M\ L^2 / T^2 / L$
Fr	Froude number	0
g	acceleration of gravity	L / T^2
h	water level	L
h_w	weir height	L
k_M	Manning coefficient	$L^{1/3} / T$
P	wetted perimeter of channel	L
Q	discharge	L^3 / T
q_{ext}	external flow density	$L^3 / T / L$
R	hydraulic radius of channel	L
S	channel bottom slope	0
s_1	slope of the channel walls	0
S_f	friction slope	0
t	time	T
u	local flow velocity	L / T
v	average flow velocity	L / T
V	volume	L^3
w_w	weir width	L
x	position along channel	L
z	height w.r.t. sea level	L
α	energy coefficient	0
β	momentum coefficient	0
ε_{ext}	external energy flow density	$M\ L^2 / T^2 / L$
ρ	water density	M / L^3

L = length
M = mass
T = time
0 = dimensionless

Appendix 2 Water level as a function of cross section

The wetted cross sectional area *A* of a prismatic channel according to equation 1 is given as a function of the water level *h*

$$A(h) = h\,(b + s_1 h)\,, \tag{46}$$

with *b* the width at the bottom, and s_1 the bank slope parameter. This is a quadratic formula in *h* that can be inverted to give the water level as a function of the cross section

$$h(A) = \frac{\sqrt{b^2 + 4As_1} - b}{2s_1}. \tag{47}$$

The second root of the quadratic equation corresponds with a negative level, and can be discarded. Note that equation 47 looks as if there is a problem for $s_1 = 0$, a rectangular channel. In fact equation 47 is regular in the limit $s_1 \to 0$ and produces the expected result for a rectangular channel

$$h(A) = \frac{A}{b}. \tag{48}$$

Hence when calculating the water level in a prismatic channel from the wetted cross section in a computer program a distinction has to be made for rectangular channels, otherwise an overflow error is generated. In practical applications often a very small value for the bank slope parameter is used in cases where the channel actually is rectangular.

Appendix 3 Analytical solution of the simple ditch scenario

For a rectangular channel in the simple ditch scenario equation 44 for the change in level reduces to

$$h'(t) = \frac{q_{\text{ext}}(t)}{b} + \frac{Q_{\text{B}}(t)}{Lb} - C_{\text{w}} w_{\text{w}} \frac{(h(t) - h_{\text{w}})^{3/2}}{Lb} . \tag{49}$$

If moreover the base flow and external flow are assumed constant the differential equation takes the form

$$h'(t) = \frac{1}{Lb}\left[q_{\text{ext}} L + Q_{\text{B}} - C_{\text{w}} w_{\text{w}} (h(t) - h_{\text{w}})^{3/2} \right]. \tag{50}$$

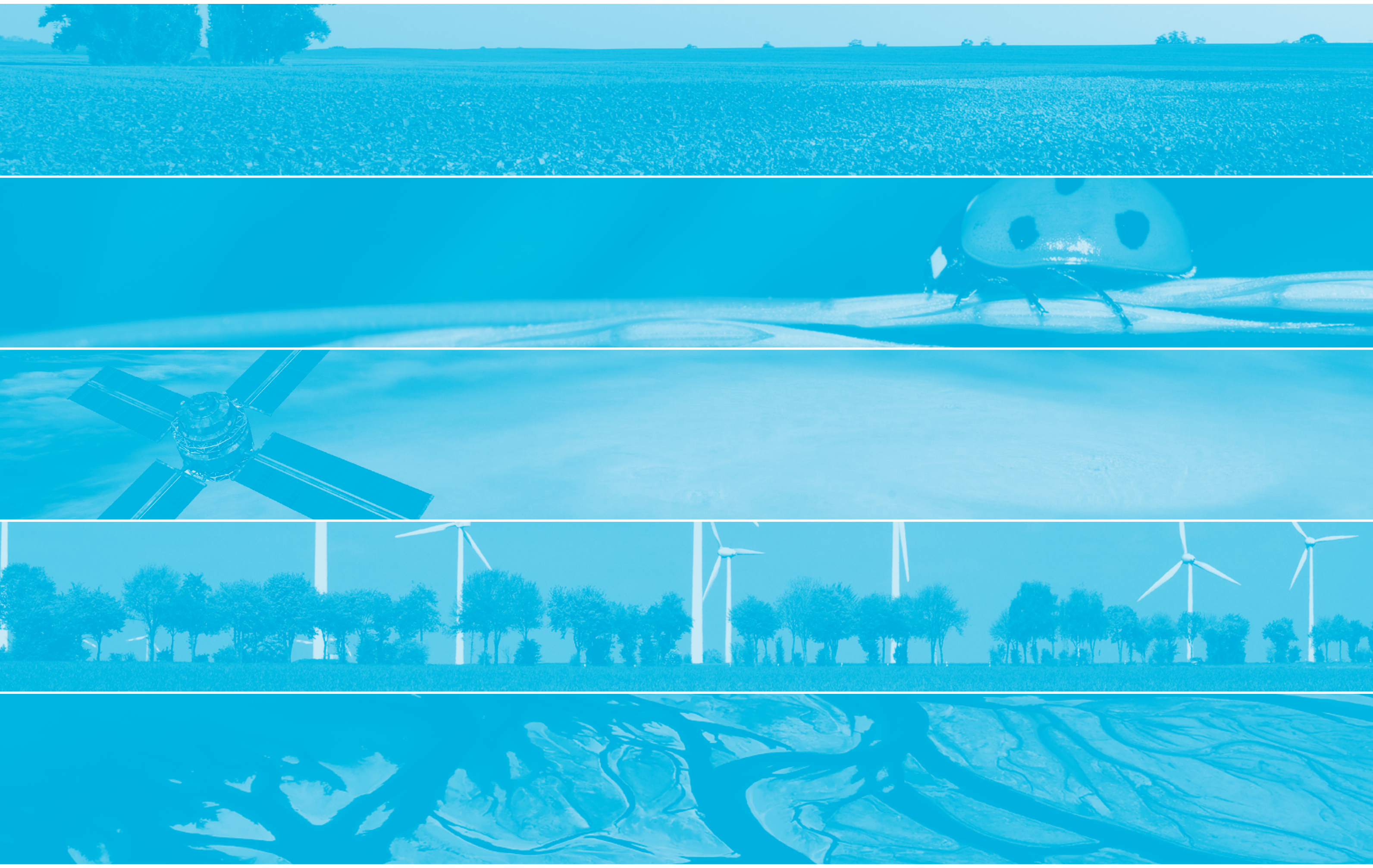
A further simplification is working with the height above the weir $h_{\text{c}}(t) = h(t) - h_{\text{w}}$

$$h_{\text{c}}'(t) = \frac{1}{Lb}\left[q_{\text{ext}} L + Q_{\text{B}} - C_{\text{w}} w_{\text{w}} h_{\text{c}}(t)^{3/2} \right]. \tag{51}$$

This equation can now formally be solved with the method of separation of variables

$$t(h_{\text{c}}) = Lb \int_{h_0}^{h_{\text{c}}} \frac{1}{q_{\text{ext}} L + Q_{\text{B}} - C_{\text{w}} w_{\text{w}} x^{3/2}} dx . \tag{52}$$

This integral can if needed, after proper rescaling, be performed analytically, for instance with a programme like Maple or Mathematica. The resulting equation is very involved, and it is still a formal solution only, as it expresses the time as function of the height, instead of vice versa.



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