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Abstract

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We study stationary flow in open discharge channels. A model is derived from basic principles, which is solved numerically for the water level and discharge as a function of position along the channel. The model describes the effect of external inflow from fields adjacent to the channel. Several scenarios are calculated, both for very slowly, and more rapidly flowing water courses.

Keywords: TOXSWA, stationary flow, open channel flow
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Inhoud

Summary

This report describes model results for stationary flow in straight discharge channels with a fixed prismatic cross section and a fixed bottom slope. A model is derived for gradually varying flow from first principles, that is conservation of mass (the water conservation equation) and energy. Using balance equations for these quantities over a short sample stretch of the channel, two ordinary differential equations are derived, one for the discharge, the second for the water level, both as a function of position along the channel.

Important is that the effect of an external flow is considered, for instance describing the runoff from fields adjacent to the channel. Given the external flow as a function of position, the equation for the discharge is readily solved. For the simplest case of zero external flow, the discharge is constant, for constant external flow the discharge increases linearly along the channel, and for more complicated functions the discharge can be found straightforwardly by numerical evaluation of the integral over that function. The only restriction is that the principle of gradually varying flow must be maintained, the external flow density must not change too rapidly along the channel.

The equation for the water level is derived from the principle of energy conservation. It is assumed the external flow leads to a net energy loss, due to mixing; other assumptions can rather easily be implemented. The friction with the channel walls is modelled using the Chézy-Manning relation for the friction slope. Viscous effects are neglected. Given the solution for the discharge, the equation for the water level is integrated numerically using a forward Euler algorithm.

The differential equations need the specification of two independent boundary conditions. Depending on the type of flow for the scenario as considered, these can be chosen at the end of the channel, or at any other position. One specific boundary condition is provided by a weir at the end of the channel, the weir relation then specifies the water level immediately in front of the weir, given the discharge over the weir. The water level upstream from the weir is calculated for different scenarios.

For very slowly moving streams without external flow, it appears the water level is almost horizontal with respect to a fixed datum, such as sea level. The water level in the stream in such cases can be found in good approximation by analytical solution of the differential equation, it increases linearly to match the decrease of the bottom of the channel along the slope. Calculating upstream, the numerical procedure gives a transition to uniform flow. That same transition occurs for more rapidly flowing streams, but in this case the water level drops along the channel, and unless a weir is present at the downstream boundary, the calculation breaks down eventually when the rate at which the level drops violates the assumption of gradually changing flow.

1 Introduction

The TOXSWA model (TOXic substances in Surface WAters) simulates the fate of pesticides in small surface waters (Adriaanse, 1996, 1997; Beltman et al., 2006). The FOCUS_TOXSWA model versions have been used for several years in the EU registration procedure, in which TOXSWA simulates watercourses with transient flow conditions. Adriaanse and Beltman (2009) described the theoretical backgrounds of the hydrology for transient flow conditions assuming a constant water depth along the length of the watercourse. They conclude with the recommendation to drop the assumption of a constant water depth in the watercourse and to consider the exact integration of the water conservation equation and discharge relation. In this report this recommendation is implemented for stationary situations.

We study the water level in a discharge channel. This level is determined by a large number of variables, such as the discharge Q, its slope, the shape of the channel, possible curves, roughness of the bed, possible vegetation, and external discharges from neighbouring fields draining into the channel. Moreover all these parameters can vary in time and position along the channel. To simplify matters we will study straight discharge channels, and assume several variables to be constant along the channel, specifically the slope, the shape, and the friction parameter. A major simplification is that we only study stationary situations, in which the water level does not change with time, and only depends on position along the channel. We will study stationary flow in straight discharge channels with a fixed cross section.

2 Model description

The water level is only one of the variables that describe open channel flow in a straight channel with a constant bottom slope. We will use a variable *x* to identify the position along the channel, the water level as a function of position then is denoted as $h(x)$. This water level is the main variable that we will study; our methods and calculations are aimed at determining this function for a number of scenarios. The other parameters that describe the channel flow are the discharge $Q(x)$, the cross sectional area $A(x)$ of the flow, perpendicular to the channel, and the flow velocity $v(x)$, averaged over the cross section. We will derive and specify four relations between these four basic system variables that will allow us to calculate the water level for a given scenario. Additionally we will need the values of two of the variables at any given point along the channel to fully specify the solution. An overview of all variables used in this report including their description is given in Appendix.

With those four relations and the two additional so called boundary conditions, we can calculate the stationary water level profile as a function of position. In this chapter we derive the relations, mention the additional assumptions, and in chapter 3 we will use them to investigate several scenarios.

2.1 Prismatic channel

We consider so called prismatic channels, whose cross section has a trapezoidal shape (Figure 1). The width at the bottom is *b*, and the slope of the banks is determined by a parameter s_1. For a rectangular channel we have $s_1 = 0$, for banks with a slope of 45 degrees $s_1 = 1$. The cross sectional area *A* of the channel is given by the formula

$$A(x) = h(x)\,(b + s_1 h(x))\,, \tag{1}$$

where $h(x)$ is the water level in the channel. The bottom width b and the inclination s1 of the banks are constant along the channel. The water level, and hence the cross sectional area of the flow, can vary with position x, as indicated in the equation. Formula (1) expresses the cross section *A(x)* in the water level *h(x).*

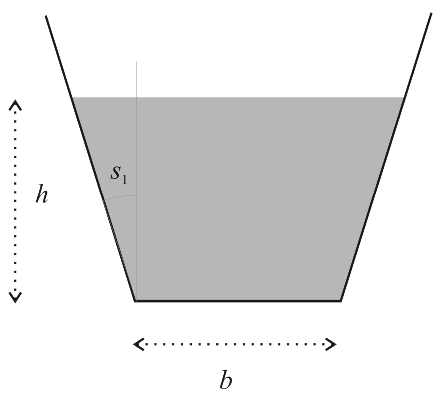

Figure 1
Cross sectional area of the channel.

2.2 Average velocity

The second relation is

$$Q(x) = A(x)\, v(x)\,, \tag{2}$$

expressing the discharge Q as the product of the cross sectional area and the average velocity

$$v(x) = \frac{\int u(x,y,z)\, dA}{A(x)}\,, \tag{3}$$

with $u(x,y,z)$ the local velocity at any point of the cross section. In general we do not know u, so we only use the average velocity v. Note that we assume the average velocity to be perpendicular to the cross section. As long as we have a channel with a small bottom slope and a water level that changes only gradually, this approximation will be quite reasonable, but if the model leads to large changes in water level over short distances, we must be especially careful. We will use equation (2) to express the average velocity $v(x)$ in the water level $h(x)$, given the discharge profile $Q(x)$.

2.3 Water conservation equation

A third relation between the system variables is given by the continuity equation, in this context also termed the *water conservation equation*. The general physics law of conservation of mass, because of the incompressibility of the water, translates into a conservation of volume. Note that this may not be correct if for instance the water contains a large amount of air bubbles or sediment. If we consider a section of the channel between points x and $x+\Delta x$ (Figure 2) the total change in volume in that section must be the result of a net inflow or outflow:

$$change\ =\ inflow\ -\ outflow\,.$$

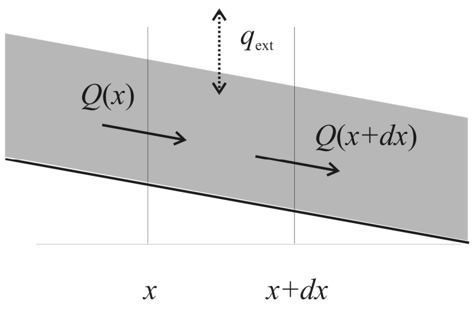

Figure 2
Volume flows for a sample section of the channel.

The discharge can vary with position, so the inflow on the left not necessarily matches the outflow on the right, and there also may be an external flow. If we consider the total change in volume between the end points of the channel section over a time interval Δt, we find

$$\Delta V = Q(x)\cdot\Delta t - Q(x+\Delta x)\cdot\Delta t + q_{\text{ext}}(x)\cdot\Delta x\cdot\Delta t\ , \tag{4}$$

where the external discharge density q_{ext} is taken per unit length along the channel, and a positive value stands for water being drained into the channel. Note that Jain uses the opposite convention. Moreover, since the distance between the two points is fixed, a change in volume must give a change in cross sectional area:

$$\Delta V = \Delta A\cdot\Delta x\,. \tag{5}$$

Combination of these two equations gives

$$\Delta A\cdot\Delta x = Q(x)\cdot\Delta t - Q(x+\Delta x)\cdot\Delta t + q_{\text{ext}}(x)\cdot\Delta x\cdot\Delta t\ , \tag{6}$$

which when divided by Δx and Δt yields

$$\frac{\Delta A}{\Delta t} = \frac{Q(x)-Q(x+\Delta x)}{\Delta x} + q_{\text{ext}}(x)\,. \tag{7}$$

In the limit of $\Delta t \rightarrow 0$ and $\Delta x \rightarrow 0$ the difference quotients become derivatives

$$\frac{dA}{dt} = -\frac{dQ}{dx} + q_{\text{ext}}(x)\,. \tag{8}$$

This is the familiar continuity equation, which relates the rate of change of the cross sectional area at a given point to the gradient of the discharge and the external discharge density at that same point. Since we are considering the stationary situation, the cross sectional area at a given position is constant with time; its rate of change with time is zero. That implies that

$$-\frac{dQ}{dx} + q_{\text{ext}} = 0 \quad\Rightarrow\quad \frac{dQ(x)}{dx} = q_{\text{ext}}(x)\,. \tag{9}$$

This is a simple differential equation that can be solved by direct integration. Integrating relation (7) results in

$$Q(x) = Q(x_0) + \int_{x_0}^{x} q_{\text{ext}}(\xi)\,d\xi\,, \tag{10}$$

starting from an arbitrary reference point x_0; a quite natural choice is to take $x_0 = 0$.

Once the external discharge density is specified, and a reference point with a specified discharge is chosen, the discharge at any other point can be calculated by numerical evaluation of the integral. In some simple cases the integral can be evaluated analytically, for instance for a constant external discharge density we find

$$Q(x) = Q(x_0) + (x - x_0)\,q_{\text{ext}}\,. \tag{11}$$

For the simplest case of zero external discharge ($q_{\text{ext}} = 0$), the discharge in the stationary situation is constant along the channel. The important conclusion is that in the stationary case the discharge $Q(x)$ is specified completely by the boundary condition and the external discharge density, no knowledge about the other variables A, h, or v is needed.

2.4 Energy conservation equation

As stated, we have four variables $Q(x)$, $A(x)$, $h(x)$, and $v(x)$, that specify the flow situation along the channel. With the three relations we have already derived, we can find the discharge, and express the cross sectional area and the average velocity in the water level, eliminating them from the equations. We need a fourth relation, next to relations (1), (2) and (9) we already have discussed, that determines the change in water level along the channel.

This fourth relation follows from the principle of energy conservation. This is a considerably more complicated concept than the three above. We will discuss the issue in detail to make clear the assumptions we make. Again we consider a short sample section of the channel between points x and $x+\Delta x$ (Figure 3). Again we consider all (energy) flows into and out off that section. To simplify matters, we assume stationarity right from the start, which implies these energy flows must match, their sum must be zero.

2.4.1 Potential and kinetic energy

The flow of energy consists of two terms, the potential energy flow and the kinetic energy flow. The potential energy flow is the potential energy density, times the discharge

$$\rho g z Q \,. \tag{12}$$

The kinetic energy flow is the energy density, times the velocity, integrated over the cross section

$$\int \tfrac{1}{2}\rho u^2 \cdot u \; dA = \tfrac{1}{2}\rho \cdot \int u^3 \; dA = \tfrac{1}{2}\rho v^2 \alpha Q \,, \tag{13}$$

Where

$$\alpha = \frac{\int u^3 \; dA}{v^3 A} = \frac{\int u^3 \; dA}{v^2 Q} \,. \tag{14}$$

The energy correction factor α indicates that the average of the velocity cubed is not the same as the cube of the average velocity. Since we do not have detailed information about the local velocity u, we again need to be satisfied with only the average velocity, as defined in (2).

Note that indeed the physical dimension of both expressions (12) and (13) is J/s. The total energy flow is simply the sum of the potential and kinetic energy flow. Energy conservation for the stationary state now implies a balance for the sample section

$$inflow + gain - losses \;=\; outflow$$

We will first work out this balance equation in detail.

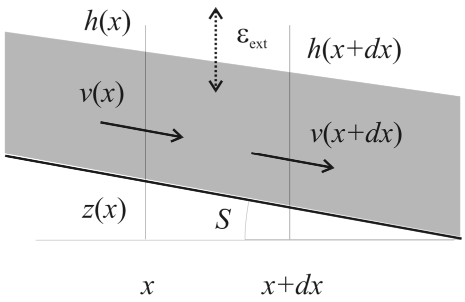

Figure 3
Energy flows for a sample section of the channel.

2.4.2 Working out the balance.

The energy balance, when worked out in detail, becomes (also see Figure 3)

$$\begin{aligned}\rho g[h(x)+z(x)]Q(x)+\tfrac{1}{2}\rho v(x)^2\alpha(x)Q(x)+\varepsilon_{\text{ext}}\cdot\Delta x-F_{\text{f}}\cdot\Delta x = \\ \rho g[h(x+\Delta x)+z(x+\Delta x)]Q(x+\Delta x)+\tfrac{1}{2}\rho v(x+\Delta x)^2\alpha(x+\Delta x)Q(x+\Delta x)\,,\end{aligned} \quad (15)$$

where ε_{ext} is the external energy flow due to water flowing sideways into (or out off) the channel, and F_{f} is the friction loss from the channel bed. We will deal with these two terms shortly. Furthermore one recognizes the first term on both sides as the potential energy flow at x and $x + \Delta x$ respectively, with a total water level measured with respect to sea level. The height of the bottom $z(x)$ with respect to a fixed reference level (*e.g.* sea level) is determined by the (constant) slope S of the channel

$$z(x+\Delta x) = z(x) - S\cdot\Delta x\,. \quad (16)$$

In all equations it is assumed that this slope is small. The second term on both sides is the kinetic energy flow at the different end surfaces of the channel section.

For the water level, the change in level over the interval is defined through

$$h(x+\Delta x) = h(x) + \Delta h \quad (17)$$

and the velocity change by

$$v(x+\Delta x) = v(x) + \Delta v\,. \quad (18)$$

For the discharge we have equation (11)

$$Q(x+\Delta x) = Q(x) + q_{\text{ext}}\,\Delta x\,.$$

Further we assume that α is a constant that only depends on the shape of the channel. In general the value of that constant for drainage channels is close to unity, so this does not establish a very strong assumption. If we substitute the above equations into (15) and divide by Δx we find in the limit $\Delta x \to 0$

$$\rho g Q \frac{dh}{dx} + \rho Q \alpha v \frac{dv}{dx} - \varepsilon_{\text{ext}} + \left(\rho g (h+z) + \tfrac{1}{2}\rho \alpha v^2\right) q_{\text{ext}} - \rho g Q S + F_{\text{f}} = 0 \,. \tag{19}$$

Again the two difference quotients become derivatives.

2.4.3 External energy flow.

The energy equation can be specified further by looking into the terms dealing with the external energy flow. The external energy flow ε_{ext} is related to the external water flow q_{ext}, as can been seen in the fourth term in equation (19). Again there is a potential energy and kinetic energy contribution. The water is added (or removed) at a total height level $h + z$ which gives a potential energy contribution

$$\varepsilon_{\text{ext,pot}} = \rho g (h+z) q_{\text{ext}} \,, \tag{20}$$

so this term drops out of the equation. The kinetic energy contribution can not be specified until we know how exactly the water flows into or out off the channel. If the water is added (or removed) at the exact average velocity as that of the main stream, for instance at a confluence point or a fork, the contribution is

$$\varepsilon_{\text{ext, kin}} = \tfrac{1}{2}\rho \alpha v^2 q_{\text{ext}} \tag{21}$$

and also this term drops out. In this case the added water behaves exactly like the incoming water in the main channel, which is what the energy terms express. In practice adding water may lead to energy losses because of the mixing with the water in the main channel. Jain suggests that the mixing gives rise to a loss term

$$\varepsilon_{\text{ext, kin}} = -\tfrac{1}{2}\rho \alpha v^2 q_{\text{ext}} \,. \tag{22}$$

If the water is added very gradually, a good approximation is to set the kinetic energy contribution to zero. We will use the assumption as made by Jain; the mixing gives a net energy loss.

2.4.4 Friction losses

The friction term also requires quite some assumptions. The Chézy-Manning relation defines the friction in a channel with a fixed water level h and a constant discharge Q, so-called uniform flow conditions (Figure 4). The discharge in this particular case is given by the empirical Manning relation

$$Q = k_{\text{M}} A R^{2/3} S^{1/2} \,, \tag{23}$$

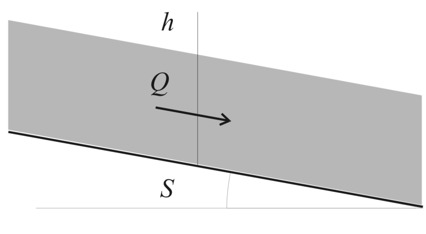

Figure 4
Uniform flow conditions.

with k_M the Manning coefficient or roughness factor of the bottom and walls of the channel. We will take this value to be constant along the length of the channel. The variable *R* is the so called hydraulic radius of the channel, which is defined by the shape of the channel

$$R = \frac{A}{P}, \tag{24}$$

with *P* the total length over which the water in the channel is in contact with the walls and the bottom, the so called *wetted perimeter*. For a trapezoidal channel the wetted perimeter is expressed in the basic variables through

$$P = b + 2h\sqrt{s_1^2 + 1}. \tag{25}$$

Uniform flow is a very specific type of flow. Since in uniform flow the water level is constant we have

$$\frac{dh}{dx} = 0. \tag{26}$$

Since also the discharge is constant, we also have

$$q_\mathrm{ext} = 0, \tag{27}$$

and since moreover the cross sectional area is constant, equation (2) implies that

$$\frac{dv}{dx} = 0, \tag{28}$$

that is, the velocity is constant. If we substitute all this into the energy relation (19) we find

$$-\rho g Q S + F_\mathrm{f} = 0 \tag{29}$$

The particular value of the slope for which uniform flow applies is called the *friction slope*, and we identify it as

$$S = S_\mathrm{f} = \frac{F_\mathrm{f}}{\rho g Q}. \tag{30}$$

The Chézy-Manning relation can be used to calculate the friction slope

$$S_\mathrm{f} = \frac{Q^2}{k_\mathrm{M}^2 A^2 R^{4/3}}. \tag{31}$$

We assume that the Manning relation also applies for non-uniform flow, and use it to calculate the frictional losses as a function of the system parameters.

2.4.5 The hydrostatic head

With the external energy flow and the frictional losses specified, the energy relation (19) becomes

$$\rho g Q \frac{dh}{dx} + \rho Q \alpha v \frac{dv}{dx} + \rho \alpha v^2 q_{\mathrm{ext}} - \rho g Q S + \rho g Q S_{\mathrm{f}} = 0 . \tag{32}$$

In hydraulics it is customary to divide all terms in the equation by the product ρg, which amounts to using a hydrostatic head instead of an energy density. If, in addition, we also divide by the discharge Q we obtain

$$\frac{dh(x)}{dx} + \frac{\alpha v(x)}{g}\frac{dv(x)}{dx} + \frac{\alpha v(x) q_{\mathrm{ext}}(x)}{gA(x)} - S + S_{\mathrm{f}}(x) = 0 , \tag{33}$$

where we again have explicitly indicated which of the variables actually are allowed to change along the length of the channel. The energy conservation principle thus results in a second differential equation, next to the very simple equation (9) we derived for the discharge from the water conservation principle. Important is that (33) is a non-linear equation, because of the product $v \cdot dv/dx$ in the second term. This non-linearity makes that in most practical situations the equations can not be solved analytically, but one must resort to numerical procedures.

2.4.6 Differential equation for the water level

We will consider the energy relation (33) as a differential equation for the water level only, and eliminate the gradient of the velocity using the other relations. When the velocity is expressed as the quotient of discharge and cross sectional area, straightforward differentiation using the quotient rule gives

$$v(x) = \frac{Q(x)}{A(x)} \quad \Rightarrow \quad \frac{dv(x)}{dx} = \frac{1}{A(x)}\frac{dQ(x)}{dx} - \frac{Q(x)}{A(x)^2}\frac{dA(x)}{dx} . \tag{34}$$

This looks a bit awkward, since the single derivative is now replaced by two derivatives, one for the discharge, and one for the cross sectional area. However, the gradient $dQ(x)/dx$ of the discharge in the stationary state is directly given by equation (9), it is simply the external discharge density q_{ext}. The gradient of the cross sectional area can be translated to that of the water level by using equation (1) and the chain rule of differentiation

$$\frac{dA(x)}{dx} = \frac{dA(h)}{dh}\frac{dh(x)}{dx} , \tag{35}$$

with $dA(h)/dh$ depending only on channel shape. It is further a matter of straightforward but tedious substitution to rewrite the energy relation as a differential equation for the water level

$$\frac{dh(x)}{dx} = \frac{S - S_{\mathrm{f}}(x) - \dfrac{2\alpha Q(x)}{gA(x)^2} q_{\mathrm{ext}}(x)}{1 - \alpha Fr(x)^2 \dfrac{h(x)}{A(x)}\dfrac{dA(h)}{dh}} , \tag{36}$$

with

$$Fr = \frac{v}{\sqrt{gh}} \tag{37}$$

the Froude number for water level h and velocity v. Note that all variables on the right hand side of (36) can be written as a function of $h(x)$, making it a differential equation for the water level only.

For a rectangular channel, with $A(x) = A(h(x)) = h(x)\, b$, the equation simplifies to

$$\frac{dh(x)}{dx} = \frac{S - S_{\mathrm{f}}(x) - \dfrac{2\alpha Q(x)}{gA(x)^2} q_{\mathrm{ext}}(x)}{1 - \alpha Fr(x)^2} \,. \tag{38}$$

Note that (38) is identical to Jain's equation 6-18, with the exception of the parameter α, that was omitted by Jain. The assumption about the energy effect of the external flow influences only the prefactor in the third term of the numerator, for an energy loss, as we assume, it has value of 2, in case this energy is zero the factor is 1.5, and when the energy is positive the factor is 1. For high external fluxes this can be an important effect.

2.5 Summary

So we have four equations for the four system variables $Q(x)$, $A(x)$, $h(x)$, and $v(x)$:

1. equation (36) that expresses the water level gradient in the system variables,
2. equation (9) that relates the discharge gradient to the external discharge density,
3. equation (2) that defines the average velocity, and
4. equation (1) that defines the cross sectional area of the channel

Two of these, equations (9) and (36), are first order differential equations. To solve the full system of equations we additionally need two boundary conditions, one for each differential equation. A boundary condition in this case implies that we need to specify the value of one of the system variables at any one point along the channel. Two such conditions can be taken at two different points, or two variables at the same point along the channel. One specific way of providing a boundary condition is if the channel ends at position L at a weir. The weir relation

$$Q = C \cdot w \cdot h_{\mathrm{crest}}^{3/2} = C \cdot w \cdot (h - h_{\mathrm{w}})^{3/2} \,, \tag{39}$$

with w and h_{w} the width and height of the weir respectively, and C the weir constant, then can be used to express the water level $h(L)$ immediately in front of the weir in terms of the local value of the discharge.

3 Results for rectangular channels

A relatively simple case is that of stationary flow in a channel with a rectangular cross section ($s_1 = 0$) without any external flow. Since there is no external flow, equation (38) reduces to

$$\frac{dh(x)}{dx} = \frac{S - S_{\mathrm{f}}(x)}{1 - \alpha Fr(x)^2}. \tag{40}$$

As stated above, both the friction slope, and the Froude number can be expressed as a function of the water level. Using the various relations we derived, we write the differential equation for the water level as

$$\frac{dh(x)}{dx} = \frac{S - \dfrac{Q^2}{k_{\mathrm{M}}^2 h(x)^2 b^2 (h(x)b/(b+2h(x)))^{4/3}}}{1 - \dfrac{\alpha Q^2}{g h(x)^3 b^2}}. \tag{41}$$

Note that the discharge *Q*, according to equation (9), is constant, as are all other system parameters occurring in this equation. Only the water level varies with position along the channel.

It turns out that even for simplest case conceivable, the differential equation for the water level can not be solved analytically. We will have to resort to numerical or model approximations to find a solution. In all numerical calculations below we have used a simple Forward Euler approximation

$$\frac{dh(x)}{dx} = f(h(x)), \qquad h(x+\Delta x) = h(x) + \Delta x \cdot f(h(x)), \tag{42}$$

where $f(h(x))$ is the complicated expression on the right hand side of equation (41). Numerical calculations have been performed using an Excel worksheet.

3.1 Slowly flowing ditch

We consider the case of a slowly flowing ditch, without external flow and we assume a rectangular cross section (see Table 1).

Table 1
Parameter values for a slowly flowing ditch

S (-)	**b** (m)	***Q*** (m^3/s)	**α** (–)	***k*$_M$** ($m^{1/3}$/s)
0.0001	1 m	0.0006 m^3/s	1.2	25

The accelaeration of gravity, g = 9.8 m/s^2 for the acceleration of gravity. We will calculate the water level along the channel with respect to a reference point. The water level at this reference point will be fixed at a value of 0.3 m. Depending on the type of flow, rapid or slow, as indicated by the Froude number, this reference point will be at the beginning or the end of the channel, respectively. In the present case the Froude number is calculated as

$$Fr = \frac{v}{\sqrt{gh}} = \frac{Q}{hb\sqrt{gh}} = 0.0012 \ , \tag{43}$$

which indeed qualifies the flow as slow.

3.1.1 Uniform flow

To explain why we have the reference point at the end of the channel for slow flow, we consider the case of uniform flow. In uniform flow, as mentioned in discussing the Manning relation, the water level, and all other system variables, are constant along the length of the channel. It is exactly from that situation that the friction slope was defined. In fact the differential equation (40) for the water level indicates again that uniform flow occurs if the system variables satisfy the relation

$$S_{\mathrm{f}} = \frac{Q^2}{k_{\mathrm{M}}^2 h^2 b^2 (hb/(b+2h))^{4/3}} = S \ . \tag{44}$$

From equation (44) we can calculate the water level for uniform flow. In the case as discussed above it is 0.027 m, considerably less than the 0.3 m at the reference point. Equation (44) also shows that the friction slope (hence the friction losses) decreases at increasing water level.

So let us have a qualitative look at the differential equation, and suppose we have a water level slightly above the depth for uniform flow. That means that the friction slope, as calculated according to equation (44), is slightly less than that for uniform flow. Equation (40) than indicates that the derivative of the water level is positive, the water level will increase along the channel, and consequently will deviate increasingly from the level for uniform flow. Since the water level increases, and the discharge remains constant, the average flow velocity will decrease, and hence the Froude number will decrease as well.

If we consider a water level slightly below that for uniform flow at the given bottom slope and discharge, the water level will decrease along the channel, and the Froude number and velocity will increase. In that sense uniform flow is unstable, any slight distortion will grow along the channel, in stationary flow. If on the hand we calculate backwards along the channel, any flow will converge to stationary flow eventually, provided of course the channel is long enough. If we have rapid flow, with $Fr > 1$, the denominator in equation (40) can become negative, and the whole discussion reverses. In fact that occurs at $Fr = 1/\sqrt{\alpha}$, which in practice will be close to unity.

In view of the above instability we always take the boundary condition to be at x = 0, the reference point, and depending on whether we have rapid or slow flow calculate forwards or backwards from that position. In all cases described below, the latter applies, positions are calculated backwards from the reference point (x < 0). The boundary condition hence is always taken at the end of the channel in the examples we investigate.

3.1.2 Analytical and numerical approximation

For our slowly flowing ditch with a water level of 0.3 m at the reference point, we calculate the friction slope at the reference point to be $S_f = 6 \cdot 10^{-8}$, completely negligible with respect to the bottom slope value *S*. Also the Froude number, as calculated above, is quite small, so the differential equation can well be approximated by $h'(x) = -S$, with an approximated analytical solution

$$h(x) = h(0) + S \cdot x = 0.3 + 0.0001\,x\,. \tag{45}$$

The water in the ditch is almost level, the increase in water level almost compensates the decrease from the bottom slope. In the approximation (45) it does so exactly, in practice there needs to be some differential head to keep the water flowing, the ditch behaves as a narrow elongated lake.

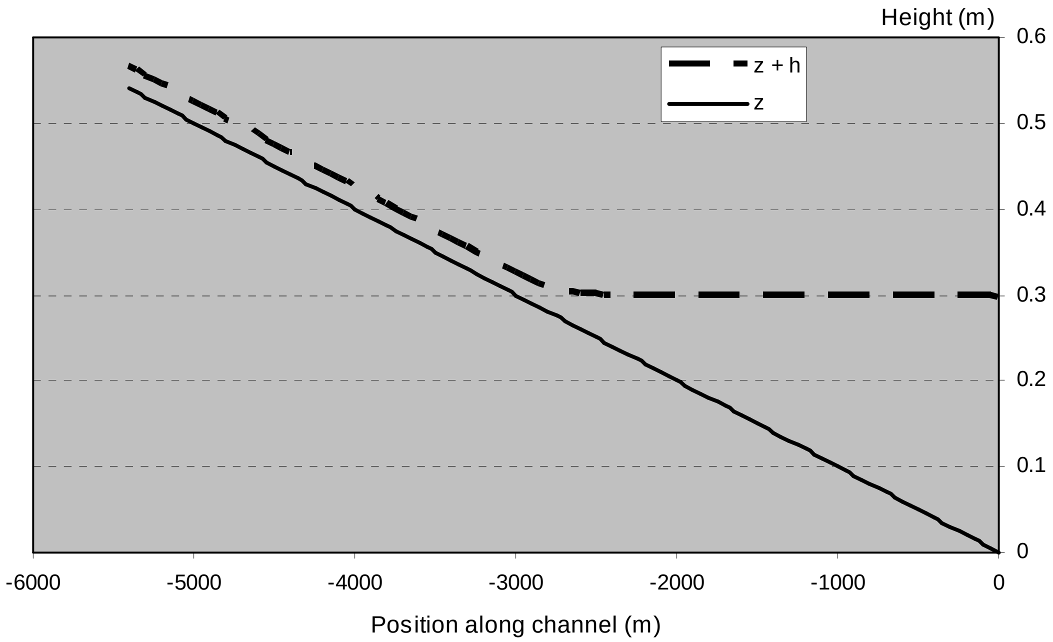

Figure 5
Water level in a slowly flowing ditch.

A numerical calculation using the Forward Euler approximation indicates that over a distance of 10 km, calculated forwards from the reference point, the water level measured with respect to a fixed horizontal datum, such as sea level, decreases by no more than 0.1 mm, implying that the depth of the ditch at that point is 1.299 m, instead of the 1.300 m we would have obtained for a fully horizontal water surface. In Figure 5 we have used the boundary condition at $x = 0$, and calculated backwards ($x < 0$). If we use a step size of $\Delta x = -50\,\mathrm{m}$, we find that at about 3km in front of the reference point uniform flow conditions are obtained. The gray line gives the level of the bottom (*z*) of the channel and the dark line the water level (z + h), both with respect to a fixed horizontal datum. At 100 meters in front of the reference point the water level (depth of the ditch) is 0.29 m, both numerically and according to equation (45). The value of 100 m is taken to be the length of the channel in this model, calculations can be extended well beyond that length, provided they are performed in the correct direction.

3.2 Rapidly flowing brook

Our next case study is a rapidly flowing brook, again with no external flow, and the following values for the system parameters

Table 2
Parameter values for rapidly flowing brook.

S *(-)*	***b*** **(m)**	***Q*** **(m^3/s)**	**α** **(–)**	***k_M*** **($m^{1/3}$/s)**
0.0002	1 m	0.15 m^3/s	1.2	11

For a calculated value of Fr = 0.3 at a water level of 0.3 m at the reference point, that does not really qualify as rapid flow, though admittedly the flow is substantially faster than for the previous case. The depth for uniform flow in the present case is 0.7 m, indicating that we may expect the water level to drop along the channel in the positive direction from the reference point. Indeed in Figure 6 the drop in water level along the channel can be observed in the numerical result calculated backwards from the reference point. At about 200 m in front of the reference point uniform flow conditions are met, and at 100 m in front of the reference point (at x = 0) the depth of the brook is a little bit above 0.6 m.

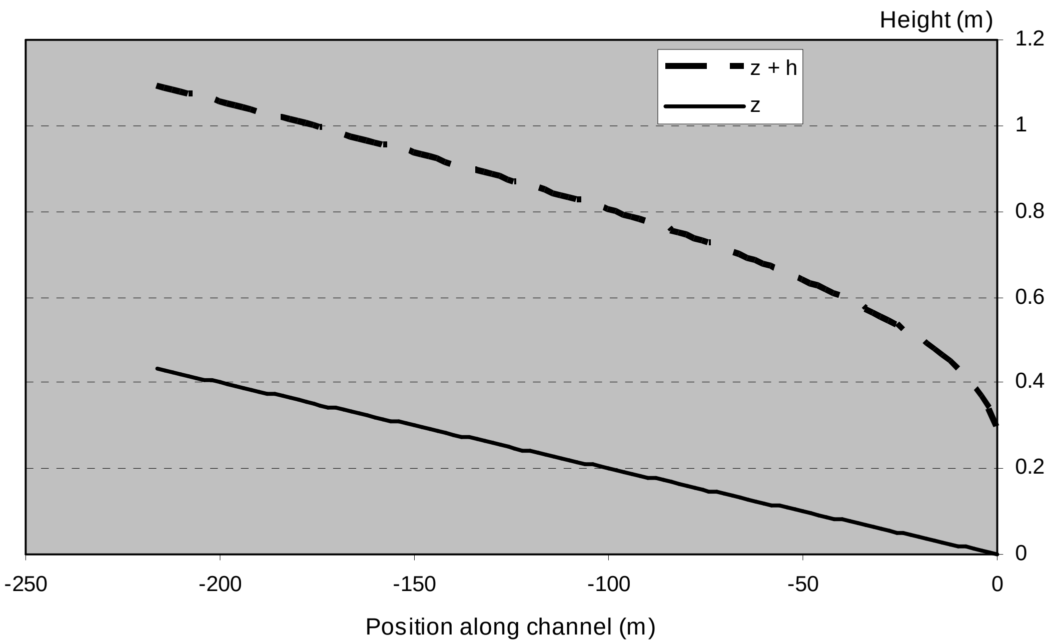

Figure 6
Water level in a more rapidly flowing brook.

In the previous case we could safely perform the calculation in the forward direction, here that is only possible for a very short distance. Because the friction slope is larger than the bottom slope, the water level decreases, but only as long as the Froude number does not become too large. As a matter of fact it does; at about 3 m beyond the reference point the denominator in the right hand side of equation (40) becomes identically zero, so the derivative $h'(x)$ become minus infinity, at which point obviously the calculation breaks down. Indeed the model breaks down, since we assumed negligible vertical acceleration of the flow, which condition is equally obviously violated.

3.3 External discharge

In cases where there is a nonzero external flow, the energy equation that describes the change in water level becomes considerably more complicated. Moreover assumptions need to be made about the energy dissipation produced by the inflow. We have used relation (38) for the rectangular channel. We studied the same two cases as above, for the slowly flowing ditch and the more rapidly flowing brook, now each with three different values for the inflow (Tables 3 and 4).

Table 3
Parameter values for the ditch.

	S (-)	b (m)	Q (m^3/s)	α (–)	k_M (m$^{1/3}$/s)	q_{ext} (m^3/m.s)
A	0.0001	1 m	$6.1 \cdot 10^{-4}$	1.2	25	$1.4 \cdot 10^{-6}$
B	0.0001	1 m	$5.9 \cdot 10^{-4}$	1.2	25	$2.6 \cdot 10^{-7}$
C	0.0001	1 m	$6.3 \cdot 10^{-4}$	1.2	25	$2.5 \cdot 10^{-6}$

The numerical results in Figure 7 show that in all three cases the same flat water level is obtained at the reference point, there is hardly any effect from the lateral inflow. Going upstream, however, the discharge decreases, and according to equation (11) actually becomes negative. At that point the model breaks down, but the numerical integration scheme does not. For the three scenarios this happens respectively at 450 m, 2350 m and 250 m in front of the reference point. Figure 7 shows that for all three cases the downstream level from the breakdown point is flat, the ditch is like a quiescent lake. The upstream result is fully artificial, and has no physical relevance whatsoever. Note that the vertical scale in figure 7 is exaggerated compared to the horizontal one by a factor of 5000.

Table 4
Parameter values for the brook.

	S (-)	b (m)	Q (m^3/s)	α (–)	k_M (m$^{1/3}$/s)	q_{ext} (m^3/m.s)
A	0.002	1 m	0.15	1.2	11	$1.3 \cdot 10^{-5}$
B	0.002	1 m	0.21	1.2	11	0
C	0.002	1 m	0.27	1.2	11	$2.5 \cdot 10^{-5}$

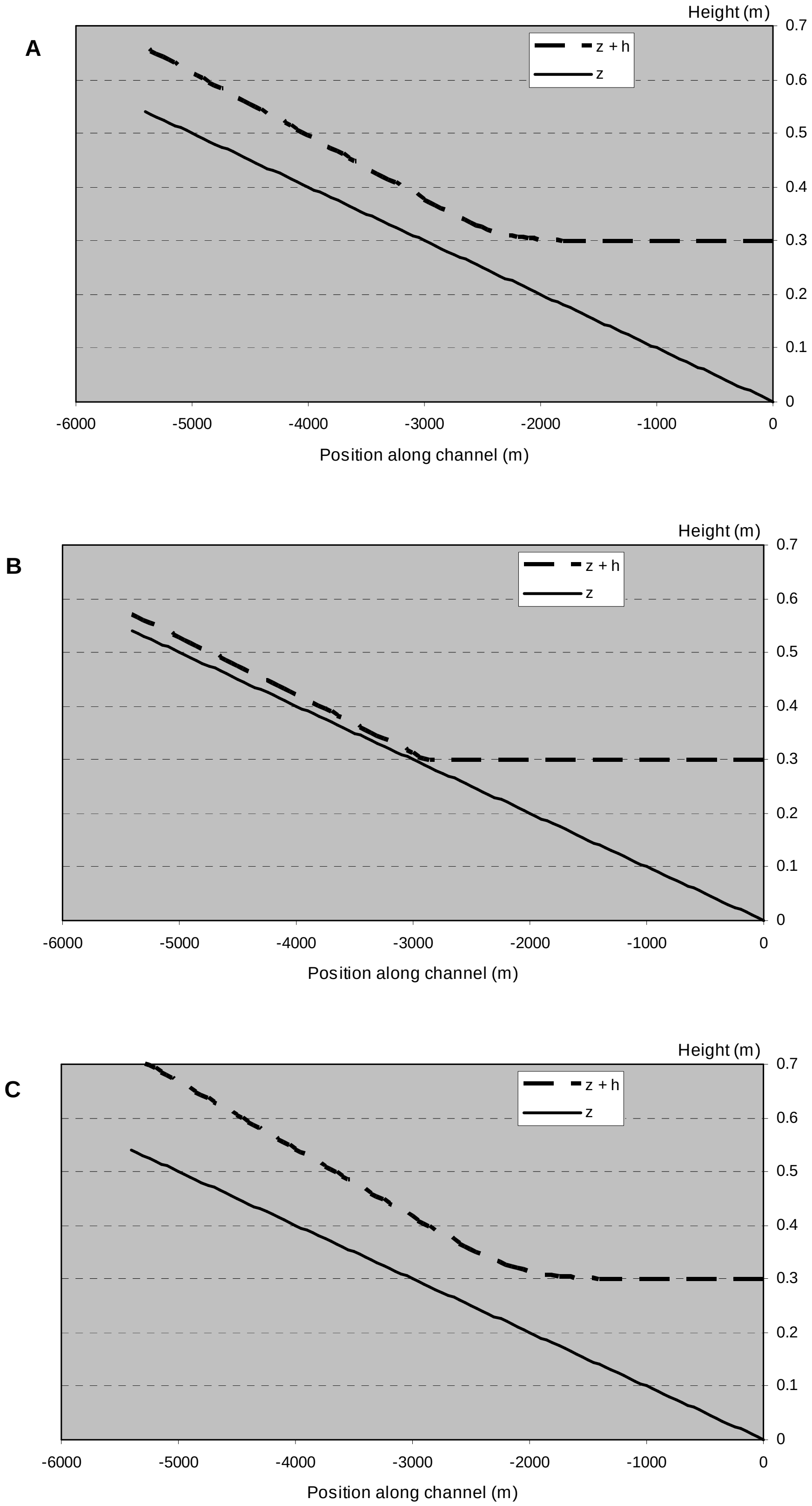

Figure 7
Water level in a slowly flowing ditch with lateral inflow.
Parameter values are given in Table 3.

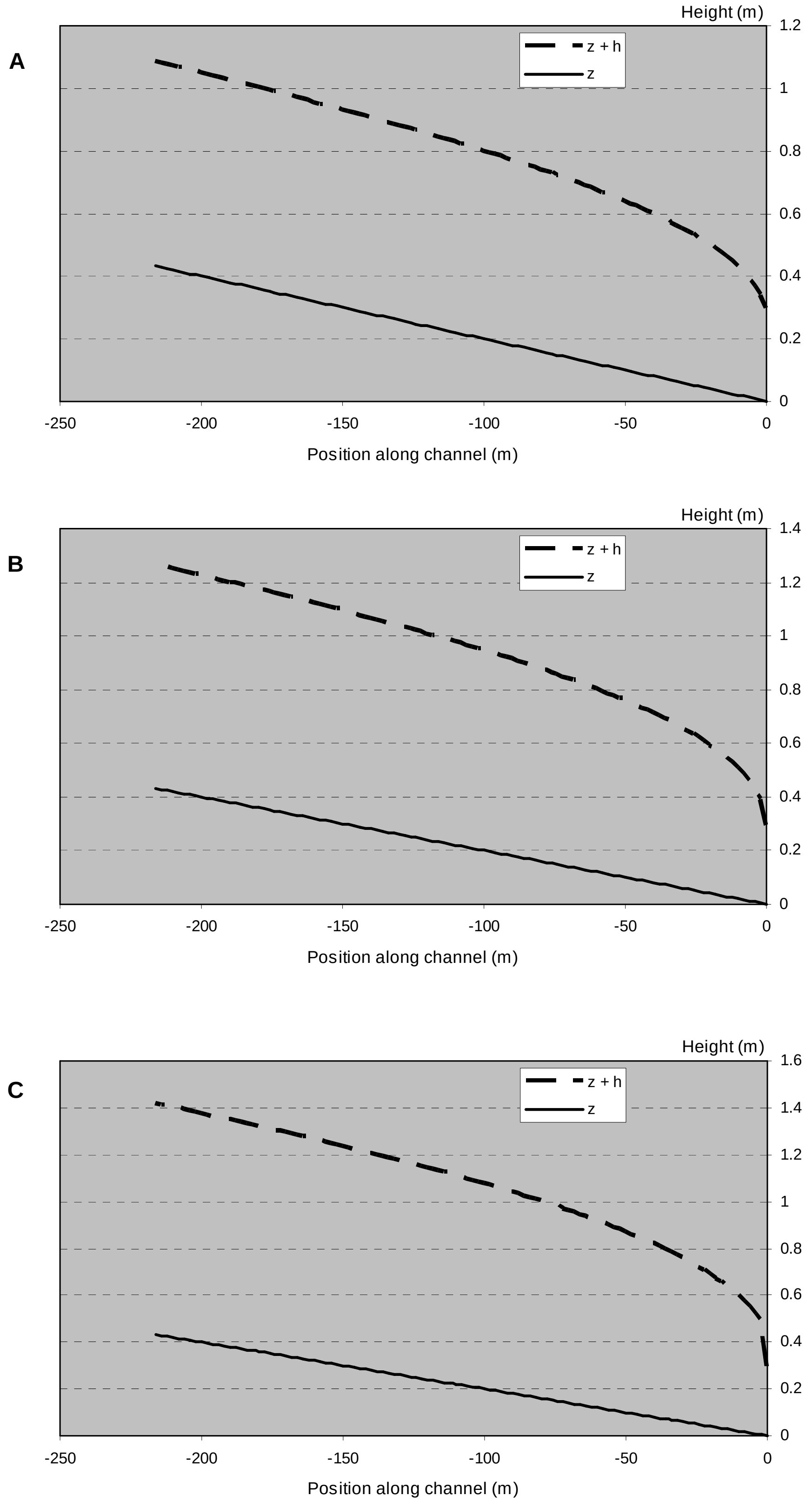

Figure 8
Water level in a more rapidly flowing brook with lateral inflow.
Parameter values are given in Table 4.

The difference between the results is quite small. Figure 8 shows clearly that a larger discharge gives a higher water level upstream, but over the distance over which the flow is followed the effect of the external discharge is minor. Note that the scaling of these figures differs from those for the ditch, the bottom slope is actually larger for the brook, but the scale difference between the vertical and horizontal axis is only a factor of 100 here.

3.4 Effect of a weir

In our method of calculation it is not necessary to assume the existence of a weir at the end of the channel, it suffices to specify discharge and water level at the reference point. Depending on the situation the calculation is performed downstream or upstream. When in fact there is a weir at the end of the channel, the water level immediately in front of the weir, in the upstream direction, can be derived from the calculated discharge at that point and the weir relation (39).

We return to the case of the ditch and the brook without external discharge. The discharge then is constant along the channel, and we can simply place the weir at the reference point and calculate backwards for the water level, using a negative step size in the numerical approximation procedure. The case of the ditch with a weir has the following parameter values

Table 5
Parameter values for the ditch with a weir.

S	B	Q	α	k_M	W	h_w	C
(-)	(m)	(m^3/s)	(–)	(m$^{1/3}$/s)	(m)	(m)	(-)
0.0001	1	0.0006	1.2	25	0.5	0.4	1.7

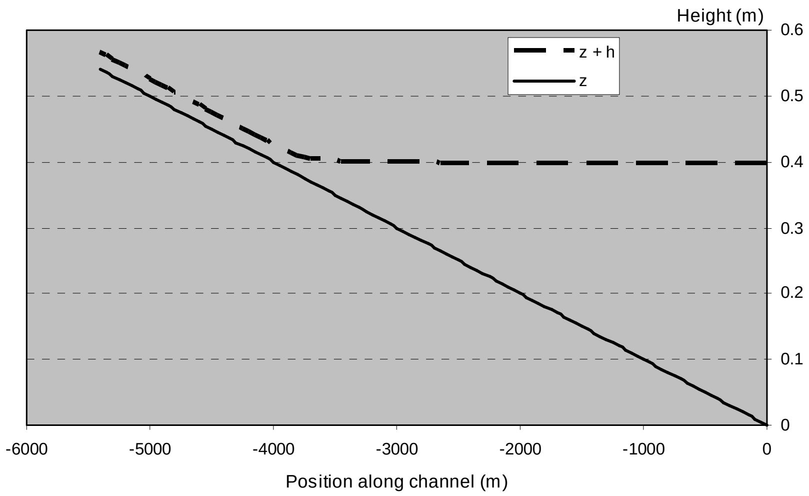

Figure 9.
Water level in slowly flowing ditch with a weir at the end.

The water level right in front of the weir is

$$h = h_{\mathrm{w}} + \left(\frac{Q}{C \cdot w} \right)^{2/3} = 0.408\,\mathrm{m}\,, \tag{46}$$

well above the level for uniform flow (0.027 m). The water surface in the ditch with a weir is almost flat, as seen in Figure 9.

The parameter values for the brook are

Table 6
Parameter values for the brook with a weir.

S	*B*	*Q*	α	k_M	*W*	h_w	*C*
(-)	(m)	(m^3/s)	(–)	($m^{1/3}$/s)	(m)	(m)	(-)
0.002	1	15	1.2	11	0.5	0.5	1.7

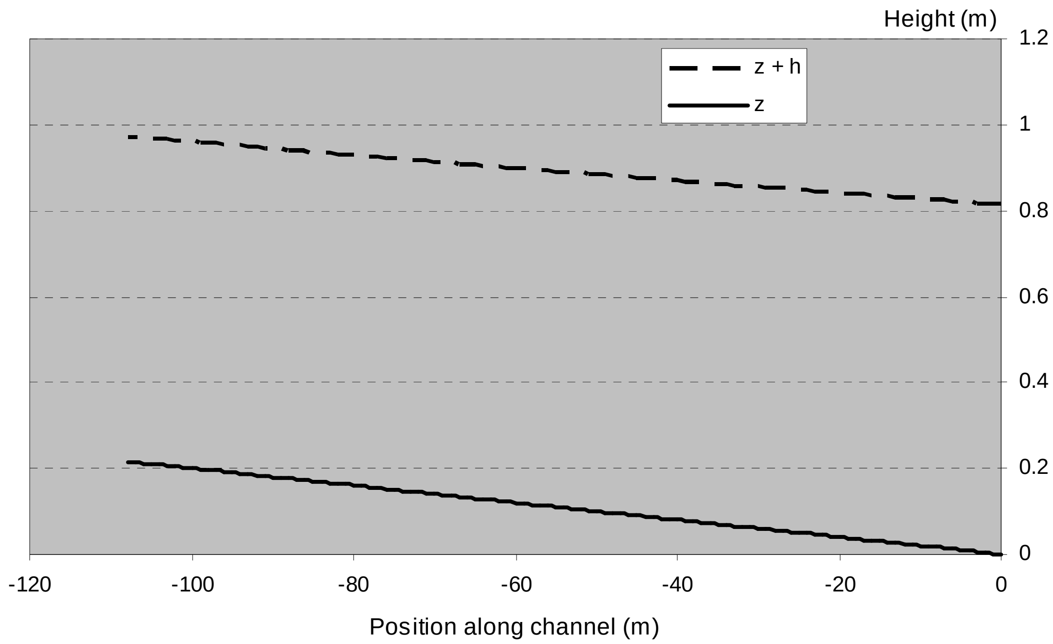

Figure 10
Water level in a more rapidly flowing brook with a weir at the end.

In this case the water level in front of the weir is 0.81 m, slightly above the level for uniform flow (0.7 m). Moving upstream we observe (Figure 10) a slow decrease in water level, at 100 m in front of the weir a level of 0.76 m is found. Note the 100 m is the (assumed) length of the channel, the calculation can be performed for any length.

4 Quasi stationary situation

Assuming the stationary situation we have investigated above qualifies as a stable state, we may expect a gradual development to such a stationary state when we start from any initial condition. Whether indeed the stability condition is observed can not be concluded from the above treatment, that would need further analysis of the problem. Given the fact that quite a large body of literature is available about this stationary state, it is quite unlikely it would be unstable, since that would imply any small perturbation of the situation would lead to a breakdown of the state, rendering the model quite useless.

So we may rather safely assume to have a stable stationary state. The question then is what can be said about the relaxation time towards such a state. In other words, suppose we have calculated the stationary state of the system for given values of the system parameters, and one or more of the parameters are changed. Eventually the system will then develop to a new stationary state, belonging to the new values of the system parameters. The question then is at what rate that process does occur. What are typical time scales in the system?

One important time scale is the time needed for the body of water in the channel to be refreshed on average, the so called hydraulic retention time. This is simply the time it takes for a small volume of water to travel along the full length of the channel

$$T = \frac{L}{v} = \frac{LA}{Q} = \frac{V}{Q}. \qquad (47)$$

For the two cases as studied in this investigation this time scale is 50000 s (fourteen hours) and 200 s (three minutes) respectively for a ditch and a brook 100 m long and a water level of 0.3 m. Another time scale that may be relevant is that of waves travelling along the surface of the channel. For slowly moving water such waves travel faster than the water itself, allowing disturbances to travel upstream. As in fact both examples involve slow flow, with a Froude number below unity, the longer time scale is that given by (47).

For the brook this would imply it takes of the order of minutes to reach a new steady state, after ten minutes the old state has changed for 95% into the new state, over the full length of the channel. Only if data sets are available with time intervals less than ten minutes it is necessary to use a fully non-stationary model description. If parameter changes occur at intervals of ten minutes or more, the stationary description will have errors of less than 5%, which in practice will be quite acceptable.

For the ditch a 14 hour relaxation time means that it will take a good part of a full day for the water to flow through the channel. In one hour the change from the old to the new stationary state, after a change of system parameters, will have proceeded for only 5-10%. That means that though the parameters may have changed, after one hour one would still observe a stationary state which is largely described by the original values of the system parameters. If one would be continuously changing the system parameters every hour, even if each change would be relatively small, after a few hours there would be little correlation between the stationary state of the channel flow at the momentary values of the system parameters, and its actual state. In plain language, the stationary approximation in such a case would be utterly useless.

5 Conclusions and recommendations

The calculational method we have described in this study is simple, straightforward, and easy to implement. It is based on direct numerical integration of two equations, one for the discharge, as follows from the water conservation equation, and one for the water level measured with respect to the bottom of the channel, as follows from energy conservation. There may be some discussion on the exact physical interpretation of the latter, specifically the energy dissipation generated by the inflowing water, the mathematical method is unwithspoken. In case the forward Euler method as used in our calculation would lead to unacceptably large numerical errors, of which the present research shows no indication whatsoever, more advanced methods could be introduced at will. The Euler method is known to become unstable for rapid changes, if one fails to adapt the integration step to the rate of change of the variable being integrated. In the present study that only occurs at the transition from subcritical to supercritical flow, when the model itself breaks down.

Direct numerical integration of the model equations is a good method to calculate stationary and quasi-stationary solutions for discharge channels with moderately rapid flow.

For subcritical flow the equation for the water level exhibits a singularity when the level at the beginning of the channel is below that of uniform flow at the given discharge and bottom slope. In that case the flow accelerates downstream, and the water level drops. Divergence occurs when the Froude number reaches a critical value, at which point the drop in water level becomes infinitely steep. This can easily occur within just a few meters beyond the beginning of the channel. In such a case one may simply specify the water level at the end of the channel, either by assuming a weir at that point, or by directly specifying the water level at that point as a boundary condition for the solution method. In our calculation the reference point can be chosen at the beginning or the end of the channel, depending on what is needed.

By choosing the reference point to be at the beginning or end of the channel, depending on the situation, in all cases the water level along the channel can be calculated directly with the method presented here.

If a weir is present, the relevant water level is found just in front of the weir, looking upstream. The water level at that position can be calculated directly from the weir properties and the specified discharge at that point, and the position of the weir can be used as the reference point. Since normally the upstream flow for a weir is subcritical, the calculation can be performed directly as indicated, without any instability.

In the case a weir is present, the described methods always allows for direct calculation of the upstream water level.

The energy dissipation due to the external inflow (eqs. 21 and 22) can not be specified on fundamental grounds only. Additional physical modelling is needed to settle the issue. In the calculations we have used the same equation (eq. 38) as specified in earlier discussions about this topic. For the lateral inflow values used in the present test cases the effect is at most small, for larger changes, or rapid changes in case of non-stationary model development, the issue should be re-addressed.

It is recommendable to reassess the effect of the lateral inflow to the energy balance before advancing to non-stationary model calculations.

For the Manning coefficient there a dependence on water level

$$k_{\mathrm{M}}(h) = k_{\mathrm{M}}(1)\, h^{m}\,, \tag{48}$$

with $m \approx \frac{1}{3}$ and h in meters, was mentioned in earlier discussions. In fact it may be expected that in most waterways the friction coefficient will change with changing water depth, for instance when there is vegetation in the water, or when the structure of the banks varies with height. The source for this formula is supposed to be the Cultuurtechnisch Vademecum, unfortunately there does not seem to be corroboration for this effect in the further literature. As a matter of fact the Manning coefficient has dimension, implying that its numerical value depends on the unit system employed. When changing for instance from meters to feet, with one foot being 0.3048 m, the Manning coefficient should be changed by a factor of $(1/\,0.3048)^{1/3} = 1.49$, with the same cube root proportionality as in the proposed seasonal correction (48). The effect on the results of the calculation is rather small actually.

It is recommendable to check the literature for this assumed effect on the Manning coefficient.

For relatively slow flow the stationary approach is not suited to describe the system behaviour under slowly changing external parameters. It is possible to develop a dynamical model, similar to the stationary one derived in this study that would be able to fully describe non stationary behaviour under moderately rapid changes in the environment. Additional modelling is needed, and additional testing of the mathematical methods to numerically solve such models is required as well, but the methods are available and can be applied as straightforwardly as for the stationary model described here. Such calculations, however, are considerably more complex than the present ones, and would require more advanced software development than the present simple Excel worksheet.

For slow flowing channels non stationary models are needed to describe the effect of moderately rapid changes. The mathematical methods to solve such models and the expertise to implement such methods are available at Biometris.

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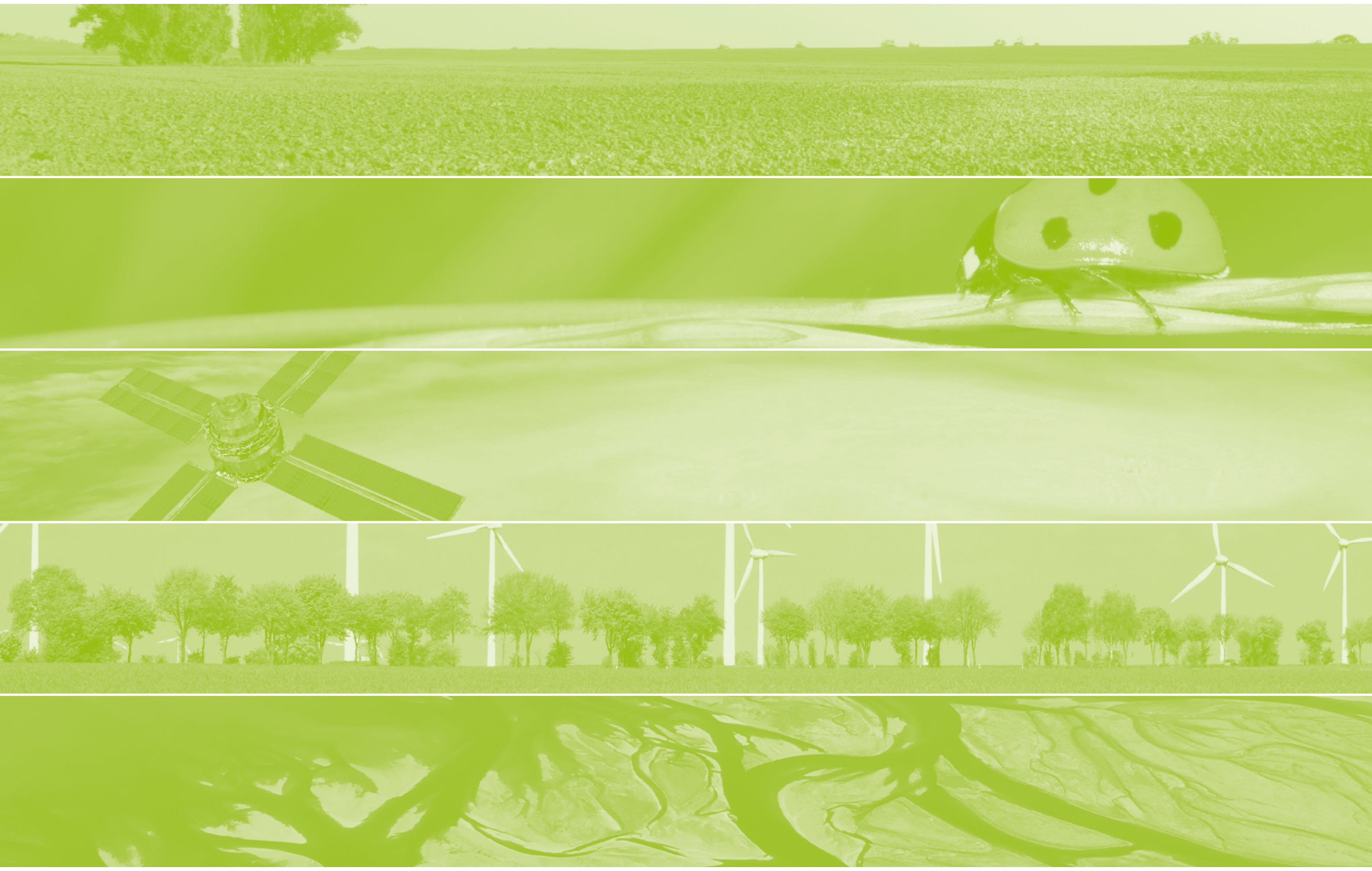
Appendix

A	cross sectional area	L^2
b	bottom width of channel	L
C	weir constant	$1 / L^{1/2} / T$
F_f	friction loss from channel bed	$M L^2 / T^2 / L$
Fr	Froude number	0
g	acceleration of gravity	L / T^2
h	water level	L
h_w	weir height	L
k_M	Manning coefficient	$L^{1/3} / T$
P	wetted perimeter of channel	L
Q	discharge	L^3 / T
q_{ext}	external flow density	$L^3 / T / L$
R	hydraulic radius of channel	L
S	channel bottom slope	0
s_1	slope of the channel walls	0
S_f	friction slope	0
t	time	T
u	local flow velocity	L / T
v	average flow velocity	L / T
V	volume	L^3
w	weir width	L
x	position along channel	L
z	height w.r.t. sea level	L
α	energy coefficient	0
ε_{ext}	external energy flow density	$M L^2 / T^2 / L$
ρ	water density	M / L^3

L = length
M = mass
T = time
0 = dimensionless



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