

On a CP approach to solve a MINLP inventory model *

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Abstract One of the most important policies adopted in inventory control is the replenishment cycle policy. Such a policy provides an effective means of dampening planning instability and coping with demand uncertainty. We describe a constraint programming approach for computing optimal replenishment cycle policy parameters under non-stationary stochastic demand, ordering, holding and shortage costs. Our solution approach exploits the convexity of the cost-function to dynamically compute during search the cost associated with a given decision variable assignment. By using our model we gauge the quality of an existing approximate mixed integer linear programming approach that exploits a piecewise linear approximation for the complex cost function. Furthermore, our computational experience shows that our approach can solve realistic instances in a fraction of a second.

Keywords: constraint programming, replenishment cycle policy, non-stationary demand, shortage cost, MINLP

1. Introduction

Much of the inventory control literature concerns the computation of optimal replenishment policies under demand uncertainty. One of the most important policies adopted is the (R,S) policy (also known as the *replenishment cycle* policy). A detailed discussion on the characteristics of (R,S) can be found in de Kok [3]. In this policy a replenishment is placed every R periods to raise the inventory position to the order-up-to-level S . An important extension for existing stochastic production/inventory control problems consists of incorporating a non-stationary demand process. Under this assumption the (R,S) policy takes the non-stationary form (R_n, S_n) where R_n denotes the length of the n^{th} replenishment cycle and S_n the corresponding order-up-to-level. To compute the *near* optimal (R_n, S_n) policy values, [5] apply a mixed integer programming (MIP) formulation using a piecewise linear approximation to a complex cost function with fixed procurement cost each time a replenishment order is placed. So far no complete approach exists for computing (R_n, S_n) policy parameters under a shortage cost scheme.

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In fact, [5] show that the cost structure is complex in this case and differs significantly from the one under a service level constraint. In this work, we give an *exact* constraint programming [1] (CP) formulation of the (R_n, S_n) inventory control problem under shortage cost scheme. Our approach embeds dedicated cost-based filtering methods [4] to improve performances of the search. Our contribution is two-fold: we can now efficiently obtain provably optimal solutions for the MINLP (R_n, S_n) inventory control problem under shortage costs and we can gauge the accuracy of the piecewise linear approximation proposed by [5]. Computational results shows the effectiveness of our approach.

2. From a stochastic to a deterministic equivalent model

We consider the single stocking location, single product inventory problem over a finite planning horizon of N periods. The demand d_t in period t is assumed to be a normally distributed random variable with known probability density function (PDF) $g_t(d_t)$. Demand is assumed to occur instantaneously at the beginning of each period. The mean rate of demand may vary from period to period. Demands in different time periods are assumed to be independent. Demands occurring when the system is out of stock are assumed to be back-ordered and satisfied as soon as the next replenishment order arrives.

In the general multi-period production/inventory problem with stochastic demands we are interested in finding the timing of the stock reviews and the size of non-negative replenishment orders, X_t in period t , which minimize the expected total cost over a finite planning horizon of N periods:

$$\min E\{TC\} = \int_{d_1} \int_{d_2} \dots \int_{d_N} \sum_{t=1}^N (a\delta_t + vX_t + hI_t^+ + sI_t^-) g_1(d_1) \dots g_N(d_N) d(d_1) \dots d(d_N) \quad (1)$$

subject to, for $t = 1 \dots N$,

$$X_t > 0 \Rightarrow \delta_t = 1 \quad (2)$$

$$I_t = \sum_{i=1}^t (X_i - d_i) \quad (3)$$

$$I_t^+ = \max(0, I_t) \quad (4)$$

$$I_t^- = -\min(0, I_t) \quad (5)$$

$$X_t, I_t^+, I_t^- \in \mathbb{R}^+ \cup \{0\}, \quad I_t \in \mathbb{R}, \quad \delta_t \in \{0, 1\} \quad (6)$$

where

- d_t : the demand in period t , a normal random variable with PDF $g_t(d_t)$,
- a : the fixed ordering cost,
- v : the proportional direct item cost,
- h : the proportional stock holding cost,
- s : the proportional shortage cost,
- δ_t : a $\{0,1\}$ variable that takes the value of 1 if a replenishment occurs in period t and 0 otherwise,
- I_t : the inventory level at the end of period t , $-\infty < I_t < +\infty$, $I_0 = 0$
- I_t^+ : the excess inventory at the end of period t carried over to the next period,
- I_t^- : the shortages at the end of period t , or magnitude of negative inventory,
- X_t : the replenishment order placed and received in period t , $X_t \geq 0$.

Let $R(i, j)$ denote a *replenishment cycle* that schedules an inventory review at period i to cover subsequent demand up to period j with cost as a function of the opening inventory level S :

$$\sum_{k=i}^j \left(h z_{\alpha(i,k)} \sigma_{i,k} + (h + s) \sigma_{i,k} [\Phi(z_{\alpha(i,k)}) - (1 - \alpha(i,k)) z_{\alpha(i,k)}] \right), \quad (7)$$

where $G_{i,k}(\cdot)$ and $\sigma_{i,k}$ denote, respectively, the cumulative distribution function and the standard deviation of $d_i + \dots + d_k$; $\alpha(i, k) = G_{i,k}(S)$; and $z_{\alpha(i,k)} = \Phi^{-1}(\alpha(i, k))$. $\Phi(\cdot)$ and $\phi(\cdot)$ denote, respectively, the standard normal CDF and PDF. Therefore we have $j - i + 1$ cost components: the holding and shortage costs at the end of period $i, i + 1, \dots, j$. Since we can

prove that the cost function to be convex, for each possible replenishment cycle we can efficiently compute the optimal S^* that minimizes it by using gradient based methods for convex optimization.

3. CP approach to solve the deterministic equivalent model

A *deterministic equivalent* (see [2]) CP formulation of the model is constructed. The expected total cost for $R(i, j)$ with opening inventory level S_i , is iteratively computed by a special-purpose constraint `objConstraint(·)` which uses a slightly extended version of Eq. (7). Intuitively, within this constraint the expected total cost for a certain replenishment plan will be computed as the sum of all the expected total costs for replenishment cycles in the solution, plus the respective ordering costs.

A *deterministic equivalent* CP formulation is then

$$\min E\{TC\} = C \quad (8)$$

subject to

$$\text{objConstraint}(C, \tilde{I}_1, \dots, \tilde{I}_N, \delta_1, \dots, \delta_N, d_1, \dots, d_N, a, h, s) \quad (9)$$

and for $t = 1 \dots N$

$$\tilde{I}_t + \tilde{d}_t - \tilde{I}_{t-1} \geq 0 \quad (10)$$

$$\tilde{I}_t + \tilde{d}_t - \tilde{I}_{t-1} > 0 \Rightarrow \delta_t = 1 \quad (11)$$

$$\tilde{I}_t \in \mathbb{R}, \quad \delta_t \in \{0, 1\}. \quad (12)$$

Decision variable \tilde{I}_t represents the expected closing inventory level at the end of period t and \tilde{d}_t represents the expected value of demand in a given period t .

The objective function (8) gives expected total cost over the planning horizon. `objConstraint(·)` dynamically computes buffer stocks and assigns to C the expected total cost related to a given assignment for replenishment decisions, depending on the demand distribution in each period. In order to propagate `objConstraint(·)`, during the search we wait for a partial assignment involving some or all δ_t variables. We look for an assignment where there exists some i s.t. $\delta_i = 1$, some $j > i$ s.t. $\delta_{j+1} = 1$ and for every k , $i < k \leq j$, $\delta_k = 0$. This uniquely identifies a replenishment cycle $R(i, j)$.

There may be more replenishment cycles associated to a partial assignment. If we consider each $R(i, j)$ identified by the current assignment, it is easy to minimize the convex cost function already discussed, and to find the optimal expected closing inventory level \tilde{I}_j for this particular replenishment cycle independently of the others.

By independently computing the optimal optimal expected closing inventory level for every replenishment cycle identified, two possible situations may arise: (i) the closing inventory levels obtained satisfies every inventory conservation constraint (Eq. (10)); or (ii) for some couple of subsequent replenishment cycles this constraint is violated. In other words, we observe an expected negative order quantity. If the latter situation arises, we can adopt a fast convex optimization procedure to compute a feasible buffer stock configuration with minimum cost.

The algorithm for computing optimal buffer stock configurations in presence of negative order quantity scenarios simply exploits the linear dependency between the opening inventory level of the second cycle and the expected closing inventory level of the first cycle. Due to this dependency the overall cost is still convex; we can apply convex optimization to find the optimal buffer stock configuration. Note that this reasoning still holds in a recursive process. Therefore, we can optimize buffer stock for two subsequent replenishment cycles, then we can treat these as a new single replenishment cycle, since their buffer stocks are linearly dependent, and repeat the process in order to consider the next replenishment cycle if a negative order quantity scenario arises. A lower bound for the expected total cost associated to the current partial assignment for δ_t , $t = 1, \dots, N$ variables is now given by the sum of all the cost components $C(S_i, i, j)$, for each replenishment cycle $R(i, j)$ identified by the assignment.

4. Numerical experience

The described approach was first compared to the one described in and the set of instances presented in [5]. For these instances, a piecewise linear approximation with seven segments usually provides a solution with a cost reasonably close to optimal (<1% difference); while using two segments produces a cost difference up to about 7%. The next experiment concerns a single problem with period demands generated from seasonal data with no trend: $\tilde{d}_t = 50[1 + \sin(\pi t/6)]$. In addition to the “no trend” case (P1) we also consider three others: (P2) positive trend case, $\tilde{d}_t = 50[1 + \sin(\pi t/6)] + t$; (P3) negative trend case, $\tilde{d}_t = 50[1 + \sin(\pi t/6)] + (52 - t)$; (P4) life-cycle trend case, $\tilde{d}_t = 50[1 + \sin(\pi t/6)] + \min(t, 52 - t)$. In each test we assume an initial null inventory level and a normally distributed demand for every period with a coefficient of variation σ_t/\tilde{d}_t for each $t \in \{1, \dots, N\}$, where N is the length of the considered planning horizon. We performed tests using four different ordering cost values $a \in \{50, 100, 150, 200\}$ and two different $\sigma_t/\tilde{d}_t \in \{1/3, 1/6\}$. The planning horizon length takes even values in the range [20, 38]. The holding cost used in these tests is $h = 1$ per unit per period. Our tests also consider two different shortage cost values $s = 15$ and $s = 25$. Direct item cost is $v = 2$ per unit produced.

Our CP approach generally requires only a fraction of a second to produce the optimal solution. Only in 6 instances the optimal solution was not produced within the given time limit of 5 seconds. Nevertheless, it should be also noted that the worst case running time for our approach over the whole test bed was 6,77 minutes. Therefore even in the few cases in which an optimal solution is not found in a less than 5 seconds, our CP model provides a reasonable running time.

5. Conclusions

We developed a constraint programming approach able to compute optimal replenishment cycle policy parameters under non-stationary stochastic demand, ordering, holding and shortage costs. In our model we exploited the convexity of the cost-function during the search to dynamically compute the cost associated with a given decision variable assignment. By using our approach we assessed the quality of an existing approximate mixed integer linear programming approach that exploits a piecewise linear approximation for the complex cost function. Our computational experience show the effectiveness of our approach.

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