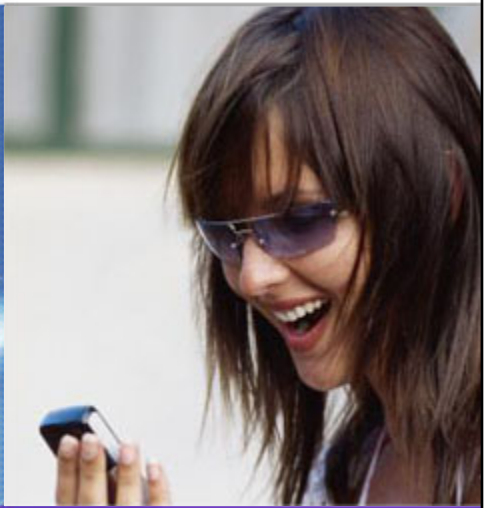
An Efficient Computational Method for Non-Stationary (R,S) Inventory Policy with Service Level Constraints



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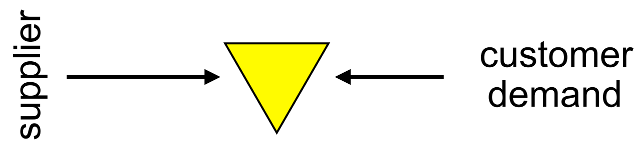
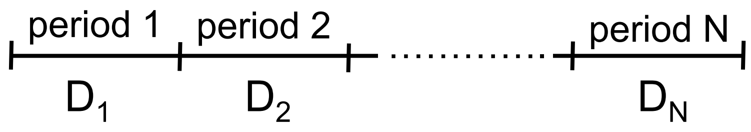
System Under Study

- Single-item, single stage lot sizing problem
- Supplier with ample capacity
- No lead time (Without loss of generality)
- Periodic review (system state and actions)
  - planning horizon: N periods
- Stochastic demand of the customer
  - non-stationary demand: $D_t \geq 0$, t=1,2,…,N
- Fixed cost of ordering (A)
- Variable cost of holding inventory (h)
- Service level constraints (P1: probability of stock outs)





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Sequence of Events

At the start of the planning horizon

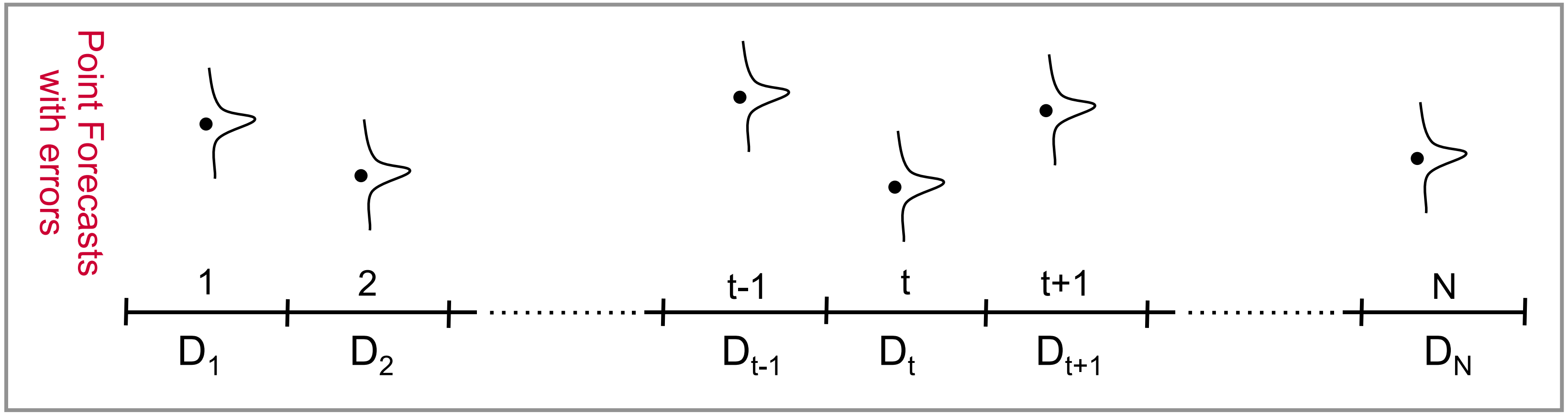
Knowing the demand forecasts, the planner determines timing of the orders, and the fixed ordering costs are incurred.

In each period

If it is a replenishment period, the planner decides on the order quantity to achieve certain service level after evaluating the current inventory position.

The order due to arrive is received, and demand occurs.

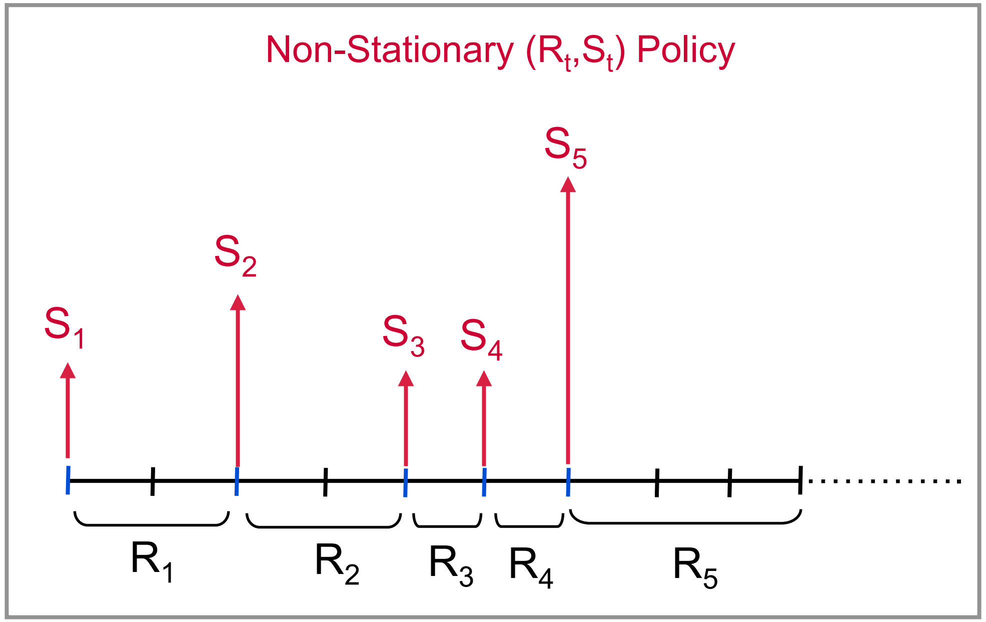
Holding cost is incurred on the period ending inventory.



Non-Stationary (R,S) Policy

Non-Stationary (R,S) Policy:

- (R,S) pairs change within the planning horizon
  - m orders → (R_i,S_i) for i=1,2,…,m



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Model

Decision Variables: Replenishment periods, $\delta_t \in \{0,1\}$

Target inventory levels $S_t \geq 0$

Objective: Minimize the expected ordering and inventory holding costs within the planning horizon while satisfying a service level constraint in each period

$$\min \mathsf{E}\left[\sum_{t=1}^{N}\left(A\delta_t + h\max\{0, I_t\}\right)\right]$$

Objective Function

$$\begin{aligned} I_t &= \max\{S_t, I_{t-1}\} - D_t \quad &\text{if} \quad \delta_t = 1 \\ I_t &= I_{t-1} - D_t \quad &\text{if} \quad \delta_t = 0 \end{aligned}$$

Inventory Balance Equations

$$\mathsf{Pr}\{I_t \geq 0\} \geq \alpha$$

Service Level Constraints

Literature & Model Assumptions

- Bookbinder and Tan (Management Science, 1988)
  - Fix the replenishment schedule with a heuristic: $\delta_t \in \{0,1\}$
  - Work with expectations rather than random variables

$$\min \sum_{t=1}^{N} \left(A\delta_t + h\tilde{I}_t \right)$$

$$\tilde{I}_t = \max\{\tilde{S}_t, \tilde{I}_{t-1}\} - \tilde{d}_t \quad \text{if} \quad \delta_t = 1$$

$$\tilde{I}_t = \tilde{I}_{t-1} - \tilde{d}_t \quad \text{if} \quad \delta_t = 0$$

$$\tilde{I}_i \geq G_i^{-1}(\alpha) - \tilde{d}_i, \quad i = 1, 2, \cdots, m$$

- Tarim and Kingsman (IJPE, 2004)
  - Under the same model assumptions of BT(1988), formulate a MIP model that also determines the replenishment schedule

MIP Formulation of Tarim and Kingsman (2004)

$$\min \mathsf{E}[TC] = \sum_{t=1}^{N} \left(A\delta_t + h\tilde{I}_t \right)$$

s.t.

$$\tilde{I}_t = \tilde{S}_t - \tilde{d}_t \qquad t = 1, \ldots, N$$

$$\tilde{S}_t \geq \tilde{I}_{t-1} \qquad t = 1, \ldots, N$$

$$\tilde{S}_t - \tilde{I}_{t-1} \leq M\delta_t \qquad t = 1, \ldots, N$$

$$\tilde{I}_t \geq \sum_{j=1}^{t} \left(G^{-1}_{d_{t-j+1}+d_{t-j+2}+\ldots+d_t}(\alpha) - \sum_{k=t-j+1}^{t} \tilde{d}_k \right) P_{tj} \qquad t = 1, \ldots, N$$

$$\sum_{j=1}^{t} P_{tj} = 1 \qquad t = 1, \ldots, N$$

$$\tilde{I}_t \geq 0 \qquad t = 1, \ldots, N$$

$$P_{tj} \geq \delta_{t-j+1} - \sum_{k=t-j+2}^{t} \delta_k \qquad j = 1, \ldots, t \quad t = 1, \ldots, N$$

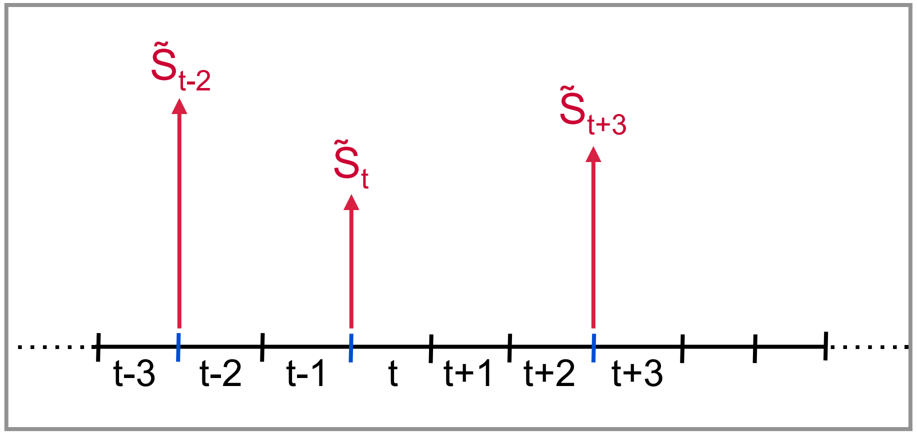
$$\delta_t, P_{tj} \in \{0, 1\} \qquad j = 1, \ldots, t \quad t = 1, \ldots, N$$

Solution of the MIP Model for a Given Replenishment Schedule

$\mathfrak{T}$ and $\overline{\mathfrak{T}}$: $\mathfrak{T} \cup \overline{\mathfrak{T}} = \{1, \ldots, N\}$, $\delta_i = 1$ for all $i \in \mathfrak{T}$ and $\delta_i = 0$ for all $i \in \overline{\mathfrak{T}}$

Lemma 1 *Given any $\mathfrak{T}$ and $\overline{\mathfrak{T}}$, the optimal solution for the MIP model has*

$$\tilde{S}_t = \begin{cases} \max\left\{\tilde{S}_{t-1} - \tilde{d}_{t-1}, G^{-1}_{d_t + d_{t+1} + \ldots + d_{\bar{t}-1}}(\alpha), \sum_{k=t}^{\bar{t}-1} \tilde{d}_k\right\} & \textit{for } t \in \mathfrak{T} \\ \\ \tilde{S}_{t-1} - \tilde{d}_{t-1} & \textit{for } t \in \overline{\mathfrak{T}}, \end{cases}$$



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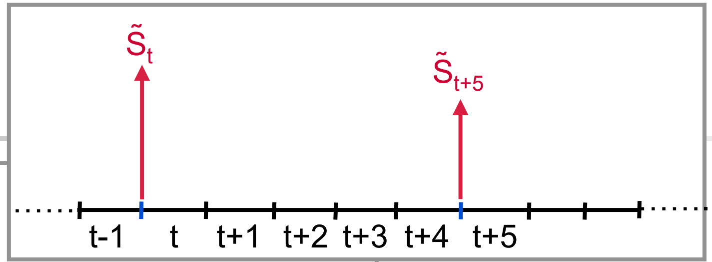
Equivalent Optimization Problem

The result of Lemma 1 can be used to formulate an equivalent optimization problem for the MIP

$$\min_{\delta_1,\ldots,\delta_N} z = \sum_{t=1}^{N} \left(a\delta_t + h(\tilde{S}_t - \tilde{d}_t) \right)$$

Relaxed MIP Model



$$\min \mathsf{E}[TC] = \sum_{t=1}^{N} \left(A\delta_t + h\tilde{I}_t\right)$$

s.t.

$$\tilde{I}_t = \tilde{S}_t - \tilde{d}_t \qquad t = 1, \ldots, N$$

$$\tilde{S}_t \geq \tilde{I}_{t-1} \qquad t = 1, \ldots, N$$

Relaxed

$$\tilde{S}_t - \tilde{I}_{t-1} \leq M\delta_t \qquad t = 1, \ldots, N$$

$$\tilde{I}_t \geq \sum_{j=1}^{t} \left(G^{-1}_{d_{t-j+1}+d_{t-j+2}+\ldots+d_t}(\alpha) - \sum_{k=t-j+1}^{t} \tilde{d}_k \right) P_{tj} \qquad t = 1, \ldots, N$$

$$\sum_{j=1}^{t} P_{tj} = 1 \qquad t = 1, \ldots, N$$

$$\tilde{I}_t \geq 0 \qquad t = 1, \ldots, N$$

Relaxed

$$P_{tj} \geq \delta_{t-j+1} - \sum_{k=t-j+2}^{t} \delta_k \qquad j = 1, \ldots, t \quad t = 1, \ldots, N$$

$$\delta_t, P_{tj} \in \{0, 1\} \qquad j = 1, \ldots, t \quad t = 1, \ldots, N$$

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Solution of the Relaxed MIP Model for a Given Replenishment Schedule

$\mathfrak{T}$ and $\overline{\mathfrak{T}}$: $\mathfrak{T} \cup \overline{\mathfrak{T}} = \{1, \ldots, N\}$, $\delta_i = 1$ for all $i \in \mathfrak{T}$ and $\delta_i = 0$ for all $i \in \overline{\mathfrak{T}}$

Lemma 2 *Given any $\mathfrak{T}$ and $\overline{\mathfrak{T}}$, the optimal solution for the relaxed MIP model has*

$$\tilde{S}_t^r = \begin{cases} G^{-1}_{d_t + d_{t+1} + \ldots + d_{\bar{t}-1}}(\alpha) \ \mathit{for}\ t \in \mathfrak{T} \\ \\ \tilde{S}^r_{t-1} - \tilde{d}_{t-1} \qquad\qquad \mathit{for}\ t \in \overline{\mathfrak{T}}, \end{cases}$$

Equivalent Optimization Problem

$$\min_{\delta_1, \ldots, \delta_N} z_r = \sum_{t=1}^{N} \left(a\delta_t + h(\tilde{S}_t^r - \tilde{d}_t) \right)$$

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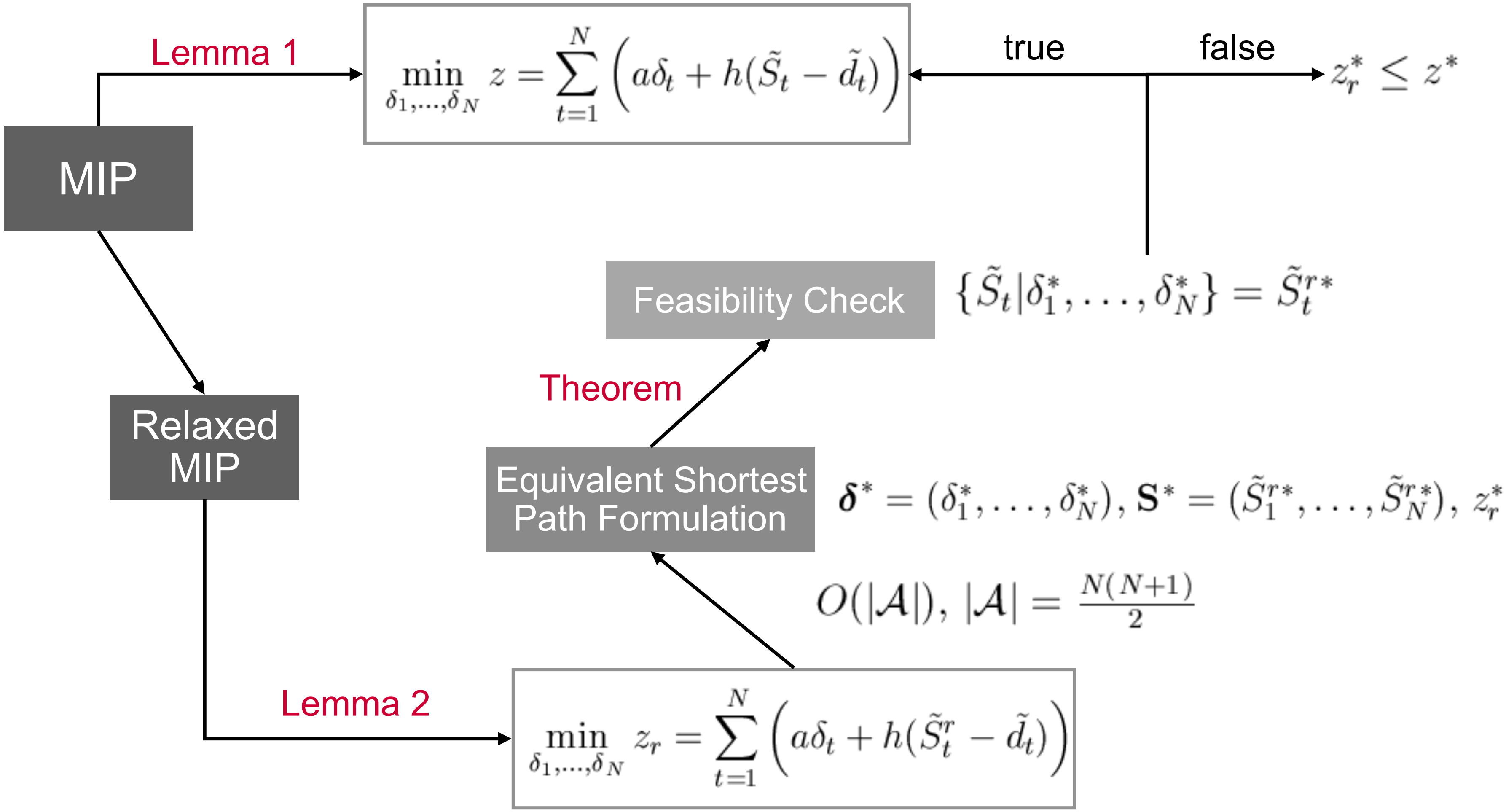
Overview of Our Results



Computational Procedure

- Solve the shortest path problem → optimal solution for the Relaxed MIP

$$\boldsymbol{\delta}^* = (\delta_1^*, \ldots, \delta_N^*), \mathbf{S}^* = (\tilde{S}_1^{r*}, \ldots, \tilde{S}_N^{r*}), \; z_r^*$$

- Check the feasibility of the solution
  - Feasible → Terminate
  - Not feasible
    - Lower Bound (z_r^*)
    - Use $\boldsymbol{\delta}^*$ to calculate $\tilde{S}_t$ → Upper bound (z)
    - Initiate a Branch & Bound search and branch on one δ_t that is not feasible

Numerical Study

We would like to address

- the percentage of the non-stationary instances solved to optimality using solely the relaxed-MIP approach, without resorting to any search effort
- the effectiveness of the bounds provided by the relaxed-MIP model if the observed solution is infeasible for the original problem
- the overall solution time performance of the proposed method
- the scalability of the proposed method

Test bed

- N=30,40,50,60
- Demand Patterns
  - μ_t: stationary (P_1), seasonal (P_2), decreasing (P_3), increasing (P_4), product life-cycle (P_5)
  - $D_t \sim Normal(r_t\mu_t, (0.25r_t\mu_t)^2)$ where r_t is sampled from U[0.4,1.6]
  - h=1, α=0.95, A is sampled from U[75,2000]

Numerical Results: Feasible Cases

Generated random instances and solved using the relaxed MIP approach (equivalent shortest path formulation) for a given N and a demand pattern (P_i)

	P_1	P_2	P_3	P_4	P_5
$N = 30$	–	124,101	238,794	304,239	49,279
$N = 40$	–	104,686	134,702	172,369	54,918
$N = 50$	–	62,780	223,889	151,011	22,251
$N = 60$	–	53,086	246,614	128,831	31,400

2,102,950 instances
160 infeasibility
99.99%

Infeasible problems

- Java Implementation, CPLEX 11.2, 2.0 GHz CPU with 32-bit machine
- Cut-off time of 1 hour

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Infeasible Cases: Solution Time

Using our method all instances are solved to optimality without any exception

- longest taking 30.1 secs, most of the time taking less than 5 seconds

MIP Model

- Time limit of 1 hour: a proven optimal solution in only 84 out of 160
  - N = 30: all the instances are solved to optimality, mean solution time is 12.5 secs
  - N = 40: 39/40 instances, mean solution time is 529.4 secs
  - N = 50: 5/40 instances, mean solution time 1631.0 secs
  - N = 60: 0/40 instances
- Demand pattern-wise, the solution time performance does not vary much, although $P5$ (life-cycle) performs slightly better than the rest

Infeasible Cases: Search Effort

Our search procedure enhanced with tight lower and upper bounds requires to visit only a small set of nodes giving an average of 241

- The average gaps for ΔLB=0.05% and ΔUB=0.02%, with worst case performances of 0.28% and 0.19%

MIP Model

- The number of nodes visited during search increases with the number of periods in the planning horizon
  - N = 30: the average number of search nodes is 20,895
  - N = 40: the average number of search nodes is 636,028
  - N = 50: the average number of search nodes is 1,117,740
- N = 60: after the search is on for 1 hour, the average optimality gap is 6:94%

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Conclusions

This study provides an efficient computational approach to solve the MIP model developed by Tarim and Kingsman (2004) for calculating the parameters of an (R,S) policy in a finite horizon with non-stationary stochastic demand

- We have developed a computational procedure with a numerically demonstrated better performance compared to a commercially available MIP solver
  - the proposed relaxation is computationally efficient and yields an optimal solution most of the time (99:99% of the time in our experiments)
  - if the relaxation produces an infeasible solution, this solution can be used as a tight lower bound during search
  - this infeasible solution can be modified easily to obtain a feasible solution, which is an upper bound for the optimal solution
- Our method is scalable and makes it possible to solve practically relevant instances in trivial time

Numerical Results (cont.)

	$N = 30$								$N = 40$							
		MIP			Our Method					MIP			Our Method			
Demand	#	Nodes	%Δ	secs	Nodes	$\%\Delta_{LB}$	$\%\Delta_{UB}$	secs	#	Nodes	%Δ	secs	Nodes	$\%\Delta_{LB}$	$\%\Delta_{UB}$	secs
$P2$	1	19200	–	12.4	1	0.00	0.00	0.1	41	1026500	–	818.1	79	0.03	0.06	1.4
	2	21900	–	12.2	59	0.12	0.16	0.7	42	646100	–	576.6	1	0.00	0.00	0.2
	3	45800	–	24.5	1	0.00	0.00	0.1	43	147100	–	115.5	79	0.03	0.03	1.2
	4	9000	–	6.7	1	0.00	0.00	0.0	44	249800	–	195.0	79	0.06	0.00	1.2
	5	36100	–	21.8	1	0.00	0.00	0.1	45	702300	–	555.1	103	0.09	0.00	1.2
	6	11500	–	7.6	151	0.08	0.00	1.4	46	561500	–	460.8	459	0.21	0.03	4.3
	7	23400	–	14.4	1	0.00	0.00	0.0	47	440300	–	367.2	199	0.12	0.00	2.3
	8	12100	–	8.6	87	0.16	0.00	0.8	48	851500	–	701.9	1	0.00	0.00	0.1
	9	7400	–	5.3	1	0.00	0.00	0.1	49	1457800	–	1154.6	469	0.09	0.00	5.7
	10	26800	–	16.3	465	0.24	0.04	3.0	50	424400	–	351.9	79	0.16	0.00	1.1
$P3$	11	22700	–	12.1	59	0.04	0.04	0.7	51	922400	–	713.5	79	0.03	0.00	1.4
	12	66500	–	38.4	91	0.08	0.08	0.8	52	5161300	1.04	–	235	0.09	0.00	2.7
	13	28300	–	18.6	107	0.12	0.08	1.0	53	247700	–	236.1	79	0.03	0.00	1.2
	14	30500	–	17.6	59	0.08	0.00	0.7	54	693400	–	583.1	247	0.06	0.00	3.2
	15	17100	–	11.1	59	0.04	0.00	0.7	55	134000	–	117.6	135	0.08	0.00	1.9
	16	22500	–	13.2	1	0.00	0.00	0.1	56	2622200	–	2136.8	79	0.03	0.11	1.3
	17	20400	–	12.4	59	0.04	0.11	0.7	57	937600	–	704.7	147	0.12	0.00	1.9
	18	26900	–	13.0	59	0.08	0.08	0.6	58	409900	–	304.9	79	0.06	0.00	1.2
	19	51700	–	26.7	59	0.04	0.00	0.7	59	866400	–	694.2	1	0.00	0.00	0.1
	20	8400	–	5.4	59	0.04	0.00	0.7	60	1703000	–	1384.0	79	0.06	0.00	1.2
$P4$	21	16800	–	10.7	1	0.00	0.00	0.1	61	991700	–	851.0	153	0.15	0.03	2.2
	22	11600	–	8.4	59	0.04	0.00	0.6	62	197000	–	178.3	79	0.15	0.03	1.4
	23	26600	–	17.0	95	0.04	0.00	0.9	63	911000	–	831.8	995	0.11	0.09	9.0
	24	15400	–	9.6	83	0.28	0.00	1.4	64	732400	–	610.0	109	0.06	0.09	1.5
	25	8000	–	5.6	1	0.00	0.00	0.0	65	152100	–	139.0	79	0.03	0.00	2.0
	26	21700	–	13.2	59	0.04	0.00	0.7	66	366200	–	374.3	97	0.09	0.00	1.6
	27	53300	–	29.6	59	0.04	0.18	0.7	67	1635600	–	1326.3	455	0.14	0.06	5.1
	28	29400	–	17.2	1	0.00	0.00	0.1	68	761800	–	706.3	139	0.03	0.00	1.8
	29	11300	–	8.3	59	0.04	0.07	0.6	69	1323800	–	1159.7	203	0.03	0.03	2.2
	30	7800	–	5.6	87	0.04	0.00	0.9	70	597900	–	513.1	1	0.00	0.00	0.7
$P5$	31	25800	–	13.0	59	0.03	0.07	0.7	71	227600	–	199.8	235	0.05	0.13	2.5
	32	17500	–	9.4	59	0.04	0.00	0.6	72	529600	–	425.2	195	0.08	0.00	2.2
	33	2200	–	1.9	59	0.03	0.07	0.6	73	172300	–	163.3	97	0.05	0.00	1.4
	34	35100	–	19.5	59	0.04	0.00	0.7	74	45600	–	42.9	89	0.08	0.00	1.2
	35	12700	–	8.8	59	0.03	0.00	0.6	75	297700	–	250.9	131	0.05	0.00	1.6
	36	6900	–	4.7	59	0.03	0.00	0.6	76	68600	–	66.3	1061	0.27	0.00	11.4
	37	8800	–	5.8	59	0.07	0.00	0.7	77	147300	–	114.1	151	0.03	0.00	2.0
	38	6400	–	4.4	59	0.03	0.19	0.7	78	318300	–	259.7	239	0.13	0.00	2.4
	39	3000	–	2.6	59	0.07	0.14	0.6	79	29900	–	27.4	79	0.03	0.00	1.2
	40	7300	–	4.9	1	0.00	0.00	0.1	80	254800	–	235.4	1	0.00	0.00	0.1

Numerical Results (cont.)

	$N = 50$								$N = 60$							
	MIP				Our Method				MIP				Our Method			
Demand	#	Nodes	%Δ	secs	Nodes	$\%\Delta_{LB}$	$\%\Delta_{UB}$	secs	#	Nodes	%Δ	secs	Nodes	$\%\Delta_{LB}$	$\%\Delta_{UB}$	secs
	81	2560500	4.22	–	1	0.00	0.00	0.3	121	1787500	7.39	–	1539	0.14	0.02	22.9
	82	2302100	3.00	–	50	0.02	0.00	2.8	122	1574200	6.29	–	1	0.00	0.00	0.3
	83	2624900	1.89	–	50	0.05	0.00	2.0	123	1693100	7.53	–	285	0.08	0.00	5.8
	84	2570800	5.89	–	50	0.05	0.07	2.0	124	1962100	6.26	–	741	0.06	0.00	13.3
$P2$	85	2249000	3.60	–	140	0.02	0.09	2.4	125	1408800	5.81	–	1	0.00	0.00	0.3
	86	2541500	4.55	–	1	0.00	0.00	0.2	126	2107000	5.75	–	331	0.04	0.00	7.6
	87	2651300	2.61	–	68	0.12	0.00	2.1	127	1499300	4.97	–	119	0.04	0.00	3.6
	88	2329900	4.22	–	95	0.02	0.00	3.2	128	1605800	5.12	–	259	0.04	0.00	5.6
	89	2323100	4.18	–	1	0.00	0.00	0.2	129	1742700	4.46	–	221	0.04	0.02	6.4
	90	2442800	3.42	–	50	0.09	0.05	2.0	130	1912800	6.40	–	1	0.00	0.00	0.3
	91	3752300	4.51	–	1	0.00	0.00	0.2	131	2341600	9.92	–	1	0.00	0.00	0.3
	92	2997800	6.99	–	50	0.02	0.00	3.4	132	2424800	10.77	–	197	0.08	0.00	4.6
	93	2747100	4.35	–	50	0.02	0.09	2.4	133	2251200	8.87	–	425	0.04	0.00	9.0
	94	2915200	3.36	–	1	0.00	0.00	0.2	134	2290900	6.85	–	1	0.00	0.00	0.3
$P3$	95	2792000	4.56	–	1	0.00	0.00	0.2	135	2119000	8.20	–	1	0.00	0.00	0.3
	96	2829500	6.62	–	1	0.00	0.00	0.1	136	2271400	8.13	–	1	0.00	0.00	0.3
	97	3074500	2.89	–	50	0.07	0.00	2.4	137	2058200	7.56	–	1	0.00	0.00	0.3
	98	2770200	3.59	–	68	0.02	0.00	3.3	138	2294200	7.30	–	1	0.00	0.00	0.3
	99	2845500	2.24	–	1	0.00	0.00	0.2	139	1763200	7.13	–	371	0.06	0.00	7.9
	100	3569400	6.02	–	73	0.09	0.02	3.9	140	1721200	7.50	–	119	0.02	0.04	4.0
	101	2231800	3.55	–	136	0.05	0.00	3.0	141	2027500	6.60	–	1	0.00	0.00	0.3
	102	2339800	3.69	–	50	0.05	0.02	2.1	142	1950800	6.52	–	1693	0.09	0.00	30.1
	103	1294100	–	2069.0	101	0.11	0.00	4.4	143	2087600	8.02	–	811	0.08	0.00	15.3
	104	2341100	4.28	–	164	0.12	0.00	7.8	144	2173100	7.88	–	669	0.08	0.00	11.9
$P4$	105	2403900	0.48	–	93	0.09	0.00	3.1	145	2013800	9.48	–	119	0.04	0.04	4.8
	106	2463500	1.71	–	50	0.04	0.00	2.9	146	1757500	5.86	–	339	0.10	0.04	7.2
	107	2540200	3.45	–	50	0.02	0.02	2.5	147	2174600	7.16	–	119	0.02	0.04	5.1
	108	2554800	3.50	–	82	0.05	0.09	3.4	148	1788200	7.59	–	1	0.00	0.00	0.3
	109	2329100	2.49	–	83	0.07	0.11	3.9	149	2046700	9.20	–	119	0.02	0.02	3.7
	110	2588600	5.08	–	216	0.07	0.00	5.0	150	1850700	7.72	–	119	0.04	0.02	4.2
	111	2802200	1.79	–	1	0.00	0.00	0.2	151	2083000	6.42	–	119	0.02	0.00	4.0
	112	2493700	0.19	–	95	0.04	0.00	2.9	152	2073300	6.09	–	227	0.02	0.00	5.3
	113	2435300	0.96	–	50	0.04	0.08	2.1	153	2038900	3.98	–	1	0.00	0.00	0.3
	114	3040700	2.37	–	1	0.00	0.00	0.2	154	1919300	7.65	–	125	0.02	0.00	3.2
$P5$	115	1692900	–	2310.3	79	0.06	0.06	3.2	155	2098800	5.83	–	207	0.03	0.02	5.0
	116	440700	–	669.6	50	0.04	0.00	1.9	156	1976300	5.84	–	231	0.05	0.03	4.5
	117	3125400	3.74	–	105	0.11	0.00	3.4	157	2294200	4.95	–	303	0.09	0.00	6.4
	118	937000	–	1375.7	96	0.10	0.00	2.5	158	1754900	5.19	–	189	0.02	0.00	4.7
	119	2892500	3.54	–	1	0.00	0.00	0.2	159	2007400	6.31	–	1	0.00	0.00	0.3
	120	1224000	–	1730.3	327	0.12	0.00	7.5	160	1891000	6.90	–	231	0.03	0.02	5.9