WORKING PAPER
MANSHOLT GRADUATE SCHOOL

On Simple Concrete Giffen Utility Functions: Old and New Results

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DISCUSSION PAPER No. 45
2009

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On Simple Concrete Giffen Utility Functions: Old and New Results.

by
WIM HEIJMAN AND PIERRE VON MOUCHE*

24 November 2008†

Abstract

We present a theoretical review of the literature on simple concrete continuous utility functions for Giffen and inferior goods within the context of the utility maximisation problem under a budget restriction and provide new functional forms. The article is organised around the specific properties such utility functions have. These properties include increasingness, quasi-concavity, and the applicability of Gossen's Second Law. (JEL: D11)

1 *Introduction*

Recently JENSEN AND MILLER [2008] have found strong evidence that for poor consumers in the Hunan province in China rice is a Giffen good. This is interesting, because it is the first empirical evidence for such goods. In fact all other 'examples' of Giffen goods, including the classic text book 'example' of the potato during the Irish famine, have been discredited because the data were not correctly interpreted or analysed. One of the problems is that price changes in the good that is supposed to be Giffen are often associated with budget changes, making it difficult to empirically isolate the Giffen effect (see NACHBAR, 1998). We will not take part into this discussion but, extra motivated by the result of Jensen and Miller, will deal with the complementary question concerning Giffen goods: to what extent are Giffen goods theoretically possible in the neoclassical framework of utility maximisation under a budget restriction.[1]

*We would like to thank R. Haagsma, W. Pijnappel and J. Rouwendal for their comments. For the upkeep of this article readers are invited to consult the home page of P. v. M.

†Revised version of 22 November 2007.

[1]But we use the opportunity to give here some more background on the empirical question. First we note that this question relates, contrary to the theoretical, to aggregate behaviour. Next we note that observed demand may in principle not be compatible with utility maximisation behaviour under a budget restriction. Reasons for the failing of empirical evidence have been suggested: for instance, aggregate demand is less likely to show Giffen behaviour than individual demand (see DOUGAIN, 1982 and references

It is generally believed that Giffen utility functions, i.e. utility functions for a Giffen good, are compatible with various reasonable properties of these functions, but rigorous proofs for such beliefs generally are lacking. The usual way to illustrate the existence of Giffen goods is by a graphical analysis for the case of two goods with a figure showing (two) indifference sets of the utility function and two budget lines. This is even the case in the advanced microeconomics text book of Mas-Colell, Whinston, and Green [1995]. But such figures do not tell much about the specific properties the utility function may have. What is needed is a simple Giffen utility function with the desired properties or just a proof that such a function exists. By 'simple' we just mean that proving that the function has the desired properties essentially can be done by hand. And for our purposes it is sufficient to agree that a function is 'concrete' when it is a (may be complicated) composition of standard functions; also piecewise definitions (like in Example 3) are allowed.

The popularity of graphical illustrations has to do with the fact that simple concrete Giffen utility functions with reasonable specific properties are not well-known. It may be noted here that in other settings of the utility maximisation problem, for example, the labour-leisure-model or in case if of other (subsistence) restrictions besides the budget restriction, it is easier to provide simple concrete Giffen utility functions with reasonable properties.[2]

Various authors dealt with the search for simple concrete Giffen utility functions. They considered two types of problems, depending on the specific reasonable properties such a function u (in the case of two goods) should have:

- u is continuous, strongly increasing[3] and quasi-concave. (The weak Giffen problem.)
- u is continuous and has the property that Gossen's Second Law is applicable. (The text book Giffen problem.)

Of course, besides the weak and text book Giffen problems one can formulate other Giffen problems. Natural is to ask for a simple concrete utility function that solves simultaneously the weak and text book Giffen problem:

- u simultaneously solves the weak and text book Giffen problem. (The strong Giffen problem.)

The weak Giffen problem was first solved by Wold [1948]. This was overlooked by other authors like Spiegel [1994] and Moffatt [2002], who also tried to solve

therein). Results for empirical evidence of inferior goods are more positive. It is known that such goods exist in abundance (at least when budgets are sufficiently high) in the real world. For example, in Baruch and Kannai [2001] evidence was found that the Japanese alcoholic beverage shochu is an inferior good and, more interestingly, that it may even be a candidate for a Giffen good. And McKenzie [2002] had 'good hope' that in Mexico after the 1995 Peso crisis tortillas are, for poor consumers, a Giffen good; but it turned out only to be inferior.

[2]For example in the labour-leisure-model a Leontief utility function $u(l, x) = \min(l, x)$ and budget restriction $wl + px \leq wT$, the demand function for leisure is strictly increasing. And Bergstrom and Varian [1996] give an example of a Giffen good in the context of subsistence restrictions: second-class carriage in the Orient Express for Agatha; see also Jensen and Miller [2008] for more on this setting.

[3]A real-valued function f on a subset A of $\mathbb{R}^n$ is strongly increasing if $f(\mathbf{x}) \leq f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in A$ with $x_i \leq y_i$ $(1 \leq i \leq n)$ and $f(\mathbf{x}) < f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in A$ with $x_i < y_i$ $(1 \leq i \leq n)$.

this problem.[4] In fact Spiegel solved the text book Giffen problem and conjectured that there does not exist a continuous quasi-concave strongly increasing Giffen utility function. This led Moffatt to refute Spiegel's conjecture by providing (a quite technical constrictive) proof that such a function does exist, without providing a concrete one. Recently a whole class of utility functions that solve the weak Giffen problem was given by SØRENSON [2006]. The strong Giffen problem still is not solved.

All this motivated us to review the literature on simple concrete Giffen utility functions and to present the explicit formula's of such functions. Because various articles that we review contain fallacies we decided to make the review self-contained and to provide in the appendix (new more modern) proofs for the claims made. Finally, we provide a new example that solves the strong Giffen problem with an arbitrary prescribed precision.

According to economic theory there is a relationship between Giffen goods and inferior goods. Under certain conditions, a Giffen good is an inferior good. Therefore the article also will deal with simple concrete utility functions for inferior goods.

We emphasise that there is no universally adopted definition of both a Giffen good and an inferior good. Definitions may differ in the details. Therefore it is important to clarify these notions and how they are related. This will be done in Section 2. Sections 3–6 then present the old and new simple concrete Giffen utility functions and Section 7 concludes.

2 Setting

The setting for the utility maximisation problems we deal with always is the following standard one: by a utility function we will understand a function $u : \mathbb{R}^n_+ \to \mathbb{R}$. And given a utility function u, (budget) $m \geq 0$ and (prices) $\mathbf{p} \in \mathbb{R}^n_{++}$, we consider the maximisation problem of u under the (budget) restriction $\mathbf{p} \cdot \mathbf{x} \leq m$. If u is upper semi-continuous, then each utility maximisation problem has at least one maximiser. Sufficient for that there is at most one maximiser is that u is strictly quasi-concave.

Given (a good) i, a budget and prices of the other goods, let I be the subset of prices p_i of good i for which the utility maximisation with this budget and these prices has a unique maximiser $\breve{\mathbf{x}}(p_i)$. By the demand function of i we understand the i-th component function $\breve{x}_i : I \to \mathbb{R}$. And given prices $\mathbf{p}$, let J be the subset of budgets m for which the utility maximisation with this budget and these prices has a unique maximiser $\hat{\mathbf{x}}(m)$. By the Engel function of i we understand the i-th component function $\hat{x}_i : J \to \mathbb{R}$.

In Theorem 1 below, we will use that each demand function $\breve{x}_i : I \to \mathbb{R}$ has the

[4] The reason why why Wold's article fell into oblivion may be due to his book WOLD AND JUREEN [1953] that did not contain the result of the article.

following property:[5] for all $p_i, {p_i}' \in I$ with $p_i < {p_i}'$ one has

$$\breve{x}_i(p_i) = 0 \;\Rightarrow\; \breve{x}_i({p_i}') = 0. \tag{1}$$

Definition 1 *We say that good i is <u>Giffen</u> if it has a demand function that is strictly increasing on a subset $\mathcal{P}$ of its domain that contains at least two points. In this case we more precisely say that good i is Giffen on $\mathcal{P}$. And we say that good i is <u>inferior</u> if it has an Engel function that is strictly decreasing on a subset of its domain that contains at least two points. In this case we more precisely say that good i is inferior on $\mathcal{M}$. We say that u is <u>Giffen</u> if there is a good that is Giffen and that u is <u>inferior</u> if there is a good that is inferior.* $\diamond$

We will refer in these definitions to $\mathcal{P}$ as <u>price section</u> and to $\mathcal{M}$ as <u>budget section</u>.

Note that Definition 1 is not only very natural but also very friendly for a utility function to be Giffen or inferior. In particular continuity or quasi-concavity of the utility function is not assumed. However, it is very easy to give, in case $n \geq 2$, a simple concrete discontinuous Giffen (inferior) utility function.[6] However, for $n = 1$ there neither exists a Giffen nor an inferior utility function as easily can be shown. Providing a simple concrete continuous utility function (in case $n = 2$) already is a little bit puzzling. Of course, more demanding examples are given below.

It is simple to prove that if good i is Giffen on a price section $\mathcal{P}$, then this price section necessarily is bounded. Also observe that the set $\mathcal{M}$ cannot contain 0 if a good is inferior on $\mathcal{M}$. Even: if good i is inferior on a budget section $\mathcal{M}$, then $\mathcal{M}$ is bounded below by a positive budget level. This does not exclude (even for continuous utility functions, as Example 5 shows) that a good may be inferior on an unbounded budget section. One should be careful by the text book wisdom that each Giffen good is inferior. Theorem 1 provides quite weak sufficient conditions for a Giffen good to be inferior.

Notation: above we have defined $\hat{\mathbf{x}}(m) \in \mathbb{R}^n_+$; $\hat{x}(m)$ there was supposed to be unique. Now, given prices, denote for each budget $m \geq 0$ by $\hat{\mathbf{x}}(m)$ the set of maximisers of the utility maximisation problem for these prices and m and let $\hat{x}_i(m) := \{x_i \mid \mathbf{x} \in \hat{\mathbf{x}}(m)\}$.

Theorem 1 *Suppose $u : \mathbb{R}^n_+ \to \mathbb{R}$ is upper semi-continuous and i is Giffen on a price section $\mathcal{P}$. Then there exists a price p'_i of good i and budgets m, m' with $m < m'$ such that $\hat{x}_i(m) > \hat{x}_i(m')$.*[7] *In particular, if for each price in $\mathcal{P}$ and each budget*

[5] We can not give a reference from the literature for this property, so we provide here a proof: we may assume that $i = 1$ and now prove the statement by contradiction. So suppose there is $p_1 \in \mathcal{P}$ with $\breve{x}_1(p_1) = 0$ and suppose ${p_1}' \in \mathcal{P}$ with $p_1 < {p_1}'$ and let $p_2, \dots, p_n$ the prices of the other goods and m the budget. Let $\mathbf{p} = (p_1, p_2, \dots, p_n)$, $\mathbf{p}' = ({p_1}', p_2, \dots, p_n)$, $\mathbf{x}$ the unique maximiser for $\mathbf{p}$ and m and $\mathbf{x}'$ the unique maximiser for $\mathbf{p}'$ and m. Now $x_1 = \breve{x}_1(p_1) = 0$ and therefore $\mathbf{p}' \cdot \mathbf{x} = \mathbf{p} \cdot \mathbf{x} \leq m$. Also $\mathbf{p} \cdot \mathbf{x}' \leq \mathbf{p}' \cdot \mathbf{x}' \leq m$. Because $\mathbf{x}'$ is unique, $u(\mathbf{x}) < u(\mathbf{x}')$ holds. And because $\mathbf{x}$ is unique, $u(\mathbf{x}') < u(\mathbf{x})$. This is a contradiction.

[6] Draw the two budget lines, define the value of the utility function to be zero outside these lines and in an appropriate way on the lines.

[7] This means that each element of the set $\hat{x}_i(m)$ is larger than each element of the set $\hat{x}_i(m')$.

each utility maximisation problem has a unique maximiser, then good i is inferior (on $\{m, m'\}$). $\diamond$

Proof.— Suppose good i is Giffen on the price section $\{p_i, p_i'\}$ with $p_i < p_i'$. Now $\breve{x}_i(p_i') > 0$. By (1), one has $\breve{x}_i(p_i) \neq 0$. Let m' be the budget for calculating the substitution and income effect (à la Slutsky): $m' - m = \breve{x}_i(p_i)(p_i' - p_i)$. One has $m < m'$. Let $\mathbf{x}''$ be a maximiser for the utility maximisation problem for budget m' and price p_i' of good i. Now

$$0 < \breve{x}_i(p_i') - \breve{x}_i(p_i) = (\breve{x}_i(p_i') - x_i'') + (x_i'' - \breve{x}_i(p_i)).$$

The second term is the substitution effect. This effect is non-positive (upper semi-continuity is sufficient for that as one can verify). Therefore $\breve{x}_i(p_i') > x_i''$. Q.E.D.

It is interesting to note that it is not clear at all whether each Giffen good always is inferior.

From now on we only will deal with the case where there are two goods, *i.e.* where $n = 2$. For this case, a utility maximisation problem with merely budget-balanced maximisers, *i.e.* maximisers $\mathbf{x}$ that satisfy $\mathbf{p} \cdot \mathbf{x} = m$, essentially comes down to maximising the function of one variable $U : [0, m/p_1] \to \mathbb{R}$ defined by

$$U(x_1) := u(x_1, \frac{m - p_1 x_1}{p_2}). \tag{2}$$

3 Text book Giffen problem

An elementary result in microeconomics is that under weak conditions, a maximiser of the utility maximisation problem, given prices (p_1, p_2) and a positive budget m, solves the following system of 2 equations in the 2 unknowns $x_1, x_2 \in \mathbb{R}_{++}$:[8]

$$p_1 x_1 + p_2 x_2 = m, \tag{3}$$

$$\frac{\partial u}{\partial x_2}(x_1, x_2)/p_2 = \frac{\partial u}{\partial x_1}(x_1, x_2)/p_1. \tag{4}$$

Indeed, each maximiser, that is interior and in which u is differentiable, solves (4) and thus is a solution of the system if it is also budget balanced. (3) is the budget-balance restriction and the equation in (4) is also sometimes referred to as Gossen's Second Law. And sufficient conditions for a solution (x_1, x_2) of the system to be a maximiser are:

u is continuous, quasi-concave and in (x_1, x_2) differentiable with positive partial derivatives (Utility Maximisation Rule).

If the Utility Maximisation Rule can not be applied, then, of course, it still might be possible that a solution of (3) and (4) is a maximiser.

By the text book Giffen problem we understand the following problem: give a simple concrete utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$, a price section $\mathcal{P}$ of good 1 and $p_2, m > 0$ such that

[8] Of course, (4) here only is present for $(x_1, x_2) \in \mathbb{R}^2_{++}$ in which u is partially differentiable.

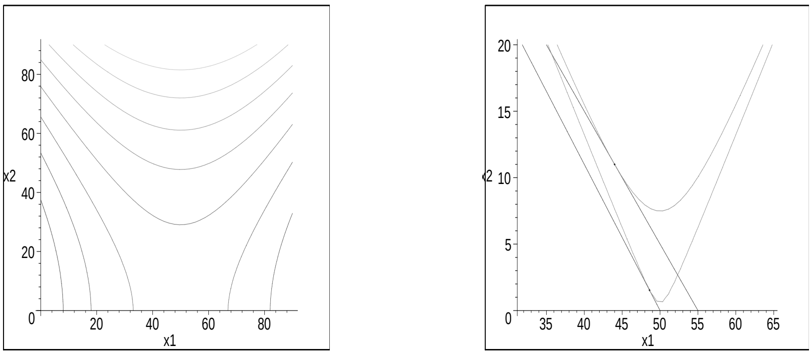

Figure 1: Indifference sets of u_1^T (left), and u_1^T is Giffen (right).

- good 1 is Giffen on $\mathcal{P}$;
- u is continuous, and on $\mathbb{R}^2_{++}$ is partially differentiable;
- for each $p_1 \in \mathcal{P}$ the unique maximiser of the maximisation problem involved is the unique solution of the system of equations (3) and (4).[9]

A modification of a utility function in Spiegel [1994] (also see Weber [1997]) solves the text book Giffen problem. The next example deals with this modified function.[10]

Example 1 *Let*

$$u_1^T(x_1, x_2) := -x_1^2 + \alpha x_1 + x_2 + \frac{x_2^2}{2},$$

where $\alpha = 100$. u_1^T *solves the text book Giffen problem. More precisely: fix* $p_2 = 1$ *and* $m \in (1, \frac{\alpha}{\sqrt{2}} - 1)$. *For* $p_1 \in \mathcal{P} = (\frac{\alpha}{m+1} - \sqrt{\frac{\alpha^2}{(m+1)^2} - 2},\ \frac{\alpha}{2} - \frac{1}{2}\sqrt{\alpha^2 - 8m})$, *the utility maximisation problem has a unique maximiser.*[11] *This maximiser is budget-balanced, interior, and given by*

$$\breve{\mathbf{x}}(p_1) = (\frac{-(m+1)p_1 + \alpha}{2 - p_1^2},\ \frac{p_1^2 - \alpha p_1 + 2m}{2 - p_1^2}).$$

Good 1 *is Giffen on the price section* $\mathcal{P}$. $\diamond$

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[9] One could wish to add here 'and that this system easily can be solved by hand', because if it is difficult to solve the system of equations, the utility function is not really appropriate for text book purposes.

[10] In fact Spiegel took $p_2 = 1$, $m = 55$ and $\mathcal{P} = \{1, 1.1\}$ and used a 'repaired' version of our u_1^T to improve the properties of the indifference sets. The repair consisted on modifying u_1^T for $x_1 > 50$. But his version only improved these properties a little bit (the modified function still is not quasi-concave) and lead to a piecewise definition, which made this function more complicated and thereby destroyed the partially differentiability of the utility function (on $\mathbb{R}^2_{++}$). Therefore Spiegel's original utility function is, strictly speaking, not a solution of the text book Giffen problem.

[11] Note that for small m a maximiser may be not unique, as can be seen from Figure 1 by looking to its south west corner.

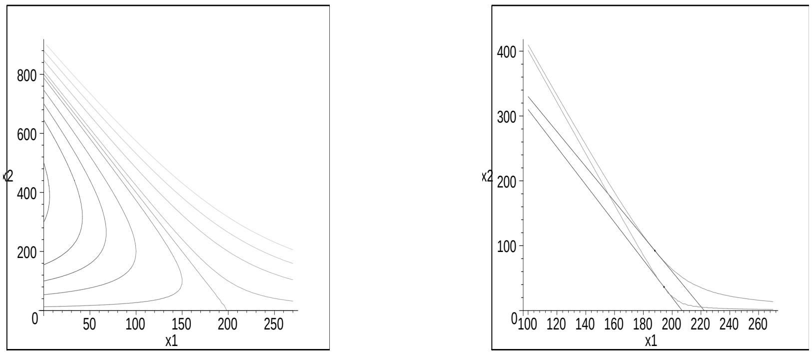

Figure 2: Indifference sets of u_2^T (left), and u_2^T is Giffen (right).

Some indifference sets of u_1^T, *i.e.* level sets, are given in Figure 1. It may be interesting to note that Example 1 shows that even additive separable functions may be Giffen. Figure 1 (where $\mathcal{P} = \{1, 11/10\}$ and $m = 55$) shows[12] that u_1^T is Giffen (by providing the two relevant indifference sets and budget lines). For this example, the analysis is easy, but proving our result is more complicated because we have to prove that $\breve{x}_1$ is strictly increasing on the whole $\mathcal{P}$ instead of only on a two points set.

As explained in footnote 10, Spiegel's original function does not solve the text book problem. But our modification of Spiegel's function does. Spiegel's function as well as our modified one have not so nice indifference sets as can be seen from Figure 1 and the figure in Spiegel's article. After a long time of puzzling we found the following, like u_1^T also quadratic, utility function that also solves the text book Giffen problem and that is, as we will explain, superior to u_1^T:

Example 2 *Let*

$$u_2^T(x_1, x_2) := x_2(x_2 + 4x_1 - a),$$

where $a > 0$. u_2^T solves the text book Giffen problem. More precisely: fix $p_2 = 1$ and $m \in (0, a)$. For $p_1 < \frac{4m}{a}$, the utility maximisation problem has a unique maximiser. This maximiser is budget-balanced, interior, and given by:

$$\tilde{\mathbf{x}}(p_1) = \left(\frac{4m - (2m - a)p_1}{2p_1(4 - p_1)}, \frac{4m - ap_1}{2(4 - p_1)}\right).$$

If $m > \frac{a}{2}$, then good 1 is Giffen on the price section $\mathcal{P} = \left(\frac{4m - 4\sqrt{am - m^2}}{2m - a}, \frac{4m}{a}\right)$. $\diamond$

Figure 2 gives some indifference sets of u_2^T (for $a = 800$) and illustrates for this case that u_2^T is Giffen for the situation $m = 600$ and $\mathcal{P} = \{27/10, 29/10\}$. Understanding by a <u>good region</u> of a utility function, a non-empty convex subset of $\mathbb{R}^2_+$ with positive measure (may be infinity) where the function is strictly quasi-concave and strongly increasing, u_2^T is superior to u_1^T in the sense that it has (as may be clear from the left panel in Figure 2), contrary to u_2^T a good region of a very simple form:

$$D := \{(x_1, x_2) \in \mathbb{R}^2_+ \mid x_2 > 0 \text{ and } x_2 + 4x_1 - a > 0\}. \tag{5}$$

[12] Please note the scaling-down on the abscissa.

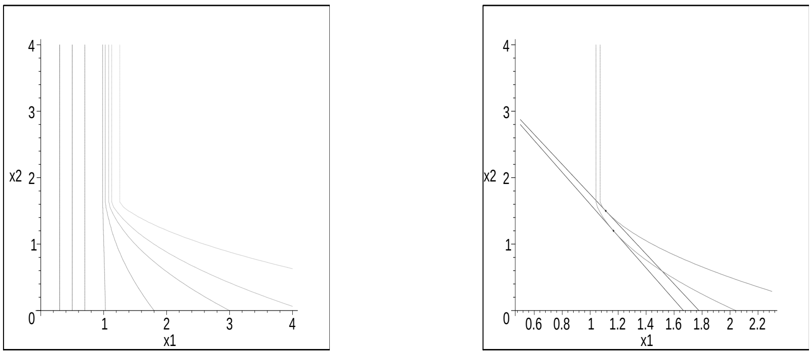

Figure 3: Indifference sets of u_1^W (left), and u_1^W is Giffen (right).

4 Weak Giffen problem

By the weak Giffen problem we understand the following problem: give a simple concrete utility function $u : \mathbb{R}_+^2 \to \mathbb{R}$, a price section $\mathcal{P}$ of good 1 and $p_2, m > 0$ such that

- good 1 is Giffen on $\mathcal{P}$;
- u is continuous, strongly increasing and quasi-concave.

Spiegel [1994] (who solved the text book Giffen problem) could not solve the weak Giffen problem, even conjectured that it does not have a solution. In fact the weak Giffen problem already many years before was solved in Wold [1948]:

Example 3 *Let*

$$u_1^W(x_1, x_2) := \begin{cases} x_1 - 1 \text{ if } x_1 \leq 1, \\ \frac{x_1 - 1}{(x_2 - 2)^2} \text{ if } x_1 > 1 \text{ and } x_2 \leq 8/5, \\ \frac{25}{4}(x_1 - 1) \text{ if } x_1 > 1 \text{ and } x_2 > 8/5. \end{cases}$$

u_1^W solves the weak Giffen problem. More precisely: each utility maximisation problem has a unique maximiser and this maximiser is budget-balanced.[13] *Let $m > \frac{9}{5}p_2$. For $p_1 \in \mathcal{P} = (m - \frac{9}{5}p_2,\ m - p_2)$, the maximiser is*

$$\breve{\mathbf{x}}(p_1) = (2 - \frac{m - 2p_2}{p_1},\ -2 + \frac{2(m - p_1)}{p_2}).$$

In case $m > 2p_2$, good 1 is Giffen on the price section $\mathcal{P}$. ⋄

Figure 3 gives some indifference sets of u_1^W and illustrates for this case that u_1^W is Giffen for the situation $m = 4$, $p_2 = 1$ and (the two points set) $\mathcal{P} = \{45/20, 48/20\}$.

Analysing the with u_1^W associated function U (given by (2)) here is straightforward but a little technical because of the piece-wise definition of U. The more efficient

[13] But not necessarily interior for positive budget.

proof we give (in the appendix) uses some theoretical arguments and the Utility Maximisation Rule from Section 3.

Note that in Example 3, the demand function $\breve{x}_1$ is strictly concave on $\mathcal{P}$. This is very nice, because demand functions that are strictly concave on some non-empty interval seem to be rare. Also nice is that the demand function $\breve{x}_1$ even is well-defined for all prices p_1. The reader may check that in case $m = 4$ and $p_2 = 1$ (by analysing U outside $\mathcal{P}$) that it is given by

$$\breve{x}_1(p_1) = \begin{cases} \frac{12}{5p_1} \; (p_1 \leq 11/5), \\ \frac{2p_1 - 2}{p_1} \; (11/5 < p_1 < 3), \\ \frac{4}{p_1} \; (p_1 \geq 3). \end{cases}$$

One problem with piecewise defined utility functions is that such functions often are not differentiable (on $\mathbb{R}^2_{++}$) which makes that such a function can not solve the text book Giffen problem.

A whole class of other utility functions that also solve the weak Giffen problem was given by SØRENSON [2006]. The interesting idea in his article is to use Leontief-like utility functions u of the form

$$u(x_1, x_2) = \min(f_1(x_1, x_2),\ f_2(x_1, x_2)) \tag{6}$$

with a widened angle at the indifference set kink. It is not so clear who first had this idea (also see DE JAEGHER, forthcoming). Possible economic interpretations of such composed utility functions can be given using the notion of characteristics of goods; see LIPSEY AND ROSENBLUTH [1971]. Due to the min-function, the utility functions here are also not partially differentiable – in fact the piece-wise definition aspect remains. We present here only one element of his class:

EXAMPLE 4 *Let*

$$u^W_2(x_1, x_2) := \min(x_2 + B, A(x_1 + x_2)),$$

where $A > 1$ *and* $B > 0$. u^W_2 *solves the weak Giffen problem. More precisely: suppose* $0 < \frac{m(A-1)}{B} < p_2$. *Then for* $p_1 \in \mathcal{P} = (0, \min(m\frac{A}{B}, p_2))$, *the utility maximisation problem has a unique maximiser. This maximiser is budget-balanced, interior, and given by*

$$\breve{\mathbf{x}}(p_1) = (\frac{Bp_2 - m(A-1)}{Ap_2 - p_1(A-1)}, \frac{Am - Bp_1}{Ap_2 - p_1(A-1)}).$$

Good 1 *is Giffen on the price section* $\mathcal{P}$. $\diamond$

Although u^W_2 is less complicated than u^W_1, u^W_1 has the advantage that each utility maximisation problem has a unique maximiser. Figure 4 gives some indifference sets of u^W_2 for $A = 2$ and $B = 10$ and illustrates for this case that u^W_2 is Giffen for the situation $m = 5$, $p_2 = 1$ and (the two points set) $\mathcal{P} = \{1/3, 3/4\}$.

In Section 3 we have seen that the text book Giffen problem already can be solved with a quadratic utility function. However, we do not know whether there is a quadratic utility function that solves the weak Giffen problem.

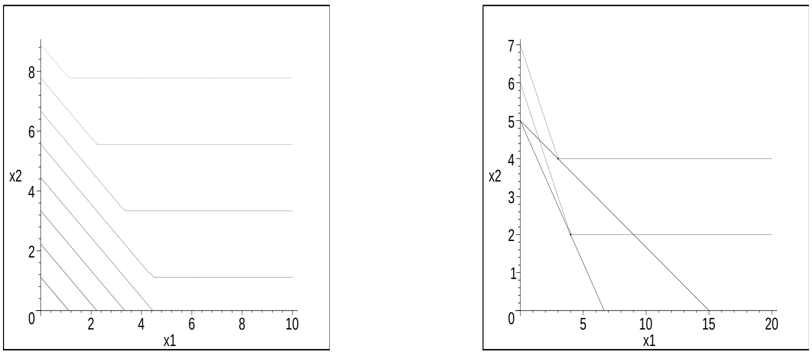

Figure 4: Indifference sets of u_2^W (left), and u_2^W is Giffen (right).

5 *Strong Giffen problem*

No utility function in the literature solves the strong Giffen problem. However, when we do not impose the conditions that the utility function should be concrete and simple, an abstract argument of Sørenson [2006] guarantees the existence of such a function: his natural idea is to approximate the function $\min(z_1, z_2)$ in u_2^W by the (strictly concave) ces-function $(z_1^\rho + z_2^\rho)^{1/\rho}$ for $\rho \to -\infty$. For solving the proper strong Giffen problem one has to work out this idea. This indeed can be done with success (von Mouche, 2008); the major problem is to handle the condition that the function should be simple.

Here now, we return to our utility function u_2^T which solves the text book Giffen problem. The interesting observation is, that its definition is such that the measure of the complement of its good region D (given by (5)), is $a^2/8$ and that thus this measure may be made arbitrary small. The conclusion is that given an arbitrary small positive number ϵ, the utility function u_2^T (for $a = \sqrt{8\epsilon}$) not only solves the text book Giffen problem but also the strong Giffen problem 'modulo a bad region of its domain with arbitrary small positive measure'.[14]

6 *Inferiority problems*

We have defined three types of Giffen problems: the weak, text book and strong Giffen problem. By modifying the definitions of these notions in an obvious way we obtain the corresponding three types of inferiority problems.

Theorem 1 guarantees that u_1^W also solves the weak inferiority problem and that u_1^S also solves the strong inferiority problem. By inspecting the parameter conditions on p_1, p_2, m in the examples on u_1^T and u_2^T, one sees that they solve the text book inferiority problem and that u_2^W solves the weak inferiority problem. Concerning (other) simple concrete inferior utility functions we found only the following relevant

[14] Furthermore it may be worthwhile to note that in Example 2 the maximisers are in the good region D (as the proof of the example shows).

example from LIEBHAFSKY [1968]:[15]

EXAMPLE 5 *Let*

$$u_1^I(x_1, x_2) := x_1^\alpha e^{x_2^2/2},$$

where $\alpha > 0$. u_1^I *is inferior. More precisely: for* $p_2 > 0$ *and* $m > 2p_2\sqrt{\alpha}$, *let*

$$y_\pm := \frac{m \pm \sqrt{m^2 - 4p_2^2\alpha}}{2}, \quad t(m) := \left(\frac{y_-}{m}\right)^\alpha e^{\frac{y_+^2}{2p_2^2}} - 1.$$

- *If* $m \leq 2p_2\sqrt{\alpha}$, *then* $\hat{\mathbf{x}}(m) = (m/p_1, 0)$ *is the unique maximiser of the utility maximisation problem.*
- *If* $m > 2p_2\sqrt{\alpha}$ *and* $t(m) \neq 0$, *then the utility maximisation problem has a unique maximiser. This maximiser is budget-balanced and given by*

$$\hat{\mathbf{x}}(m) = \begin{cases} (y_-/p_1, y_+/p_2) \text{ if } t(m) > 0, \\ (m/p_1, 0) \text{ if } t(m) < 0. \end{cases}$$

- *If* $m > 2p_2\sqrt{\alpha}$ *and* $t(m) = 0$, *then* $\{(y_-/p_1, y_+/p_2),\ (m/p_1, 0)\}$ *is the set of maximisers.*

Finally, given $\alpha, p_1, p_2 > 0$, *there exists* $m_0 > 2p_2\sqrt{\alpha}$ *such that good* 1 *is inferior on the budget section* $[m_0, \infty)$. $\diamond$

Figure 5 gives some indifference sets of u_1^I in case $\alpha = 1/10$ and illustrates for this case that u_1^I is inferior by for the situation $p_1 = 1, p_2 = 1$ and $\mathcal{M} = \{1, 2\}$. u_1^I is not quasi-concave and therefore does not solve the weak inferiority problem. In order to show that u_1^I does not solve the text book inferiority problem either, we consider the system of equations (3) and (4). They only have a solution in case $m \geq 2p_2\sqrt{\alpha}$. In case $m = 2p_2\sqrt{\alpha}$, the unique solution is $(\frac{m}{2p_1}, \frac{m}{2p_2})$, but the result in Example 5 shows that this is not a maximiser of the utility maximisation problem. And in case $m > 2p_2\sqrt{\alpha}$, there is not a unique solution, but there are two different solutions: $(y_-/p_1, y_+/p_2)$ and $(y_+/p_1, y_-/p_2)$.

7 Conclusion

Depending on the specific properties a Giffen utility function should have, finding a simple concrete one may not be so easy (if possible at all), although verification that a given function has the desired properties is in general straightforward enough. The situation is similar to finding an antiderivative of a function: finding one may be difficult, but once found its verification is in general easily done.

With respect to the specific properties of a Giffen utility function we introduced the weak, text book and strong Giffen problem. We reviewed the literature by

[15] In fact LIEBHAFSKY [1968] does not prove the correctness of his example and refers for that to a book of him where the analysis then is not correct. He states that the formula for $\hat{\mathbf{x}}(m)$ in Example 5 for the case $m > 2p_2\sqrt{\alpha}$ and $t(m) > 0$ also is correct in case $t(m) \leq 0$. However, for $\alpha = 1/10,\ p_1 = 1,\ p_2 = 1,\ m = \frac{21}{10}p_2\sqrt{a}$, one has $t(m) < 0$ which (now referring to our proof in the appendix) implies $U(m/p_1) > U(y_-/p_1)$ from which it follows that $(0, m/p_1)$ is the unique maximiser instead of $(y_-/p_1, y_+/p_2)$.

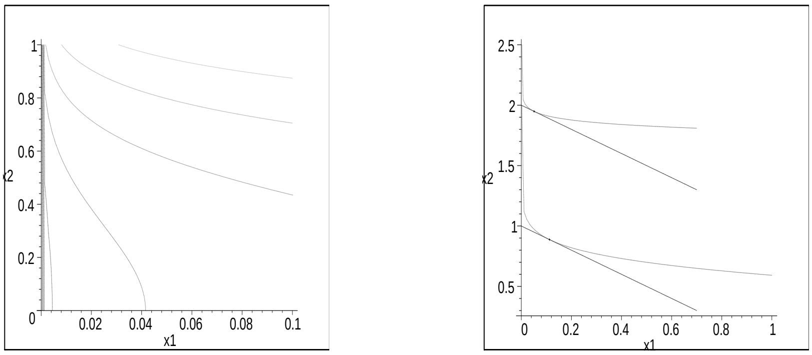

Figure 5: Indifference sets of u_1^I (left), and u_1^I is inferior (right).

providing the relevant examples of simple concrete utility functions (of Wold [1948], Spiegel [1994] and Sørenson, 2006) that concern these problems. The example of Spiegel had to be modified to solve the text book Giffen problem in the proper sense; we also extended his example by providing a price section with more than only two points on which the good is Giffen. For Wold's example we provided a new short proof. Also, with our utility function u_2^T, we provided a more simple new solution of the text book Giffen problem.

The strong Giffen problem has been solved by Sörenson when we ignore the condition that the utility function should be concrete and simple. However, our simple concrete utility function u_2^T not only solves the text book Giffen problem but at the same time (by adjusting a parameter) the strong Giffen problem 'modulo a bad region of the domain of the utility function' with an arbitrary small positive measure.

We provided quite weak sufficient conditions for a Giffen good to be inferior. But it is not clear whether each Giffen good is inferior. Concerning inferior utility functions we reviewed an example of Liebhafsky [1968] and showed that his result only holds for quite specific values of the prices and budget.

A *Proofs of the examples*

A.1 *Proof of Example 1*

Since u_1^T is strictly increasing in x_2, each maximiser is budget-balanced. Fix $p_2 = 1$ and $1 \leq m \leq \frac{\alpha}{\sqrt{2}} - 1$. Now

$$c_1^{\pm}(m) := \frac{\alpha}{m+1} \pm \sqrt{\frac{\alpha^2}{(m+1)^2} - 2}, \quad c_2(m) := \frac{\alpha}{2} - \frac{1}{2}\sqrt{\alpha^2 - 8m}$$

are well defined real numbers, and for $m \neq 1$ one has that $0 < c_1^-(m)$ and $c_2(m) < \sqrt{2}$. Also one has $c_1^-(m) < c_2(m)$,[16] thus $\mathcal{P} = (c_1^-(m), c_2(m))$ is a price section.

[16]Here is a proof: let f be the function $c_1^- - c_2$. Suppose m is a zero of f. Then $(\alpha/(m+1) - \alpha/2)^2 = \left(\sqrt{\alpha^2/(m+1)^2 - 2} - \sqrt{(\alpha^2 - 8m)/2}\right)^2$. Evaluating this equation, next simplifying and then squaring

Now consider the function U given by (2) where $p_1 \in \mathcal{P}$:

$$U(x_1) = (\frac{1}{2}p_1^2 - 1)x_1^2 + (\alpha - (m+1)p_1)x_1 + \frac{1}{2}m^2 + m.$$

U is a quadratic concave function. Considered as a function on the whole $\mathbb{R}$ it has

$$\breve{x}_1(p_1) = \frac{-(m+1)p_1 + b}{2 - p_1^2}$$

as unique maximiser. However, the inequalities $m < \frac{\alpha}{\sqrt{2}} - 1$ and $p_1 < \sqrt{2}$ imply that $\breve{x}_1(p_1) > 0$ and $p_1 < c_2(m)$ implies that $\breve{x}_1(p_1) < \frac{m}{p_1}$. Thus we can conclude that $\breve{x}_1(p_1)$ is a maximiser of U. Its derivative is

$$\breve{x}_1'(p_1) = -\frac{m+1}{(p_1^2-2)^2}(p_1^2 - \frac{2\alpha}{m+1}p_1 + 2)$$

$$= -\frac{m+1}{(p_1^2-2)^2}(p_1 - (\frac{\alpha}{m+1} + \sqrt{\frac{\alpha^2}{(m+1)^2} - 2}))(p_1 + (\frac{\alpha}{m+1} - \sqrt{\frac{\alpha^2}{(m+1)^2} - 2}))$$

$$= -\frac{m+1}{(p_1^2-2)^2}(p_1 - c_1^+(m))(p_1 - c_1^-(m))$$

and we see that $\breve{x}_1'(p_1) > 0$ on $\mathcal{P}$, as desired.

In order to see that u_1^T solves the text book Giffen problem, one finally has to check simply that $\breve{\mathbf{x}}(p_1)$ is the unique solution of the system of equations (3) and (4). Q.E.D.

A.2 Proof of Example 2

Suppose $p_2 = 1$. For the function U given by (2) we have

$$U(x_1) = (m - p_1x_1)((m - p_1x_1) + 4(x_1 - \frac{a}{4})) = (m - p_1x_1)((4 - p_1)x_1 + m - a).$$

In case $m/p_1 > a/4$, we see there is a left punctured neighbourhood of m/p_1 where $U(x_1)$ is positive. And in case $m > a$, we see that there is a right punctured neighbourhood of 0 where $U(x_1)$ is positive. This implies that in both these cases the maximum of the utility maximisation problem is positive. This in turn implies that in these both cases for each maximiser $(x_1^\star, x_2^\star)$ of the utility maximisation problem one has $x_2^\star > 0$ and $x_2^\star + 4x_1^\star - a > 0$ and therefore $(x_1^\star, x_2^\star) \in D$ (with D given by (5)). Because on D the utility function is strongly increasing, it follows that $(x_1^\star, x_2^\star)$ is budget-balanced and therefore that we can use U also for further analysis in these cases.

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the result after appropriate rearranging, leads to $2m^4 - (\alpha^2+4)m^2 + 2m\alpha^2 - \alpha^2 + 2 = 0$. Because $2m^4 - (\alpha^2+4)m^2 + 2m\alpha^2 - \alpha^2 + 2 = 2(m-1)^2(m - \alpha/\sqrt{2} - 1)(m + \alpha/\sqrt{2} + 1)$, we can conclude that the zeros of f are 1 and $\alpha/\sqrt{2} - 1$. Noting that $f'(1) = (\sqrt{\alpha^2 - 8} - a) < 0$ and that f is continuous, it follows that $f < 0$ on the interval $(1, \alpha/\sqrt{2} - 1)$, as desired.

It would be interesting to understand the deeper meaning why the inequality $c_1^-(m) < c_2(m)$, that in fact came up in a natural way, holds.

Now suppose further that $m/p_1 > a/4$ and $m < a$. Noting that

$$U(x_1) = (p_1^2 - 4p_1)x_1^2 + (4m - 2mp_1 + p_1 a)x_1 + m^2 - ma,$$

studying U quickly gives that the utility maximisation problem has a unique maximiser $\breve{\mathbf{x}}(p_1)$ as desired. We have

$$\frac{d\breve{x}_1}{dp_1}(p_1) = -(m - \frac{a}{2})\frac{(p_1 - \frac{4m+4\sqrt{am-m^2}}{2m-a})(p_1 - \frac{4m-4\sqrt{am-m^2}}{2m-a})}{p_1^2(p_1-4)^2}.$$

Now further supposing that $m > a/2$, on $\mathcal{P}$ this derivative is positive and thus good 1 is Giffen on $\mathcal{P}$.

In order to see that u_2^T solves the text book Giffen problem one has to check simply that $\breve{\mathbf{x}}$ is the unique solution of the system of equations (3) and (4). Q.E.D.

A.3 Proof of Example 3

It is straightforward to check that u_1^W is strongly increasing and continuous. Quasi-concavity of u_1^W may now be clear from the shape of the upper level sets in Figure 3; we omit a formal proof. Because u_1^W is strongly increasing each maximiser is budget-balanced. Next we prove by contradiction that each utility maximisation problem has a unique maximiser. So suppose $\mathbf{a}, \mathbf{b}$ would be two different maximisers. Because u_1^W is quasi-concave, the set of maximisers is convex and therefore each point on the segment with $\mathbf{a}$ and $\mathbf{b}$ as end points would be a maximiser too. Because all these points are budget-balanced, we would have a non-vertical line segment on which u is constant, which clearly is (by Figure 3) not the case.

Next, let $m > \frac{9}{5}p_2$, and consider the utility maximisation problem for $p_1 \in \mathcal{P}$. Let $\breve{\mathbf{x}}(p_1)$ as given. Then $p_1\breve{x}_1(p_1) + p_2\breve{x}_2(p_1) = m$. Because $p_1 \in \mathcal{P}$ it follows that $0 < \breve{x}_1(p_1) < m/p_1$. Therefore $\breve{\mathbf{x}}(p_1) \in \mathbb{R}^2_{++}$ and thus $\breve{\mathbf{x}}(p_1)$ is a solution of (3). Noting that also $\breve{x}_1(p_1) > 1$ and $\breve{x}_2(p_1) < 8/5$, u is in $\breve{\mathbf{x}}(p_1)$ differentiable and one easily checks that $\breve{\mathbf{x}}(p_1)$ also is a solution of (4), and even that $\frac{\partial u}{\partial x_2}/p_2 = \frac{\partial u}{\partial x_1}/p_1 > 0$. Because u is quasi-concave, the Utility Maximisation Rule guarantees that $\breve{\mathbf{x}}(p_1)$ is a maximiser. If $m > 2p_2$, then u_1^W is Giffen on $\mathcal{P}$. Q.E.D.

A.4 Proof of Example 4

Of course, u_2^W is continuous and strongly increasing. Therefore each utility maximisation problem only has budget-balanced maximisers. And u_2^W is (quasi-) concave because it is a minimum of two concave functions.

For the utility maximisation problem we study again the function U defined by (2). We have

$$U(x_1) = \min\left(-\frac{p_1}{p_2}x_1 + \frac{m}{p_2} + B,\ A(1 - \frac{p_1}{p_2})x_1 + \frac{Am}{p_2}\right).$$

Further let $0 < m\frac{A-1}{B} < p_2,\ p_1 \in \mathcal{P} = (0, \min(m\frac{A}{B}, p_2))$. Now $0 < \frac{Bp_2 - m(A-1)}{Ap_2 - p_1(A-1)} < \frac{m}{p_1}$, $p_1 < p_2$ and

$$U(x_1) = \begin{cases} A(1 - \frac{p_1}{p_2})x_1 + \frac{Am}{p_2} \text{ if } 0 \leq x_1 \leq \frac{Bp_2 - m(A-1)}{Ap_2 - p_1(A-1)}, \\ -\frac{p_1}{p_2}x_1 + \frac{m}{p_2} + B \text{ if } \frac{Bp_2 - m(A-1)}{Ap_2 - p_1(A-1)} \leq x_1 \leq \frac{m}{p_1}. \end{cases}$$

We see that the unique maximiser of U is $\frac{Bp_2-m(A-1)}{Ap_2-p_1(A-1)}$. Thus good 1 is Giffen on $\mathcal{P}$. Q.E.D.

A.5 Proof of Example 5

Because u_1^I is strongly increasing, each utility maximisation problem only has budget-balanced maximisers and therefore we can study the function U given by (2). We have

$$U(x_1) = x_1^\alpha e^{\frac{(m-p_1x_1)^2}{2p_2^2}}.$$

Now let $m \neq 0$, It is clear that 0 is not a maximiser of U. U is on $(0, m/p_1]$ differentiable and there

$$U'(x_1) = \frac{U(x_1)}{x_1} q(x_1),$$

where

$$q(x_1) := \alpha - \frac{p_1}{p_2^2} x_1(m - p_1x_1).$$

Note that in case $m > 2p_2\sqrt{\alpha}$, we have $0 < y_-/p_1 < y_+/p_1 < m/p_1$ and

$$q = \frac{p_1^2}{p_2^2}(x_1 - \frac{y_-}{p_1})(x_1 - \frac{y_+}{p_1}).$$

Case $(0 <)m < 2p_2\sqrt{\alpha}$. Now $q > 0$ and therefore $U' > 0$ on $(0, m/p_1]$. Thus U is strictly increasing on $(0, m/p_1]$ and therefore m/p_1 is the unique maximiser of U.

Case $m = 2p_2\sqrt{\alpha}$. Now again U is strictly increasing on $(0, m/p_1]$ and thus m/p_1 is the unique maximiser of U.

Case $m > 2p_2\sqrt{\alpha}$. Now on $(0, m/p_1]$, $U'(x_1) = \frac{p_1^2}{p_2^2}\frac{U(x_1)}{x_1}(x_1 - \frac{y_-}{p_1})(x_1 - \frac{y_+}{p_1})$. So only $y_-/p_1, y_+/p_1$ and m/p_1 may be a maximiser of U. U is on $(0, m/p_1]$ twice differentiable:

$$U''(x_1) = \frac{p_1^2}{p_2^2}\frac{U(x_1)}{x_1^2}\Big((x_1 - \frac{y_-}{p_1})^2(x_1 - \frac{y_+}{p_1})^2 - (x_1 - \frac{y_-}{p_1})(x_1 - \frac{y_+}{p_1})$$
$$+x_1(x_1 - \frac{y_-}{p_1}) + x_1(x_1 - \frac{y_+}{p_1})\Big),$$

it follows that $U''(y_-/p_1) < 0$ and $U''(y_+/p_1) > 0$. This implies that only y_-/p_1 and m/p_1 may be a maximiser of U. Now we have to compare $U(y_-/p_1)$ and $U(m/p_1)$. The condition $t(m) > 0$ is equivalent with $U(y_-/p_1) > U(m/p_1)$ and therefore in this case y_-/p_1 is the unique maximiser. The condition $t(m) < 0$ is equivalent with $U(y_-/p_1) < U(m/p_1)$ and therefore in this case m/p_1 is the unique maximiser. The condition $t(m) = 0$ is equivalent with $U(y_-/p_1) = U(m/p_1)$ and therefore in this case the set of maximisers is $\{y_-/p_1, m/p_1\}$.

In order to prove the last statement we note that $\frac{dy_-}{dm} < 0$ and analyse the function $t : (2p_2\sqrt{\alpha}, \infty) \to \mathbb{R}$. We have

$$t = \Big(\frac{1}{2}(1 - \sqrt{1 - \frac{4\alpha p_2^2}{m^2}})\Big)^\alpha e^{\frac{\left(m+\sqrt{m^2-4p_2^2\alpha}\right)^2}{8p_2^2}} - 1.$$

This implies that $\lim_{t\to\infty} t(m) = \infty$. So there exist $m_0 > 2p_2\sqrt{\alpha}$ such that $t(m) > 0$ $(m \geq m_0)$. By the above good 1 is inferior on $[m_0, \infty)$.[17] Q.E.D.

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[17] A straightforward calculation shows that

$$t' = \frac{1}{4}\frac{(\sqrt{m^2-4p_2^2\alpha}(m+\sqrt{m^2-4p_2^2\alpha}))e^{\frac{\left(m+\sqrt{m^2-4p_2^2\alpha}\right)^2}{8p_2^2}}(m+\sqrt{m^2-4p_2^2\alpha})\left(\frac{m-\sqrt{m^2-4p_2^2\alpha}}{2m}\right)^{\alpha}}{mp_2^2\sqrt{m^2-4p_2^2\alpha}}.$$

This implies that $t' > 0$ and thus that t is strictly increasing. Therefore for m_0 even each $m_0 > 2p_2\sqrt{\alpha}$ with $t(m_0) > 0$ can be taken.

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