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Comment on "Influence of capillarity on a simple harmonic oscillating water table: Sand column experiments and modeling" by Nick Cartwright et al.

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1. Introduction

[1] *Cartwright et al.* [2005] and, in a preceding paper, *Nielsen and Perrochet* [2000] conducted an investigation to assess the effects of an oscillating water table on the effective porosity. Both studies present results from a thorough laboratory experiment and numerical simulations. We wish to comment on some of the findings and conclusions of *Cartwright et al.* [2005] with regard to the effect of capillarity and the response at very high and very low oscillation frequencies. We also present some results from a simple numerical solution to a nonhysteretic one-dimensional Richards equation model and compare these to the original data and the modeling results presented by *Cartwright et al.* [2005].

2. Frequency Response and the Influence of Capillarity

[2] *Cartwright et al.* [2005] defined a complex effective porosity, n_ω, as

$$n\frac{\mathrm{d}h_{tot}}{\mathrm{d}t} = n_\omega \frac{\mathrm{d}h}{\mathrm{d}t} \tag{1}$$

$$|F| = \frac{|\eta|}{|\eta_0|} \tag{2}$$

and

$$F(\omega) = |F|e^{-i\phi} = \frac{1}{1 + i\omega\frac{n_\omega D}{K}} \tag{3}$$

where h_{tot} is the equivalent height of the total storage of soil moisture, h is the water table height (calculated according to equation (12) from *Nielsen and Perrochet* [2000]), t is time, $n = (\theta_s - \theta_r)$, θ_s is the saturated soil moisture content, θ_r is the residual soil moisture content, $|\eta|$ and $|\eta_0|$ are respectively the amplitudes of the water table height and driving head, ϕ is the phase shift between measured water table height $h(t)$ and driving head $h_0(t)$, D is the average water table height, K is the saturated hydraulic conductivity, $\omega = 2\pi/T$ is the angular frequency, T is the period of the oscillation, $i = \sqrt{-1}$, and n_ω is the effective porosity which is a complex number. Equation (3) can be reformulated as

$$n_\omega = \left(\frac{1}{F(\omega)} - 1\right)\frac{K}{i\omega D} \tag{4}$$

From equation (1) of *Cartwright et al.* [2005] the authors draw the conclusion that for low frequencies, n_ω should converge toward $n = (\theta_s - \theta_r)$ since the phase lag will approach 0, thereby eliminating the imaginary part of n_ω. An alternative motivation for the same conclusion follows from the limit case behavior of equation (5) of *Cartwright et al.* [2005], however, this function is the result of a function fit to the data which is extrapolated outside the range of measurements, and no physical basis for the function form is given. For high frequencies, they note that $|n_\omega|$ decays linearly on a double-logarithmic scale and they mention a curious relationship between the decay and the van Genuchten parameters. In this comment we state that, under the influence of capillarity, n_ω can converge to other values than n for low frequencies. For high frequencies we attempt to explain the sharp decay of $|n_\omega|$ with ω, and we present a quantitative analysis of the processes involved.

2.1. Low-Frequency Response

[3] First, the conclusion that n_ω converges to $(\theta_s - \theta_r)$ for low frequencies is only expected to hold when the water table is not in the proximity of the soil surface: for these frequencies it can be expected that h_0 and h will be in phase with equal amplitude, indeed causing the complex n_ω to converge toward n. However, when the water table fluctuations occur sufficiently close to the soil surface, the temporal changes in total storage h_{tot} will be very small compared to the changes in h, causing the value of n_ω in (1) to approach a value lower than $(\theta_s - \theta_r)$ [e.g., *Hilberts et al.*, 2005]. *Cartwright et al.* [2005] mentioned that the presented experiments are unaffected by the proximity of the water table to the soil surface. However, their Table 2 and Figure 3 indicate that for the glass bead soil this assumption is dubious. Table 2 shows that the minimal average driving head level is about 50 cm above the base of the column

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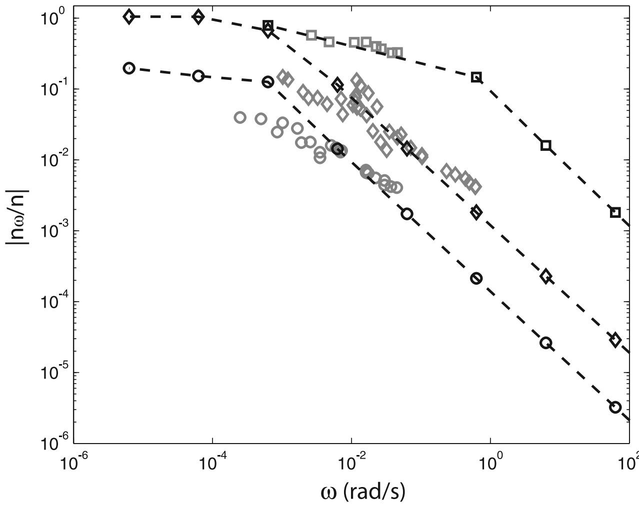

Figure 1. Frequency response of $|n_\omega/n|$ for the three soil types used in this study (namely, glass beads (circles), fine sand (diamonds), and coarse sand (squares)). The gray symbols are the data from *Cartwright et al.* [2005], and the black lines and symbols are the results of the modeling exercise in this paper.

(total height of 180 cm) which leaves approximately 130 cm unsaturated zone depth on average. Figure 3 shows that the glass bead soil (i.e., d_{50} = 0.082 mm) at pressure head $\psi = -130$ cm is still not fully drained. We expect therefore that the proximity to the soil surface is quite strongly affecting the values of n_ω for the glass bead soil. This also seems to be indicated by the results in Table 2, from which we can conclude that the highest value for $|n_\omega|$ of the glass bead soil is 0.013 even though the value of the oscillation frequency is very low (namely, 2.49 10^{-4} s^{-1}, which corresponds to an oscillation period T = 7 hours).

[4] A second issue is that the limit case $\lim_{\omega\to 0}(n_\omega)$ cannot be determined from (4), since it has no solution. Also equation (1) offers no solution for the limit case where frequencies approach 0, since the derivatives dh_{tot}/dt and dh/dt will approach 0, leaving the relationship between n_ω and n undetermined.

[5] In relation to the response at very low frequencies, Figure 7 of *Cartwright et al.* [2005] indicates that the value of $|n_\omega/n|$ for a numerical solution to Richards equation for the glass bead soil (i.e., solid circles in their figure) converges to 1 relatively quickly, i.e., $|n_\omega|$ converges to n. Here, we compare the results of a simple numerical solution of a nonhysteretic Richards equation model to the results of *Cartwright et al.* [2005]. For our simulations we use identical soil parameter settings as in *Cartwright et al.* [2005], (i.e., for the glass bead soil $K = 2.8 \times 10^{-5}$ m/s, $\theta_s = 0.38$, $\theta_r = 0.06$, $\alpha = 0.68$ 1/m, and $\beta = 10$, where α and β are van Genuchten parameters), and we used an oscillation amplitude $|\eta_0| = 0.15$ m, an average driving head D = 0.50 m, and a column depth of 1.8 m. In our Figure 1 the results of these simulations, together with the original data listed in Table 2 of *Cartwright et al.* [2005], are plotted as a function of ω. Note that the data that are derived from the paper by *Nielsen and Perrochet* [2000] are left out because the average water table height significantly deviates from that of *Cartwright et al.* [2005] and this paper.

[6] Figure 1 illustrates that for the glass bead soil (black circles) even for very low frequencies (e.g., $\omega = 6.2832 \times 10^{-6}$ rad/s or $T \simeq 11.6$ days), the value for $|n_\omega/n|$ based on our modeling exercise is still clearly lower than 1, namely, $|n_\omega/n| = 0.20$. If we compare this to Figure 7 of *Cartwright et al.* [2005], it can be seen that for the same frequency (i.e., $n\omega H_\psi/K = 0.1091$, where H_ψ is the equivalent capillary fringe height) the indicated value of $|n_\omega/n|$ based on their model result is much higher (approximately 0.9). However, for the fine and course sand (diamonds and squares), $|n_\omega/n|$ converges to 1 as indicated by *Cartwright et al.* [2005].

[7] Figure 2 shows the same data as Figure 1, however, the x axis is scaled according to *Cartwright et al.* [2005]. When we compare Figure 2 to Figure 7 of *Cartwright et al.* [2005], we notice a clear distinction: where Figure 7 shows a clear overestimation of $|n_\omega/n|$ for approximately $n\omega H_\psi/K < 10^1$ for all three soil types, Figure 2 shows lower drainable porosity values for the coarse sand and the glass bead soil than presented in the Richards model results of *Cartwright et al.* [2005]. For the coarse sand, the fit to the measurements is better than for the other soil types and better than that presented by *Cartwright et al.* [2005]. A satisfactory explanation for the good performance for the coarse sand is yet to be found. On the basis of our model results, we cannot dismiss a nonhysteretic Richards equation model for all soil types, since it performs reasonably well when confronted with the measurements for the coarse sand. However, the authors agree with Cartwright et al. in that an improved match can be obtained by incorporation of the effects of hysteresis into modeling practice [e.g., *Werner and Lockington*, 2003].

2.2. High-Frequency Response

[8] Also for high frequencies, equations (1) and (4) prove troublesome: for the limit case $\lim_{\omega\to\infty}(n_\omega)$, equation (4) has no solution. For high-frequency oscillations of the driving head, the amplitude of the water table oscillations will converge to 0, causing the right hand side of equation (1) to become 0 which causes n_ω to be undefined.

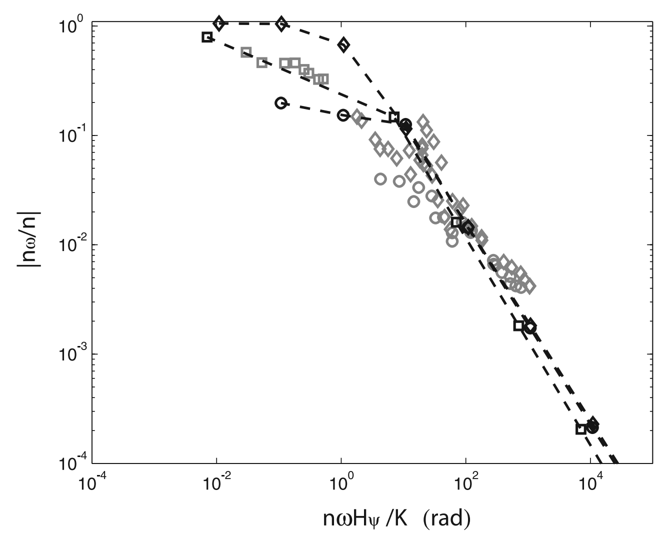

Figure 2. Frequency response of $|n_\omega/n|$ for the three soil types used in this study, scaled according to *Cartwright et al.* [2005]. The symbols are as defined in Figure 1.

[9] The data obtained by *Cartwright et al.* [2005] show that the calculated values of $|n_\omega/n|$ decay apparently linearly (on a double-logarithmic scale) for higher frequencies, with a computed slope of $-2/3$. The results of the Richards equation modeling exercise in the same paper show a slope of approximately -1 (see their Figure 6). The high-frequency response of our modeling exercise shown in Figures 1 and 2 indicates that all three slopes show a very similar decay, which ranges from -0.90 to -0.93.

[10] The decay of $|n_\omega|$ for increasing frequencies can be ascribed to the inability of a part of the unsaturated zone to reach a new equilibrium situation for high-frequency oscillations. If there is an equilibrium initial condition in the unsaturated zone and h_0 is increased rapidly, merely a small part of the unsaturated zone will become saturated before h_0 starts to drop. However, at higher elevations above the water table, where conductivities get increasingly smaller, the response to changes in h_0 gradually become negligible. For increasing frequencies of $h_0(t)$, $|\eta|$ will become smaller, as will the changes in the unsaturated zone storage close to the soil surface. For small $|\eta|$ and large ω, the fluctuations of $h(t)$ will mainly occur within the nearly saturated part of the retention curve (i.e., the capillary fringe), causing the value of $|n_\omega|$ to drop rapidly. *Cartwright et al.* [2005, paragraph 21] stated that this drop occurs due to an increased influence of capillarity for higher frequencies, and therefore $|n_\omega| \ll n$. Here, we propose a more quantitative explanation for the observed decay of $|n_\omega|$ with increasing frequencies.

2.3. Attempt to Explain the Modeled and Measured High-Frequency Decay of $|n_\omega|$

[11] We shall assume that the response of the unsaturated zone is restricted to the domain over which water table fluctuations take place (namely, over a range $2|\eta|$). This assumption implies that above $D + |\eta|$ no changes in soil moisture conditions take place (i.e., $\theta(z > D + |\eta|, t) = \theta(z > D + |\eta|, t + \mathrm{d}t)$, see Figure 3). The assumption is expected to become more valid for increasing frequencies. The integrated soil moisture in the profile (or total storage) at time t can be calculated as

$$s(t) = h(t)(\theta_s - \theta_r) + \int_{h(t)}^{Z} (\theta(t,z) - \theta_r)\mathrm{d}z \tag{5}$$

where Z is the location of the soil surface. Substitution of $t + \mathrm{d}t$ into equation (5) allows us to express the change in storage from time t to $t + \mathrm{d}t$

$$ds = \mathrm{d}h(\theta_s - \theta_r) - \int_{h(t)}^{h(t+\mathrm{d}t)} (\theta(t,z) - \theta_r)\mathrm{d}z \tag{6}$$

where $\mathrm{d}h = h(t + \mathrm{d}t) - h(t)$ is the water table change over a time increment $\mathrm{d}t$. As stated before, we assume that there is an equilibrium soil moisture profile above the lowest water table height $h = D - |\eta|$ that returns each cycle (at $t = kT$, where $k = 0, 1, 2, \ldots$). According to the integration described by *Hilberts et al.* [2005] which uses an alternative van Genuchten parameterization (namely, $m = 1 + 1/\beta$), equation (6) can then be expressed as

$$ds = \mathrm{d}h(\theta_s - \theta_r)\left[1 - \left(1 + (\alpha'\mathrm{d}h)^{\beta'}\right)^{-1/\beta'}\right] \tag{7}$$

where α', and β' are the modified van Genuchten parameters. The parameter values for the three soils used in this study are listed in Table 1.

[12] Equation (7) describes the total storage change over a time interval $\mathrm{d}t$, which is equal to $n\mathrm{d}h_{tot}$ from equation (1).

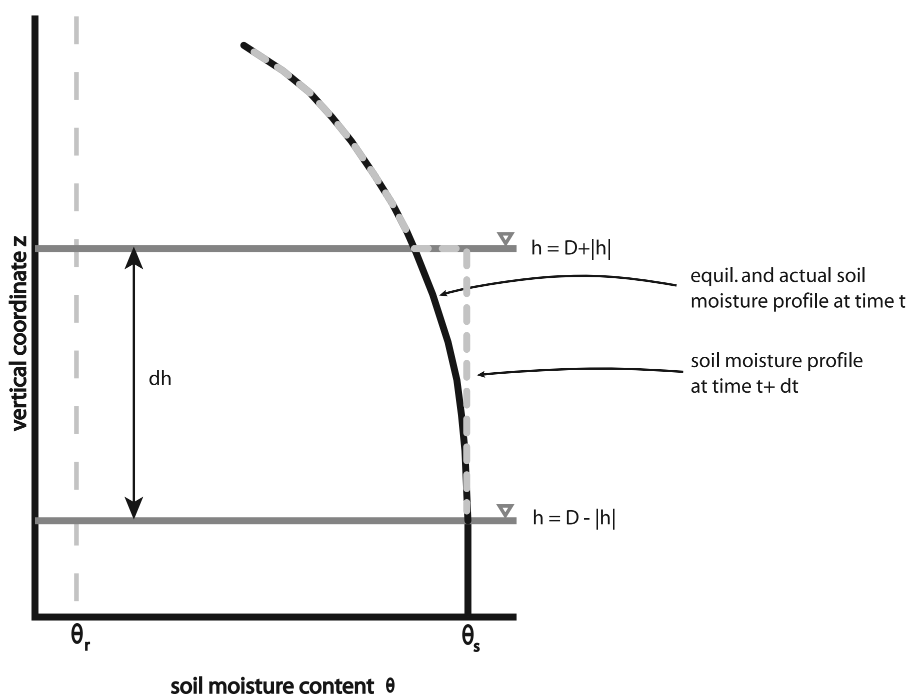

Figure 3. Sketch of the assumed soil moisture profile response (from time t to t + dt) as a result of oscillations in the pressure head with amplitude $|\eta|$.

Table 1. Modified van Genuchten Parameters for the Three Soil Types Used in This Study[a]

Parameter	Glass Bead	Fine Sand	Coarse Sand
θ_s	0.38	0.38	0.41
θ_r	0.06	0.09	0.08
α', 1/cm	−0.0066	−0.0163	−0.11
β'	9.33	8.27	20

[a]Modified indicates $m = 1 + 1/\beta$.

Combination of equations (1) and (7) allows us to derive a simple expression for $|n_\omega|$ for high frequencies:

$$|n_\omega| = (\theta_s - \theta_r)\left[1 - \left(1 + (\alpha' \mathrm{d}h)^{\beta'}\right)^{-1/\beta'}\right] \quad (8)$$

[13] Given that the total water table change dh = 2$|\eta|$, the value of $|n_\omega|$ can be calculated using equation (8) and the results of the numerical solution to Richards equation. The results are shown in Figure 4.

[14] We note that the value of $|n_\omega/n|$ based on equation (8) (in gray in Figure 4) are clearly underestimated compared to the values obtained using equation (4) (in black in Figure 4). This indicates that the assumption of a static soil water profile above $D + |\eta|$ is too conservative: a larger part of the unsaturated zone than assumed (probably a significant part of the capillary fringe) will respond to the oscillations in the driving head. However, the slope of the gray curves show convergence toward a slope of approximately 0.90 that we observed in our modeling results. Also, *Cartwright et al.* [2005] make note of slopes equal to $(1 - 1/\beta)$ in their modeling results, which is very close to 0.90 for the given parameters. Their section 7 is devoted to this curious relationship between the van Genuchten parameters and the slopes of the curves for high frequencies, for which no apparent cause was found. On the basis of equation (8), we expect a power law relationship between dh and $|n_\omega|$, where the slope of this curve is $(-1/\beta')$. Furthermore, from analysis of the results of our modeling exercise we find that ω and dh also have an approximate power law relationship for high frequencies. This implies that ω and the modeled $|n_\omega|$ also have a power law relationship, of which the slope is determined by β'. However, since we lack a functional relationship between dh and ω, we cannot present an analytical expression for the modeled $|n_\omega|$ as a function of ω.

3. Criteria to Determine the Influence of Capillarity

[15] *Cartwright et al.* [2005, paragraph 26] mentions that the effects of capillarity become important when the distance from the water table to the sand surface is approximately half the value of the equivalent saturated capillary fringe height (H_ψ) for the cases that were investigated. Table 1 of Cartwright et al. lists 1.50 m, 0.60 m, and 0.085 m for H_ψ for respectively the glass bead soil, the fine sand, and the course sand. For deep water tables, the uptake of water as a result of an increase of the pressure head dh (or yield in case of drainage) will be $(\theta_s - \theta_r)$dh. The effect of capillarity (or rather, the unsaturated storage effect) becomes noticeable when the change in water table height dh does no longer lead to constant uptake or yield of water but is reduced instead. This effect is reflected in the shape of the differential soil moisture capacity function, which is defined as the derivative of the soil moisture profile with respect to the pressure head (i.e., dθ/dψ). The point where the uptake or yield is no longer constant (namely, $\theta_s - \theta_r$) is the point at which dθ/dψ is no longer 0, because at this point the yield changes depend on the actual suction head as well as the suction head changes. The differential soil moisture capacity for the three soil types investigated in this study is depicted in Figure 5.

[16] The suction heads for which the capillarity effect were noticeable according to *Cartwright et al.* [2005] were

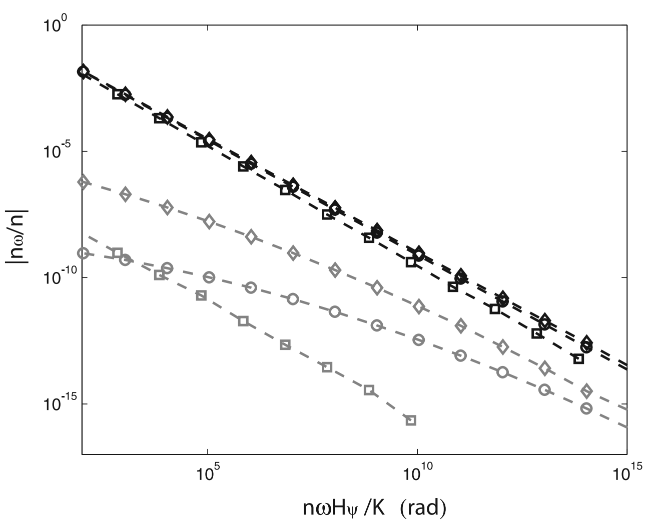

Figure 4. High-frequency response of $|n_\omega/n|$ for the three soil types used in this study, scaled according to *Cartwright et al.* [2005]. The symbols are as defined in Figure 1, but the black lines indicate results based on equation (4), and gray lines indicate results based on equation (8).

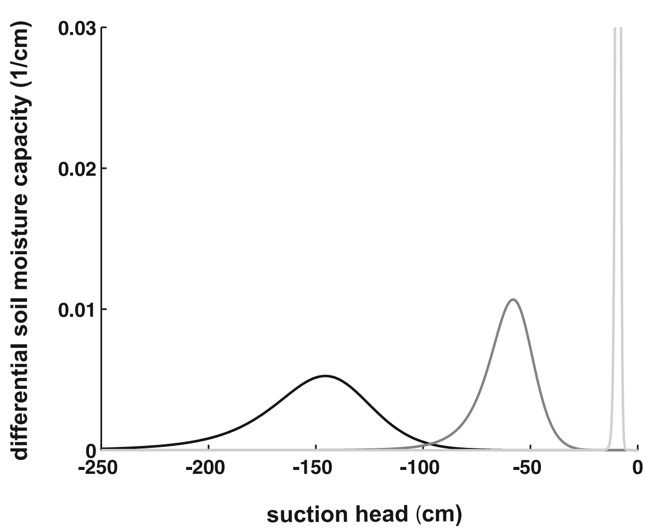

Figure 5. Differential moisture capacity functions for the three soil types used by *Cartwright et al.* [2005] (black, glass bead soil; dark gray, fine sand; light gray, coarse sand).

approximately 0.75 m, 0.30 m, and 0.043 m. These values correspond nicely to the suction heads at which the differential soil moisture capacity starts to rise in Figure 5 (going from ψ = 0 toward smaller ψ, i.e., the drying direction). Inspection of the differential soil moisture capacity may form an easy and elegant way to assess the expected effects of unsaturated storage on water table changes.

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