Analytical solution of the linearized hillslope-storage Boussinesq equation for exponential hillslope width functions

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[1] This technical note presents an analytical solution to the linearized hillslope-storage Boussinesq equation for subsurface flow along complex hillslopes with exponential width functions and discusses the application of analytical solutions to storage-based subsurface flow equations in catchment studies. *INDEX TERMS:* 1829 Hydrology: Groundwater hydrology; 1860 Hydrology: Runoff and streamflow; 1824 Hydrology: Geomorphology (1625); 1832 Hydrology: Groundwater transport; *KEYWORDS:* analytical solutions, Boussinesq equation, groundwater modeling, hillslope drainage, linearization, subsurface flow

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1. Introduction

[2] Recently, *Troch et al.* [2003] introduced the hillslope-storage Boussinesq (hsB) equation to describe subsurface flow and saturation along complex hillslopes. They demonstrated that (numerical) solutions of the hsB equation account explicitly for plan shape of the hillslope, by introducing the hillslope width function, and for profile curvature through bedrock slope angle and the hillslope soil depth function [see also *Paniconi et al.*, 2003; *Hilberts et al.*, 2004]. They also presented several simplified versions of the hsB equation (including linearizations) and discussed to what extend these reduced versions are able to capture the dynamic response of hillslopes.

[3] One reduced version of the hsB equation is of particular interest for further investigation, namely, a linearized formulation under the additional assumption of an exponential hillslope width function. In this case the hsB equation reduces to a linear partial differential equation (PDE) with constant coefficients and a nonhomogeneous forcing term, while maintaining a mathematical description of the convergence and divergence rate of the plan shape.

[4] In this paper we present an exact solution to equation (12) of *Troch et al.* [2003]. We then discuss the applicability of this and similar solutions in catchment studies.

2. Derivation of an Exact Solution

[5] The starting point of our analysis is equation (12) given by *Troch et al.* [2003]:

$$f\frac{\partial S}{\partial t} = kpD\cos i\left[\frac{\partial^2 S}{\partial x^2} - a\frac{\partial S}{\partial x}\right] + k\sin i\frac{\partial S}{\partial x} + Nwf, \qquad (1)$$

where x is flow distance from the outlet of the hillslope, t is time, $S(x, t)$ is saturated storage, f is drainable porosity, k is hydraulic conductivity, i is bedrock slope angle, $w(x)$ is hillslope width function defined perpendicular to x, N is recharge, D is constant soil depth, p is a linearization parameter, and a is defined as

$$w(x) = ce^{ax}. \qquad (2)$$

Equation (2) is the assumed exponential width function, and c defines the width at the outlet of the hillslope. For the derivation of equation (1) the reader is referred to *Troch et al.* [2003].

[6] Solution of equation (1) can be sought for the following initial and boundary conditions:

$$\begin{aligned} S(x,t) &= \gamma fce^{ax} \quad 0 \le x \le L \quad t = 0, \\ S(x,t) &= 0 \quad x = 0 \quad t > 0, \\ \frac{kpD\cos i}{f}\frac{\partial S}{\partial x} + \frac{k\sin i - akpD\cos i}{f}S &= 0 \quad x = L \quad t > 0. \end{aligned}$$

The initial condition assumes an initially constant water table height ($0 < \gamma \le D$) along the hillslope. The first boundary condition fixes the storage at the outlet at zero, while the second boundary condition assumes a no-flow boundary at the topographic divide. These boundary conditions are common in subsurface flow hydrology [*Brutsaert*, 1994; *Verhoest and Troch*, 2000].

[7] Equation (1) can be written as

$$\frac{\partial S}{\partial t} = K\frac{\partial^2 S}{\partial x^2} + U\frac{\partial S}{\partial x} + Nw, \qquad (3)$$

with

$$K = \frac{kpD\cos i}{f} \qquad (4)$$

$$U = \frac{k\sin i - akpD\cos i}{f}. \qquad (5)$$

If P denotes the Laplace transform of S and s denotes the Laplace variable, then equation (3), taking into account the above specified initial condition and equation (2), becomes

$$-sP + UP_x + KP_{xx} = ce^{ax}\left(\frac{-N}{s} - \gamma f\right), \qquad (6)$$

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where P_x denotes the first-order derivative of P with respect to x and P_{xx} denotes the second-order derivative. Equation (6) is an ordinary differential equation, with constant coefficients and a nonhomogeneous forcing term, which is subject to the following boundary conditions in the Laplace domain:

$$P = 0 \qquad x = 0 \tag{7}$$

$$KP_x + UP = 0 \qquad x = L. \tag{8}$$

It is easy to show that the particular solution of equation (6) is given by

$$P_{\mathrm{part}} = \frac{-c(N + \gamma f\, s)}{s(Ka^2 + Ua - s)} e^{ax}. \tag{9}$$

The homogeneous solution is found by introducing the trial solution $P = e^{\lambda x}$ in equation (6) without forcing term, which results in

$$K\lambda^2 + U\lambda - s = 0.$$

Solving this equation for λ results in

$$\lambda_{1,2} = d \pm b, \tag{10}$$

where d and b are

$$d = \frac{-U}{2K} \tag{11}$$

$$b = \sqrt{d^2 + \frac{s}{K}}. \tag{12}$$

Now the homogeneous solution is

$$P_{\mathrm{hom}} = C_1 e^{(d+b)x} + C_2 e^{(d-b)x}. \tag{13}$$

Summation of equations (9) and (13) yields the general solution of equation (6):

$$P = C_1 e^{(d+b)x} + C_2 e^{(d-b)x} - \frac{c(N + \gamma fs)}{s(Ka^2 + Ua - s)} e^{ax}. \tag{14}$$

The constants C_1 and C_2 are found by accounting for the boundary conditions (7) and (8):

$$C_1 = \frac{c(N + \gamma fs)}{s(Ka^2 + Ua - s)} \frac{(b + d)e^{dL}e^{-bL} + (a - 2d)e^{aL}}{2e^{dL}[b\cosh(bL) - d\sinh(bL)]} \tag{15}$$

$$C_2 = \frac{c(N + \gamma f\, s)}{s(Ka^2 + Ua - s)} \frac{(b - d)e^{dL}e^{bL} - (a - 2d)e^{aL}}{2e^{dL}[b\cosh(bL) - d\sinh(bL)]}. \tag{16}$$

Substituting these expressions of C_1 and C_2 into equation (14) results in the solution of the differential equation in the Laplace domain:

$$P = \frac{c\gamma f e^{d(x-L)}\{(a - 2d)e^{aL}\sinh(bx) + be^{dL}\cosh(b(L - x)) - de^{dL}\sinh[b(L - x)]\}}{(Ka^2 + Ua - s)[b\cosh(bL) - d\sinh(bL)]} + \frac{cNe^{d(x-L)}\{(a - 2d)e^{aL}\sinh(bx) + be^{dL}\cosh[b(L - x)] - de^{dL}\sinh[b(L - x)]\}}{s(Ka^2 + Ua - s)[b\cosh(bL) - d\sinh(bL)]} - \frac{c(N + \gamma fs)e^{ax}}{s(Ka^2 + Ua - s)}. \tag{17}$$

The solution of equation (3) is obtained by transforming equation (17) back to the time domain by means of the inverse Laplace transform. The inverse Laplace transform of a function $f(s)$ can be found by solving the Bromwich integral [e.g., *Arfken*, 1985], which can be evaluated by the regular methods of contour integration. However, for $t > 0$ the inverse Laplace transform of a function $f(s)$ can be generated according to the residue theorem [*Arfken*, 1985], which states that the inverse Laplace transform of a function $f(s)$ equals the sum of the residues (R) at the poles of $f(s)e^{st}$.

[8] The first term of equation (17) (P_1) has one pole at $s = Ka^2 + Ua$ and poles at $s = s_n$ for which $f = b\cosh(bL) - d\sinh(bL) = 0$. Replacing bL by jz in the latter equation, this transforms into

$$\frac{z}{dL} = \tan(z), \tag{18}$$

which has an infinity of roots at a mutual distance converging to π. According to *Brutsaert* [1994] the residues at $s = s_n$ for $P_1 e^{s_n t}$ can be calculated with

$$R_{1,1} = \frac{S_{1,1}(s_n)e^{s_n t}}{T'_{1,1}(s_n)} \tag{19}$$

in which $S_{1,1}$ denotes the numerator and $T_{1,1}$ denotes the denominator of P_1.

[9] The numerator of equation (19) can be written as

$$S_{1,1}(s_n)e^{s_n t} = c\gamma\, f e^{s_n t - d(L-x)} j\sin\left(\frac{z_n x}{L}\right) \cdot \left\{e^{dL}\left[\frac{z_n}{L}\sin(z_n) + d\cos(z_n)\right] + (a - 2d)e^{aL}\right\}. \tag{20}$$

The derivative of the denominator of equation (19) with respect to s_n, $T'_{1,1}$, is

$$T'_{1,1}(s_n) = j\frac{\cos(z_n)}{2z_n dL^2}\left(-dL + z_n^2 + d^2L^2\right)\left(a^2L^2 - 2daL^2 + z_n^2 + d^2L^2\right). \tag{21}$$

So, finally,

$$R_{1,1}(n) = \eta_n \gamma f \sin\left(\frac{z_n x}{L}\right) e^{s_n t - d(L-x)}, \tag{22}$$

with

$$\eta_n = \frac{2z_n dL^2 c\left\{e^{dL}\left[\frac{z_n}{L}\sin(z_n) + d\cos(z_n)\right] + (a - 2d)e^{aL}\right\}}{\cos(z_n)\left(-dL + z_n^2 + d^2L^2\right)\left(a^2L^2 - 2daL^2 + z_n^2 + d^2L^2\right)}. \tag{23}$$

The computation of the residue of P_1 at $s = Ka^2 + Ua$ ($R_{1,2}$) results in

$$R_{1,2} = fc\gamma e^{Kat(a-2d)+ax}. \qquad (24)$$

According to the residues theorem the inverse Laplace transform of P_1 is now given by

$$S_1 = \sum_{n=0}^{n=\infty} \eta_n \gamma f \sin\left(\frac{z_n x}{L}\right) e^{s_n t - d(L-x)} + fc\gamma e^{Kat(a-2d)+ax}. \qquad (25)$$

For the derivation of the inverse Laplace transform of the second term of equation (17), P_2 is rewritten into the form $P_2 = f(s)g(s)$, where

$$f(s) = \frac{cNe^{d(x-L)}\left\{(a-2d)e^{aL}\sinh(bx) + be^{dL}\cosh[b(L-x)] - de^{dL}\sinh[b(L-x)]\right\}}{[b\cosh(bL) - d\sinh(bL)](Ka^2 + Ua - s)}$$

$$g(s) = \frac{1}{s}.$$

The inverse Laplace transform of the product of two functions ($f(s)$ and $g(s)$) equals the convolution product of the Laplace-original of both functions ($F(t)$ and $G(t)$):

$$\mathcal{L}^{-1}[f(s)g(s)] = (F * G)(t) = \int_0^t F(t-\tau)G(\tau)d\tau. \qquad (26)$$

The inverse Laplace transform of $g(s)$ is simply 1, and $\mathcal{L}^{-1}f(s)$ is found in a similar manner as above and results in

$$\mathcal{L}^{-1}[f(s)] = \sum_{n=0}^{\infty} \eta_n N \sin\left(\frac{z_n x}{L}\right) e^{s_n t - d(L-x)} + Nce^{Kat(a-2d)+ax}. \qquad (27)$$

Now, S_2 can be derived by applying the convolution product:

$$S_2 = \int_0^t 1\left[\sum_{n=0}^{\infty} \eta_n N \sin\left(\frac{z_n x}{L}\right) e^{s_n \tau - d(L-x)} + cNe^{aK\tau(a-2d)+ax}\right] d\tau$$
$$= \sum_{n=0}^{\infty} \frac{\eta_n N}{s_n} \sin\left(\frac{z_n x}{L}\right) e^{-d(L-x)}\left(e^{s_n t} - 1\right)$$
$$+ \frac{cNe^{ax}}{aK(a-2d)}\left(e^{aKt(a-2d)} - 1\right). \qquad (28)$$

[10] Finally, the third term of equation (17) (P_3) has to be transformed back to the time domain. Therefore P_3 is rewritten as

$$P_3 = -\frac{c\gamma f e^{ax}}{Ka^2 + Ua - s} - \frac{cNe^{ax}}{s(Ka^2 + Ua - s)} = P_{3,1} + P_{3,2}.$$

$P_{3,1}$ has one single pole at $s = Ka^2 + Ua$, so

$$S_{3,1} = \lim_{s \to Ka^2 + Ua} -\left(c\gamma f e^{ax} e^{st}\right) = -c\gamma f e^{ax} e^{aK(a-2d)t}. \qquad (29)$$

$S_{3,2}$ is found again by applying the convolution product:

$$S_{3,2} = \int_0^t 1\left(-cNe^{ax + \left(Ka^2 + Ua\right)\tau}\right) d\tau = \frac{-cNe^{ax}}{Ka(a-2d)}\left(e^{Ka(a-2d)t} - 1\right). \qquad (30)$$

[11] Adding equations (25), (28), (29), and (30) and eliminating some terms gives the solution of the PDE describing the storage dynamics of hillslopes with an exponential width function, during free drainage after initially partial saturated conditions:

$$S = \sum_{n=0}^{n=\infty} \eta_n \gamma f \sin\left(\frac{z_n x}{L}\right) e^{s_n t - d(L-x)}$$
$$+ \sum_{n=0}^{n=\infty} \frac{\eta_n N \sin\left(\frac{z_n x}{L}\right) e^{-d(L-x)}}{s_n}\left(e^{s_n t} - 1\right) \qquad (31)$$

with $s_n = -K(\frac{z_n^2}{L^2} + d^2)$. With the definition of η_n, equation (23), this last equation can be written as

$$S(x,t) = \sum_{n=0}^{n=\infty} \frac{2\gamma f z_n dL^2 c\left\{e^{dL}\left[\frac{z_n}{L}\sin(z_n) + d\cos(z_n)\right] + (a-2d)e^{aL}\right\}}{\cos(z_n)\left(-dL + z_n^2 + d^2L^2\right)\left(a^2L^2 - 2daL^2 + z_n^2 + d^2L^2\right)} \sin\left(\frac{z_n x}{L}\right) e^{s_n t - d(L-x)}$$
$$+ \sum_{n=0}^{n=\infty} \frac{2z_n dL^2 c\left\{e^{dL}\left[\frac{z_n}{L}\sin(z_n) + d\cos(z_n)\right] + (a-2d)e^{aL}\right\}N}{\cos(z_n)\left(-dL + z_n^2 + d^2L^2\right)\left(a^2L^2 - 2daL^2 + z_n^2 + d^2L^2\right)s_n} \sin\left(\frac{z_n x}{L}\right) e^{-d(L-x)}\left(e^{s_n t} - 1\right). \qquad (32)$$

[12] Applying the continuity equation for $x = 0$, the drainage flux dynamics are given by

$$Q(0) = \sum_{n=0}^{n=\infty} (\gamma f s_n + N) \frac{-Lz_n}{z_n^2 + L^2 d^2} \frac{2z_n dL^2 c\left\{e^{dL}\left[\frac{z_n}{L}\sin(z_n) + d\cos(z_n)\right] + (a-2d)e^{aL}\right\}e^{s_n t - dL}}{\cos(z_n)\left(-dL + z_n^2 + d^2L^2\right)\left(a^2L^2 - 2daL^2 + z_n^2 + d^2L^2\right)} + \frac{Nc}{a}\left(e^{aL} - 1\right) \qquad (33)$$

The steady state solutions corresponding to a constant recharge N can be obtained from equations (32) and (33):

$$\lim_{t\to\infty} S = -\sum_{n=0}^{n=\infty} \frac{2z_n dL^2 cN\left\{e^{dL}\left[\frac{z_n}{L}\sin(z_n) + d\cos(z_n)\right] + (a-2d)e^{aL}\right\}\sin\left(\frac{z_n x}{L}\right)}{\left(-dL + z_n^2 + d^2L^2\right)\left(a^2L^2 - 2daL^2 + z_n^2 + d^2L^2\right)s_n} \quad \lim_{t\to\infty} Q = \frac{Nc}{a}\left(e^{aL} - 1\right). \qquad (34)$$

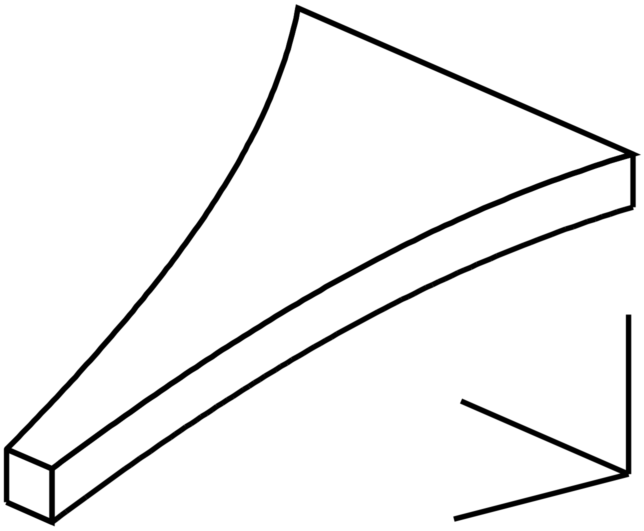

Figure 1. Three-dimensional view of the convergent hillslope that is used to validate the analytical solution. The length of the hillslope $L = 100$ m, the soil depth D equals 2 m, the slope steepness i equals 5%, and the width function is given by $w = ce^{ax}$, where $c = 6.77$ m and $a = 0.02$ 1/m. The storage $S(x, t)$ in equation (1) is defined perpendicular to the bedrock along which the x axis (positive upslope) is aligned.

3. Comparison With Numerical Results

[13] To show the correctness of our analytical solution, we compare equations (32) and (33) with a numerical solution of equation (1) applied to a converging hillslope as shown in Figure 1. The length L is 100 m, the bedrock slope angle i is 5%, the soil depth D is 2 m, and the hillslope width function parameters a and c are 0.02 1/m and 6.77 m, respectively. Further, we assume the drainable porosity f equals 0.30, the linearization parameter p equals 1, and hydraulic conductivity k = 1 m/hr.

[14] Equation (1) is solved numerically by discretization in space by finite differences and by applying a multistep ordinary differential equation solver in time. The code is written in MATLAB. Figure 2 compares the storage and drainage flux dynamics during free drainage after initially partial (20%) saturation of the hillslope computed with equations (32) and (33) and our numerical solution. The number of roots used to produce Figure 2 is 600. As can be seen from Figure 2, the analytical solution matches the numerical solution perfectly. Apparently, numerical dispersion is insignificant in the numerical hsB model.

[15] An additional check of the correctness of our solution is to introduce unit-width hillslope characteristics, that

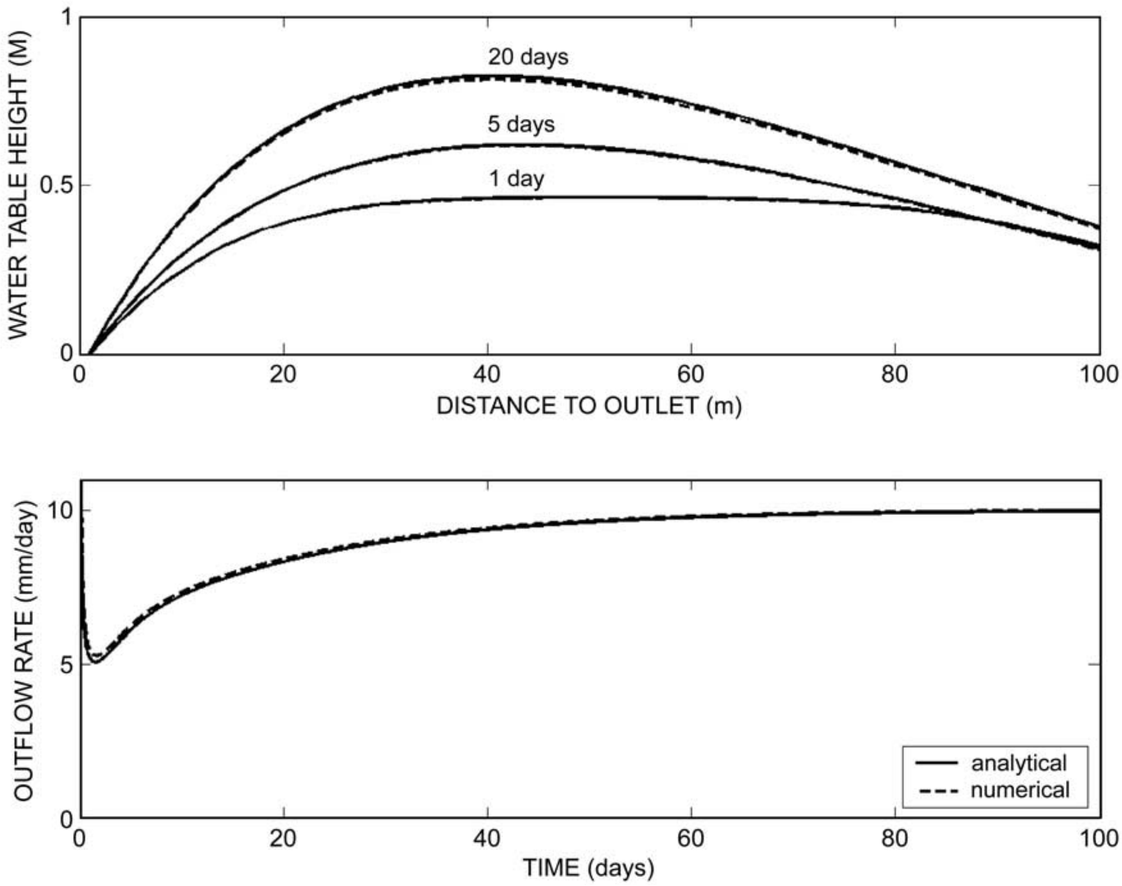

Figure 2. (top) Water tables values and (b) outflow rates of the analytical (solid lines) and numerical (dashed lines) solutions of the hsB equation, when applied to the convergent hillslope of Figure 1, using a porosity of 30% and a saturated conductivity of 1 m/hr. The initial condition is that of a partial saturation of 20% (i.e., $\gamma = 0.40$ m), and the rainfall intensity N is 10 mm/d. For these simulations the parameter p is set to 1.

is, a = 1 1/m and c = 0 m, and see if our solution converges to the exact solution given by *Verhoest and Troch* [2000]:

$$h = -\frac{N\left\{-1 + e^{2dx} - 2d\left[\left(e^{2dx} - 1\right)L + x\right]\right\}}{4d^2 f K} + \sum_{n=1}^{\infty} \frac{2z_n dL^2\left(D + \frac{L^2 N}{f s_n}\right)\left\{e^{dL}\left[\frac{z_n}{L}\sin(z_n) + d\cos(z_n)\right] - 2d\right\}\sin\left(\frac{z_n x}{L}\right)e^{s_n t - d(L-x)}}{\cos(z_n)\left(-dL + z_n^2 + d^2L^2\right)\left(z_n^2 + d^2L^2\right)}. \quad (35)$$

[16] After multiplication with the drainable porosity, equation (35) transforms into an equation describing the storage dynamics. Easily, it can be seen that the second term of equation (35) corresponds with the dynamical part of our solution. However, rewriting the first term of equation (35) into the term describing the steady state situation in our solution is not straightforward. It is, however, easy to show by direct computation of the steady state that the first term of equation (35) is equal to our steady state equation (34).

4. Discussion

[17] Now that an exact solution to the linearized hsB equation for exponential width functions is available, it is important to discuss the applicability of the instantaneous unit hydrograph of different complex hillslopes (convergent-divergent and bedrock slope angle) for catchment studies. It has since long been recognized that the hydraulic groundwater theory applied to idealized hillslopes (i.e., for unit width hillslopes) is a useful tool for low-flow hydrology [e.g., *Brutsaert and Nieber*, 1977; *Troch et al.*, 1993; *Brutsaert*, 1994]. The hsB concept offers the possibility to extend the applicability of the hydraulic groundwater theory to rainfall-runoff processes, as this equation accounts for topographic and geomorphologic controls on subsurface flow and the dynamics of variable source areas during extreme rainfall events [*Paniconi et al.*, 2003]. In this respect, the question under which conditions the linearized hsB equation is able to capture the storage and flow dynamics needs to be answered.

[18] Of course, the limitation of relatively small soil depth to hillslope length ratios, relatively high hydraulic conductivity, and sharp draining soils with small capillary fringe inherent to the Boussinesq concept also restricts the applicability of the hsB equation. When these conditions are met, *Paniconi et al.* [2003] showed that the hsB equation accurately reproduces the total storage profiles and hydrographs for both convergent and divergent hillslopes as computed by a three-dimensional Richard's equation–based numerical model. We therefore can use the hsB equation as a benchmark model for analyzing the performance of its linearization. From Figures 9 and 10 of *Troch et al.* [2003] it can be seen that for small bedrock slope angles (of the order of 5%) the linearized hsB equation underestimates the total storage profiles but preserves the general shape of the storage profiles quite well. The underestimation almost completely vanishes when the bedrock slope angle increases (of the order of 30%).

[19] It is also interesting to compare the performance of the linearized hsB equation with the hillslope-storage kinematic wave (hsKW) model for which exact analytical solutions were derived by *Troch et al.* [2002]. Unlike the linearized hsB equation, the hsKW model does not include diffusion drainage processes. Comparing both models for convergent hillslopes, *Troch et al.* [2003] demonstrate that the hsKW model does not work well for small bedrock slope angles (5%) and improves only slightly for steeper slopes (30%). The general shape of the storage profiles and the hydrographs never matches the true dynamics of convergent hillslopes. On the contrary, for divergent hillslopes the hsKW works rather well for all bedrock slopes.

[20] In view of the question of usefulness of recently derived analytical solutions to storage-based subsurface flow equations for complex hillslopes, on the basis of the recently conducted model studies of *Troch et al.* [2002, 2003], *Paniconi et al.* [2003], and *Hilberts et al.* [2004], we can conclude the following: (1) It suffices to adopt the kinematic wave formulation to model the dynamic response of divergent hillslopes, independent of bedrock slope angle. (2) For uniform hillslopes with small bedrock slope angle (of the order of 5%) the introduction of the diffusion-driven drainage, as in the linearized hsB equation, significantly improves the description of the dynamic response, but for steeper slopes (of the order of 30%), again the kinematic wave approximation suffices [*Beven*, 1982]. (3) For convergent (channel head) hillslopes the use of the linearized hsB solution over the hsKW solution is recommended for all bedrock slope angles.

[21] These are useful insights into the functioning of basic hillslope types that might prove important for upscaling hillslope flow processes to the catchment scale. For example, from these analytical solutions it is easy to derive expressions for the moments of the travel time distribution of subsurface flow in complex hillslopes. These moment equations thus can be related to geomorphologic and hydraulic characteristics of the hillslopes composing the landscape, which offers opportunities for hydrological parameterization in poorly gauged basins.

[22] The above results can also be illustrated by making use of the dimensionless parameter introduced by *Brutsaert* [1994]. *Brutsaert* [1994] proposed $-dL = UL/2K$ as a characteristic dimensionless parameter of uniform hillslopes which determines the relative magnitude of gravity drainage versus diffusion drainage. For complex hillslopes discussed here, this parameter becomes

$$-dL = \tan(i)L/2pD - aL/2. \quad (36)$$

For uniform hillslopes (a = 0), equation (36) reduces to the definition of *Brutsaert* [1994]. For convergent hillslopes ($a > 0$) the value of the dimensionless parameter decreases with respect to a uniform hillslope of similar geometry (i, D, and L), and this reduction depends on the degree of convergence. This means that the relative magnitude of gravity drainage versus the diffusion term is decreased, which explains why we need to take into account diffusion effects on groundwater outflow to describe the characteristic response of convergent hillslopes independent of slope angle. This supports the conclusions reached above. In a similar way the value of this dimensionless parameter increases for divergent hillslopes, which again explains the conclusion that the kinematic wave approximation works fine for divergent hillslopes, even for small values of i.

[23] In groundwater transport studies the dimensionless parameter $-dL$ is also referred to as the Peclet (Pe) number. The Peclet number is defined as the ratio of the timescales of dispersive and advective transport from the middle of the hillslope. Our analytical solution provides a physical basis for the Peclet number for complex hillslopes. Assuming that the bulk advection through the hillslopes occurs at a fixed velocity [*Kirchner et al.*, 2001], determined by the hydraulic conductivity and the slope of the underlying impermeable layer, the Peclet number is given by

$$Pe = vL/2\delta, \tag{37}$$

where $v = (k/f)\sin(i)$, the kinematic velocity, and δ is the dispersion coefficient. From equation (36) it follows that the dispersion coefficient for a given hillslope is equal to

$$\delta = \frac{kpD\sin(i)}{f(\tan(i) - apD)}. \tag{38}$$

For uniform hillslopes ($a = 0$) the dispersion coefficient is simply $\delta = kpD\cos(i)/f$. For convergent hillslopes ($a > 0$) the dispersion coefficient always increases, and therefore the Peclet number decreases with respect to the uniform case. For divergent hillslopes ($a < 0$) the dispersion coefficient decreases, and the Peclet number increases relative to the uniform case.

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