Limit-cycle oscillators subject to a delayed feedback

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The coexistence of two stable limit cycles exhibiting different periods is examined for a nonlinear oscillator subject to a delayed feedback. For the case of a weakly nonlinear oscillator, we discuss the validity of a previously determined phase equation. For the case of a strongly nonlinear oscillator, we derive a phase equation and analyze its bifurcation diagram. Our analysis is motivated by previous experimental studies of chemical oscillators controlled by a delayed feedback.

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I. INTRODUCTION

In 2003, Beta *et al.* [1] investigated an oscillatory surface chemical reaction (CO oxidation on platinum) and studied the effect of a delayed feedback by controlling the partial pressure of one of the reactants. The control was of the form $p=p_0+\alpha[I-I(t-\tau)]$, where p and I denote the pressure of CO and the integral intensity of a photoemission electron microscope image, respectively. The delay τ was of the same order of magnitude as the period T of the homogeneous limit-cycle oscillations ($T=2-10$ s). By progressively increasing the delay, they observed that the period T exhibits a jump from $T>\tau$ to $T<\tau$ at a critical value of τ that suggests a Z-shaped branch for the period $T=T(\tau)$. However, the coexistence of two stable regimes for the same value of τ (birhythmicity [2]) could not be demonstrated because of technical difficulties. In an earlier study, Weiner *et al.* [3] were more successful. They examined the effect of delay on the oscillations of the minimal bromate oscillator in a continuous stirred tank reactor. These authors controlled the flow rate as $k=k_0\{1+\beta[C(t-\tau)-C_{av}]C_{av}^{-1}\}$, where C denotes the concentration of ceric ions Ce^{4+}, C_{av} is a constant reference value, and τ is ranging from zero to three times the period of the oscillations without delay ($T\sim 10^2$ s). They recorded the period of the oscillations by progressively increasing and then decreasing τ and found three successive regions where low and large period oscillations may coexist.

Anticipating the Z-shaped bifurcation diagram for $T=T(\tau)$, Beta *et al.* [1] provided a first theoretical explanation of the bistability phenomenon. They considered the amplitude equation of a supercritical Hopf bifurcation and derived a delay differential equation (DDE) for the phase ϕ of the oscillations. Looking for constant frequency solutions ($\phi=\sigma t$), they obtained a Z-shaped branch for $\sigma=\sigma(\tau)$.

We propose to reexamine this bistability problem for two reasons. First, the theory of Beta *et al.* [1] is valid in the limit of very weak feedback. In this case, the amplitude of the limit-cycle oscillations is not modified by the feedback and only the phase of the oscillations is perturbed. But as the feedback progressively increases from zero, we wonder how large the feedback rate must be in order to modify the amplitude of the oscillations. Second, known chemical oscillators such as the CO and the bromate oscillators are strongly nonlinear relaxation oscillators which cannot be analyzed by weakly nonlinear theories. The question is whether singular perturbation techniques appropriate for relaxation oscillators can be used for the DDE problem. Relaxation oscillators are limit-cycle oscillators that differ from the nearly conservative oscillators modeling mechanical vibrations or laser pulsating outputs. As the feedback amplitude is progressively increased from zero, both the amplitude and the phase of the oscillations are modified for a nearly conservative oscillator [30,31]. In the case of a limit-cycle oscillator, only the phase of the oscillations is modified by a very weak feedback.

In order to investigate both a weakly nonlinear and a strongly nonlinear limit-cycle oscillator, we consider the Van der Pol (VDP) equation [4,5] with a delayed feedback. The free oscillator depends on only one parameter and admits simple analytical solutions for both its weakly and strongly nonlinear oscillation limits [6]. Historically, the VDP limit cycle motivated several two-variable reductions of the Hudgkin Huxley equations [7] (Nagumo [8], FitzHugh [9], Morris-Lecar [10]). Moreover, its phase-plane description has guided several studies of two-variable chemical models [see Keener and Tyson [11] for the Belousov-Zhabotinsky (BZ) reaction and Lengyel *et al.* [12] for the chlorine dioxide-iodine-malonic acid (CIMA) reaction].

Atay [13] analyzed the weakly nonlinear VDP equation with a delayed feedback and highlighted the stabilizing effect of the delay. More recently, Pyragiené and Pyragas [14] analyzed the same weakly nonlinear VDP oscillator subject to a periodic modulation and a delayed feedback. They showed how the delay may stabilize unstable periodic orbits. Other studies of the VDP equation concentrated on different effects of a delayed feedback [15–18] or on two delayed coupled VDP oscillators [19,20]. More recently, Jiang and Wei [21] investigate VDP DDE close to a triple zero eigenvalue and a slight modification of VDP DDE is examined by Benner *et al.* [22] in the context of a delayed control.

In this paper, we derive slow time amplitude equations for the weakly nonlinear VDP equation under different conditions and discuss the validity of the phase DDE as the feedback rate progressively increases. We then consider the case of a strongly nonlinear VDP oscillator and adapt the technique used in Ref. [23], to determine a phase DDE. This equation is analyzed in terms of the delay τ. The quantitative validity of all our analytical results are tested by simulating numerically the original VDP DDE. The benefits of our asymptotic analyses compared to numerical continuation techniques is that it makes visible how the delayed feedback acts on the oscillator.

The paper is organized as follows. Section II treats the case of the weakly nonlinear DDE and considers two distinct cases depending on the strength of the feedback compared to the natural damping of the free oscillations. For very weak feedback, the bistability phenomenon is possible provided the delay is sufficiently large. The amplitude of the oscillations is unperturbed in first approximation. Gradually increasing the feedback, however, leads to a progressive change of the amplitude. The case of a strongly nonlinear DDE is then described in Sec. III where a phase equation is derived that takes into account the phase-response curve of the unperturbed relaxation oscillator. Finally, Sec. IV briefly discusses the interest of new experiments.

II. WEAKLY NONLINEAR OSCILLATOR

In this section, we analyze the case of weakly nonlinear oscillators. Specifically, we consider the following VDP equation with a delayed feedback

$$x'' + \varepsilon(x^2 - 1)x' + x = \mu[x(t-\tau) - x], \tag{1}$$

where $0 < \varepsilon \ll 1$.

A. Weak feedback: Phase equation

The simplest case is when the order of magnitude of the feedback matches the deviation of the frequency from the value that the oscillator has for $\varepsilon = 0$ and $\mu = 0$. Since this frequency correction is proportional to ε^2 [24], we take

$$\mu = \varepsilon^2 \mu_2. \tag{2}$$

We plan to construct the solution of Eq. (1) by using a multiple time scale method (see the Appendix). When constructing a formal asymptotic expansion of a solution in the usual manner, one is confronted with so-called secular terms that grow without bounds as $t \to \infty$. These terms can be removed by applying solvability conditions that determine the unknown amplitude R of the oscillations. We obtain the solution

$$x = R \exp\{i[t + \phi(t)]\} + \text{c.c.} + O(\varepsilon), \tag{3}$$

where $R=1$ and the evolution of the phase ϕ is described by the following DDE:

$$\phi' = -\frac{\varepsilon^2}{16} - \frac{\mu}{2}\{\cos[-\tau + \phi(t-\tau) - \phi] - 1\}. \tag{4}$$

Looking for a constant frequency solution of the form $\phi = \sigma t$, we find that σ satisfies the transcendental equation

$$\sigma = -\frac{\varepsilon^2}{16} + \frac{\mu}{2}[1 - \cos(\tau + \sigma\tau)]. \tag{5}$$

We determine σ and compute the period as $T = 2\pi/(1+\sigma)$. Figure 1 shows the gradual emergence of hysteresis as τ progressively increases from zero. The conditions for the appearance of a Z-shaped curve in the $\sigma = \sigma(\tau)$ diagram depends on μ and are given by $d\tau/d\sigma = d^2\tau/d\sigma^2 = 0$, together with Eq. (5). These conditions are analyzed in the Appendix and lead to the critical values $(\mu, \tau, \sigma) = [\mu_c(n), \tau_c(n), \sigma_c(n)]$. They are defined by

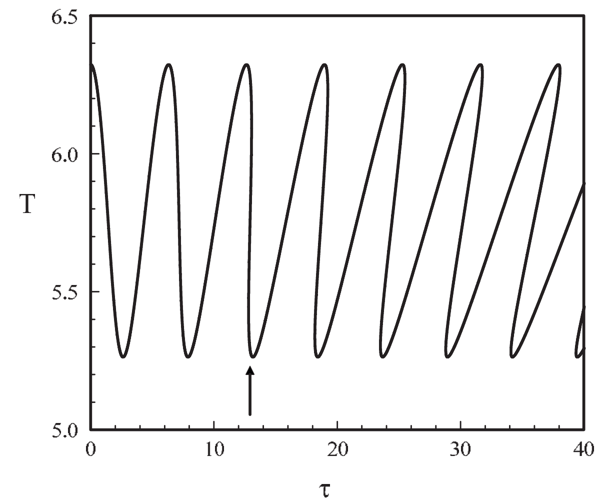

FIG. 1. Weak feedback. Period $T = 2\pi/(1+\sigma)$ as a function of τ. σ is determined from Eq. (5) with $\varepsilon^2 = 0.1$ and $\mu = 0.2$. The arrow indicates the emergence of the first Z-shaped diagram close to $\tau \simeq 13$ ($n=2$).

$$\mu_c(n) = \frac{2}{\pi/2 - 1 + 2\pi n}\left(1 - \frac{\varepsilon^2}{16}\right), \tag{6}$$

$$\tau_c(n) = \frac{\pi/2 - 1 + 2\pi n}{\left(1 - \frac{\varepsilon^2}{16}\right)}, \tag{7}$$

where $n = 0, 1, 2, \ldots$, and $\sigma_c(n)$ is obtained from Eq. (5). Computing the expressions (6) and (7) for increasing values of n, we note that for $\varepsilon^2 = 0.1$ and $\mu = 0.2$, only the Z-shaped curves corresponding to $n \geq 2$ are possible.

B. Stronger feedback: Amplitude and phase equations

The validity of our previous analysis stems from the fact that the feedback was sufficiently weak so that only the phase of the VDP oscillations is perturbed by the delay. However, if the feedback rate is progressively increased, we expect that both the amplitude and the phase of the oscillations will be affected. To investigate this possibility, we consider in Eq. (1)

$$\mu = \varepsilon \mu_1 \tag{8}$$

and seek a solution by again using the method of multiple time scales (see the Appendix). Instead of Eq. (3), the solution now is

$$x = R(t)\exp\{i[t + \phi(t)]\} + \text{c.c.} + O(\varepsilon) \tag{9}$$

where R and ϕ satisfy

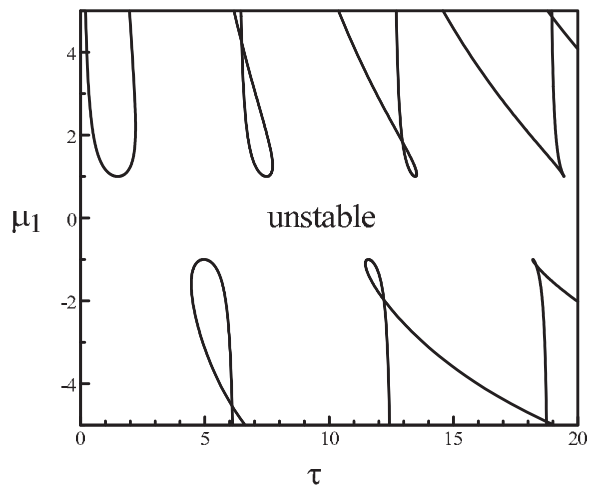

FIG. 2. Stronger feedback. Islands of stability for the zero solution appear and are bounded by Hopf bifurcation boundaries. $\varepsilon=0.1$.

$$R' = -\frac{\varepsilon}{2}R(R^2-1) + \frac{\mu}{2}R(s-\varepsilon\tau)\sin[-\tau+\phi(s-\varepsilon\tau)-\phi], \tag{10}$$

$$\phi' = -\frac{\mu}{2}\left[\frac{R(s-\varepsilon\tau)}{R}\cos[-\tau+\phi(s-\varepsilon\tau)-\phi]-1\right]. \tag{11}$$

From the expression (9), we note that a time-periodic solution for $x(t)$ corresponds to the solution $R=\text{cst}\neq 0$ and $\phi=\omega_1\varepsilon t$. From Eqs. (10) and (11), we then obtain the following conditions for R and ω_1:

$$0 = R^2 - 1 + \mu_1\sin(\tau+\omega_1\varepsilon\tau), \tag{12}$$

$$2\omega_1 = -\mu_1[\cos(\tau+\omega_1\varepsilon\tau)-1]. \tag{13}$$

These equations are analyzed in the Appendix and lead to the solution $R^2=R^2(\tau)$ in the parametric form (A23) and (A24). The Hopf bifurcation points of the basic steady state $R=0$ satisfy Eqs. (12) and (13) with $R=0$. These conditions lead to the lines in the (τ, μ_1) parameter space and are given by Eqs. (A25) and (A23). In the case of weak feedback, the zero solution was always unstable. However, if the feedback is stronger, stable regions limited by Hopf bifurcation boundaries are possible (see Fig. 2).

We compare the asymptotic approximation for $R^2=R^2(\tau)$ with the solution obtained numerically from Eq. (1). If $|\mu_1|<1$, the periodic solutions coexist with the unstable steady state and bistable branches of solutions are possible provided τ is sufficiently large [see Fig. 3(a)]. On the other hand if $|\mu_1|\geqslant 1$, these periodic solutions bifurcate from the zero solution [see Fig. 3(b)].

III. STRONGLY NONLINEAR OSCILLATOR

Most of the experimentally studied chemical oscillators exhibit strongly pulsating relaxation oscillations. In this section, we consider a piecewise linear version of the Van der Pol oscillator.

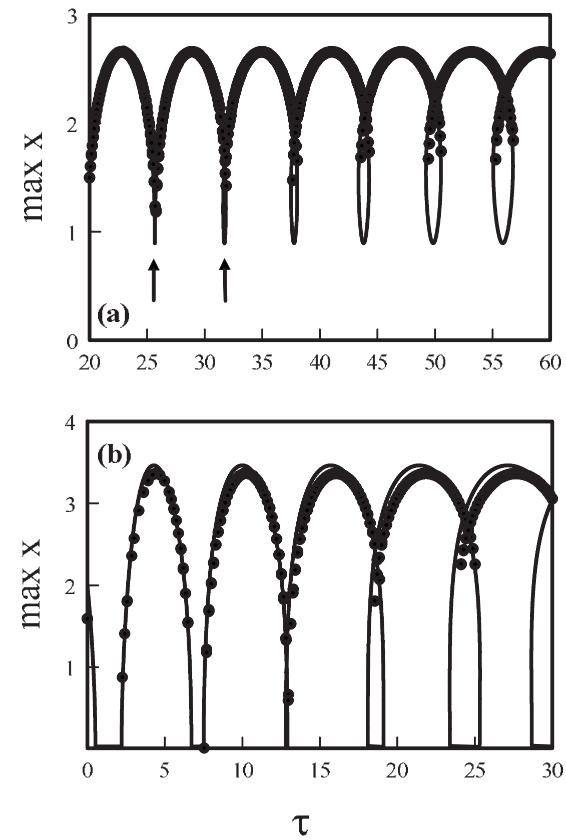

FIG. 3. Stronger feedback. Bifurcation diagram of the maxima of x as a function of τ. The dots correspond to stable periodic solutions of the VDP DDE (1) for $\varepsilon=0.1$, (a) $\mu=0.08$, and (b) $\mu=0.2$. They have been obtained numerically by progressively increasing and then decreasing τ. The full line is the asymptotic approximation representing $\max(x)=2R$ as a function of τ [they are obtained using Eqs. (A23) and (A24) with $\varepsilon=0.1$ and (a) $\mu_1=0.8$ or (b) $\mu_1=2$]. The arrows in Fig. 3(a) indicate the emergence of the first two bistable curves for $\tau\simeq 25.7$ and $\tau\simeq 32$, respectively. The two x axes do not show the same range of τ.

A. Formulation

Specifically, we wish to analyze the following DDE:

$$x'' + \nu\,\text{sgn}(x^2-1)x' + x = \delta[x(t-\nu\tau)-x], \tag{14}$$

where $\nu\gg 1$ is a large parameter. For $\delta=0$, the period admits the approximation $T\simeq 2\nu\ln(3)$. Since we are interested on how the period changes with the delay of the same order of magnitude, the parameter ν is included in the delay term. In the Liénard representation, Eq. (14) is rewritten as a system of two coupled first order differential equations. Changing time as $t\to t/\nu$ and switching to the new parameter $\varepsilon=\nu^{-2}\ll 1$, we arrive at

$$\varepsilon x' = y - f(x), \tag{15}$$

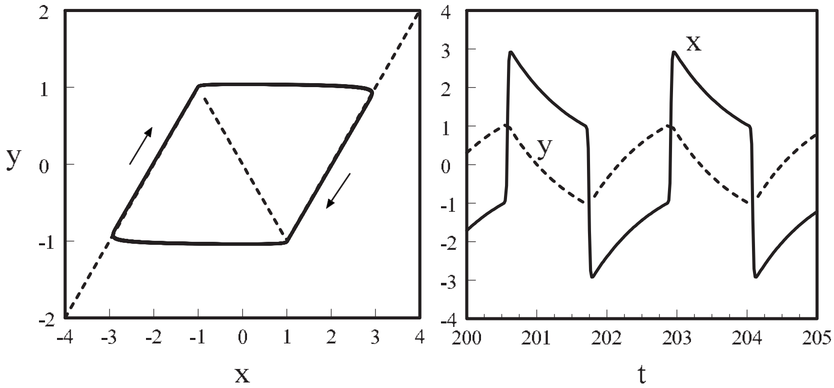

FIG. 4. Limit-cycle solution of the free oscillator (δ=0 and ε $=10^{-2}$). Left: limit-cycle orbit in the phase-plane (x,y). The broken line represents the function $y=f(x)$. Right: relaxation oscillations. The period of the oscillations is $T_1(\varepsilon)=2.32\simeq T_0=2\ln(3)$.

$$y' = -x + \delta[x(t-\tau) - x], \tag{16}$$

where $\varepsilon \ll 1$ and $f(x)$ is a piecewise linear function of x given by

$$\begin{aligned} f(x) &= -x(|x| \leq 1), \\ &= x - 2(x > 1), \\ &= x + 2(x < -1). \end{aligned} \tag{17}$$

Figure 4 represents the limit-cycle solution of Eqs. (15) and (16) in the case of no feedback (δ=0). If $\varepsilon \to 0$, the limit-cycle oscillations approach a discontinuous limit satisfying

$$y_0 = f(x_0) \text{ and } y_0' = -x_0. \tag{18}$$

Using Eq. (17), the solution of Eq. (18) is easily obtained as

$$x_0 = 3\exp(-t),\ y_0 = x_0 - 2[0 < t \leq t_0 = \ln(3)],$$

$$x_0 = -3\exp[-(t-t_0)],\ y_0 = x_0 + 2(t_0 < t \leq T_0 = 2t_0). \tag{19}$$

This analytical solution will be useful in our analysis of the DDE problem.

If $0 < \delta \ll 1$ we note that the amplitude of the oscillations does not change very much but the period T as a function of the delay admits an interesting behavior (see Fig. 5). The period varies between two extrema and exhibits bistability if τ is sufficiently large. The two extrema of the period can be determined as follows.

B. Maximum period

First, we seek a particular solution satisfying

$$x(t-\tau) = x(t). \tag{20}$$

Inserting Eq. (20) into Eqs. (15) and (16), we obtain VDP equations without feedback. It admits a limit-cycle solution of period $T_1(\varepsilon)$ [note that $T_1(\varepsilon) \to T_0$ as $\varepsilon \to 0$]. We then conclude that Eq. (20) is satisfied if

$$\tau = nT_1 \tag{21}$$

for $n=0,1,2,\ldots$.

C. Minimum period

Second, we seek another particular solution satisfying

$$x(t-\tau) = -x(t). \tag{22}$$

Inserting Eq. (22) into Eqs. (15) and (16), we obtain

$$\varepsilon x' = y - f(x), \quad y' = -x(1+2\delta). \tag{23}$$

These equations admit a limit cycle of period $T_2(\varepsilon) < T_1(\varepsilon)$ [note that $T_2(\varepsilon) \to T_0/(1+2\delta)$ as $\varepsilon \to 0$]. Because

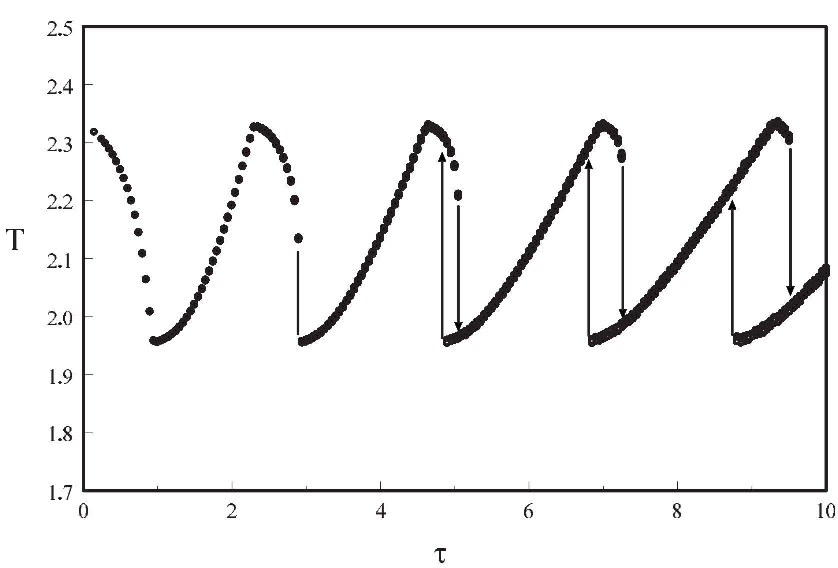

FIG. 5. Relaxation oscillations. Progressive emergence of bistable curves for T as a function of τ. The figure has been obtained by progressively increasing and then decreasing τ ($\varepsilon=10^{-2}$ and δ $=0.1$).

$x(t-T_2/2)=-x(t)$, we conclude that Eq. (22) is satisfied if

$$\tau=(1+2n)\frac{T_2}{2} \tag{24}$$

for $n=0,1,\ldots$.

D. Bistability

In order to demonstrate the bistability phenomenon, we now apply a technique developed by Grasman [23]. In the limit $\varepsilon\to 0$, Eqs. (15) and (16) reduce to

$$0=y-f(x), \tag{25}$$

$$y'=-x+\delta[x(t-\tau)-x] \tag{26}$$

holding over two sections of the orbit: the left branch with $y=x-2$ for $x<-1$ and the right one with $y=x+2$ for $x>1$ (see Fig. 4). Being at the left branch, the solution monotonically increases until it arrives at the value $(x,y)=(-1,1)$. From there, it jumps instantaneously to the landing point $(x,y)=(3,1)$ at the right branch. For δ sufficiently small, x' is negative at this point. Consequently, the solution follows this branch downwards until the leaving point $(x,y)=(1,-1)$. From there it jumps again to the left branch. Clearly, the orbit is independent of δ in first approximation. However, the phase of the oscillator may change. Eliminating y, Eqs. (25) and (26) reduces to the single DDE for x only given by

$$x'=-x+\delta[x(t-\tau)-x] \tag{27}$$

supplemented by the conditions

$$x=-3 \text{ if } x<1 \text{ and } x=3 \text{ if } x>-1. \tag{28}$$

Introducing

$$x=x_0[\phi(t)] \tag{29}$$

into Eq. (27) and using the fact that $dx_0/d\phi=-x_0$, we obtain

$$\frac{d\phi}{dt}\frac{dx_0}{d\phi}=-x_0+\delta\{x_0[\phi(t-\tau)]-x_0\},$$

$$\frac{d\phi}{dt}=1+\delta-\delta\frac{x_0[\phi(t-\tau)]}{x_0}. \tag{30}$$

We now seek a solution of Eq. (30) of the form $\phi=\phi_0(t,s)+\delta\phi_1(t,s)+\ldots$, where $s\equiv\delta t$ is defined as a slow time variable. The leading problem, $\phi_{0t}=1$, admits the solution

$$\phi_0=t+\psi(s), \tag{31}$$

where $\psi(s)$ is unknown. The next problem for ϕ_1 then reduces to

$$\phi_{1t}=-\psi_s+a-\frac{x_0[t+\psi(s-\delta\tau)-\tau]}{x_0[t+\psi(s)]}, \tag{32}$$

where we assume $\delta\tau=O(1)$. We wish that ϕ_1 remains bounded with respect to the fast time t. This implies the solvability condition

$$\frac{d\psi}{ds}=a-F(\Delta), \tag{33}$$

where

$$F(\Delta)\equiv\frac{1}{T}\int_0^T\frac{x_0(\zeta+\Delta)}{x_0(\zeta)}d\zeta \tag{34}$$

and $\Delta=\psi(s-\delta\tau)-\psi-\tau$ is assumed constant in the ζ integral. We next seek a solution of Eq. (33) of the form

$$\psi=\sigma s+\psi_0 \tag{35}$$

and compute $F(\Delta)$, where Δ is reducing to

$$\Delta=-(1+\delta\sigma)\tau \tag{36}$$

and is assumed to be negative. We find

$$F=\frac{\exp(-\Delta)}{t_0}\left[t_0+\frac{4\Delta}{3}\right](0<-\Delta<t_0), \tag{37}$$

$$=\frac{\exp(-\Delta)}{3t_0}\left[-\frac{4\Delta}{3}-\frac{7t_0}{3}\right](t_0<-\Delta<2t_0), \tag{38}$$

where $t_0=\ln(3)$. The function is represented in Fig. 6. Injecting the expression (35) into Eq. (33), we find that σ satisfies the following equation:

$$\sigma=1-F(\Delta), \tag{39}$$

where Δ is defined by the expression (36). This equation admits an analytical solution in parametric form. From (36), we obtain

$$\tau=\frac{-\Delta}{1+\delta\sigma}=\frac{-\Delta}{1+\delta[1-F(\Delta)]}. \tag{40}$$

Equations (39) and (40) provide a parametric solution for $\sigma=\sigma(\tau)$, where Δ is the parameter. The period is then computed as $T=2\pi/(1+\delta\sigma)$. It is shown by a full line in Fig. 7 and agrees quantitatively with the numerical solution obtained from the original equations.

IV. DISCUSSION

We have shown that a delayed feedback applied to a nonlinear limit-cycle oscillator may heavily interfere with the mode of free oscillation. Its period may change substantially even for a weak feedback. Furthermore, this change may be accompanied by the so-called birhythmicity phenomenon [2]: stable high and low period oscillations may coexist provided the delay is sufficiently large. This phenomenon may occur for both weakly and strongly nonlinear oscillators. For weakly nonlinear oscillations, we improved the approximation method of Ref. [1] that was based on the adiabatic approximation in which the delay was neglected in the equation for the amplitude. In addition, we considered the case of a progressively stronger feedback for the weakly nonlinear oscillator which leads to a larger perturbation of the amplitude and possibly to successive Hopf bifurcations. It is not clear whether this effect will also be present for the strongly non-

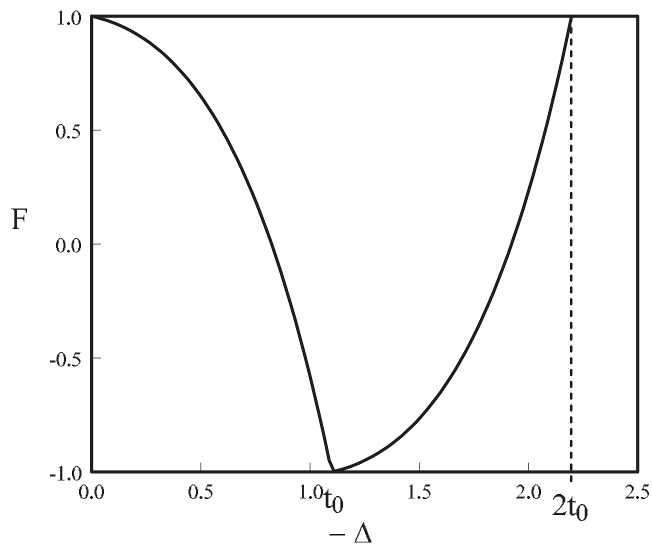

FIG. 6. The phase-shift function F as a function of $-\Delta > 0$.

linear oscillator. In the latter case, we introduced a method [23] that, at a first glance, looks completely different form the one for the weakly nonlinear case. However, when comparing the objective of the two techniques, we note that they both perform a type of averaging over the period of the oscillator.

Our investigation concentrated on the dynamics of the Van der Pol oscillator. We need to be careful if we wish to generalize our conclusions for any nonlinear oscillator. In the literature, the Van der Pol oscillator has frequently been used as a prototype for the analysis of periodic phenomena in chemical and biological processes, see Refs. [2,6,23]. However, further numerical and experimental studies are desired in which a feedback with a progressively increasing strength is considered. The illuminated Belousov-Zhabotinsky reaction [25,26] or the illuminated chlorine dioxide-iodine-malonic-acid reaction [27,28], under spatially uniform conditions, are good candidates for systematic experimental studies. A theoretical analysis of the BZ patterns subject to a delayed feedback control was recently proposed in Ref. [29]. In a next step, we shall examine the case of strongly nonlinear oscillators subject to both a weak delayed feedback and weak diffusion.

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APPENDIX: THE WEAKLY NONLINEAR THEORY

1. Weak feedback

We seek a solution of Eq. (1) with Eq. (2) of the form

$$x = x_0(t,s) + \varepsilon x_1(t,s) + \varepsilon^2 x_2(t,s) + \cdots, \quad \text{(A1)}$$

where $s \equiv \varepsilon^2 t$ is defined as a slow time variable. The assumption of two independent time variables implies the chain rules

$$x' = x_t + \varepsilon^2 x_s,$$

$$x'' = x_{tt} + 2\varepsilon^2 x_{ts} + \varepsilon^4 x_{ss}, \quad \text{(A2)}$$

where subscripts mean partial derivatives. After inserting the expressions (2), (A1), and (A2) into Eq. (1), we equate to zero the coefficients of each power of ε. We then obtain a sequence of linear problems for the unknown functions $x_0, x_1, x_2, \ldots$, which are given by

$$x_{0tt} + x_0 = 0, \quad \text{(A3)}$$

$$x_{1tt} + x_1 = -(x_0^2 - 1)x_{0t}, \quad \text{(A4)}$$

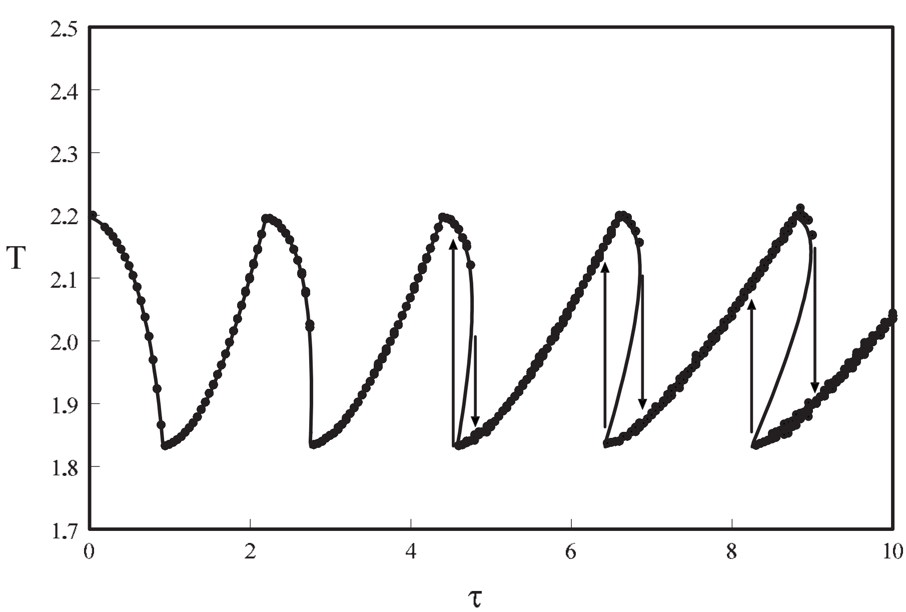

FIG. 7. Period as a function of τ for the reduced DDE problem ($\varepsilon=0$). The dots represents the period of the numerical solutions (δ=0.1). The line is the analytical approximation valid in the limit δ small.

$$x_{2tt}+x_2=-2x_0x_1x_{0t}-(x_0^2-1)x_{1t}+\mu_2[x_0(t-\tau)-x_0]-2x_{0ts}. \tag{A5}$$

The solution of Eq. (A3) is

$$x_0=A(s)\exp(it)+\text{c.c.}, \tag{A6}$$

where A is an unknown function of s. A bounded solution for x_1 then requires the solvability condition

$$\int_0^{2\pi}\text{RHS}\exp(\pm it)dt=0, \tag{A7}$$

where RHS means the right-hand side of Eq. (A4) and s is kept constant in the t integral. Eq. (A7) leads to the condition

$$(AA^*-1)A=0. \tag{A8}$$

The solution of Eq. (A4) then is

$$x_1=[A_1(s)\exp(it)+\text{c.c.}]+\left(\frac{1}{8}iA^3\exp(3it)+\text{c.c.}\right), \tag{A9}$$

where the first and second terms represent the solution of the homogeneous problem and the particular solution, respectively. A_1 is a new unknown amplitude. Finally, the solvability condition (A7) applied to Eq. (A5) leads to an equation for A_1 given by

$$2iA'=-iA(AA_1^*+2A^*A_1)+iA_1+\frac{1}{8}A^3A^{*2}$$
$$+\mu_2[A(s-\varepsilon^2\tau)\exp(-i\tau)-A], \tag{A10}$$

where we have kept the slow time delay assuming $\varepsilon^2\tau=O(1)$ or larger.

We are now ready to analyze the conditions on the amplitudes A and A_1. Introducing $A=R\exp(i\phi)$ and $A_1=R_1\exp(i\phi)$ into Eqs. (A8) and (A10), we find $R=1$ and the following two conditions for R_1 and ϕ:

$$-2\phi'=\frac{1}{8}+\mu_2\{\cos[-\tau+\phi(s-\varepsilon^2\tau)-\phi]-1\}, \tag{A11}$$

$$0=-2R_1+\mu_2\sin[-\tau+\phi(s-\varepsilon^2\tau)-\phi]. \tag{A12}$$

Equation (A11) is a DDE for ϕ while Eq. (A12) provides R_1 as a function of ϕ. The constant frequency solutions are found by introducing $\phi=\sigma_2 s$ into Eq. (A11). We then obtain a transcendental equation for σ_2 of the form

$$\sigma_2=-\frac{1}{16}+\frac{\mu_2}{2}[1-\cos(\tau+\varepsilon^2\sigma_2\tau)]. \tag{A13}$$

The conditions for the emergence of hysteresis in the $\sigma_2=\sigma_2(\tau)$ diagram depends on μ_2 and are given by $d\tau/d\sigma_2=d^2\tau/d\sigma_2^2=0$, together with Eq. (A13). We successively find equations for μ_2, τ, and σ_2 given by

$$1-\frac{\mu_2\varepsilon^2\tau}{2}\sin(\tau+\varepsilon^2\sigma_2\tau)=0,$$

$$\cos(\tau+\varepsilon^2\sigma_2\tau)=0,$$

$$\sigma_2+\frac{1}{16}-\frac{\mu_2}{2}=0. \tag{A14}$$

Solving for μ_2, we obtain

$$\mu_2=\frac{2}{\varepsilon^2(\pi/2+n2\pi-1)}\left(1-\frac{\varepsilon^2}{16}\right) \tag{A15}$$

while τ and σ_2 are related to μ_2 by

$$\tau=\frac{2}{\varepsilon^2\mu_2}\text{ and }\sigma_2=-\frac{1}{16}+\frac{\mu_2}{2}. \tag{A16}$$

2. Stronger feedback

We now seek a solution of the form (A1) with Eq. (8), where the slow time is $s\equiv\varepsilon t$. The problem for x_0 is still given by Eq. (A3) but Eq. (A4) is now modified as

$$x_{1tt}+x_1=-(x_0^2-1)x_{0t}+\mu_1[x_0(t-\tau)-x_0]-2x_{0ts}. \tag{A17}$$

The solution of Eq. (A3) is (A6) and the solvability condition (A7) for Eq. (A17) now requires that

$$2iA'=-iA(AA^*-1)+\mu_1[A(s-\varepsilon\tau)\exp(-i\tau)-A]. \tag{A18}$$

Introducing $A=R\exp(i\phi)$ into Eq. (A18), we obtain

$$2R'=-R(R^2-1)+\mu_1R(s-\varepsilon\tau)\sin[-\tau+\phi(s-\varepsilon\tau)-\phi], \tag{A19}$$

$$2\phi'=-\mu_1\left[\frac{R(s-\varepsilon\tau)}{R}\cos[-\tau+\phi(s-\varepsilon\tau)-\phi]-1\right]. \tag{A20}$$

Periodic solutions of Eq. (A17) correspond to solutions of Eqs. (A19) and (A20) with $R=cst\neq 0$ and $\phi=\omega_1 s$. From Eqs. (A19) and (A20), we find

$$0=R^2-1+\mu_1\sin(\tau+\omega_1\varepsilon\tau), \tag{A21}$$

$$\omega_1=-\frac{\mu_1}{2}[\cos(\tau+\omega_1\varepsilon\tau)-1]. \tag{A22}$$

We wish to determine R and ω_1 as functions of τ. From Eq. (A22) we obtain $\omega_1=\omega_1(\tau)$ in the implicit form

$$\tau=\frac{1}{1+\omega_1\varepsilon}\arccos\left(1-\frac{2\omega_1}{\mu_1}\right), \tag{A23}$$

where $0\leqslant\omega_1\leqslant\mu_1$. Having ω_1, R^2 is found from Eq. (A21). Using (A23), the expression of R^2 simplifies as

$$R^2 = 1 - 2\sqrt{\omega_1(\mu_1 - \omega_1)} \geqslant 0. \quad \text{(A24)}$$

The amplitude R does no more plays a passive role and may even become zero at bifurcation points. Setting $R=0$ into (A24) and solving for ω_1 give

$$\omega_1 = \frac{\mu_1 \pm \sqrt{\mu_1^2 - 1}}{2} \quad \text{(A25)}$$

provided that $|\mu_1| \geqslant 1$.

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