Calculation Methods for Two-dimensional Groundwater Flow

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P. van der Veer

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for

Two-dimensional Groundwater Flow

PROEFSCHRIFT

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PETER VAN DER VEER

CIVIEL-INGENIEUR
GEBOREN TE APELDOORN

DIT PROEFSCHRIFT IS GOEDGEKEURD DOOR DE PROMOTOR

PROF. DR. IR. J. C. VAN DAM

To my wife Puck
and
my daughter Nienke

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Contents

Page **Part III: Analytical Function Method**

Introduction

The motion of groundwater plays an important role in many civil engineering projects. Low-situated polders receive groundwater from the surroundings, while from water reservoirs there may be a groundwater flow towards the surroundings if the water table in the reservoir is higher than the groundwater head in the surroundings. In cases of groundwater recovery for public water supply or at a trench, it is important to know the consequences of the abstraction of groundwater; a sharp fall of the groundwater head may cause damage to structures and to the vegetation. If it is desired to confine the adverse effects of groundwater recovery, it is important to be able to predict the consequences of various actions. Knowledge of the groundwater motion plays a role in problems of groundwater pollution and groundwater management too; the stability of dikes and shore constructions also depends on the motion of groundwater (e.g., the behaviour at a seepage surface and overpressures under an impermeable dike revetment). It is thus seen that it is important to have a good knowledge of the motion of groundwater.

Although in many cases it is possible to find mathematical solutions for groundwater flow problems, it is often difficult to define exactly the relevant parameters in a flow problem. Generally the average coefficient of permeability can be determined rather well in situ, but it is much more complicated to determine the anisotropy, especially when the direction of the anisotropy is not known previously. It is almost impossible to determine in situ the exact location and permeability of thin layers of clay or silt; in addition, these layers may have sharply differing properties. The same problems are encountered in determining the properties of resistance layers on the talus and bottom of a canal. Generally in a calculation these lines are not equipotential lines; in other words, the effect of the resistance layers (clay or silt) may not be neglected in the calculation. An extra complication in these problems is the alteration of the resistance properties of the layer when it bursts off as a result of overpressures. All this shows how unsure a description of groundwater flow may be. This does not mean, however, that there is no need for reliable calculation techniques. The availability of effective mathematical tools can give a good insight into the effects of any alteration of parameters of the problem. In many cases, therefore, geohydrologic calculations consist of some calculations of the same problem with varying parameters. In that way an insight is obtained into the consequences of the lack of knowledge of parameters of the problem. That experience can be useful for the decision whether additional information has to be compiled, for example, by field measurements.

The performance of reliable geohydrologic calculations may result in a lower safety coefficient for the relevant work because more knowledge has been obtained about the phenomenon.

This treatise on calculation methods has been restricted to two-dimensional groundwater flow. Although in reality groundwater flow is always three-dimensional, in many cases the flow can be assumed to be two-dimensional.
Such cases are, for example, a horizontal flow region without replenishment of water from above or below; in vertical flow regions the flow can be assumed to be two-dimensional if the relevant section is present over a great length (dikes, canals, rivers and long structures).
Some basic assumptions have been made:
– The porous medium is incompressible.
– The coefficient of permeability is sectionally constant; in other words, the coeffecient of permeability may vary if the variation is not continuous but step-wise. There may be resistance layers between different sand layers. The soil may be anisotropic.
– The fluids are incompressible and there is an abrupt alteration of density going from one fluid to another.
– There is only groundwater in a saturated zone.
– Anywhere and at any moment the flow is laminar. This means that Darcy's relationship between groundwater head-gradient and specific discharge always holds.

I Basic Theory

1 Basic Laws

The groundwater head ϕ is defined by

$$\phi = \frac{p}{\rho g} + y \tag{I.1}$$

where

p – pressure with respect to a reference pressure (atmospheric pressure)
ρ – density
g – acceleration of gravity
y – elevation with respect to a reference level.

The specific discharge v_s in a direction s is given by Darcy's Law:

$$v_s = -k\frac{\partial\phi}{\partial s}$$

where k is the coefficient of permeability.

The condition of continuity for an element $dxdy$ is given by the following expression (see figure I.1): $d(v_x dy + v_y dx) + P(x, y, t)dxdy = 0$.

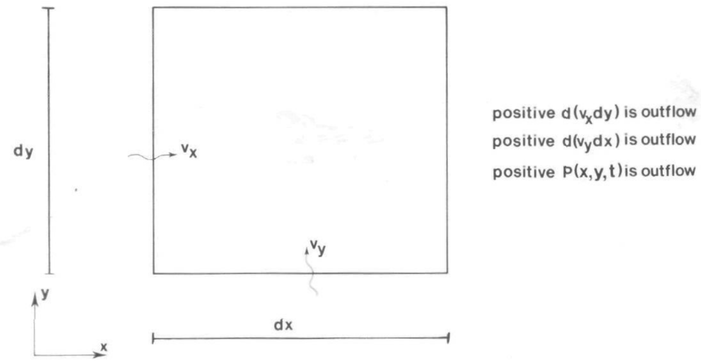

Figure I.1 Flow through an element.

In this expression $P(x, y, t)$ is the amount of water abstraction per unit of time and per unit of area.
From the previous expression the continuity equation is derived:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = -P(x, y, t) \tag{I.2}$$

If Darcy's Law is combined with the continuity equation (I.2) a differential equation is obtained that defines the flow problem together with the boundary conditions. For a homogeneous isotropic porous medium the differential equation becomes:

$$k\frac{\partial^2 \phi}{\partial x^2} + k\frac{\partial^2 \phi}{\partial y^2} = P(x, y, t) \tag{I.3}$$

In the case that $P(x, y, t) = 0$, for each x, y, t (I.3) reduces to the well-known differential equation of Laplace:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{I.4}$$

In that case the function $\phi(x, y)$ is apparently a harmonic function (see Appendix 1).

2 Definitions

Potential

The relationship between specific discharge and groundwater head is given by Darcy's Law; for a homogeneous isotropic porous medium the following expressions hold:

$$v_x = -k\frac{\partial \phi}{\partial x} \quad \text{and} \quad v_y = -k\frac{\partial \phi}{\partial y}$$

The potential Φ is defined as the product of the groundwater head and the coefficient of permeability: $\Phi = k\phi$. With this variable, Darcy's Law can be written in a form where the specific discharges are derived from the potential:

$$v_x = -\frac{\partial \Phi}{\partial x} \quad \text{and} \quad v_y = -\frac{\partial \Phi}{\partial y} \tag{I.5}$$

Because the potential Φ is equal to $k\phi$ with constant k, also Φ satisfies the differential equation of Laplace (in the case that $P(x, y, t) = 0$). So the potential is a harmonic function (see appendix 1).

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

Stream function

Another important function is the stream function Ψ. This function has the property that the amount of water that per unit of time passes a line between two points is equal to the stream function difference between those points.
This follows from the definition of the stream function:

$$v_x = -\frac{\partial \Psi}{\partial y} \text{ and } v_y = \frac{\partial \Psi}{\partial x} \tag{I.6}$$

(For example: the amount of water that passes a vertical line between two points at a mutal distance Δy is equal to $v_x \Delta y$. According to (I.6), this is equal to $-\Delta\Psi$).
If the expressions (I.6) are substituted in the continuity equation (I.2) with $P(x, y, t) = 0$, it is seen that the stream function also satisfies the Laplace differential equation; so Ψ is a harmonic function too:

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0$$

The relationship between the potential and the stream function is found by combining the expressions (I.5) and (I.6). This yields:

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y} \text{ and } \frac{\partial \Phi}{\partial y} = - \frac{\partial \Psi}{\partial x}$$

These are just the Cauchy-Riemann relationships (see Appendix 1), which means that the functions Φ and Ψ are conjugate harmonic functions. So in a homogeneous isotropic porous medium, equipotential lines (Φ = constant) and stream lines (Ψ = constant) are perpendicular.

Complex potential

Using the potential Φ and the stream function Ψ, another important function can be defined:

The complex potential Ω is defined by:

$$\Omega = \Phi + i\Psi$$

Because Φ and Ψ are conjugate harmonic functions of x and y, the complex potential Ω is an analytical function of the complex variable $z = x + iy$.

3 Anisotropy

If the coefficient of permeability has not the same magnitude in any direction, the porous medium is anisotropic. By a simple geometric transformation the anisotropic region (coordinates x and y; $z = x + iy$) can be transformed into another region (coordinates $\underline{x}$ and $\underline{y}$; $\underline{z} = \underline{x} + i\underline{y}$) that can be considered as an isotropic region:

Therefore a region with coordinates x and y is considered in which the maximum and the minimum coefficients of permeability are k_{max} and k_{min}. The maximum coefficient of permeability occurs in a direction x^* that has an angle α with the x-axis, as shown in figure I.2:

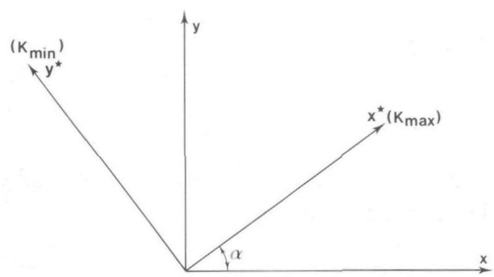

Figure I.2 Main directions of permeability.

The relationship between $z = x + iy$ and $z^* = x^* + iy^*$ is given by:

$$z^* = z \exp(-i\alpha)$$

In other notation:

$$x^* + iy^* = (x + iy)(\cos\alpha - i\sin\alpha)$$

From this it follows:

$$x^* = x\cos\alpha + y\sin\alpha$$

$$y^* = y\cos\alpha - x\sin\alpha$$

According to Darcy's Law the following expressions hold:

$$v_{x^*} = -k_{\max}\frac{\partial\phi}{\partial x^*} \text{ and } v_{y^*} = -k_{\min}\frac{\partial\phi}{\partial y^*} \tag{I.7}$$

The stream function Ψ is defined by:

$$v_{x^*} = -\frac{\partial\Psi}{\partial y^*} \text{ and } v_{y^*} = \frac{\partial\Psi}{\partial x^*} \tag{I.8}$$

By eliminating of v_{x^*} and v_{y^*} from (I.7) and (I.8) it follows:

$$\frac{\partial\Psi}{\partial x^*} = -k_{\min}\frac{\partial\phi}{\partial y^*} \qquad \frac{\partial\Psi}{\partial y^*} = k_{\max}\frac{\partial\phi}{\partial x^*} \tag{I.9}$$

Now the coordinates $z^* = x^* + iy^*$ are transformed into $\underline{z} = \underline{x} + i\underline{y}$ according to:

$$x^* = \underline{x}\sqrt{k_{\max}} \qquad y^* = \underline{y}\sqrt{k_{\min}} \tag{I.10}$$

Substitution of (I.10) in (I.9) yields:

$$\frac{\partial\Psi}{\partial\underline{x}} = -\sqrt{k_{\min}k_{\max}}\frac{\partial\phi}{\partial\underline{y}} \text{ and } \frac{\partial\Psi}{\partial\underline{y}} = \sqrt{k_{\min}k_{\max}}\frac{\partial\phi}{\partial\underline{x}} \tag{I.11}$$

A fictive coefficient of permeability for the $\underline{z}$ - plane is defined by $k = \sqrt{k_{\min}k_{\max}}$. Then (I.11) can be written in the following form:

$$\frac{\partial\Psi}{\partial\underline{x}} = -k\frac{\partial\phi}{\partial\underline{y}} \text{ and } \frac{\partial\Psi}{\partial\underline{y}} = +k\frac{\partial\phi}{\partial\underline{x}}$$

If the potential Φ is defined as $\Phi = k\phi$ for the $\underline{z}$ - plane there come the following expressions:

$$\frac{\partial\Psi}{\partial\underline{x}} = -\frac{\partial\Phi}{\partial\underline{y}} \text{ and } \frac{\partial\Psi}{\partial\underline{y}} = +\frac{\partial\Phi}{\partial\underline{x}}$$

or:

$$\frac{\partial\Phi}{\partial\underline{x}} = \frac{\partial\Psi}{\partial\underline{y}} \text{ and } \frac{\partial\Phi}{\partial\underline{y}} = -\frac{\partial\Psi}{\partial\underline{x}} \tag{I.12}$$

The relationships (I.12) between the stream function Ψ and the potential Φ in the

$\underline{z}$ – plane are just the Cauchy-Riemann relationships. So Φ and Ψ are conjugate harmonic functions in the $\underline{z}$ – plane. So the functions Φ and Ψ satisfy the Laplace differential equation and $\Omega = \Phi + i\Psi$ is an analytical function of $\underline{z} = \underline{x} + i\underline{y}$.
So the flow in an anisotropic region can be calculated in the same way as for an isotropic region if the following transformations are applied (where the direction of the maximum coefficient of permeability has an angle α with the positive x-axis):

$$z^* = z \exp(-i\alpha)$$

$$\underline{x} = x^*/\sqrt{k_{\max}}$$
$$\underline{y} = y^*/\sqrt{k_{\min}}$$

In other notation, where $\underline{z}$ is expressed directly in z, this is:

$$\underline{z} = \frac{1}{\sqrt{k_{\max}}} \mathrm{Re}\{z \exp(-i\alpha)\} + \frac{i}{\sqrt{k_{\min}}} \mathrm{Im}\{z \exp(-i\alpha)\}$$

The fictive coefficient of permeability for the $\underline{z}$-plane is:

$$k = \sqrt{k_{\max} k_{\min}}$$

4 Boundary Conditions

A groundwater flow problem is defined by:
- continuity condition,
- darcy's Law, and
- boundary conditions (of space and time).

In addition, factors as anisotropy and inhomogeneities have to be taken into account and also the variation of the density of the fluids. In this Chapter a schematic survey of the most important boundary conditions is given, as they are relevant for the treatise of the existing solution methods in Part II. A more comprehensive formulation is given in Chapter 14 in a form that is convenient for the Analytical Function Method of Part III.

Equipotential line

An equipotential line is a line of constant potential Φ. For a homogeneous porous medium, also the groundwater head ϕ is constant along such a line.

Stream line

A stream line is a line along which a fictive water particle moves (where the physical porestructure is neglected). No water passes a stream line. From the definition of Ψ (see (I.6)) it follows that the stream function is constant along a stream line.

Seepage line

A seepage line is a line along which the water leaves the soil and freely streams off. The thickness of the seepage layer is generally very small. Therefore at the seepage line the pressure may be assumed to be equal to the atmospheric pressure (which is the reference pressure that is set to zero). From the definition of the groundwater head ϕ (see (I.1)), it follows:

$$p = g(\rho\phi - \rho y).$$

Substitution of $p = 0$ in this expression gives the boundary condition at the seepage line:

$$\phi = y.$$

So a seepage line is a line along which the groundwater head is equal to the known height y.

Phreatic line

In a steady state a phreatic line without precipitation or evaporation is a stream line. Then the stream function is constant. If there is precipitation, the stream function Ψ increases per unit of length with an amount that is equal to the precipitation on that unit of length. In any case (steady or non-steady), the condition holds that the pressure along the phreatic line is equal to the atmospheric pressure. Then the same condition is arrived at as for the seepage line:

$$\phi = y.$$

So a phreatic line is also a line for which the groundwater head is equal to the height y (although generally y is not previously known).

Interface

An interface is a separation line between two fluids with different densities. In a steady state an interface is a stream line; then the stream function Ψ is constant. In any case (steady or non-steady), the condition holds that the pressure on both sides of the interface is equal.

From the definition of the groundwater head ϕ (see (I.1)), it follows:

$$p = g(\rho\phi - \rho y).$$

If this expression is used to define the pressure on one side of the interface and the pressure on the other side is defined by:

$$p_c = g(\rho_c\phi_c - \rho_c y),$$

it follows from $p = p_c$ that

$$\rho\phi - \rho y = \rho_c\phi_c - \rho_c y,$$

from which:

$$\phi = \frac{\rho - \rho_c}{\rho} y + \frac{\rho_c\phi_c}{\rho} \qquad \text{(I.13)}$$

So an interface is a line along which the groundwater head ϕ varies in a prescribed way with y (although y generally is not previously known).

If $\rho_c = 0$ is substituted in (I.13) or, in other words, on one side of the interface the density is zero, the expression reduces to:

$$\phi = y,$$

which is just the condition for the phreatic line.

So a phreatic line may be considered as the special case of an interface when one of the fluids has no density.

II Existing Solution Methods

In this Part a general survey is given of the most important solution methods. The principles of each method are discussed briefly and without a complete derivation, and the methods are illustrated with examples. The following sub-division is used:

- Analytical methods. In these methods a solution in a closed form is obtained.
- Approximative methods. In some of these methods an approximative closed form is obtained (e.g., in the method of fragments, see Chapter 8), while in other approximative methods a solution is found where the relationship between the parameters that define the problem is not directly visible (e.g., the finite element method, see Chapter 11).

Analytical Methods

5 Direct Methods

Direct methods are methods where the solution is obtained by direct operations of the differential equation that defines the flow.

5.1 Direct Integration

Generally direct integration of a differential equation is only possible for one-dimensional flow. A very simple example with one-dimensional flow in a confined aquifer is given in figure II.1:

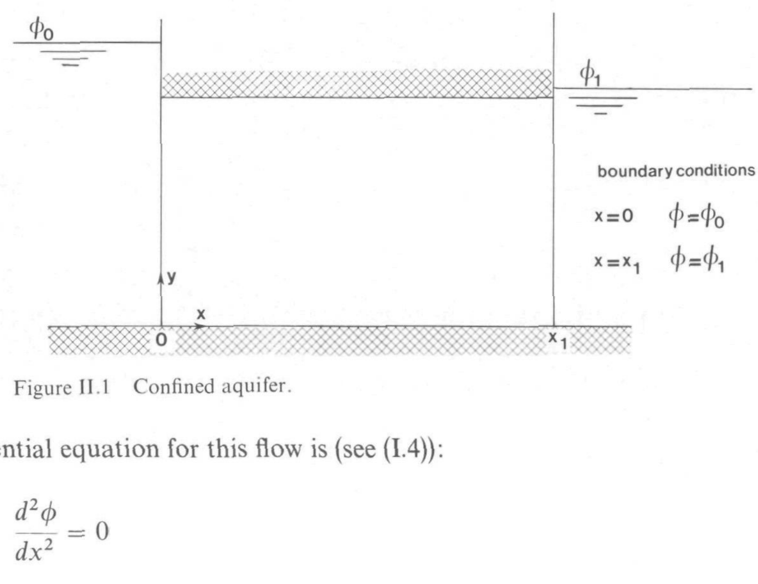

Figure II.1 Confined aquifer.

The differential equation for this flow is (see (I.4)):

$$\frac{d^2\phi}{dx^2} = 0$$

From direct integration it follows:

$$\phi = c_1 x + c_0$$

Substitution of the boundary conditions yields the solution:

$$\phi = \frac{(\phi_1 - \phi_0)}{x_1} x + \phi_0$$

5.2 Separation of Variables

This method has been comprehensively treated in literature (see, for example, Wylie, 1966, Arfken, 1970).
The solution of the differential equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{II.1}$$

is assumed to be the sum or the product of two functions of respectively x and y

$$\phi = F(x) + G(y), \tag{II.2}$$

$$\text{or: } \phi = F(x)\, G(y). \tag{II.3}$$

If the form (II.2) is used, then from substitution in the differential equation (II.1) it follows:

$$\frac{\partial^2 F(x)}{\partial x^2} + \frac{\partial^2 G(y)}{\partial y^2} = 0 \tag{II.4}$$

Because $F(x)$ and $G(y)$ are functions of respectively x and y, (II.4) can only be satisfied if $F''(x)$ as well as $G''(y)$ are constant. Then it follows from direct integration:

$$\begin{aligned} F(x) &= a_1 x^2 + b_1 x + c_1 \\ G(y) &= a_2 y^2 + b_2 y + c_2 \end{aligned} \tag{II.5}$$

By substitution of (II.5) in (II.4) it follows:

$$a_1 = -a_2$$

If $a_1 = a$ and $c_1 + c_2 = c$, then the solution has the following form:

$$\phi = a(x^2 - y^2) + b_1 x + b_2 y + c$$

The four degrees of freedom in this expression depend on the boundary conditions of the problem.

If the product of the two functions $F(x)$ and $G(y)$ is used, see (II.3), the following expression is found by substitution of (II.3) in (II.1):

$$\frac{F(x)}{F''(x)} = -\frac{G(y)}{G''(y)}$$

This expression can only be satisfied if both members of the equation are constant:

$$\frac{F(x)}{F''(x)} = c \text{ and } \frac{G(y)}{G''(y)} = -c$$

Examples of functions that satisfy these equations are $\sin(x)$, $\cos(x)$, $\sinh(x)$, $\cosh(x)$, $\exp(x)$, etc.
For example, consider the following functions:

$$F(x) = \phi_1 \exp(ax) \text{ and}$$
$$G(y) = \cos(ay)$$

The flow problem is defined by:

$$\phi = \phi_1 \exp(ax) \cos(ay)$$

This function generates the groundwater flow through a dam of variable width (see figure II.2). The width of the dam is seen from $\phi_1 = \phi_1 \exp(ax) \cos(ay)$ or $y = \frac{1}{a} \arccos(\exp(-ax))$. Then the width of the dam is:

$$b = \frac{\pi}{2a} - \frac{1}{a}\arccos(\exp(-ax))$$

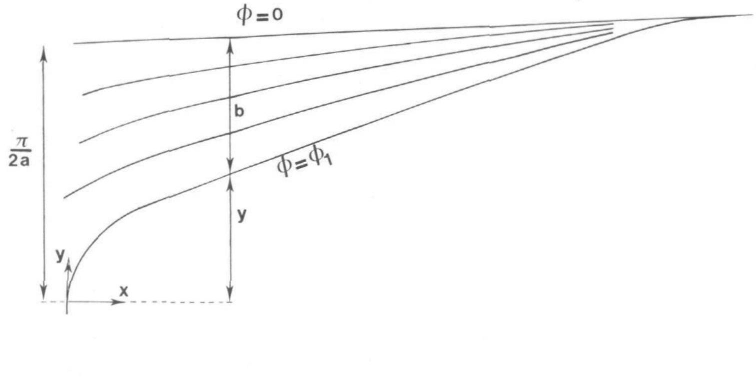

Figure II.2 Horizontal flow through a dam of variable width.

5.3 Integral Transforms

An integral transform is an operation of the differential equation that generally results in an equation or differential equation that is easier to solve. By an inverse transformation of the solution of that equation, the solution of the flow problem is found. For the transformations useful tables are available (see, e.g., Erdélyi, 1954). In the literature the theory of integral transforms has been given (Churchill, 1958, Sneddon, 1972, Ditkin and Prudnikov, 1965). Applications are given by Bruggeman, 1972.

In this section some important integral transforms are mentioned and some applications are shown. The integral transforms that will be discussed are those of Laplace and Fourier.

Generally the Laplace transform is used to transform the time variable and the Fourier transform is used for one of the space variables x, y.

Generally a linear integral transform has the following form:

$$T\{F(t)\} = \int_a^b K(t, s)\, F(t)\, dt$$

The various integral transforms differ in the type of $K(t, s)$ and the integration boundaries a and b.

LAPLACE TRANSFORM

The Laplace transform of a function $F(t)$ results in a function $f(s)$ and is defined by:

$$L\{F(t)\} = \int_0^\infty \exp(-st)F(t)dt = f(s)$$

A more convenient form is:

$$L\{h(x, t)\} = \int_0^\infty \exp(-st)h(x, t)dt = \bar{h}(x, s)$$

Some properties are:

$$\left.\begin{aligned} L\left\{\frac{\partial h(x, t)}{\partial t}\right\} &= s\,\bar{h}(x, s) - h(x, 0) \\ L\left\{\frac{\partial^2 h(x, t)}{\partial x^2}\right\} &= \frac{\partial^2 \bar{h}(x, s)}{\partial x^2} \\ L\{a\} &= \frac{a}{s} \end{aligned}\right\} \quad \text{II.6}$$

As an example, the problem of Edelman is chosen (see Bruggeman, 1972): horizontal groundwater flow towards a canal in which the water table is lowered suddenly:

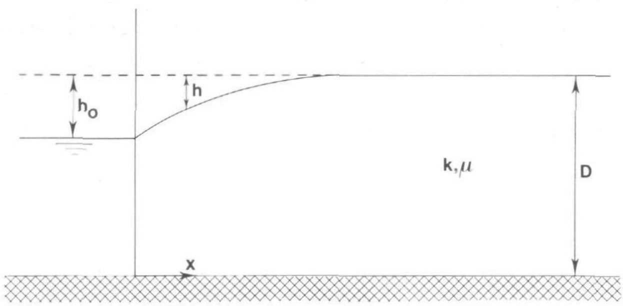

Figure II.3 The problem of Edelman.

The differential equation that defines two-dimensional groundwater flow is (see (I.3)):

$$k\frac{\partial^2\phi}{\partial x^2} + k\frac{\partial^2\phi}{\partial y2} = P(x, y, t)$$

As the flow is assumed to be horizontal, the variation of ϕ in y-direction may be neglected. Then the differential equation becomes:

$$k\frac{\partial^2\phi}{\partial x^2} = P(x, t)$$

Along the phreatic line $\phi = y$ (see Chapter 4). The differential equation will be written in a form where the original water table is the reference level and h denotes the lowering of the phreatic line (then $\phi = -h$ at the phreatic line).
The term $P(x, t)$ was defined as the abstraction of water in an element at location x at time t (see figure I.1); here it is the storage increase that is caused by the increase of the elevation of the phreatic line.
The term $P(x, t)$ is a quantity per unit of area and per unit of time (see Chapter 1).

Here the storage increase $\left(-\mu\frac{\partial \mathrm{h}}{\partial t}\right)$ occurs along the phreatic line (where μ is the storage coefficient). To obtain the storage increase per unit of area, $\left(-\mu\frac{\partial h}{\partial t}\right)$ has to be divided by the height of the water column. It is assumed that $h \ll D$, and so D may be used for the height of the water column.

According to this assumption, the term $P(x, t)$ may be written as:

$$P(x,t) = -\frac{\mu}{D}\frac{\partial h}{\partial t}$$

(negative $\partial h/\partial t$ means outflow from a vertical element $h\,dx$; so negative $\partial h/\partial t$ conforms to positive $P(x, t)$).

The differential equation now becomes (where $\phi = -h$):

$$k\frac{\partial^2 h}{\partial x^2} = \frac{\mu}{D}\frac{\partial h}{\partial t}$$

or:

$$\frac{\partial^2 h}{\partial x^2} = \frac{\mu}{kD}\frac{\partial h}{\partial t} \tag{II.7}$$

The boundary conditions are:

$$\begin{aligned} h(x,0) &= 0 \\ h(0,t) &= h_0 \\ h(\infty,t) &= 0. \end{aligned}$$

Application of (II.6) to the differential equation (II.7) yields the transformed equation:

$$\frac{\partial^2 \bar{h}}{\partial x^2} = \frac{\mu}{kD}(s\bar{h} - h(x,0)) \tag{II.8}$$

In this equation the first boundary condition ($h(x, 0) = 0$) can be substituted; and the two other boundary conditions can be written in transformed form if (II.6) is used:

$$\left.\begin{aligned} &\bar{h}(0,s) = h_0/s \text{ and} \\ &\bar{h}(\infty,s) = 0 \end{aligned}\right\} \tag{II.9}$$

If $h(x, 0) = 0$ is substituted in (II.8), the following form is found:

$$\frac{\partial^2 \bar{h}}{\partial x^2} - \frac{\mu}{kD}s\bar{h} = 0 \tag{II.10}$$

A general solution for this differential equation is:

$$\bar{h} = A\exp(\alpha x) + B\exp(-\alpha x) \tag{II.11}$$

By substitution of this solution in the differential equation (II.10) and inserting the boundary conditions (II.9) in (II.11), the following expression is found:

$$\bar{h} = \frac{h_0}{s} \exp\left(-x\sqrt{\frac{\mu s}{kD}}\right)$$

This is a well-known form. The inverse transformation can be found in tables. The solution of the problem is then found to be:

$$h = h_0 \operatorname{erfc}\left(\frac{x}{2}\sqrt{\frac{\mu}{kDt}}\right)$$

The function erfc(x) is defined by 1-erf(x) and:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2)du$$

FOURIER TRANSFORM

Depending on the type of problem, some kinds of Fourier transforms can be applied. In the following example the infinite Fourier-sine transform is used:
The infinite Fourier-sine transform of a function $F(x)$ results in a function $f(r)$; it is defined by:

$$S\{F(x)\} = \int_0^\infty F(x) \sin(rx)dx = f(r)$$

Written in a more convenient form:

$$S\{h(x,t)\} = \int_0^\infty h(x,t) \sin(rx)dx = \bar{h}(r,t)$$

Some properties are:

$$\left.\begin{aligned}
&S\left\{\frac{\partial^2 h(x,t)}{\partial x^2}\right\} = -r^2\bar{h}(r,t) + rh(0,t) \\
&(\text{if: } h(\infty, t) = \frac{\delta}{\delta x} h(\infty, t) = 0) \\
&S\left\{\frac{\partial}{\partial t} h(x,t)\right\} = \frac{\partial \bar{h}(r,t)}{\partial t} \\
&S\{0\} = 0
\end{aligned}\right\} \quad \text{(II.12)}$$

The inverse transformation is given by:

$$h(x,t) = \frac{2}{\pi} \int_0^\infty \bar{h}(r,t) \sin(rx) dr \tag{II.13}$$

The same example as for the Laplace transform is shown here to illustrate this Fourier transform (see Bruggeman, 1972). The differential equation that generates the flow is (see (II.7)):

$$\frac{\partial^2 h}{\partial x^2} = \frac{\mu}{kD} \frac{\partial h}{\partial t} \tag{II.14}$$

The boundary conditions are:

$$\begin{aligned} h(x,0) &= 0 \\ h(0,t) &= h_0 \\ h(\infty,t) &= 0 \end{aligned}$$

The boundary condition $h(\infty, t) = 0$ has already to be satisfied for application of the Fourier-sine transform (it is a condition in II.12; our problem satisfies it). Application of (II.12) to the differential equation (II.14) yields the transformed equation:

$$-r^2\bar{h} + r h(0,t) = \frac{\mu}{kD} \frac{\partial \bar{h}}{\partial t}$$

The boundary condition $h(0,t) = h_0$ can be substituted in this equation.
The remaining condition $h(x,0) = 0$ is given in transformed form by:

$$\bar{h}(x,0) = 0 \tag{II.15}$$

By inserting the boundary condition for $h(0,t)$ in the transformed differential equation, the following expression is found:

$$\frac{\partial \bar{h}}{\partial t} + r^2 \frac{kD}{\mu} \bar{h} = r h_0 \frac{kD}{\mu} \tag{II.16}$$

A general solution for this equation is:

$$\bar{h} = A \exp(\alpha t) + B \exp(-\alpha t) + C \tag{II.17}$$

By substitution of (II.17) in (II.16) and inserting the boundary condition (II.15) in the general solution (II.17), the following expression for $\bar{h}$ is found:

$$\bar{h} = h_0\left[\frac{1}{r} - \exp(-kDtr^2/\mu)\,/r\right]$$

Using (II.13) the inverse transform h can be found:

$$h = \frac{2}{\pi}\int_0^\infty h_0\left[\frac{1}{r} - \exp(-kDtr^2/\mu)/r\right]\sin(rx)dr$$

or:

$$h = \frac{2h_0}{\pi}\int_0^\infty \frac{\sin(rx)}{r}dr - \frac{2h_0}{\pi}\int_0^\infty \frac{\exp(-kDtr^2/\mu)}{r}\sin(rx)dr$$

The first integral is equal to $\pi/2$, and the second can be reduced to a well-known function. Then the solution of the problem is found:

$$h = h_0\left[1 - \operatorname{erf}\left(\frac{x}{2}\sqrt{\frac{\mu}{kDt}}\right)\right]$$

or:

$$h = h_0 \operatorname{erfc}\left(\frac{x}{2}\sqrt{\frac{\mu}{kDt}}\right)$$

6 Indirect Methods

6.1 Superposition

The superposition principle holds for the Laplace differential equation:
If ϕ_1 and ϕ_2 are solutions of $\nabla^2\phi = 0$, then $\phi_3 = a\phi_1 + b\phi_2$ is also a solution. This can be shown simply:

$$\nabla^2\phi_3 = \nabla^2 (a\phi_1 + b\phi_2) = a\nabla^2\phi_1 + b\nabla^2\phi_2$$

$$\nabla^2\phi_1 = 0; \ \nabla^2\phi_2 = 0 \text{ then } \nabla^2\phi_3 = 0$$

Examples of application of the superposition method are flow in an infinite plane with sources and sinks, and the method of images.

Example 1: The flow pattern due to a sink and a source in a linear flow field in an infinite plane is found by a superposition of three separate flow patterns (see figure II.4).

Linear flow field : $\phi_1 = ax + b$

Source in (x_2, y_2) with discharge Q: $\phi_2 = \dfrac{-Q}{2\pi k} \ln[\sqrt{(x - x_2)^2 + (y - y_2)^2}]$

Sink in (x_3, y_3) with discharge Q : $\phi_3 = \dfrac{+Q}{2\pi k} \ln[\sqrt{(x - x_3)^2 + (y - y_3)^2}]$

Example 2: The method of images is used for problems with sources and sinks where some special boundary conditions can be satisfied if extra fictive sinks or sources are located outside the region of interest.
In the case of a straight boundary, the property is used that in the middle between a sink and a source the straight line perpendicular to the connecting line is an equipotential line. In the case of two sinks or two sources, that line is a stream line, (see figure II.5).

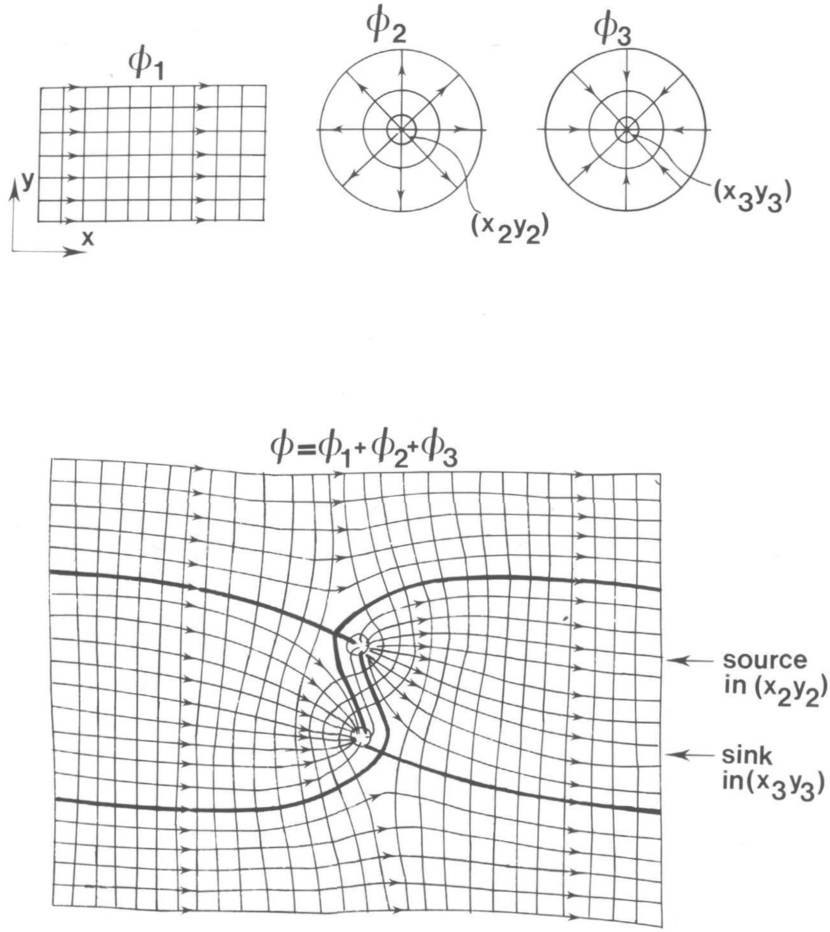

Figure II.4 Superposition of solutions.

Another application of the method of images is the flow field of an excentrically located sink in a circular island, see figure II.6. Here the property is used that in the case of a source and a sink in an infinite plane, the equipotential lines are circles. This can be shown by a simple derivation:
Consider a sink and a source, both with discharge Q, in respectively (x_1, y_1) and (x_2, y_2). The groundwater head in the plane is given by:

$$\phi = \frac{Q}{2\pi k} \ln\left[\frac{[(x - x_1)^2 + (y - y_1)^2]^{\frac{1}{2}}}{[(x - x_2)^2 + (y - y_2)^2]^{\frac{1}{2}}}\right] \qquad \text{(II.18)}$$

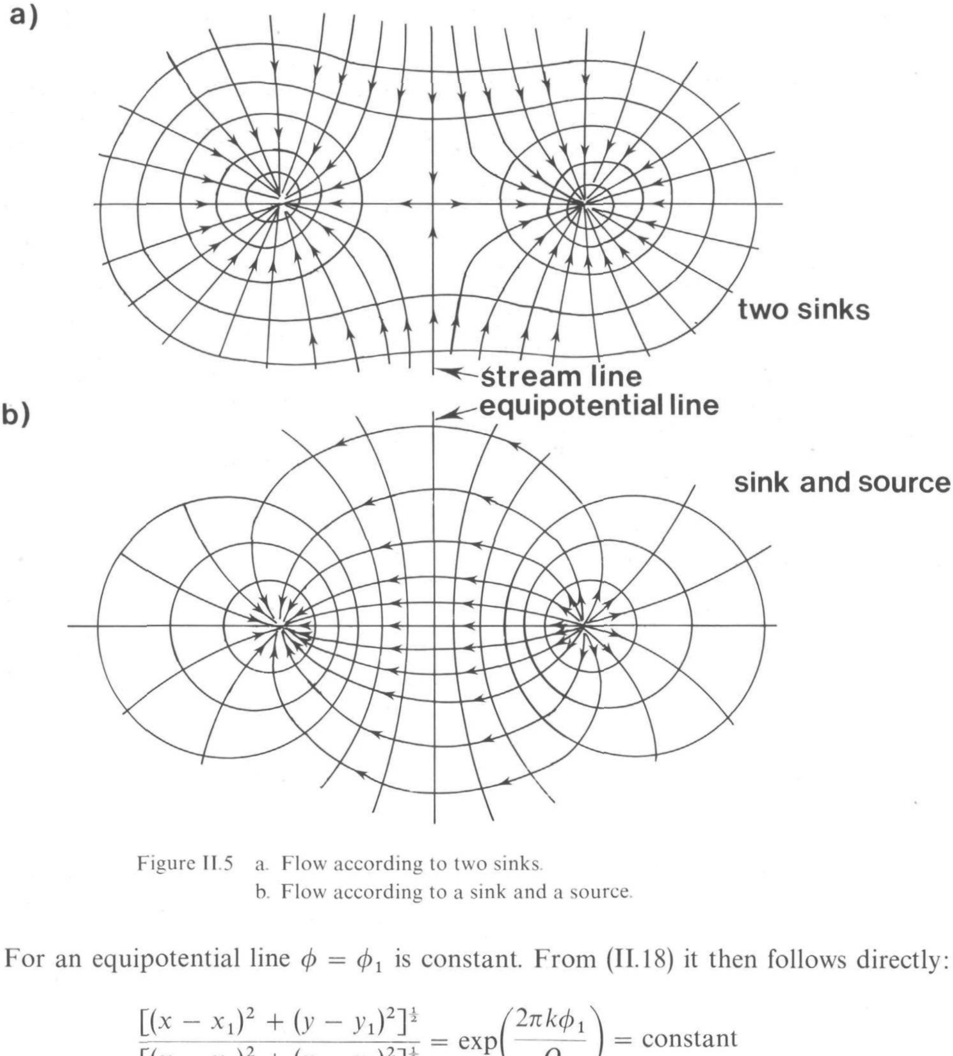

Figure II.5 a. Flow according to two sinks.
b. Flow according to a sink and a source.

For an equipotential line $\phi = \phi_1$ is constant. From (II.18) it then follows directly:

$$\frac{[(x-x_1)^2+(y-y_1)^2]^{\frac{1}{2}}}{[(x-x_2)^2+(y-y_2)^2]^{\frac{1}{2}}} = \exp\left(\frac{2\pi k\phi_1}{Q}\right) = \text{constant}$$

or:

$$(x-x_1)^2+(y-y_1)^2 = \exp\left(\frac{4\pi k\phi_1}{Q}\right)\left[(x-x_2)^2+(y-y_2)^2\right]$$

This can be written as:

$$\left(x-\frac{x_1-ax_2}{1-a}\right)^2+\left(y-\frac{y_1-ay_2}{1-a}\right)^2 = \left(\frac{x_1-ax_2}{1-a}\right)^2+\left(\frac{y_1-ay_2}{1-a}\right)^2-$$

$$-\left(\frac{{x_1}^2-a{x_2}^2}{1-a}\right)-\left(\frac{{y_1}^2-a{y_2}^2}{1-a}\right) \tag{II.19}$$

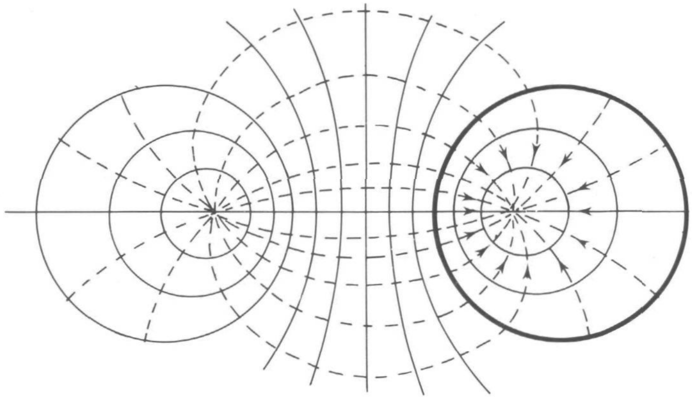

Figure II.6 Excentrically located sink in a circular island.

where: $a = \exp(4\pi k\,\phi_1/Q)$

The expression (II.19) is the equation of a circle with its centre at:

$$x_0 = \frac{x_1 - ax_2}{1 - a} \quad \text{and} \quad y_0 = \frac{y_1 - ay_2}{1 - a} \tag{II.20}$$

and with radius R according to:

$$R^2 = \left(\frac{x_1 - ax_2}{1 - a}\right)^2 + \left(\frac{y_1 - ay_2}{1 - a}\right)^2 - \left(\frac{{x_1}^2 - a{x_2}^2}{1 - a}\right) - \left(\frac{{y_1}^2 - a{y_2}^2}{1 - a}\right)$$

Generally only (x_1, y_1) is known and the location of the centre of the island. From (II.20) the location of the fictive source can be calculated, yielding:

$$x_2 = (x_1 - x_0(1 - a))/a \quad \text{and } y_2 = (y_1 - y_0(1 - a))/a$$

With these known values of (x_2, y_2), the groundwater head distribution over the island can be calculated using (II.18).

6.2 Green's Function

The solution method that uses Green's function has been described in literature (see, for example, Bear, Zaslavski, Irmay, 1968 and De Wiest, 1969). The method will be illustrated here with a simple example.

The flow is considered in a region D that has a smooth boundary C (see Bear, Zaslavski, Irmay, 1968). The values of the head in the points $A(x, y)$ in D are $\phi(A)$; ϕ satisfies $\nabla^2\phi = 0$; so ϕ is a harmonic function. Suppose that the values of ϕ are known in all points of C as a function of the location s on the boundary according to a function $f(s)$. Then it is possible to express $\phi(A)$ in another harmonic function $G(A, A')$ that is called a Green's function. The function $G(A, A')$ is a function of the coordinates of two points ($A(x, y)$ and $A'(x', y')$) such that for each point $A(x, y)$ the value $\phi(A)$ can be expressed as a function of the values of $f(s)$ at the points A' on C and the partial derivate of G with respect to n (n is the inner normal on C):

$$\phi(A) = \int_C f(s) \frac{\partial G(A, A')}{\partial n(A')} ds(A') \quad A' \text{ on } C \tag{II.21}$$

In literature (see, e.g. Bear, Zaslavski, Irmay, 1968) for several rather simply shaped regions the Green's functions can be found. For example, for the right half plane Green's function is given by:

$$G(x, y; x', y') = \frac{1}{4\pi} \ln\left[\frac{(x - x')^2 + (y - y')^2}{(x + x')^2 + (y - y')^2}\right] \tag{II.22}$$

As an illustration, this method is used to describe the flow in a half plane with the following conditions:

$$\phi(x', y')_{y'<0} = \phi_1/2 \qquad \phi(x', y')_{y'>0} = -\phi_1/2$$

Figure II.7 gives a sketch of the flow pattern:

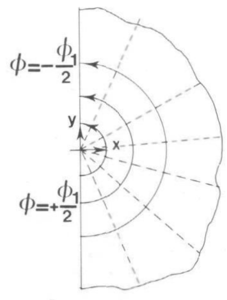

Figure II.7 Vortex flow.

The solution is the flow according to a vortex in the origin:

$$\phi = -\frac{\phi_1}{\pi} \operatorname{arctg}\left(\frac{y}{x}\right) \tag{II.23}$$

This solution can also be found by means of the Green's function method, using (II.21) and (II.22).
The combination of (II.21) and (II.22) yields an expression for the groundwater head in the right half plane:

$$\phi = \frac{1}{4\pi}\int_{\infty}^{0}\left(\frac{-\phi_1}{2}\right)\frac{\partial}{\partial x'}\ln(F)\,\partial y' + \frac{1}{4\pi}\int_{0}^{-\infty}\frac{\phi_1}{2}\frac{\partial}{\partial x'}\ln(F)\partial y' \qquad \text{(II.24)}$$

where $\quad F = \dfrac{(x - x')^2 + (y - y')^2}{(x + x')^2 + (y - y')^2}$

After differentiation and substitution of $x' = 0$, (II.24) becomes:

$$\phi = \frac{\phi_1}{2\pi}x\int_{y'\to\infty}^{0}\frac{\partial y'}{x^2 + (y - y')^2} - \frac{\phi_1}{2\pi}x\int_{y'=0}^{-\infty}\frac{\partial y'}{x^2 + (y - y')^2}$$

or:

$$\phi = \frac{\phi_1}{2\pi}\,x\int_{(y'-y)\to\infty}^{-y}\frac{\partial(y' - y)}{x^2 + (y' - y)^2} - \frac{\phi_1}{2\pi}\,x\int_{(y'-y)=-y}^{-\infty}\frac{\partial y'}{x^2 + (y - y')^2}$$

$$\phi = \frac{\phi_1}{2\pi}\left[\operatorname{arctg}\left(\frac{y' - y}{x}\right)\right]_{(y'-y)\to\infty}^{-y} - \frac{\phi_1}{2\pi}\left[\operatorname{arctg}\left(\frac{y' - y}{x}\right)\right]_{(y'-y)=-y}^{-\infty}$$

$$\phi = -\frac{\phi_1}{\pi}\operatorname{arctg}\left(\frac{y}{x}\right)$$

With this expression the result (II.23) is obtained.

7 Methods based on Complex Function Theory

7.1 Method of Pavlovskii

This method is well-known in literature (Bear, Zaslavski, Irmay, 1968, Verruyt 1970). A region in the z-plane can be conformally mapped upon a region in another plane, e.g., by elementary functions such as $\sin(z)$, $\ln(z)$, z^n, etc. A region with a polygon boundary can be mapped on the upper half plane Im $\{\zeta\} \geqslant, 0$ by the well-known Schwarz-Christoffel integral:

$$z = \alpha \int_0^\zeta (\lambda - \xi_1)^{-k_1} (\lambda - \xi_2)^{-k_2} \ldots (\lambda - \xi_n)^{-k_n} d\lambda + \beta \qquad \text{(II.25)}$$

In this expression ξ_j are the images of the edge points of the polygon on the real axis ξ. The factors k_j correspond to the alteration of direction in each edge point according to $k_j = \alpha/\pi$, see figure II.8; α en β are complex constants.

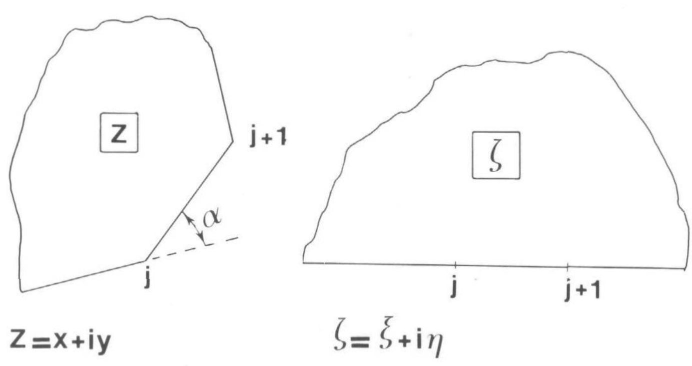

Figure II.8 Conformal mapping by the Schwarz-Christoffel integral.

A large number of examples of conformal representations is given by Kober (1957). In literature (see, for example, Verruyt 1970) a more comprehensive treatise of the Schwarz-Christoffel integral can be found.

In the method of Pavlovskii some problems are solved by conformal mapping of Ω on z, while other problems are solved by mapping the z-plane on another plane and solving the problem in that plane. For both kinds of problems, an example is given:

Example 1: Flow in a confined aquifer to a low-situated polder, see figure II.9.

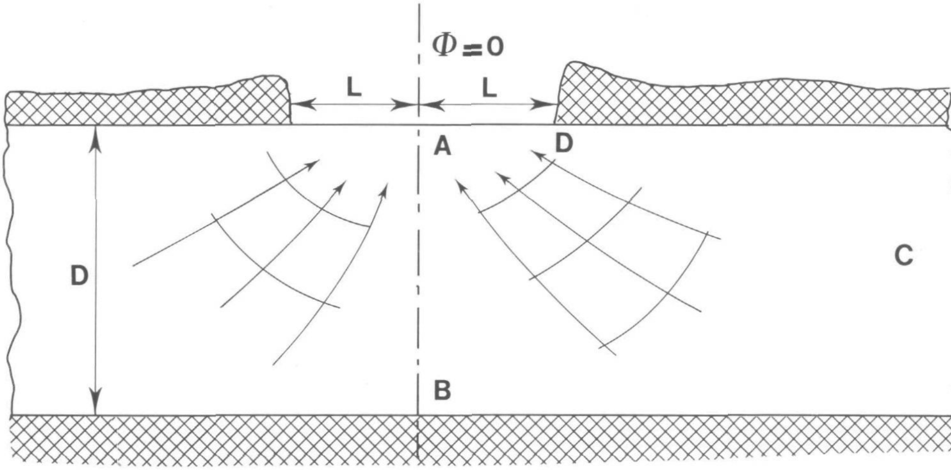

Figure II.9 Polder aquifer.

The problem is symmetrical; therefore it is sufficient to consider half the flow region. Figure II.10 gives the z-plane and the Ω-plane:

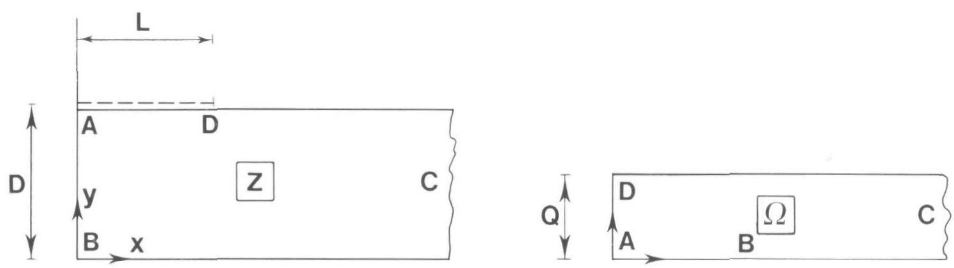

Figure II.10 z-plane and Ω-plane.

The application of the sine transform (see Appendix 2) for both the Ω- and z-regions yields the mappings as represented in the upper half planes ζ and ζ^*, see figure II.11:

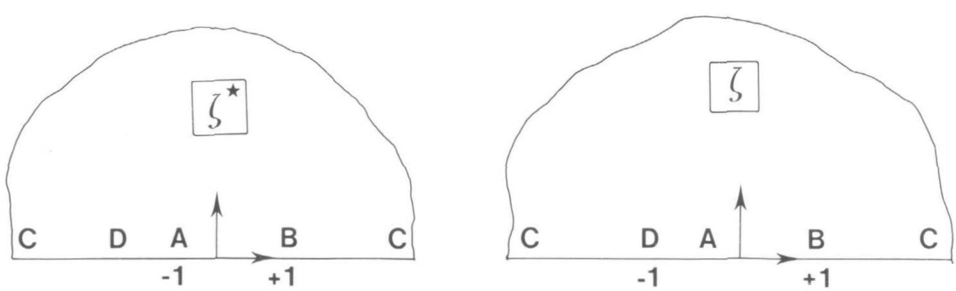

Figure II.11 Upper half planes ζ^* and ζ.

For this mapping the z-plane and the Ω-plane are first rotated over $\pi/2$. Then they are linearly transformed, and translated over $\pi/2$. So they are first mapped upon a half infinite strip with width π (figure II.12).

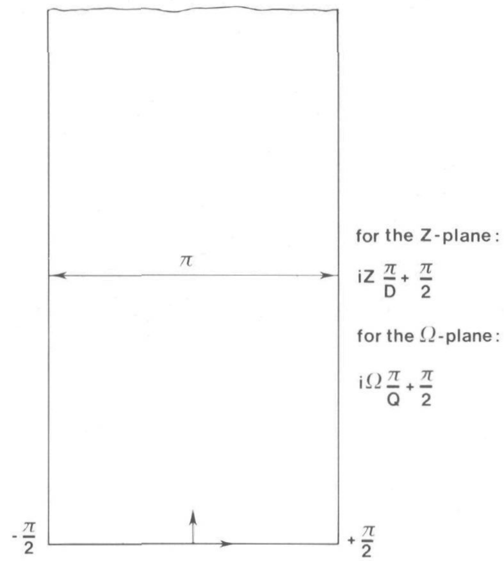

Figure II.12 Mapping upon a half infinite strip.

The mapping functions are:

$$\zeta^* = \sin\left(iz\frac{\pi}{D} + \frac{\pi}{2}\right)$$

$$\zeta = \sin\left(i\Omega\frac{\pi}{Q} + \frac{\pi}{2}\right)$$

This can be reduced to:

$$\zeta^* = \cos\left(iz\frac{\pi}{D}\right) = \cosh\left(\frac{z\pi}{D}\right) \tag{II.26}$$

$$\zeta = \cos\left(\frac{i\Omega\pi}{Q}\right) = \cosh\left(\frac{\Omega\pi}{Q}\right) \tag{II.27}$$

The relationship between the ζ^*- and the ζ-plane is found by a transformation, so that the points D and B of the ζ^*-plane are mapped upon the points -1 and $+1$;

$$\zeta = 1 - \frac{2(1 - \zeta^*)}{1 - \zeta_D^*}$$

From figure II.10 it follows: $z_D = L + iD$

Then according to (II.26):

$$\zeta_D^* = \cos\left(i\frac{\pi}{D}(L + iD)\right) = \cos\left(-\pi + i\frac{L}{D}\pi\right) = \cos\left(\pi - iL\frac{\pi}{D}\right) =$$

$$= -\cos\left(\frac{iL\pi}{D}\right) = -\cosh\left(\frac{L\pi}{D}\right)$$

Then ζ is given by:

$$\zeta = 1 - \frac{2(1 - \zeta^*)}{1 + \cosh\left(\frac{L\pi}{D}\right)} \tag{II.28}$$

A combination of the expressions (II.26), (II.27) and (II.28) yields the relationship between Ω and z:

$$\Omega = \frac{Q}{\pi}\operatorname{arccosh}\left\{1 - 2\frac{\left[1 - \cosh\left(\frac{z}{D}\pi\right)\right]}{\left[1 + \cosh\left(\frac{L}{D}\pi\right)\right]}\right\} \tag{II.29}$$

This expression is the solution of the flow problem. Because only half the flow region is considered, Q represents half the seepage discharge per m'. Q can be calculated if at some distance from the polder the groundwater head with respect to the polder level is known:
From (II.29) it follows using $\Phi = k\phi$:

$$\phi = \operatorname{Re}\left\{\frac{Q}{k\pi}\operatorname{arccosh}\left[1 - 2\frac{\left[1 - \cosh\left(\frac{z}{D}\pi\right)\right]}{\left[1 + \cosh\left(\frac{L}{D}\pi\right)\right]}\right]\right\}$$

or:

$$Q = \frac{\pi k \phi}{\operatorname{Re}\left\{\operatorname{arccosh}\left[1 - 2\frac{\left[1 - \cosh\left(\frac{z}{D}\pi\right)\right]}{\left[1 + \cosh\left(\frac{L}{D}\pi\right)\right]}\right]\right\}}$$

With this formula Q can be calculated; ϕ is the difference between the groundwater head at the point z and the polder level.

Example 2: A number of sinks in the vicinity of a contraction of a canal, see figure II.13:

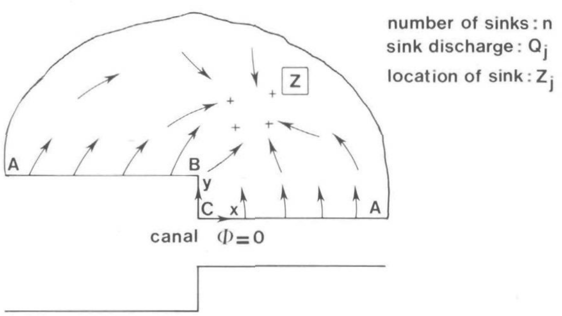

Figure II.13 Sinks near a canal.

The boundary *ABCA* is assumed to be an equipotential line.
The z-plane is mapped on the upper half plane using the Schwarz-Christoffel integral (II.25), see figure II.14:

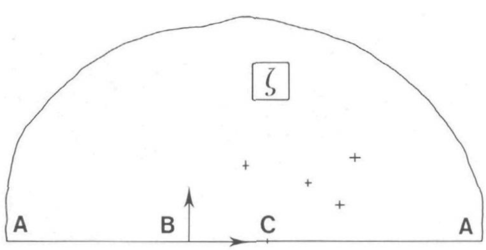

Figure II.14 Upper half plane ζ.

The point A is mapped onto infinity and does not appear in the integral (see Verruyt, 1970).
The alterations of direction in the points B and C of the polygon are $-\pi/2$ and $\pi/2$.

Then: $k_B = -\frac{1}{2} \qquad k_C = +\frac{1}{2}$

The points B and C are chosen to be mapped on $\zeta = 0$ and $\zeta = 1$. Then the Schwarz-Christoffel integral gets the following form for this problem:

$$z = \alpha \int_0^\zeta \left(\frac{\lambda}{(\lambda - 1)} \right)^{\frac{1}{2}} d\lambda + \beta$$

After integration the constants α and β are determined in such a way that the points B and C correspond with the points $z = i$ and $z = 0$ in the z-plane, see figure II.13. That leads to:

$$z = \frac{1}{\pi} \{ \sqrt{\zeta(\zeta - 1)} + \ln[\sqrt{\zeta} + \sqrt{(\zeta - 1)}] \} \tag{II.30}$$

In the ζ-plane the problem can be solved simply by use of the method of images:

$$\Omega = \sum_{j=1}^{n} \frac{Q_j}{2\pi} \ln \left[\frac{\zeta - \zeta_j}{\zeta - \bar{\zeta}_j} \right] \tag{II.31}$$

The expressions (II.30) and (II.31) form together the solution of the flow problem. Therefore ζ is eliminated from (II.30) with known z. Then from (II.31) the complex potential Ω can be evaluated.

7.2 Method of Vedernikov-Pavlovskii

This method can be found in literature (Bear, Zaslavski, Irmay, 1968; Verruyt, 1970).
The function Z is defined by:

$$Z = \Omega + ikz$$

From this definition it follows by separating real and imaginary parts (using $Z \equiv Z_1 + iZ_2$):

$$Z_1 = \Phi - ky$$

$$Z_2 = \Psi + kx$$

Using the function Z (Zhukovski's function) problems can be solved with boundaries

that consist solely of horizontal equipotential lines, vertical stream lines and a phreatic line.
For a horizontal equipotential line Z_1 is constant:

$$Z_1 = \Phi_1 - ky_1 = \text{constant}$$

For a vertical stream line Z_2 is constant:

$$Z_2 = \Psi_1 + kx_1 = \text{constant}$$

For a phreatic line without precipitation $Z_1 = 0$. This can be shown by substituting the boundary condition $\phi = y$ and thus $\Phi = ky$ in the expression for Z_1:

$$Z_1 = \Phi - ky = ky - ky = 0$$

The properties just mentioned, can be used in the problem of Nelson-Skornakov (see Verruyt, 1970). The problem deals with a flow under a dike that separates a low-lying polder from higher surroundings, see figure II.15:

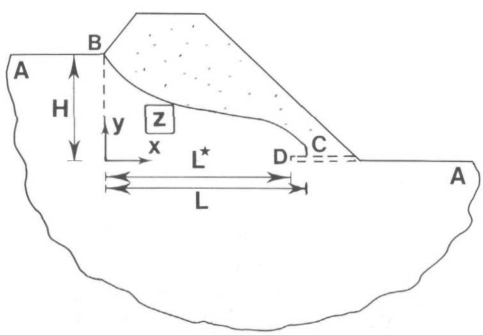

Figure II.15 Problem of Nelson-Skornakov.

In figure II.16 the Z-plane and the Ω-plane are given for this problem. The position of the origin is shown in the figure. The potential in the polder and that in the surroundings just correspond with the elevation of the relevant ground level. The image of the flow region correspond with the right half plane of Z. In the Ω-plane, the image of the flow region is a half infinite strip.

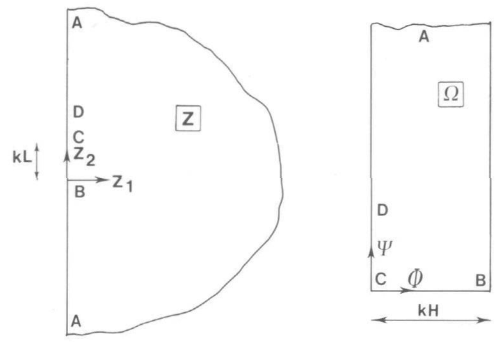

Figure II.16 Z-plane and Ω-plane.

The relationship between Z and Ω can be found by application of a sine transform (see Appendix 2):
First the half infinite strip in the Ω-plane is mapped on a half infinite strip with width π in the Ω^*-plane. The Z-plane is rotated over $\frac{\pi}{2}$ and transformed linearly so that the points C and B are mapped in the ζ-plane on the points $\zeta = -1$ and $\zeta = +1$, see figure II.17.

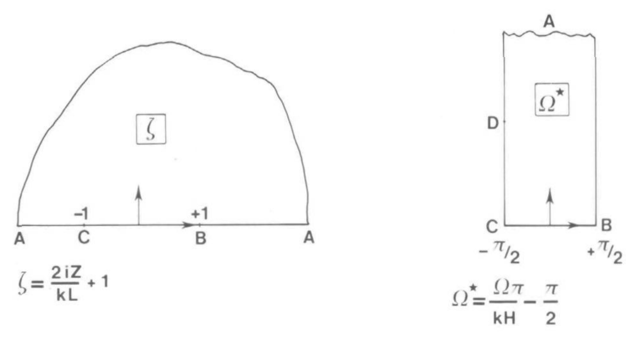

Figure II.17 ζ-plane and Ω^*-plane.

Therefore it is considered that for point C Z_2 is given by:

$$Z_2 = \Psi + kx = kL$$

By application of a sine transform to the Ω^*-plane, the relationship between Z and Ω is found (using the relationship between Ω and Ω^*, see figure II.17):

$$\frac{2iZ}{kL} + 1 = \sin\left(\frac{\Omega\pi}{kH} - \frac{\pi}{2}\right)$$

$$\frac{2iZ}{kL} = -\cos\left(\frac{\Omega\pi}{kH}\right) - 1$$

or:

$$Z = \frac{ikL}{2}\left[2\cos^2\left(\frac{\Omega\pi}{2kH}\right)\right]$$

Substitution of $Z = \Omega + i\,k\,z$ leads to:

$$z = L\cos^2\left(\frac{\pi\Omega}{2kH}\right) + \frac{i\Omega}{k} \tag{II.32}$$

In this expression L is unknown (the position where the phreatic surface meets the drain is not known). The parameter L can be calculated by using the condition that for point D the x-coordinate is the smallest on the traject CDA.
From expression (II.32) it follows:

$$x = L\,\mathrm{Re}\left\{\cos^2\left(\frac{\pi\Omega}{2kH}\right)\right\} - \frac{\Psi}{k}$$

Along CDA $\Phi = 0$; then:

$$\cos^2\left(\frac{\pi\Omega}{2kH}\right) = \cos^2\left(\frac{i\pi\Psi}{2kH}\right) = \cosh^2\left(\frac{\pi\Psi}{2kH}\right)$$

Then at CDA the following condition holds:

$$x = L\cosh^2\left(\frac{\pi\Psi}{2kH}\right) - \frac{\Psi}{k} \tag{II:33}$$

The condition that x has its minimum at point D for all points on the traject CDA can be formulated in the following way:

$$\frac{\partial x}{\partial\Psi} = 0 \quad or: \quad 2L\cosh\left(\frac{\pi\Psi}{2kH}\right)\sinh\left(\frac{\pi\Psi}{2kH}\right)\frac{\pi}{2kH} - \frac{1}{k} = 0$$

For point D then follows:

$$\sinh\left(\frac{\pi\Psi_D}{kH}\right) = \frac{2H}{\pi L}$$

and th us:

$$\Psi_D = \frac{kH}{\pi}\operatorname{arcsinh}\left(\frac{2H}{\pi L}\right) \tag{II.34}$$

The relationship between Ψ_D and L^* is found from (II.33):

$$L^* = L\cosh^2\left(\frac{\pi\Psi_D}{2kH}\right) - \frac{\Psi_D}{k} \tag{II.35}$$

By elementary operations one term in (II.35) can be rewritten:

$$\cosh^2\left(\frac{\pi\Psi_D}{2kH}\right) = \tfrac{1}{2}\left[1 + \cosh\left(\frac{\pi\Psi_D}{kH}\right)\right] = \tfrac{1}{2}\left[1 + \left(1 + \sinh^2\left(\frac{\pi\Psi_D}{kH}\right)\right)^{\frac{1}{2}}\right]$$

Substitution of this expression and (II.34) in (II.35) yields:

$$L^* = \tfrac{1}{2}L\left\{1 + \left(1 + \left(\frac{2H}{\pi L}\right)^2\right)^{\frac{1}{2}}\right\} - \frac{H}{\pi}\operatorname{arcsinh}\left(\frac{2H}{\pi L}\right)$$

or in dimensionless form:

$$\frac{L^*}{H} = \frac{1}{2}\frac{L}{H}\left\{1 + \left(1 + \left(\frac{2H}{\pi L}\right)^2\right)^{\frac{1}{2}}\right\} - \frac{1}{\pi}\operatorname{arcsinh}\left(\frac{2H}{\pi L}\right)$$

From this expression the parameter L can be calculated if L^* and H are given, and from (II.32) the complex potential can then be evaluated for any point.

7.3 Hodograph Method

This method can be found in literature (see e.g. Verruyt 1970). The complex function Ω is a function of the complex variable $z = x + iy$. The partial derivative with respect to x can be written as follows:

$$\frac{\partial\Omega}{\partial x} = \frac{d\Omega}{dz}\frac{\partial z}{\partial x} = \frac{d\Omega}{dz}$$

The function w is defined by:

$$w = -\frac{d\Omega}{dz} = -\frac{\partial\Omega}{\partial x} = -\left[\frac{\partial\Phi}{\partial x} + i\frac{\partial\Psi}{\partial x}\right]$$

According to the Cauchy-Riemann relationships (see Chapter 2), the following condition holds:

$$\frac{\partial\Psi}{\partial x} = -\frac{\partial\Phi}{\partial y}$$

Then:

$$w = -\frac{\partial\Phi}{\partial x} + i\frac{\partial\Phi}{\partial y}$$

After substitution of Darcy's Law, expressed in derivatives of the potential, it follows:

$$w = v_x - i\, v_y \tag{II.36}$$

The inverse form of $w = -\dfrac{d\Omega}{dz}$ is given by:

$$z = -\int w^{-1}\, d\Omega \tag{II.37}$$

From the linear appearance of v_x and v_y in (II.36), it is seen that straight equipotential lines and straight stream lines in the flow region are represented by straight lines in the w-plane. Because for those lines v_y/v_x = constant they are going through the origin. A phreatic stream line is represented in the hodograph-plane (the plane of specific discharges), and thus in the w-plane too, by a circle which can easily be derived in the following way:

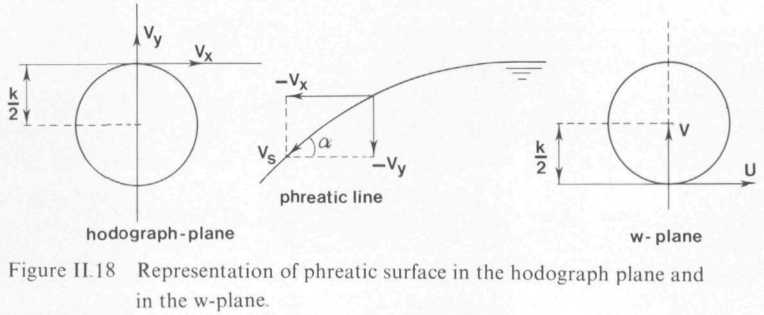

Figure II.18 Representation of phreatic surface in the hodograph plane and in the w-plane.

$$v_s = -\frac{\partial \Phi}{\partial s} = -k\frac{\partial y}{\partial s} = -k\sin\alpha = -k\frac{v_y}{v_s} \rightarrow$$

$$\rightarrow v_s^{\,2} + kv_y = 0 \rightarrow v_x^{\,2} + v_y^{\,2} + kv_y = 0 \rightarrow v_x^{\,2} + \left(v_y + \frac{k}{2}\right)^2 = \left(\frac{k}{2}\right)^2$$

The latter expression is the equation of a circle with radius $k/2$, and the centre located at $(-i\,k/2)$, see figure II.18.
In the w^{-1}-plane the phreatic line is represented by a straight line parallel to the real axis at a distance $-1/k$. That can easily be shown (using $v_s^{\,2} = -kv_y$; from the previous derivation):

$$w^{-1} = \frac{1}{v_x - iv_y} = \frac{v_x + iv_y}{v_x^{\,2} + v_y^{\,2}} = \frac{v_x + iv_y}{v_s^{\,2}} = \frac{v_x + iv_y}{-kv_y} = -\frac{v_x}{kv_y} - \frac{i}{k}$$

So the imaginary part is constant and equal to $-1/k$.

The function w is a function of Ω. The solution of a problem is found by expressing w^{-1} in Ω. Therefore the conformal mapping technique is used and (II.37) is integrated. This procedure is illustrated by an example given in figure II.19. It is the problem of Vreedenburgh (see De Vos, 1929, Kozeny, 1931 and Verruyt, 1970).

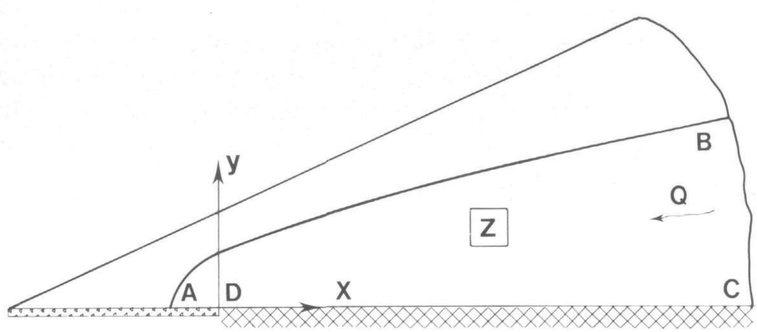

Figure II.19 Problem of Vreedenburgh.

In figure II.20 the hodograph plane: the w-plane, the w^{-1}-plane and the Ω-plane are given:

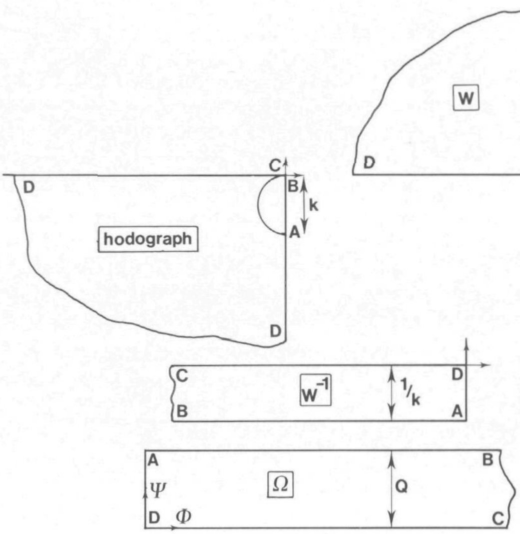

Figure II.20 Conformal mappings for the problem of Vreedenburgh.

The relationship between w^{-1} and Ω is found directly from figure II.20:

$$w^{-1} = -\Omega/kQ$$

Substitution of this expression in (II.37) yields the solution of the problem:

$$z = \frac{1}{kQ} \int \Omega d\Omega = \frac{\Omega^2}{2kQ} + c$$

The integration constant c is determined by the condition that for point D : $z = 0$, and $\Omega = 0$. From that it follows that $c = 0$. Then the solution is:

$$z = \frac{\Omega^2}{2kQ} \quad \text{or: } \Omega = \sqrt{2kQz}$$

Approximative Methods

8 Method of Fragments

In the method of fragments the flow region is sub-divided into a number of sub-regions. This sub-division is made in such a way that for each sub-region a more simple solution can be found. For example, such a sub-division can be made if one or several known straight lines may be assumed to be an equipotential line.
The method is illustrated with an example:
In the problem of figure II.21 the discharge is calculated:

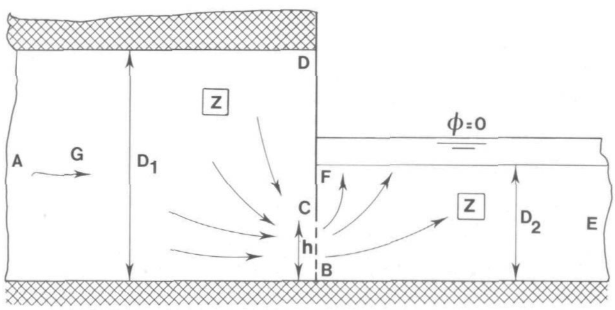

Figure II.21 Seepage towards a basin.

The solution of the problem can be given in closed form using conformal mapping of the region $AEFCD$ upon the upper half plane $\mathrm{Im}\{\zeta\} \geq 0$ (method Pavlovskii). However, the numerical evaluation of this solution may give numerical problems for some values of the quotients h/D_2 and D_1/D_2.
A more simple and approximative solution is obtained if the line segment BC is assumed to be an equipotential line (with unknown potential Φ_1). The flow problem is solved for the sub-regions on both sides of the line segment BC, with the unknown potential Φ_1 being eliminated for both. Then the approximative solution of the flow is found. In figure II.22 the conformal mapping process is given schematically.

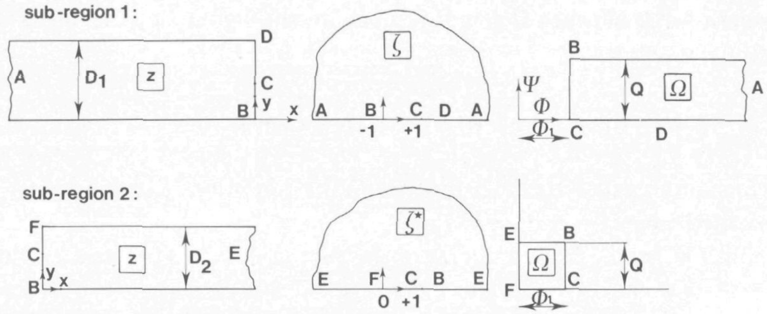

Figure II.22 Conformal mappings for the basin problem.

Sub-region 1 (ABDA):
Conformal mapping of the z-plane on the ζ-plane and of the ζ-plane on the Ω-plane yields:

$$\zeta = \frac{2\left(1 - \cos\left(i\dfrac{z\pi}{D_1}\right)\right)}{1 - \cos\left(\dfrac{h}{D_1}\pi\right)} - 1$$

$$\zeta = \cos\left[\frac{(\Omega - \Phi_1)}{iQ}\pi\right]$$

A combination of these expressions leads to:

$$\Omega = \Phi_1 + \frac{iQ}{\pi}\arccos\left\{\frac{2\left(1 - \cos\left(iz\dfrac{\pi}{D_1}\right)\right)}{1 - \cos\left(\dfrac{h}{D_1}\pi\right)} - 1\right\} \tag{II.38}$$

Sub-region 2 (EFBE):
The relationship between ζ^* and z is found by means of a sine transform:

$$\zeta^* = \tfrac{1}{2}\cos\left(\frac{z\pi}{iD_2}\right) + \tfrac{1}{2} \tag{II.39}$$

The relationship between Ω and ζ^* is given for subregion 2 by the following Schwarz-Christoffel integral:

$$\Omega = \alpha \int_0^{\zeta^*} \lambda^{-\frac{1}{2}}(\lambda - c)^{-\frac{1}{2}}(\lambda - 1)^{-\frac{1}{2}}\, d\lambda + \beta$$

The constants α and β are determined with the conditions that the points F and C are represented correctly in the z-plane.
The solution is an elliptic integral of the first kind (see Abramowitz, Stegun, 1968).

$$\frac{\Omega - \Phi_1}{\Phi_1} = -\frac{F(w|c)}{K(c)} \tag{II.40}$$

where:

$$w = \arccos\left\{\sqrt{\frac{\zeta^*(1 - c)}{c(1 - \zeta^*)}}\right\} \tag{II.41}$$

$$c = \sin^2\left(\frac{\pi(D_2 - h)}{2D_2}\right)$$

$$F(w|c) = \int_0^w (1 - c\sin^2\theta)^{-\frac{1}{2}}\, d\theta$$

$$K(c) = F\left(\frac{\pi}{2}|c\right)$$

The discharge quantity Q is found by calculation of $\mathrm{Im}\{\Omega\}$ in the point B (see figure II.22), using (II.39), (II.40) and (II.41).
This yields:

$$\frac{Q}{\Phi_1} = \frac{K(1 - c)}{K(c)} \tag{II.42}$$

By eliminating Φ_1 from (II.38) and (II.42), the relationship between Q, Ω and z in subregion 1 is found:

$$\Omega = Q\left[\frac{K(c)}{K(1 - c)} + \frac{i}{\pi}\arccos\left\{\frac{2\left(1 - \cos\left(iz\frac{\pi}{D_1}\right)\right)}{1 - \cos\left(\frac{h}{D_1}\pi\right)} - 1\right\}\right] \tag{II.43}$$

If the potential is supposed to be known at a point z_0, the discharge Q can be calculated from (II.43). Then the following expression is found:

$$Q = \frac{\Phi_0}{\dfrac{K(c)}{K(1-c)} + \operatorname{Re}\left\{\dfrac{i}{\pi}\arccos\left\{\dfrac{2\left(1-\cos\left(iz_0\dfrac{\pi}{D_1}\right)\right)}{1-\cos\left(\dfrac{h}{D_1}\pi\right)} - 1\right\}\right\}}$$

where Φ_0 is the potential at the point z_0.

9 Graphical Flow Net Method

The flow net method is a graphical method based on the property that stream lines and equipotential lines are perpendicular if the flow region is homogeneous and isotropic. Generally $\Delta\Phi = \Delta\Psi$ is chosen; then a figure is found where the stream lines and the equipotential lines form elementary squares.
Starting from a first sketch in which the expected flow pattern is drawn, a process of improving the drawn pattern leads after some iterations to a convenient flow pattern. Then the quantity $\Delta\Phi$ is the same over each square: also $\Delta\Psi$ is the same over each square. With that knowledge the discharge quantity Q and the potential at any point in the flow region can be found from the drawn flow pattern.
As an example, a flow net is roughly sketched for the groundwater flow in a confined aquifer, see figure II.23. (The analytical solution for this problem can be found using the method of Pavlovskii and has a form with an elliptic integral of the first kind).

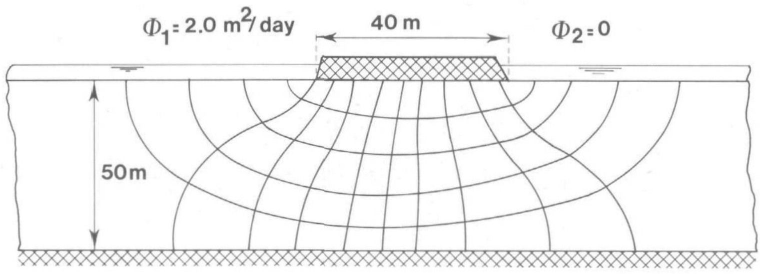

Figure II.23 Flow net under a dike.

The number of squares between two stream lines going from $\Phi_1 = 2.0$ m^2/day to $\Phi_2 = 0$ is 10. So each square represents a decrease of the potential of 0.20 m^2/day. To calculate the discharge Q, the discharge per square ($\Delta\Psi = \Delta\Phi = 0.20$ m^2/day) is multiplied by the number of squares between two equipotential lines:

$$Q = 5 \times 0.20 = 1 \text{ m}^2/\text{day}$$

Generally the discharge is expressed in the coefficient of permeability. Then Φ_1 may be given by, for example, $\Phi_1 = 2\,k$ (using the definition $\Phi = k\phi$). Then in the same way $Q = 1\,k$ m^2/day is found.

10 Finite Difference Method

In the finite difference method the flow region is sub-divided into a number of rectangular or square elements. According to a basic assumption about the variation of the groundwater head in the elements, the solution can be calculated. Generally the function $\phi(x, y, t)$ is calculated for a moment $t = t_1$. If the flow is non-steady, a discretisation in time is used and the position of the boundary (e.g., a phreatic line) after a step in time is calculated from the flow pattern. Then the function $\phi(x, y, t)$ is calculated for the moment $t = t_1 + \Delta t$, etc.
The variation of the groundwater head is assumed to be linear within each element.
In figure II.24 square elements are considered:

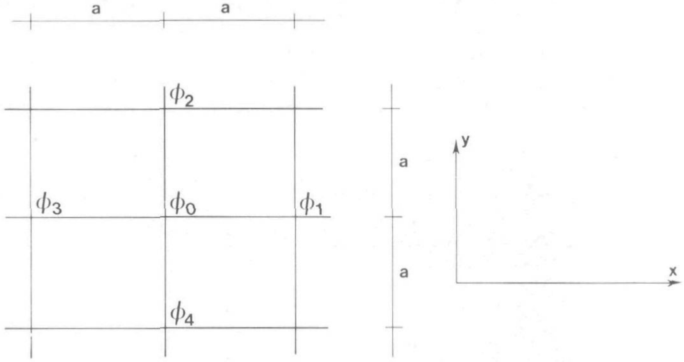

Figure II.24 Square elements.

The function $\phi(x, y, t)$ is continuous everywhere but not differentiable everywhere because at the sides of each element it is generally nodded.
The partial second derivatives of ϕ with respect to x, and y can be expressed as follows:

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{a}\left[\frac{\phi_1 - \phi_0}{a} - \frac{\phi_0 - \phi_3}{a}\right] = \frac{1}{a^2}[\phi_1 + \phi_3 - 2\phi_0]$$

and

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{1}{a}\left[\frac{\phi_2 - \phi_0}{a} - \frac{\phi_0 - \phi_4}{a}\right] = \frac{1}{a^2}[\phi_2 + \phi_4 - 2\phi_0]$$

Substitution of these expressions in the differential equation (I.3) that generates two-dimensional groundwater flow, yields:

$$\frac{1}{a^2}\left[\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_0\right] = P(x, y, t_1)$$

or:

$$\phi_0 = \tfrac{1}{4}[\phi_1 + \phi_2 + \phi_3 + \phi_4 - a^2 \; P(x, y, t_1)] \qquad \text{(II.44)}$$

This expression can also be derived from $\frac{\partial^2 \phi}{\partial x^2}$ and $\frac{\partial^2 \phi}{\partial y^2}$ from a Taylor series:

The Taylor series is defined by:

$$f(x) = f(a) + f'(a)(x - a) + f''(a)\frac{(x - a)^2}{2!} + f'''(a)\frac{(x - a)^3}{3!} + \dots$$

When this formula is applied to a grid with sides Δx and Δy (see figure II.25)

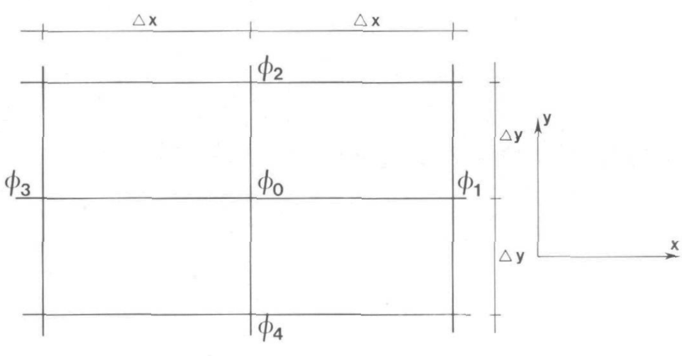

Figure II.25 Rectangular elements.

it yields:

$$\phi_1 = \phi_0 + \frac{\partial \phi}{\partial x}\Delta x + \frac{\partial^2 \phi}{\partial x^2}\frac{(\Delta x)^2}{2} + \frac{\partial^3 \phi}{\partial x^3}\frac{(\Delta x)^3}{3!} + \frac{\partial^4 \phi}{\partial x^4}\frac{(\Delta x)^4}{4!} + \dots$$

$$\phi_3 = \phi_0 - \frac{\partial \phi}{\partial x}\Delta x + \frac{\partial^2 \phi}{\partial x^2}\frac{(\Delta x)^2}{2} - \frac{\partial^3 \phi}{\partial x^3}\frac{(\Delta x)^3}{3!} + \frac{\partial^4 \phi}{\partial x^4}\frac{(\Delta x)^4}{4!} - \dots$$

Addition of these expressions leads to:

$$\phi_1 + \phi_3 = 2\phi_0 + \frac{\partial^2 \phi}{\partial x^2}(\Delta x)^2 + 2\frac{\partial^4 \phi}{\partial x^4}\frac{(\Delta x)^4}{4!} + \ldots \tag{II.45}$$

If all terms from $2\dfrac{\partial^4 \phi}{\partial x^4}\dfrac{(\Delta x)^4}{4!}$ are neglected, then (II.45) can be written in the following form:

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\phi_1 + \phi_3 - 2\phi_0}{(\Delta x)^2} \tag{II.46}$$

In an analogous way an expression for $\dfrac{\partial^2 \phi}{\partial y^2}$ is obtained:

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\phi_2 + \phi_4 - 2\phi_0}{(\Delta y)^2} \tag{II.47}$$

Substitution of (II.46) and (II.47) in the differential equation (I.3) leads to:

$$\frac{\phi_1 + \phi_3 - 2\phi_0}{(\Delta x)^2} + \frac{\phi_2 + \phi_4 - 2\phi_0}{(\Delta y)^2} = P(x, y, t_1)$$

From this expression it follows that:

$$\phi_0 = \frac{(\Delta x)^2 \ (\Delta y)^2}{2[(\Delta x)^2 + (\Delta y)^2]}\left\{\frac{\phi_1 + \phi_3}{(\Delta x)^2} + \frac{\phi_2 + \phi_4}{(\Delta y)^2} - P(x, y, t_1)\right\} \tag{II.48}$$

If the sides Δx and Δy are chosen of the same magnitude $\Delta x = \Delta y = a$, then (II.48) reduces to the previously-found formula (II.44):

$$\phi_0 = \tfrac{1}{4}[\phi_1 + \phi_2 + \phi_3 + \phi_4 - a^2\, P(x, y, t_1)]$$

The error made by this approximation is of the order $2\dfrac{\partial^4 \phi}{\partial x^4} \cdot \dfrac{a^4}{4!}$

Expression (II.44) is the basic expression that is used for this finite difference method. All terms appear linearly in that expression. A set of linear equations will result when the groundwater heads in the nodal points are related to each other and to the boundary conditions. That set of equations can be solved by well-known techniques, resulting in known heads in the nodal points. For points within the elements, the head is calculated by linear interpolation according to the basic assumption.

11 Finite Element Method

In the finite element method generally the flow region is sub-divided into a number of triangular elements. Using an assumption about the variation of the groundwater head in the elements, the solution can be calculated. Generally the function $\phi(x, y, t)$ is calculated for a moment $t = t_1$. If the flow is non-steady, a discretisation in time is used and the position of the boundary (e.g., a phreatic line) after a step in time is calculated using the specific discharges at the moment $t = t_1$. Then the function $\phi(x, y, t)$ is calculated for $t = t_1 + \Delta t$, etc.

The finite element method is based on a variational principle. The basic formulas can be derived using a mathematical variational analysis. Here it is shown that a somewhat alternative derivation using an energy concept leads to the same result.

For groundwater flow an energy flux function can be defined by $e = m^* g\phi$. The dimension of e for two-dimensional flow is energy per unit of length and per unit of time. The parameter m^* is given by $m^* = \rho v$. Thus:

$$e = \rho g v \phi \tag{II.49}$$

The dimension of m^* is mass per unit of length and per unit of time. Note that the density ρ has the dimension of mass per unit of area in our two-dimensional case. From the appearance of the specific discharge v in the definition of e it is seen that e is a vector function. The energy flux per unit of time in a direction n (through a unit of length with direction l perpendicular to n) is given by e_n where e_n is the component of e in the direction n. Consequently the energy transport per unit of time, dE, through a line segment dl is given by:

$$dE = e_n \, dl = \rho g v_n \phi dl$$

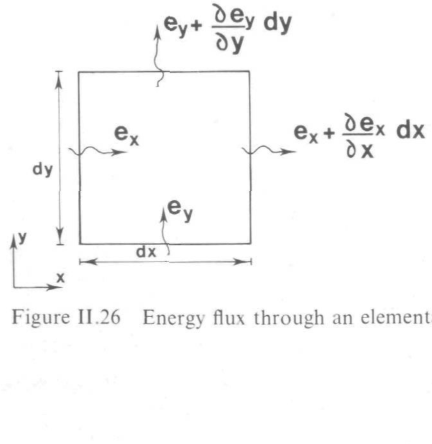

Figure II.26 Energy flux through an elementary rectangle.

The concept used in this Chapter says that in a flow region there will be such a groundwater head distribution that the total loss of energy transport per unit of time due to the flow is a minimum. (Of course this is also a variational principle).

In figure II.26 the energy flux e per unit of time through an elementary rectangle is considered; e is composed of the following components:

in x-direction: $e_x = \rho v_x g\phi$
in y-direction: $e_y = \rho v_y g\phi$

The loss of energy transport dE, in the rectangle is given by (see figure II.26):

$$-\left\{\frac{\partial e_x}{\partial x}dx\,dy + \frac{\partial e_y}{\partial y}dy\,dx\right\}$$

or, after substitution of (II.49) and Darcy's Law and considering the case with constant density ρ, constant coefficient of permeability k and $P(x, y, t) = 0$ (see Chapter 1):

$$\rho gk\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]dx\,dy + \rho g\phi k\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right]dx\,dy$$

The second term of this expression is zero (see (I.4)). Thus the expression becomes:

$$\rho gk\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]dx\,dy$$

For the whole flow region R the loss of energy transport per unit of time E is then given by:

$$A^* = \rho gk\iint_R\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]dx\,dy \tag{II.50}$$

If a quantity A is defined according to

$$A = \frac{A^*}{2\rho g}$$

then (II.50) becomes:

$$A = \tfrac{1}{2}k\iint_R\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]dx\,dy \tag{II.51}$$

This expression is well-known in literature (see, for example, Verruyt, 1970). Expression (II.51) is used to solve groundwater flow problems. According to the mentioned energy concept, there will be such a groundwater head distribution that the total loss of E is a minimum which occurs if in each point ϕ has such a value that

$$\frac{\partial A}{\partial \phi} = 0 \qquad \text{(II.52)}$$

Approximative solutions are obtained by sub-dividing the flow region in a number of elements. An assumption is made about the variation of the groundwater head in an element and the unknown heads in the nodal points are calculated by application of (II.52). It will be shown that this leads to a set of linear equations.
The solution of this set of equations gives an approximative solution of the problem. Generally triangular elements are used (see figure II.27) and the variation of the groundwater head in an element is assumed to be linear in x and y, see figure II.28.

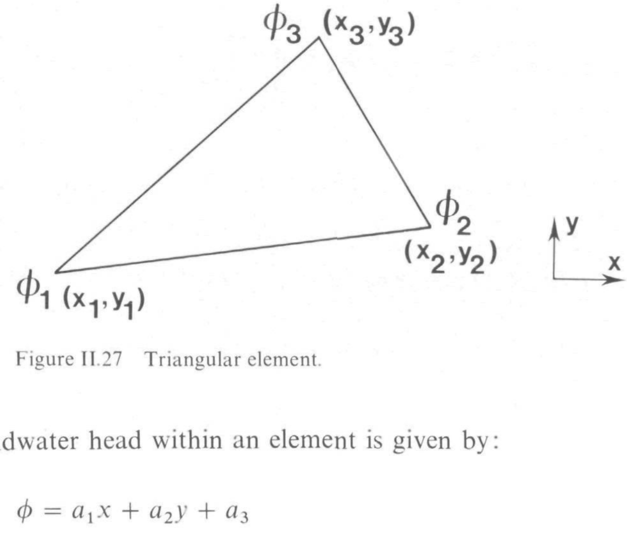

Figure II.27 Triangular element.

The groundwater head within an element is given by:

$$\phi = a_1 x + a_2 y + a_3 \qquad \text{(II.53)}$$

The constants a_1, a_2, a_3 are chosen in such a way that in (x_1, y_1), (x_2, y_2) and (x_3, y_3) the heads just are respectively ϕ_1, ϕ_2 and ϕ_3.
The constants a are given by:

$$\left.\begin{aligned} a_1 &= (b_1\phi_1 + b_2\phi_2 + b_3\phi_3)/\Delta \\ a_2 &= (c_1\phi_1 + c_2\phi_2 + c_3\phi_3)/\Delta \\ a_3 &= (d_1\phi_1 + d_2\phi_2 + d_3\phi_3)/\Delta \end{aligned}\right\} \qquad \text{(II.54)}$$

where:

$$\left.\begin{aligned}
b_1 &= y_2 - y_3\\
b_2 &= y_3 - y_1\\
b_3 &= y_1 - y_2\\
c_1 &= x_3 - x_2\\
c_2 &= x_1 - x_3\\
c_3 &= x_2 - x_1\\
d_1 &= x_2y_3 - x_3y_2\\
d_2 &= x_3y_1 - x_1y_3\\
d_3 &= x_1y_2 - x_2y_1\\
\Delta &= x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)
\end{aligned}\right\} \quad \text{(II.55)}$$

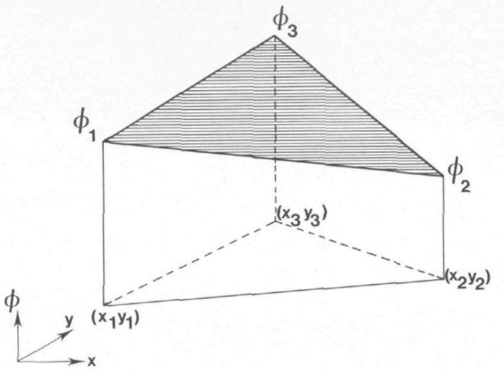

Figure II.28 Linear variation of groundwater head in an element.

To determine the contribution to the quantity A (see (II.51)) of an element, the partial derivatives of ϕ with respect to x and y are obtained from (II.53):

$$\frac{\partial\phi}{\partial x} = a_1 \quad \frac{\partial\phi}{\partial y} = a_2 \tag{II.56}$$

After substitution of (II.53) and (II.56) in (II.51), the following expression is obtained for the contribution of the element with number j:

$$A_j = \tfrac{1}{2}k\iint_{R_j}[a_1^2 + a_2^2]dx\,dy \tag{II.57}$$

where R_j denotes the area of this element.
As a_1, a_2, a_3 are constant within an element, (II.57) becomes:

$$A_j = \tfrac{1}{2}k(a_1^2 + a_2^2)\iint_{R_j}dx\,dy$$

The integral $\iint_{R_j} dx\,dy$ is equal to the area of the triangular element:

$$|\Delta|/2.$$

Where Δ is given in (II.55).
Then the mentioned contribution of the element j is given by:

$$A_j = \frac{k|\Delta|}{4}(a_1{}^2 + a_2{}^2) \tag{II.58}$$

According to the energy concept, the correct head distribution is such that for each head ϕ_i (in nodal point with number i) the condition holds that A has its minimum. According to (II.52):

$$\frac{\partial A}{\partial \phi_i} = \sum^{n^i} \frac{dA_j}{\partial \phi_i} = 0$$

where $\sum^{n^i}$ denotes summation over all elements of which ϕ_i is an angular point. Considering that a_1, a_2, a_3 are functions of the nodal heads (see (II.54)) this expression can be written in the following form, using (II.58):

$$\sum^{n^i}\left[\frac{k|\Delta|}{4}\left[2a_1\frac{\partial a_1}{\partial \phi_i} + 2a_2\frac{\partial a_2}{\partial \phi_i}\right]\right] = 0$$

or, considering that k is a constant:

$$\sum^{n^i}\left[|\Delta|\left[a_1\frac{\partial a_1}{\partial \phi_i} + a_2\frac{\partial a_2}{\partial \phi_i}\right]\right] = 0 \tag{II.59}$$

The derivatives that appear in this expression are obtained from (II.54). For the nodal point $i = 1$ that is used in the following as an example they are:

$$\frac{\partial a_1}{\partial \phi_1} = \frac{b_1}{\Delta} \qquad \frac{\partial a_2}{\partial \phi_1} = \frac{c_1}{\Delta}$$

Then (II.59) becomes:

$$\sum^{n^i}\left[|\Delta|\left[\frac{b_1}{\Delta}a_1 + \frac{c_1}{\Delta}a_2\right]\right] = 0 \tag{II.60}$$

Substitution of (II.54) in (II.60) leads to:

$$\sum^{n^i}\left[(b_1{}^2\,\phi_1 + b_1 b_2\,\phi_2 + b_1 b_3\,\phi_3 + c_1{}^2\,\phi_1 + \right.$$

$$\left. + c_1 c_2\,\phi_2 + c_1 c_3\,\phi_3)\frac{1}{|\Delta|}\right] = 0 \qquad \text{(II.61)}$$

In other form (II.61) is given by:

$$\sum^{n^i}[P_1\phi_1 + P_2\phi_2 + P_3\phi_3] = 0 \qquad \text{(II.62)}$$

where:

$$P_1 = \frac{1}{|\Delta|}(b_1{}^2 + c_1{}^2)$$

$$P_2 = \frac{1}{|\Delta|}(b_1 b_2 + c_1 c_2)$$

$$P_3 = \frac{1}{|\Delta|}(b_1 b_3 + c_1 c_3)$$

and b_1, b_2, b_3, c_1, c_2, c_3 being given by (II.55).
The expression (II.62) is a linear equation in the head of the nodal point 1 and the heads of the surrounding nodal points, as shown in figure II.29.

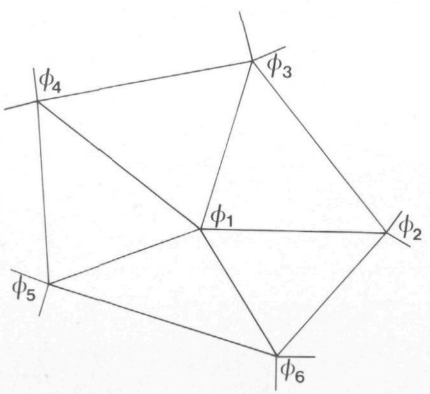

Figure II.29 Elements arond a nodal point.

The condition (II.62) for all nodal points leads to a set of linear equations where the unknowns are the nodal groundwater heads. The boundary conditions have to be taken into account. The set of equations leads to an approximative solution of the problem. For points within the elements, the head is calculated by linear interpolation according to the basic assumption (II.53).

III Analytical Function Method

12 General Description

In part III a calculation method is described for two-dimensional groundwater flow problems, using complex functions. Anisotropy is taken into account according to the theory that is mentioned in Chapter 3 (simple geometric transformation and defining a fictive coefficient of permeability). Inhomogeneities with respect to the coefficient of permeability or the density of the fluid are accounted by sub-division of the flow region in corresponding sub-regions that all have a constant coefficient of permeability and density.
The sub-regions are coupled in the calculation (see Chapters 14 and 16) by means of special connecting conditions at the separation lines.

In the following text the flow is considered in regions or sub-regions that already have been transformed for their anisotropy, and are then further considered as if they were isotropic. So in Part III anisotropy is not mentioned further in formulas. In the computer program at the end of Part III it is accounted according the theory of Chapter 3.
Only pure two-dimensional flow is considered here. Multi-layer systems, where in each aquifer the flow is assumed to be two-dimensional and between the aquifers there are semi-pervious layers, are semi-three-dimensional. They are not dealt with here.
According to the theory mentioned in Chapter 1, two-dimensional groundwater flow in an isotropic region is given by:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{P(x, y, t)}{k}$$

The analytical function method (A.F.M.) is based on the calculation of flow patterns, which means that non-steady flow is calculated as the non-steady behaviour for a sequence of steps in time. So there is a discretisation in time. At a particular moment the flow pattern is calculated, and then from the specific discharge distribution a new position of the boundary is calculated after a step in time (for example, the changed position of an interface). Then the flow pattern is calculated at the following moment: and so on (see Chapter 16). The non-steady character of the flow is accounted in the boundary conditions. The differential equation for the flow pattern is:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{P(x, y)}{k}$$

The term $P(x, y)$ represents the abstraction of water per unit of time and per unit of area in the interior of the region at the considered moment. If such an abstraction area reduces to zero where the product of $P(x, y)$ and the area remains constant, then there is a sink in the region.
In two-dimensional flow problems supply or abstraction is generally concentrated in sinks or sources. For cases with a line abstraction (drain) a better generation of the flow pattern is usually made by assuming that line as an equipotential line rather than as a line where the specific discharge per unit of length is constant (although that can be accounted by the formula of a distribution of sinks).
It is shown in Chapter 14 (dealing with the boundary conditions) and in Chapter 16 (concerning the procedure used in solving groundwater flow problems) that the storage alteration along the boundaries need not be accounted in a supply or abstraction term in the differential equation when flow patterns are calculated. The storage alteration is accounted when the position of the boundary after a step in time is calculated.
If only abstraction or supply at sinks and sources is present, then for the whole region (except these singular points) the flow is described by the Laplace differential equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{(III.1)}$$

Complex functions are very suitable for the solution of flow problems that satisfy the Laplace equation (III.1). The real and imaginary parts of any analytical function satisfy the Laplace equation (see Appendix 1). This means that an analytical function that satisfies the boundary conditions at all points of the boundary is the exact solution of the flow probiem. A calculation method that uses the complex potential $\Omega = \Phi + i\Psi$ (see Chapter 2) has the advantage that the solution also contains the stream function.
For the analytical function method (A.F.M.) analytical functions with degrees of freedom are used so that a flow pattern is generated that satisfies the boundary conditions at a number of points of the boundary. Because generally such a solution does not satisfy the boundary conditions at all points of the boundary, the solution is not exact. However, because the complex functions are analytical, the solution satisfies the Laplace equation everywhere except at singular points. Therefore the solution is exact within an approximative boundary. The solutions obtained in this way are different from those of the numerical methods like the finite difference method and the finite element method (see respectively Chapters 10 and 11). In those methods a solution is found that is approximative over the whole region.

13 Composition of the Solution

For each sub-region there is an analytical function that generates the flow in it, and is defined in the sub-region and on its boundary.
The presence of sinks and sources can be accounted by addition of the well-known logarithmic expressions for sinks and sources (according to the principle of superposition (see Chapter 6)).

The general solution for the flow in a sub-region has the following form:

$$\Omega(z) = \Omega_2(z) + \Omega_1(z) + \Omega_0 \tag{III.2}$$

where:

$\Omega_2(z)$: approximative part,
$\Omega_1(z)$: exact part for sinks and sources, and
Ω_0 : reference constant.

The expression $\Omega(z) = \Omega_2(z) + \Omega_1(z)$ is the basic solution of the flow problem. The complex constant Ω_0 is a reference constant that produces a translation of the values of the potential and the stream function. The use of a reference constant is allowed because the flow pattern does not depend on absolute values of the potential and the stream function but on potential and stream function gradients. The complex reference constant can be used to define the potential and the stream function at a point, which then becomes a reference point and the complex potential is defined with respect to that point.
It is not neccessary that there is one and the same reference point for the stream function and the potential. There may be two different points. The reference conditions with respect to Φ and Ψ in two points then yield the values of the complex reference constant if the flow problem is otherwise defined.
The complex function $\Omega_1(z)$ generates the flow due to sinks and sources in the flow region.
By the principle of superposition the complex potential due to sinks and sources is given by:

$$\Omega_1(z) = \sum_{j=1}^{m} \frac{Q_j}{2\pi} \ln(z - z_j) \tag{III.3}$$

where:

m : number of sources and sinks
Q_j: discharge of sink ($Q > 0$) or source ($Q < 0$)
z_j : $z_j = x_j + iy_j$; position of sink or source

The complex function $\Omega_2(z)$ is an approximative function that is superimposed on the flow according to sinks and sources.
The function $\Omega_2(z)$ is defined in such a way that at a number of points of the boundary the boundary conditions are satisfied (or the connecting conditions with other subregions). To achieve this, the function $\Omega_2(z)$ contains a number of degrees of freedom that are so defined that the boundary conditions are satisfied by the solution $\Omega = \Omega_2 + \Omega_1 + \Omega_0$. For the choice of an appropriate function Ω_2 there are many possibilities.
It is essential that the function Ω_2 is an analytical function because only then does it satisfy the Laplace equation. In addition, the appearance of the degrees of freedom in the formula has to be such that the process of calculation is as simple as possible. The most simple suitable form is used when the degrees of freedom appear linearly in the solution:

$$\Omega_2(z) = \sum_{j=1}^{n} \gamma_j \, F_j(z) \qquad \text{(III.4)}$$

where:

n : number of terms of Ω_2
γ_j : constant
F_j: analytical function of z

The degrees of freedom γ_j in $\Omega_2(z)$ may be real, imaginary or complex. The use of complex γ_j has the advantage that there are two degrees of freedom per term $\gamma_j F_j(z)$ This reduces the amount of calculation work because a maximum number of degrees of freedom is used with a minimum number of complex functions $F_j(z)$. Therefore here the choice is made using degrees of freedom of complex type.

14 Boundary Conditions

14.1 General

In this Chapter the boundary conditions are drawn up in a form that is convenient for use in the analytical function method (A.F.M.). Expressions are derived from formulas for a 'general boundary' by varying some of the parameters.
The 'general boundary' is shown in figure III.1. There is a thin silt layer between two sub-regions, and the density of the fluids in both sub-regions may be different. Parameters for one of the two sub-regions are denoted by the subscript c. The resistance of the silt layer is c_s, and its thickness is set to zero.

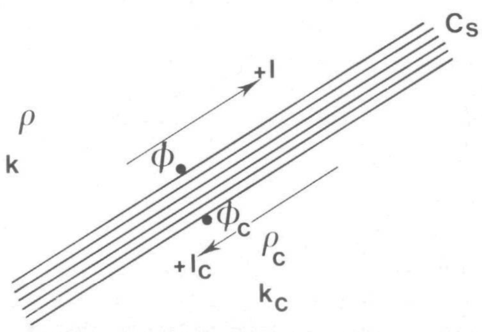

Figure III.1 'General boundary'.

The position of a point of the silt layer is denoted by the real variable l and by l_c for the other sub-region. The positive directions for l and l_c are opposite, having been chosen in such a way that an anti-clockwise rotation is carried out when the whole boundary of a sub-region is followed.
In the situation shown in figure III.1 there will be a flow through the silt layer.
Application of Darcy's Law to an element of the silt layer yields:

$$\frac{\partial \Psi}{\partial l} = \frac{-\phi + \phi_c^*}{c_s}$$

or:

$$\phi - \phi_c^* + c_s \frac{\partial \Psi}{\partial l} = 0 \qquad \text{(III.5)}$$

The fictive groundwater head $\phi_c{}^*$ is a groundwater head that is derived from the actual ϕ_c. The relationship between $\phi_c{}^*$ and ϕ_c is the following well-known expression (which expresses the relationship between two groundwater heads that hold the same pressure at a point with height y):

$$\phi_c{}^* = \frac{\rho_c}{\rho}\phi_c - \frac{\rho_c - \rho}{\rho} y$$

Then (III.5) becomes:

$$\phi - \frac{\rho_c}{\rho}\phi_c + c_s\frac{\partial \Psi}{\partial l} = \frac{\rho - \rho_c}{\rho} y \qquad \text{(III.6)}$$

In addition, the continuity condition has to be satisfied:

$$\frac{\partial \Psi}{\partial l} + \left(\frac{\partial \Psi}{\partial l}\right)_c = 0 \qquad \text{(III.7)}$$

Expression (III.7) is only relevant at the separation line between two sub-regions. The conditions (III.6) and (III.7), if relevant, are sufficient to arrive at an exact solution if they are posed at all points of the boundaries. Here, however, they are posed at selected points of the boundary ('boundary points'). The accuracy of the solution can be improved by increasing the number of these boundary points, as well as by using the condition that the expressions (III.6) and (III.7) remain valid in the close neighbourhood of the boundary points (then the variation of parameters in the direction l and resp. l_c are considered). Here the choice is made to use also these 'derivative conditions' for the boundary points. The expressions with respect to the variations in the direction l are given by (where it is mentioned that the positive directions for l and l_c are opposite):

$$\frac{\partial \phi}{\partial l} - \frac{\rho_c}{\rho}\frac{\partial \phi_c}{\partial l} + c_s\frac{\partial^2 \Psi}{\partial l^2} = \frac{\rho - \rho_c}{\rho}\frac{\partial y}{\partial l}$$

or:

$$\frac{\partial \phi}{\partial l} + \frac{\rho_c}{\rho}\left(\frac{\partial \phi}{\partial l}\right)_c + c_s\frac{\partial^2 \Psi}{\partial l^2} = \frac{\rho - \rho_c}{\rho}\frac{\partial y}{\partial l} \qquad \text{(III.8)}$$

and:

$$\frac{\partial^2 \Psi}{\partial l^2} + \frac{\partial}{\partial l}\left(\frac{\partial \Psi}{\partial l}\right)_c = 0$$

or:

$$\frac{\partial^2 \Psi}{\partial l^2} - \left(\frac{\partial^2 \Psi}{\partial l^2}\right)_c = 0 \qquad \text{(III.9)}$$

14.2 Specific Expressions to be used

Equipotential line with silt layer

In Chapter 4 it was mentioned that along an equipotential line the groundwater head is constant. Here the situation is considered where there is a thin resistance layer between the equipotential line and the flow region, as shown in figure III.2:

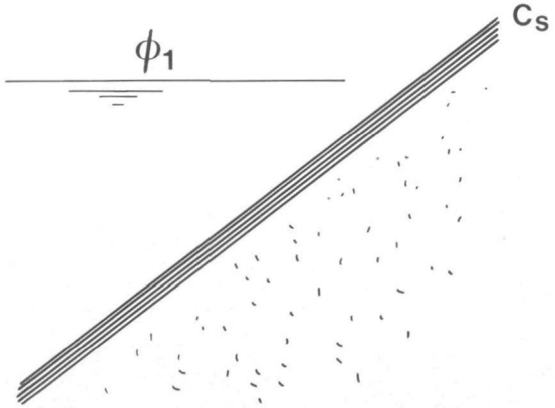

Figure III.2 Equipotential line with silt layer.

This situation corresponds to many practical situations where thin silt or clay layers are present at the bottom and talus of canals, lakes, etc.

The expressions to be derived here are found from (III.6) and (III.8) by putting $\rho = \rho_c$ and $\phi_c = \phi_1$ (which is a known constant). Then the result is:

$$\phi + c_s \frac{\partial \Psi}{\partial l} = \phi_1 \qquad \text{(III.10)}$$

and:

$$\frac{\partial \phi}{\partial l} + c_s \frac{\partial^2 \Psi}{\partial l^2} = 0 \qquad \text{(III.11)}$$

In calculations Ω and $\frac{d\Omega}{dz}$ are used instead of Φ, Ψ and their derivatives with respect to l. Therefore (III.10) and (III.11) are written in complex form:

$$\mathrm{Re}\left\{\frac{\Omega}{k} - ic_s \frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right\} = \phi_1 \qquad \text{(III.12)}$$

$$\mathrm{Re}\left\{\frac{d\Omega}{dz}\left[\frac{1}{k}\frac{\partial z}{\partial l} - ic_s\frac{\partial^2 z}{\partial l^2}\right] - \frac{d^2\Omega}{dz^2}\left[ic_s\left(\frac{\partial z}{\partial l}\right)^2\right]\right\} = 0 \tag{III.13}$$

Stream line boundary

In Chapter 4 it is mentioned that a stream line boundary is impervious, see figure III.3:

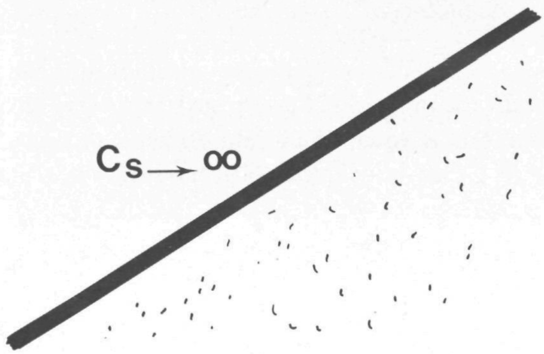

Figure III.3 Stream line boundary.

The expressions are found from (III.6) and (III.8) by substitution of the condition that the resistance of the silt layer is infinitely large:

$$c_s \to \infty$$

Division by c_s in (III.6) and (III.8) yields:

$$\frac{\phi}{c_s} - \frac{\rho_c}{\rho}\frac{\phi_c}{c_s} + \frac{\partial \Psi}{\partial l} = \frac{\rho - \rho_c}{\rho}\frac{y}{c_s}$$

and
$$\frac{1}{c_s}\frac{\partial \phi}{\partial l} + \frac{\rho_c}{\rho c_s}\left(\frac{\partial \phi}{\partial l}\right)_c + \frac{\partial^2 \Psi}{\partial l^2} = \frac{\rho - \rho_c}{\rho}\frac{1}{c_s}\frac{\partial y}{\partial l}$$

For $c_s \to \infty$ these expressions become:

$$\frac{\partial \Psi}{\partial l} = 0$$

and
$$\frac{\partial^2 \Psi}{\partial l^2} = 0$$

In complex denotation they are given by:

$$\operatorname{Im}\left\{\frac{d\Omega}{dz}\,\frac{\partial z}{\partial l}\right\} = 0 \tag{III.14}$$

and

$$\operatorname{Im}\left\{\frac{d\Omega}{dz}\,\frac{\partial^2 z}{\partial l^2} + \frac{d^2\Omega}{dz^2}\left(\frac{\partial z}{\partial l}\right)^2\right\} = 0 \tag{III.15}$$

Seepage line with silt layer

In Chapter 4 it is mentioned that at a seepage line the groundwater can freely leave the soil and run off. Generally the thickness of a seepage layer is very small, and then the pressure at the seepage line is equal to the atmospheric pressure, which is a reference pressure, set to zero. Using the definition of groundwater head, the condition for a seepage line is:

$$\phi = y$$

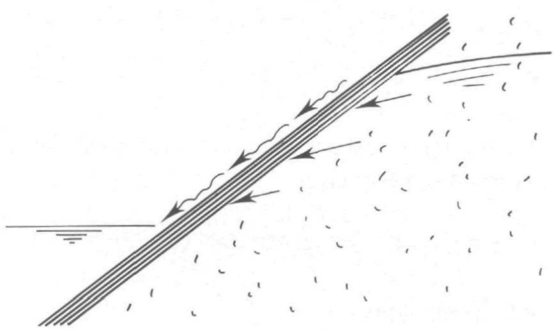

Figure III.4 Seepage line with silt layer.

So at a seepage line the groundwater head varies in a prescribed way. In this study the general case of a seepage line with a silt layer was considered, see figure III.4.
If the resistance of the silt layer is set to zero in the expressions that will be given, the conditions are obtained for a common seepage line.
The seepage line expressions can be found from (III.6) and (III.8) by the substitution of $\rho_c = 0$.
This yields:

$$\phi + c_s \frac{\partial \Psi}{\partial l} = y$$

and

$$\frac{\partial \phi}{\partial l} + c_s \frac{\partial^2 \Psi}{\partial l^2} = \frac{\partial y}{\partial l}$$

These expressions are given in complex denotation by:

$$\mathrm{Re}\left\{\frac{\Omega}{k} - ic_s \frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right\} = \mathrm{Im}\{z\} \tag{III.16}$$

and:

$$\mathrm{Re}\left\{\frac{d\Omega}{dz}\left[\frac{1}{k}\frac{\partial z}{\partial l} - ic_s \frac{\partial^2 z}{\partial l^2}\right] - \frac{d^2\Omega}{dz^2}\left[ic_s\left(\frac{\partial z}{\partial l}\right)^2\right]\right\} = \mathrm{Im}\left\{\frac{\partial z}{\partial l}\right\} \tag{III.17}$$

It should be noted here that generally the position of a seepage line is known, as it follows from the geometry of the flow region.
However, generally the upper point of the seepage is not known; this point is also a point of a phreatic line, the position of which is usually not known previously. So the problem of the unknown upper point of a seepage line has to be considered as a phreatic line problem.

Phreatic line
In Chapter 4 it has been mentioned that at a phreatic line the pressure is equal to the atmospheric pressure (that was set to zero). Using the definition of groundwater head, the following condition was derived:

$$\phi = y$$

So a phreatic line can be considered as a line at which the head varies in a prescribed way.
If the position of the phreatic line in steady flow is known and there is no precipitation, the stream line conditions (III.14) and (III.15) may be used. If there is precipitation, the condition should be used that at the phreatic line Ψ varies so that its variation per unit of length in horizontal direction corresponds to the precipitation. In A.F.M. calculations the condition $\phi = y$ and the derivative condition $\frac{\partial \phi}{\partial l} = \frac{\partial y}{\partial l}$ are used because those conditions may be used for steady flow as well as for non-steady flow and are independent of the presence of precipitation.
Generally the position of the phreatic line is not known previously. The steady position of the phreatic line is calculated from the non-steady behaviour where other boundary conditions are invariable. After a number of steps in time it is seen

that there is practically no further movement of the phreatic line. In the solution it is seen then that the specific discharge components normal to the phreatic line have become very small and the stream function variation along the phreatic line is very small too, if there is no precipitation. In the presence of precipitation the stream function varies so that this variation just corresponds to the precipitation.
The precipitation is taken into account in the following way:
At the moment $t = 0$, the flow pattern is calculated, using a known or assumed position of the phreatic line. Displacement of the phreatic line in a step in time is calculated using the specific discharges along the phreatic line. The rise according to the precipitation has to be superimposed. The rise (or fall) of the phreatic line also depends on the storage coefficient μ of the soil. The effect of the precipitation N in a step in time Δt is:

$$\Delta y = \frac{N \Delta t}{\mu}$$

A combination of the effects of the precipitation and specific discharges leads to the position of the phreatic line after one step in time (see Chapter 16).
The new position of the phreatic line is then used to calculate the problem again, and so on.

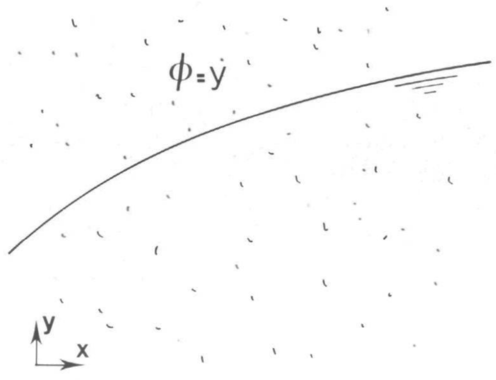

Figure III.5 Phreatic line.

The phreatic line expressions to be used in flow pattern calculations can be found from (III.6) and (III.8) by the substitution of $\rho_c = 0$ and $c_s = 0$, which yields:

$$\phi = y$$

and

$$\frac{\partial \phi}{\partial l} = \frac{\partial y}{\partial l}$$

In complex form these expressions are given by:

$$\mathrm{Re}\left\{\frac{\Omega}{k}\right\} = \mathrm{Im}\{z\} \tag{III.18}$$

and:

$$\mathrm{Re}\left\{\frac{d\Omega}{dz}\,\frac{1}{k}\,\frac{\partial z}{\partial l}\right\} = \mathrm{Im}\left\{\frac{\partial z}{\partial l}\right\} \tag{III.19}$$

Interface

It is mentioned in Chapter 4 that an interface is a separation line between two fluids with different densities. In steady state an interface is a stream line. If the steady position of the interface is known, the expressions (III.14) and (III.15) can be used. In any case (steady or non-steady), the condition holds that the pressure is equal on both sides of the interface; it has already been shown that this can be expressed by the following condition:

$$\phi = \frac{\rho - \rho_c}{\rho} y + \frac{\rho_c \phi_c}{\rho} \tag{III.20}$$

So an interface can be considered as a line along which the groundwater head varies in a prescribed way with the height y. Here (III.20) is chosen to be used because it holds for steady as well as for non-steady flow. Generally the position of the interface is not known previously. The steady position of the interface is calculated from the non-steady behaviour where other boundary conditions are invariable. After a number of steps in time, the interface will be practically at rest. Then it is seen in the solution that the normal components of the specific discharges at the interface have become very small. The calculation procedure starts from a known or assumed position of the interface, and then the flow pattern is calculated. The specific discharges at the interface are used to calculate the interface position after one step in time. Then the problem is calculated again; and so on.

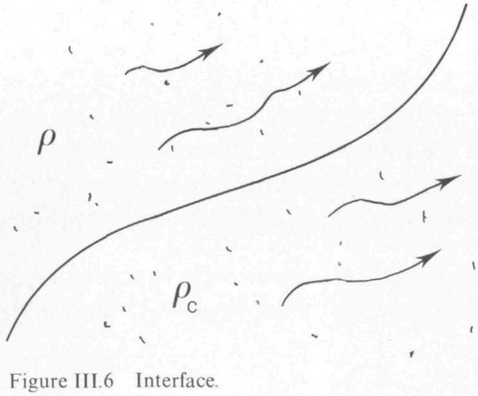

Figure III.6 Interface.

Two interface expresions are found from (III.6) and (III.8) using the condition that $c_s = 0$, yielding:

$$\phi - \frac{\rho_c}{\rho}\phi_c = \frac{\rho - \rho_c}{\rho} y$$

and:

$$\frac{\partial \phi}{\partial l} + \frac{\rho_c}{\rho}\left(\frac{\partial \phi}{\partial l}\right)_c = \frac{\rho - \rho_c}{\rho}\frac{\partial y}{\partial l}$$

In addition, the continuity condition (III.7) and its derivative expression (III.9) hold:

$$\frac{\partial \Psi}{\partial l} + \left(\frac{\partial \Psi}{\partial l}\right)_c = 0$$

and:

$$\frac{\partial^2 \Psi}{\partial l^2} - \left(\frac{\partial^2 \Psi}{\partial l^2}\right)_c = 0$$

In complex form, these expressions are given (after multiplication by ρ) by:

$$\mathrm{Re}\left\{\frac{\rho\Omega}{k} - \left(\frac{\rho\Omega}{k}\right)_c\right\} = (\rho - \rho_c)\,\mathrm{Im}\{z\} \tag{III.21}$$

$$\mathrm{Re}\left\{\frac{\rho}{k}\frac{d\Omega}{dz}\frac{\partial z}{\partial l} + \left(\frac{\rho}{k}\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right)_c\right\} = (\rho - \rho_c)\mathrm{Im}\left\{\frac{\partial z}{\partial l}\right\} \tag{III.22}$$

$$\mathrm{Im}\left\{\frac{d\Omega}{dz}\frac{\partial z}{\partial l} + \left(\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right)_c\right\} = 0 \tag{III.23}$$

$$\mathrm{Im}\left\{\frac{d\Omega}{dz}\frac{\partial^2 z}{\partial l^2} - \left(\frac{d\Omega}{dz}\frac{\partial^2 z}{\partial l^2}\right)_c + \frac{d^2\Omega}{dz^2}\left(\frac{\partial z}{\partial l}\right)^2 - \left(\frac{d^2\Omega}{dz^2}\left(\frac{\partial z}{\partial l}\right)^2\right)_c\right\} = 0 \tag{III.24}$$

In the special case that there is a stationary fluid on one side of the interface, then $\phi_c = \phi_1 = \text{constant}$ and $\dfrac{\partial \phi_c}{\partial l} = 0$, the conditions (III.23) and (III.24) are not relevant as these formulas are continuity conditions for the case with flow on both sides of a separation line between sub-regions. In that special case, the interface is not a separation line between sub-regions.

The expressions (III.21) and (III.23) then become:

$$\mathrm{Re}\left\{\rho\frac{\Omega}{k}\right\} = (\rho - \rho_c)\,\mathrm{Im}\{z\} + \rho_c\phi_1$$

and:

$$\mathrm{Re}\left\{\frac{\rho}{k}\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right\} = (\rho - \rho_c)\,\mathrm{Im}\left\{\frac{\partial z}{\partial l}\right\}$$

Inhomogeneity line with silt layer

An inhomogeneity line separates two sub-regions with different coefficients of permeability or anisotropy (direction or magnitude). The general case discussed here is that in which there is a silt layer at the inhomogeneity line. This agrees with many natural situations.

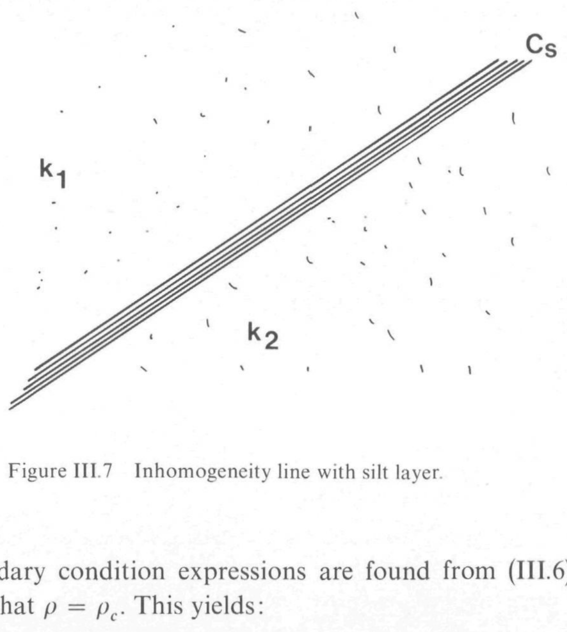

Figure III.7 Inhomogeneity line with silt layer.

Two boundary condition expressions are found from (III.6) and (III.8) using the condition that $\rho = \rho_c$. This yields:

$$\phi - \phi_c + c_s\frac{\partial \Psi}{\partial l} = 0$$

and:

$$\frac{\partial \phi}{\partial l} + \left(\frac{\partial \phi}{\partial l}\right)_c + c_s\frac{\partial^2 \Psi}{\partial l^2} = 0$$

In addition, the continuity condition (III.7) and its derivative expression (III.9) hold:

$$\frac{\partial \Psi}{\partial l} + \left(\frac{\partial \Psi}{\partial l}\right)_c = 0$$

and:

$$\frac{\partial^2 \Psi}{\partial l^2} - \left(\frac{\partial^2 \Psi}{\partial l^2}\right)_c = 0$$

These four expressions are given in complex form by:

$$\mathrm{Re}\left\{\frac{\Omega}{k} - \left(\frac{\Omega}{k}\right)_c - ic_s \frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right\} = 0 \tag{III.25}$$

$$\mathrm{Re}\left\{\frac{d\Omega}{dz}\left[\frac{1}{k}\frac{\partial z}{\partial l} - ic_s\frac{\partial^2 z}{\partial l^2}\right] + \left(\frac{1}{k}\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right)_c - \frac{d^2\Omega}{dz^2}\left[ic_s\left(\frac{\partial z}{\partial l}\right)^2\right]\right\} = 0 \tag{III.26}$$

$$\mathrm{Im}\left\{\frac{d\Omega}{dz}\frac{\partial z}{\partial l} + \left(\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right)_c\right\} = 0 \tag{III.23}$$

$$\mathrm{Im}\left\{\frac{d\Omega}{dz}\frac{\partial^2 z}{\partial l^2} - \left(\frac{d\Omega}{dz}\frac{\partial^2 z}{\partial l^2}\right)_c + \frac{d^2\Omega}{dz^2}\left(\frac{\partial z}{\partial l}\right)^2 - \left(\frac{d^2\Omega}{dz^2}\left(\frac{\partial z}{\partial l}\right)^2\right)_c\right\} = 0 \tag{III.24}$$

SUMMARY OF BOUNDARY CONDITION EXPRESSIONS

Equipotential line with silt layer

$$\mathrm{Re}\left\{\frac{\Omega}{k} - ic_s\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right\} = \phi_1 \tag{III.12}$$

$$\mathrm{Re}\left\{\frac{d\Omega}{dz}\left[\frac{1}{k}\frac{\partial z}{\partial l} - ic_s\frac{\partial^2 z}{\partial l^2}\right] - \frac{d^2\Omega}{dz^2}\left[ic_s\left(\frac{\partial z}{\partial l}\right)^2\right]\right\} = 0 \tag{III.13}$$

Stream line boundary

$$\mathrm{Im}\left\{\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right\} = 0 \tag{III.14}$$

$$\mathrm{Im}\left\{\frac{d\Omega}{dz}\frac{\partial^2 z}{\partial l^2} + \frac{d^2\Omega}{dz^2}\left(\frac{\partial z}{\partial l}\right)^2\right\} = 0 \tag{III.15}$$

Seepage line with silt layer

$$\mathrm{Re}\left\{\frac{\Omega}{k} - ic_s\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right\} = \mathrm{Im}\{z\} \tag{III.16}$$

$$\mathrm{Re}\left\{\frac{d\Omega}{dz}\left[\frac{1}{k}\frac{\partial z}{\partial l} - ic_s\frac{\partial^2 z}{\partial l^2}\right] - \frac{d^2\Omega}{dz^2}\left[ic_s\left(\frac{\partial z}{\partial l}\right)^2\right]\right\} = \mathrm{Im}\left\{\frac{\partial z}{\partial l}\right\} \tag{III.17}$$

Phreatic line

$$\mathrm{Re}\left\{\frac{\Omega}{k}\right\} = \mathrm{Im}\{z\} \tag{III.18}$$

$$\mathrm{Re}\left\{\frac{d\Omega}{dz}\frac{1}{k}\frac{\partial z}{\partial l}\right\} = \mathrm{Im}\left\{\frac{\partial z}{\partial l}\right\} \tag{III.19}$$

Interface

$$\mathrm{Re}\left\{\frac{\rho\Omega}{k} - \left(\frac{\rho\Omega}{k}\right)_c\right\} = (\rho - \rho_c)\,\mathrm{Im}\{z\} \tag{III.21}$$

$$\mathrm{Re}\left\{\frac{\rho}{k}\frac{d\Omega}{dz}\frac{\partial z}{\partial l} + \left(\frac{\rho}{k}\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right)_c\right\} = (\rho - \rho_c)\mathrm{Im}\left\{\frac{\partial z}{\partial l}\right\} \tag{III.22}$$

$$\mathrm{Im}\left\{\frac{d\Omega}{dz}\frac{\partial z}{\partial l} + \left(\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right)_c\right\} = 0 \tag{III.23}$$

$$\mathrm{Im}\left\{\frac{d\Omega}{dz}\frac{\partial^2 z}{\partial l^2} - \left(\frac{d\Omega}{dz}\frac{\partial^2 z}{\partial l^2}\right)_c + \frac{d^2\Omega}{dz^2}\left(\frac{\partial z}{\partial l}\right)^2 - \left(\frac{d^2\Omega}{dz^2}\left(\frac{\partial z}{\partial l}\right)^2\right)_c\right\} = 0 \tag{III.24}$$

Inhomogeneity line with silt layer

$$\mathrm{Re}\left\{\frac{\Omega}{k} - \left(\frac{\Omega}{k}\right)_c - ic_s\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right\} = 0 \tag{III.25}$$

$$\mathrm{Re}\left\{\frac{d\Omega}{dz}\left[\frac{1}{k}\frac{\partial z}{\partial l} - ic_s\frac{\partial^2 z}{\partial l^2}\right] + \left(\frac{1}{k}\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right)_c - \frac{d^2\Omega}{dz^2}\left[ic_s\left(\frac{\partial z}{\partial l}\right)^2\right]\right\} = 0 \tag{III.26}$$

$$\operatorname{Im}\left\{\frac{d\Omega}{dz}\,\frac{\partial z}{\partial l}+\left(\frac{d\Omega}{dz}\,\frac{\partial z}{\partial l}\right)_c\right\}=0 \tag{III.23}$$

$$\operatorname{Im}\left\{\frac{d\Omega}{dz}\,\frac{\partial^2 z}{\partial l^2}-\left(\frac{d\Omega}{dz}\,\frac{\partial^2 z}{\partial l^2}\right)_c+\frac{d^2\Omega}{dz^2}\left(\frac{\partial z}{\partial l}\right)^2-\left(\frac{d^2\Omega}{dz^2}\left(\frac{\partial z}{\partial l}\right)^2\right)_c\right\}=0 \tag{III.24}$$

15 The Approximative Function Ω_2 (z)

15.1 General

It is mentioned in Chapter 13 that the following form was chosen for the approximative part of the solution:

$$\Omega_2(z) = \sum_{j=1}^{n} \gamma_j F_j(z) \tag{III.4}$$

where:

n : number of terms in $\Omega_2(z)$
γ_j : complex constant
$F_j(z)$: analytic function of z

There are many possibilities for the choice of the function $F_j(z)$. Analogous to a sub-division for functions of a real variable (see Rektorys, 1969), a difference is made between algebraic functions and transcendental functions. Algebraic functions are polynomials (rational functions) or quotients of polynomials (fractional rational functions). Functions that are not algebraic are transcendental functions in which distinction is made between elementary and higher transcendental functions. Examples of elementary transcendental functions are z^a, trigonometric, hyperbolic and exponential functions and the related inverse functions. Higher transcendental functions are defined by differential equations (for example, Bessel functions) or integrals (e.g., elliptic integrals).

15.2 Algebraic Functions

It was mentioned in Section 15.1 that algebraic functions are polynomials or quotients of polynomials. A polynomial contains its degrees of freedom in the form of constants. If the quotient of two polynomials were used, the degrees of freedom would not appear linearly in the expressions for the boundary conditions, thus making the calculation more complicated. For example, the function:

$$\Omega_2(z) = \frac{a_1 z + a_2 z^2}{b_1 z + b_2 z^2}$$

has the first derivative:

$$\Omega_2'(z) = \frac{(b_1 z + b_2 z^2)(a_1 + 2a_2 z) - (a_1 z + a_2 z^2)(b_1 + 2b_2 z)}{(b_1 z + b_2 z^2)^2}$$

As constants a_1, a_2, b_1, b_2 do not all appear linearly in these expressions, the use of this kind of function, therefore, would result in complicated non-linear boundary condition expressions. So the polynomial is a better choice:

$$\Omega_2(z) = a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n \tag{III.27}$$

In this case there is not a constant in the polynomial because in the solution there is already a constant (Ω_0).
Expression (III.27) agrees with the previously chosen form (III.4):

$$\Omega_2(z) = \sum_{j=1}^{n} \gamma_j F_j(z)$$

This is seen by writing (III.27) in another form:

$$\Omega_2(z) = \sum_{j=1}^{n} a_j z^j \tag{III.28}$$

Generally, (III.28) may be suitable; the function $F_j(z) = z^j$ is analytic.
When (III.28) is combined with the general solution (III.2) and the boundary conditions (see Chapter 14), a set of linear equations is obtained in the unknowns a_j and Ω_0. After solving this set of equations, the solution is known.
In general it may be a disadvantage of polynomials that there may arise numerical problems when there are many terms (then $|z|^j$ becomes very large).

A special kind of polynomial is the well-known Lagrange interpolation polynomial, which in complex form is given by the following expression:

$$\Omega_2(z) = \sum_{j=1}^{n} \gamma_j \prod_{\substack{i=1 \\ i \neq j}}^{n} \frac{(z - z_i)}{(z_j - z_i)}$$

This Lagrange polynomial has the special property that the complex constants γ_j are just the values of the function in the points $z = z_j$:

$$\Omega_2(z_j) = \gamma_j$$

In principle, the interpolation polynomial of Lagrange may be used for the solution of flow problems. There is, however, an important practical disadvantage: Kosten (see Kuipers en Timman, 1969 chapter XII) notes that the Lagrange polynomial is not convenient for practical calculations because of the considerable amount of calculation work necessary for evaluating values of the function.

15.3 Transcendental Functions

It is mentioned in Section 15.1 that there are elementary and higher transcendental functions, the latter being defined by integrals or differential equations. Because the amount of calculation work required for evaluating values of functions is generally much greater for higher transcendental functions than for elementary transcendental functions, only the latter functions are discussed.
In principle, there are many possibilities that may be used for $F_j(z)$ in (III.4):

$$\Omega_2(z) = \sum_{j=1}^{n} \gamma_j F_j(z)$$

Some examples of the use of elementary transcendental functions in the function $\Omega_2(z)$ are:

$$\Omega_2(z) = \sum_{j=1}^{n} \gamma_j \sin(z - z_j)$$

$$\Omega_2(z) = \sum_{j=1}^{n} \gamma_j [\sin(z - z_j) + \cos(z - z_j)]$$

$$\Omega_2(z) = \sum_{j=1}^{n} \gamma_j \ln(z - z_j) \tag{III.29}$$

where z_j are the boundary points (points where the boundary conditions are posed, see Chapter 14). It is not neccessary that the positions of the boundary points (z_j) appear in the expression for $\Omega_2(z)$. The form (III.29) with $(z - z_j)$ was chosen for its analogy with the appearance of $(z - z_j)$ in the interpolation polynomial of Lagrange. Although in principle the expressions mentioned for $\Omega_2(z)$ and many others might be useful, a further selection is made between elementary transcendental functions:
When the computer time needed for the evaluation of values of some elementary transcendental functions are compared, it is seen that the time needed for complex logarithms is about 25% less than for complex sines and complex cosines (see IBM Systems Reference Library 360S-LM-501). This is shown in the following table, where the time needed for a complex sine evaluation is set to 100%:

Function	Name	Time (%)
$\sin(z)$	CSIN(Z)	100
$\cos(z)$	CCOS(Z)	100
$\exp(z)$	CEXP(Z)	87
$\ln(z)$	CLOG(Z)	75

On account of this fact, the use of complex logarithms is preferable. For example, there could be used (III.29):

$$\Omega_2(z) = \sum_{j=1}^{n} \gamma_j \ln(z - z_j) \tag{III.29}$$

For the case that γ_j is complex, this expression represents a flow due to sources and sinks with strength α_j and vortices with strength β_j (where $\gamma_j = \alpha_j + i\beta_j$) in the points z_j. In principle, (III.29) can be used to approximate flow patterns, but this has the disadvantage that the logarithmic function is singular at the boundary points z_j. That difficulty can be avoided by choosing the points z_j outside the flow region and posing the boundary conditions at other points on the boundary. Although this can be done, it is not efficient because points have to be defined at the boundary as well as points outside the region. In addition, when the points z_j of (III.29) are chosen close to the boundary, many terms would be needed to prevent inaccuracy caused by the singular behaviour of the complex logarithms. Then the calculation would require a considerable amount of work.

The situation with many sinks, sources and vortices can be replaced by a system of continuous distributions of sinks, sources and vortices over the boundaries. The use of those distributions has the advantage that the expression for $\Omega_2(z)$ is not singular in the boundary points and a flow pattern is generated that is more smooth in the surroundings of a boundary point.

The boundary points are chosen at the middles of the boundary segments over which there is a distribution of sinks, sources and vortices.

To ensure the smoothness of the flow pattern at the boundary and for efficiency the boundary segments are chosen in sequence without gaps. Then only the end points and the properties of these boundary segments form the boundary input for a calculation program.

In the following the distribution strength has been chosen to be constant for each boundary segment.

The expression for the complex potential due to a distribution of complex strength is derived first. The potential due to a sink (or source) and a vortex in the point z_a is given by:

$$\Omega(z) = \frac{Q}{2\pi} \ln(z - z_a)$$

where Q is the complex strength.
When the quantity Q is distributed over a line segment x_1x_2 on the x-axis according to:

$$q = \frac{dQ}{dx}$$

where $q = r + is$, then the complex potential is given by:

$$\Omega(z) = \frac{q}{2\pi} \int_{x1}^{x2} \ln(z - \lambda)d\lambda$$

After integration, and neglecting the integration constant, it can be written as:

$$\Omega(z) = -\frac{q}{2\pi} \{(z - x_2)\ln(z - x_2) - (z - x_1)\ln(z - x_1)\} \tag{III.30}$$

When the end points of the segment are chosen at arbitrary positions in the complex plane, the complex potential can be simply derived from (III.30) by a translation b and a rotation θ of the coordinates as shown in figure III.8.

$$z^* = ze^{i\theta} + b$$

where b is complex.
The points $z_1 = x_1$ and $z_2 = x_2$ have become the following positions in the transformed plane:

$$z_1{}^* = x_1e^{i\theta} + b \text{ and}$$
$$z_2{}^* = x_2e^{i\theta} + b$$

where:

$$\theta = \arg(z_2{}^* - z_1{}^*)$$

A constant c^* per line segment $z_1^*z_2^*$ is defined by $c^* = \exp(-i\theta)$ or:

$$c^* = \exp(-i \arg(z_2{}^* - z_1{}^*))$$

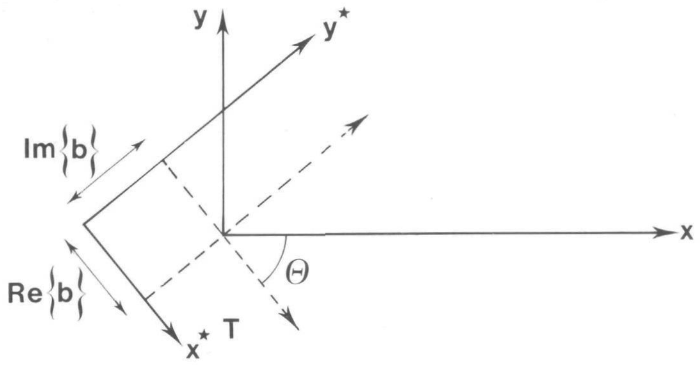

Figure III.8 Rotation and translation.

Otherwise written, using $(z_2{}^* - z_1{}^*) = |z_2{}^* - z_1{}^*| \exp(i \arg(z_2{}^* - z_1{}^*))$ (III.31)

$$c^* = \frac{|z_2{}^* - z_1{}^*|}{(z_2{}^* - z_1{}^*)}$$

When the inverse expressions (for z, $z_1 = x_1$ and $z_2 = x_2$):

$$z = c^*(z^* - b)$$

$$x_1 = c^*(z_1{}^* - b)$$

$$x_2 = c^*(z_2{}^* - b)$$

are substituted in (III.30), there comes:

$$\Omega(z^*) = -\frac{q}{2\pi}\{(c^*z^* - c^*z_2{}^*)\ln[c^*z^* - c^*z_2{}^*] + \\ - (c^*z^* - c^*z_1{}^*)\ln[c^*z^* - c^*z_1{}^*]\}$$

So the complex potential due to a distribution of complex strength q over a line segment z_1z_2 is given by:

$$\Omega(z) = -\frac{qc}{2\pi}\{(z - z_2)\ln[c(z - z_2)] - (z - z_1)\ln[c(z - z_1)]\} \qquad \text{(III.32)}$$

where c is defined by $c = \dfrac{|z_2 - z_1|}{(z_2 - z_1)}$

The use of (III.32) in the formula for the approximative part $\Omega_2(z)$ of the solution yields (where each distribution is denoted by the subscript j):

$$\Omega_2(z) = \sum_{j=1}^{n} - \frac{q_j c_j}{2\pi} \{(z - z_{2j})\ln[c_j(z - z_{2j})] - (z - z_{1j})\ln[c_j(z - z_{1j})]\} \tag{III.33}$$

where: $$c_j = \frac{|z_{2j} - z_{1j}|}{(z_{2j} - z_{1j})} \tag{III.34}$$

The constant $-1/2\pi$ is not relevant in (III.33) because it can be accounted in the constants q_j. Because of the visibility that q_j represents a distribution strength, the form (III.33) with $-q_j/2\pi$ is chosen to be used in the general solution (III.2).

In the literature the use of singularities and distributions of singularities is well-known. Lamb (1932) showed that a flow that satisfies the Laplace differential equation can be generated by an appropriate distribution of sinks and sources or vortices over the boundaries.
In aerodynamics, applications are given by Von Kármán (1927) and Mc. Nown and Hsu (1949). Von Kármán calculated the pressure distribution on the surface of air ships. Therefore he located distributions of sinks and sources over a number of line segments at the axis of symmetry. At a corresponding number of points on the air ship surface, he posed the condition that the normal velocity is zero, which resulted in linear equations with the unknown distribution strengths. The solution was used to calculate the pressure distribution on the air ship surface.
Mc. Nown and Hsu generated an inlet flow between two walls, using separate vortices outside the flow region. The way of solving the problem was analogous to that of Von Kármán.
De Josselin de Jong (1960, 1969) used singularity distributions for generating flow through a porous medium with varying properties of fluid or porous medium. He located singularity distributions at interfaces. These distributions were chosen in such a way that the problem reduced to a homogeneous problem that could be solved by usual methods. The solution of a flow problem then consists of two parts: one part that is the flow due to the singularity distributions and a second part that makes the solution (including the first part) satisfy the boundary conditions.

16 The Calculation of Flow Problems

It was mentioned in Chapter 13 that the general solution, generating the flow pattern in a sub-region, is given by:

$$\Omega(z) = \Omega_2(z) + \Omega_1(z) + \Omega_0 \tag{III.2}$$

where Ω_0 is a reference constant, $\Omega_1(z)$ is the exact part that generates the flow due to the sinks and sources, and $\Omega_2(z)$ is the approximative part that contains parameters that are used to meet the boundary conditions at a number of points.
Substitution of (III.33) and (III.3) in (III.2) yields:

$$\Omega(z) = \sum_{j=1}^{n} -\frac{q_j c_j}{2\pi}\{(z - z_{2j})\ln[c_j(z - z_{2j})] - (z - z_{1j})\ln[c_j(z - z_{1j})]\} +$$

$$+ \sum_{j=1}^{m} \frac{Q_j}{2\pi}\ln(z - z_j) + \Omega_0 \tag{III.35}$$

where c_j is given by (III.34):

$$c_j = \frac{|z_{2j} - z_{1j}|}{(z_{2j} - z_{1j})} \tag{III.34}$$

The constant $q_j = r_j + is_j$ is the complex distribution strength over the line segment from $z_{1j} = x_{1j} + iy_{1j}$ to $z_{2j} = x_{2j} + iy_{2j}$.

The first and the second derivatives of $\Omega(z)$ with respect to z also appear in the boundary condition expressions mentioned in Chapter 14.
These are found by differentiation of (III.35):

$$\frac{d\Omega(z)}{dz} = \sum_{j=1}^{n} -\frac{q_j c_j}{2\pi}\{\ln[c_j(z - z_{2j})] - \ln[c_j(z - z_{1j})]\} + \sum_{j=1}^{m} \frac{Q_j}{2\pi(z - z_j)} \tag{III.36}$$

$$\frac{d^2\Omega(z)}{dz^2} = \sum_{j=1}^{n} -\frac{q_j c_j}{2\pi}\left\{\frac{1}{(z - z_{2j})} - \frac{1}{(z - z_{1j})}\right\} + \sum_{j=1}^{m} \frac{-Q_j}{2\pi(z - z_j)^2} \tag{III.37}$$

The complex reference constant Ω_0 in (III.35) permits a translation of the values of the potential and the stream function.

The value of Ω_0 can be defined by two equations. The equation with respect to $\mathrm{Im}\{\Omega_0\}$ may be given by the condition that at a point z_{01} the stream function is set to zero:

$$\mathrm{Im}\{\Omega(z_{01})\} = 0$$

The equation with respect to $\mathrm{Re}\{\Omega_0\}$ is a similar equation for the potential in a point z_{02}; however, it is not permitted to choose an arbitrary value of the potential at the point z_{02} because values of the potential are involved in the boundary conditions. Therefore it is convenient to choose the point z_{02} at the boundary. Then for the equation with respect to $\mathrm{Re}\{\Omega_0\}$ the potential at the point z_{02} is related to the potential at the boundary or just outside the boundary (e.g., if there is an equipotential line with a silt layer). This equation has the same form as that of the previously discussed boundary condition equations (e.g., for an equipotential line with a silt layer). Because boundary conditions are posed at the middles of the boundary segments, the point z_{02} may not be chosen at the middle of a boundary segment (then the equation with respect to $\mathrm{Re}\{\Omega_0\}$ would be the same as a boundary condition equation; consequently, the set of equations would be undetermined).

The calculation procedure for solving flow problems will now be outlined. The general solution is given by (III.35) and a combination of (III.35), (III.36) and (III.37) with the boundary condition expressions (see Chapter 14) leads to a set of linear equations. The unknowns are the complex distribution strengths $q_j (q_j = r_j + is_j)$ and the complex reference constant Ω_0. So when there are n boundary segments the result of posing boundary conditions and reference conditions is a set of $(2n + 2)$ linear equations for each sub-region. After solving this set, the complex potential can be evaluated at any point by substitution of q_j and Ω_0 in (III.35).

The specific discharges are given by:

$$v_x = -\frac{\partial \Phi}{\partial x} = -\mathrm{Re}\left\{\frac{\partial \Omega}{\partial x}\right\} = -\mathrm{Re}\left\{\frac{d\Omega}{dz}\,\frac{\partial z}{\partial x}\right\} = -\mathrm{Re}\left\{\frac{d\Omega}{dz}\right\}$$

$$v_y = -\frac{\partial \Phi}{\partial y} = -\mathrm{Re}\left\{\frac{d\Omega}{\partial y}\right\} = -\mathrm{Re}\left\{\frac{d\Omega}{dz}\,\frac{\partial z}{\partial y}\right\} = -\mathrm{Re}\left\{\frac{id\Omega}{dz}\right\} = \mathrm{Im}\left\{\frac{d\Omega}{dz}\right\} \tag{III.38}$$

If the flow problem is non-steady where one or more parts of the boundary are moving (phreatic line or interface), then that behaviour is calculated in the following way. (The expressions hold for a phreatic line. Similar expressions hold for an interface but then the storage coefficient μ has to be replaced by the effective porosity η). Starting from a known or assumed position of the boundary, the flow pattern is calculated in the way described, followed by the calculation of the specific discharges at the moving boundaries. These specific discharges are used to calculate the position of the boundary after one step in time. The alteration of the position of a point in a step in time is given by (using (III.38)):

$$\left.\begin{aligned} \Delta x &= -\operatorname{Re}\left\{\frac{d\Omega}{dz}\right\}\frac{\Delta t}{\mu} \\ \Delta y &= \operatorname{Im}\left\{\frac{d\Omega}{dz}\right\}\frac{\Delta t}{\mu} \end{aligned}\right\} \qquad \text{(III.39)}$$

where μ is the storage coefficient of the soil and Δt is the time step size. If the moving boundary is a phreatic line with precipitation, the rise due to the precipitation has also to be accounted. In Chapter 14 it was mentioned that the rise according to precipitation N is given by:

$$\Delta y = N\frac{\Delta t}{\mu} \qquad \text{(III.40)}$$

where N is the precipitation per unit of length and time. Then the position alteration of a point of the boundary (phreatic line) in a step in time is given by the sum of (III.40) and (III.39):

$$\left.\begin{aligned} \Delta x &= -\operatorname{Re}\left\{\frac{d\Omega}{dz}\right\}\frac{\Delta t}{\mu} \\ \Delta y &= \left[\operatorname{Im}\left\{\frac{d\Omega}{dz}\right\} + N\right]\frac{\Delta t}{\mu} \end{aligned}\right\} \qquad \text{(III.41)}$$

After correction of the boundary position, the flow pattern after a step in time is calculated and then the procedure is repeated.
Generally the position of a phreatic line is not known previously, but it is pointed out in Chapter 14 that the steady position of that line can be found by calculation of the non-steady behaviour of the flow: After some steps in time the boundary will reach a position that is practically at rest. In the calculation the normal component of the specific discharge at the phreatic line will then tend to zero.
If the normal direction is denoted by n' (see figure III.9), the specific discharge in n'-direction is given by:

$$v_{n'} = -\frac{\partial \Phi}{\partial n'} = \frac{\partial \Psi}{\partial l} \qquad \text{(III.42)}$$

or:

$$v_{n'} = \operatorname{Im}\left\{\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right\} \qquad \text{(III.43)}$$

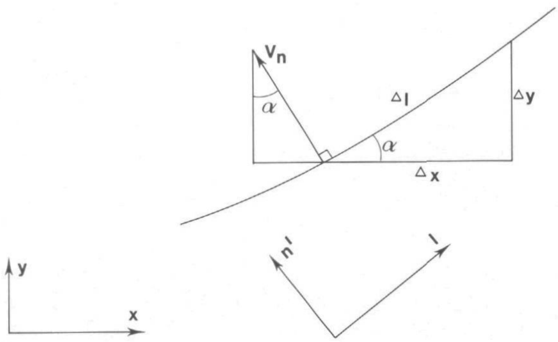

Figure III.9 Tangential and normal direction at the boundary.

When large steps in time are chosen, the specific discharges at the end of the step in time may differ considerably from those at the beginning of the step in time. In such cases the average velocity of the specific discharge over the step in time should have been used; then some iterations might be necessary. In most cases, however, it will be more convenient to choose the time step size so that these effects are small. The time step size that is convenient depends on the kind of problem.

The equations that are used for generating flow patterns and the expressions for correcting the position of a moving boundary will now be detailed, with the expressions (III.35), (III.36) and (III.37) being written in a more convenient form:
The following notations are used:

$$\left.\begin{aligned}
F0_j(z) &= -\frac{c_j}{2\pi}\{(z - z_{2j})\ln[c_j(z - z_{2j})] - (z - z_{1j})\ln[c_j(z - z_{1j})]\}\\
F1_j(z) &= -\frac{c_j}{2\pi}\{\ln[c_j(z - z_{2j})] - \ln[c_j(z - z_{1j})]\}\\
F2_j(z) &= -\frac{c_j}{2\pi}\left\{\frac{1}{(z - z_{2j})} - \frac{1}{(z - z_{1j})}\right\}\\
G0(z) &= \sum_{j=1}^{m}\frac{Q_j}{2\pi}\ln(z - z_j)\\
G1(z) &= \sum_{j=1}^{m}\frac{Q_j}{2\pi(z - z_j)}\\
G2(z) &= \sum_{j=1}^{m}\frac{-Q_j}{2\pi(z - z_j)^2}.
\end{aligned}\right\} \quad \text{(III.44)}$$

Then the expressions (III.35), (III.36) and (III.37) may be given by:

$$\Omega(z) = \sum_{j=1}^{n} q_j\, F0_j(z) + G0(z) + \Omega_0 \tag{III.45}$$

$$\frac{d\Omega(z)}{dz} = \sum_{j=1}^{n} q_j\, F1_j(z) + G1(z) \tag{III.46}$$

$$\frac{d^2\Omega(z)}{dz^2} = \sum_{j=1}^{n} q_j\, F2_j(z) + G2(z) \tag{III.47}$$

The specific discharges can be expressed by the following formulas (using (III.38) and (III.46)):

$$\left.\begin{aligned} v_x &= -\operatorname{Re}\left\{\sum_{j=1}^{n} q_j F1_j(z) + G1(z)\right\} \\ v_y &= \operatorname{Im}\left\{\sum_{j=1}^{n} q_j\, F1\,(z) + G1(z)\right\} \end{aligned}\right\} \tag{III.48}$$

The expressions (III.41) for the alteration of the position of a point of a phreatic line can be expressed using (III.46).

$$\left.\begin{aligned} \Delta x &= -\operatorname{Re}\left\{\sum_{j=1}^{n} q_j\, F1_j(z) + G1(z)\right\}\frac{\Delta t}{\mu} \\ \Delta y &= \left[\operatorname{Im}\left\{\sum_{j=1}^{n} q_j\, F1_j(z) + G1(z)\right\} + N\right]\frac{\Delta t}{\mu} \end{aligned}\right\} \tag{III.49}$$

The reference equation Im $\Omega(z_{0\,1}) = 0$ for the stream function can be written in the following form, using (III.45):

$\operatorname{Im}\left\{\sum_{j=1}^{n} q_j\, F0_j(z_0) + G0(z_0) + \Omega_0\right\} = 0$, or otherwise arranged (using $q_j = r_j + is_j$):

$$\sum_{j=1}^{n}\left[r_j \operatorname{Im}\{F0_j(z_0)\} + s_j \operatorname{Re}\{F0_j(z_0)\}\right] + \Psi_0 = \operatorname{Im}\{-G0(z_0)\} \tag{III.50}$$

This is a linear equation in r_j, s_j and Ψ_0.

It has already been mentioned in this Chapter that the equation for Re$\{\Omega_0\}$ has the

same form as a boundary condition equation.
The boundary condition equations are based on the boundary condition expressions of Chapter 14 in which the terms $\frac{\partial z}{\partial l}$ and $\frac{\partial^2 z}{\partial l^2}$ appear, representing the form of the boundary. For a straight line, the following conditions hold:

$$\frac{\partial z}{\partial l} = \text{constant} \quad \frac{\partial^2 z}{\partial l^2} = 0$$

In this text only straight line segments of the boundary are used, thus $\frac{\partial^2 z}{\partial l^2} = 0$.

This is permitted when the boundary is not sharply curved. A sharply-curved boundary can be approximated by a number of straight line segments.
The following boundary condition equations refer to boundary conditions for the middle of a boundary segment with number j^*, the end points of which are z_{1j^*} and z_{2j^*}. The term $\frac{\partial z}{\partial l}$ can be discretisized for that segment by:

$$\frac{\partial z}{\partial l} \triangleq \frac{(z_{2j^*} - z_{1j^*})}{|z_{2j^*} - z_{1j^*}|}$$

Analogous to (III.34), this can be written with c_{j^*}, thus:

$$\frac{\partial z}{\partial l} \triangleq \frac{(z_{2j^*} - z_{1j^*})}{|z_{2j^*} - z_{1j^*}|} = \frac{1}{c_{j^*}}$$

or:

$$c_{j^*} = \frac{|z_{2j^*} - z_{1j^*}|}{(z_{2j^*} - z_{1j^*})} \tag{III.51}$$

Im some of the following expressions $\operatorname{Im}\left\{\frac{\partial z}{\partial l}\right\}$ appears.

In a similar way this can be discretised by:

$$\operatorname{Im}\left\{\frac{\partial z}{\partial l}\right\} \triangleq \frac{\operatorname{Im}\{z_{2j^*} - z_{1j^*}\}}{|z_{2j^*} - z_{1j^*}|}$$

Further, this is denoted by e_{j^*}:

$$e_{j^*} = \frac{\operatorname{Im}\{z_{2j^*} - z_{1j^*}\}}{|z_{2j^*} - z_{1j^*}|} \tag{III.52}$$

Equipotential line with silt layer

The first boundary condition expression is (III.12):

$$\mathrm{Re}\left\{\frac{\Omega}{k} - ic_s \frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right\} = \phi_1$$

Multiplying by k and combining it with (III.45), (III.46) and (III.51) yields:

$$\mathrm{Re}\left\{\sum_{j=1}^{n} q_j\, F0_j(z) + G0(z) + \Omega_0 - i\frac{kc_s}{c_{j^*}}\left[\sum_{j=1}^{n} q_j\, F1_j(z) + G1(z)\right]\right\} = k\phi_1.$$

After some rearranging, it can be written in the following form:

$$\sum_{j=1}^{n}\left[r_j \mathrm{Re}\{H1_j(z)\} + s_j\, \mathrm{Im}\{-H1_j(z)\}\right] + \Phi_0 =$$

$$= \mathrm{Re}\{k\phi_1 - G0(z) + i\frac{kc_s}{c_{j^*}} G1(z)\} \qquad \text{(III.53)}$$

where:

$$H1_j(z) = F0_j(z) - i\frac{kc_s}{c_{j^*}} F1_j(z) \qquad \text{(III.54)}$$

The second boundary expression is (III.13):

$$\mathrm{Re}\left\{\frac{d\Omega}{dz}\left[\frac{1}{k}\frac{\partial z}{\partial l} - ic_s\frac{\partial^2 z}{\partial l^2}\right] - \frac{d^2\Omega}{dz^2}\left[ic_s\left(\frac{\partial z}{\partial l}\right)^2\right]\right\} = 0$$

By substitution of $\dfrac{\partial^2 z}{\partial l^2} = 0$ and multiplication by k, and after combining with (III.46), (III.47) and (III.51), there is obtained:

$$\mathrm{Re}\left\{\frac{1}{c_{j^*}}\left[\sum_{j=1}^{n} q_j F1_j(z) + G1(z)\right] - i\frac{kc_s}{c_{j^*}^{\;2}}\left[\sum_{j=1}^{n} q_j F2_j(z) + G2(z)\right]\right\} = 0$$

Some rearranging yields:

$$\sum_{j=1}^{n}\left[r_j \mathrm{Re}\{H2_j(z)\} + s_j \mathrm{Im}\{-H2_j(z)\}\right] = \mathrm{Re}\left\{-\frac{1}{c_{j^*}} G1(z) + i\frac{kc_s}{c_{j^*}^{\;2}} G2(z)\right\} \qquad \text{(III.55)}$$

where:

$$H2_j(z) = \frac{1}{c_{j^*}} F1_j(z) - i\frac{kc_s}{c_{j^*}{}^2} F2_j(z) \tag{III.56}$$

Stream line boundary

The first boundary condition expression is (III.14):

$$\mathrm{Im}\left\{\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right\} = 0$$

Substitution of (III.46) and (III.51) yields:

$$\mathrm{Im}\left\{\frac{1}{c_{j^*}}\left[\sum_{j=1}^{n} q_j F1_j(z) + G1(z)\right]\right\} = 0$$

Rearranged, this becomes:

$$\sum_{j=1}^{n}\left[r_j \mathrm{Im}\left\{\frac{1}{c_{j^*}} F1_j(z)\right\} + s_j \mathrm{Re}\left\{\frac{1}{c_{j^*}} F1_j(z)\right\}\right] = \mathrm{Im}\left\{-\frac{1}{c_{j^*}} G1(z)\right\} \tag{III.57}$$

The second boundary condition expression is (III.15):

$$\mathrm{Im}\left\{\frac{d\Omega}{dz}\frac{\partial^2 z}{\partial l^2} + \frac{d^2\Omega}{dz^2}\left(\frac{\partial z}{\partial l}\right)^2\right\} = 0$$

Substitution of $\frac{\partial^2 z}{\partial l^2} = 0$ yields:

$$\mathrm{Im}\left\{\frac{d^2\Omega}{dz^2}\left(\frac{\partial z}{\partial l}\right)^2\right\} = 0$$

Substitution of (III.47) and (III.51) gives:

$$\mathrm{Im}\left\{\frac{1}{c_j{}^2{}_*}\left[\sum_{j=1}^{n} q_j F2_j(z) + G2(z)\right]\right\} = 0$$

In rearranged form, this is:

$$\sum_{j=1}^{n}\left[r_j \mathrm{Im}\left\{\frac{1}{c_{j^*}{}^2} F2_j(z)\right\} + s_j \mathrm{Re}\left\{\frac{1}{c_{j^*}{}^2} F2_j(z)\right\}\right] = \mathrm{Im}\left\{-\frac{1}{c_{j^*}{}^2} G2(z)\right\} \tag{III.58}$$

Seepage line with silt layer

The first boundary condition expression is (III.16):

$$\mathrm{Re}\left\{\frac{\Omega}{k} - ic_s \frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right\} = \mathrm{Im}\{z\}$$

This expression is almost identical with the expression for an equipotential line with silt layer (III.12). The term ϕ_1 here is replaced by $\mathrm{Im}\{z\}$. By such a replacement in (III.53), the first equation for a seepage line with silt layer is obtained, where it is noted that $\mathrm{Im}\{z\} = \mathrm{Re}\{-iz\}$:

$$\sum_{j=1}^{n}\left[r_j\mathrm{Re}\{H1_j(z)\} + s_j\mathrm{Im}\{-H1_j(z)\}\right] + \Phi_0 =$$

$$= \mathrm{Re}\left\{-iz - G0(z) + \frac{ikc_s}{c_{j^*}} G1(z)\right\} \tag{III.59}$$

where $H1_j(z)$ is given by (III.54).

The second boundary condition expression is (III.17):

$$\mathrm{Re}\left\{\frac{d\Omega}{dz}\left[\frac{1}{k}\frac{\partial z}{\partial l} - ic_s\frac{\partial^2 z}{\partial l^2}\right] - \frac{d^2\Omega}{dz^2}\left[ic_s\left(\frac{\partial z}{\partial l}\right)^2\right]\right\} = \mathrm{Im}\left\{\frac{\partial z}{\partial l}\right\}$$

By substituting $\dfrac{\partial^2 z}{\partial l^2} = 0$ and multiplying by k, there results:

$$\mathrm{Re}\left\{\frac{d\Omega}{dz}\frac{\partial z}{\partial l} - ikc_s\frac{d^2\Omega}{dz^2}\left(\frac{\partial z}{\partial l}\right)^2\right\} = k\,\mathrm{Im}\left\{\frac{\partial z}{\partial l}\right\}$$

The left-hand part of this equation is the same as the expression for an equipotential line with silt layer (III.13) with $\dfrac{\partial^2 z}{\partial l^2} = 0$ and multiplied by k. By putting $k\ \mathrm{Im}\left\{\dfrac{\partial z}{\partial l}\right\}$ in the right-hand side of the equation and noting that $\mathrm{Im}\left\{\dfrac{\partial z}{\partial l}\right\}$ is given by c_{j^*} (see (III.52)), the result is:

$$\sum_{j=1}^{n}\left[r_j\,\mathrm{Re}\{H2_j(z)\} + s_j\,\mathrm{Im}\{-H2_j(z)\}\right] =$$

$$= \mathrm{Re}\left\{k\,e_{j^*} - \frac{1}{c_{j^*}} G1(z) + \frac{ikc_s}{{c_{j^*}}^2} G2(z)\right\} \tag{III.60}$$

where $H2_j(z)$ is given by (III.56).

Phreatic line
The first boundary condition expression is (III.18):

$$\mathrm{Re}\left\{\frac{\Omega}{k}\right\} = \mathrm{Im}\{z\}$$

Multiplying by k and combining with (III.45) yields:

$$\mathrm{Re}\left\{\sum_{j=1}^{n} q_j F0_j(z) + G0(z) + \Omega_0\right\} = k\,\mathrm{Im}\{z\}$$

After some rearranging and noting that $k\,\mathrm{Im}\{z\} = \mathrm{Re}\{-ikz\}$ there is obtained:

$$\sum_{j=1}^{n}\left[r_j \mathrm{Re}\{F0_j(z)\} + s_j\,\mathrm{Im}\{-F0_j(z)\}\right] + \Phi_0 = \mathrm{Re}\{-ikz - G0(z)\} \tag{III.61}$$

The second boundary condition expression is (III.19):

$$\mathrm{Re}\left\{\frac{d\Omega}{dz}\,\frac{1}{k}\,\frac{\partial z}{\partial l}\right\} = \mathrm{Im}\left\{\frac{\partial z}{\partial l}\right\}$$

After multiplying by k and combining with (III.46) and (III.51) and noting that $\mathrm{Im}\left\{\frac{\partial z}{\partial l}\right\}$ is given by e_{j^*} according to (III.52), there comes:

$$\mathrm{Re}\left\{\frac{1}{c_{j^*}}\left[\sum_{j=1}^{n} q_j F1_j(z) + G1(z)\right]\right\} = k\,e_{j^*}$$

In rearranged form, this becomes:

$$\sum_{j=1}^{n}\left[r_j \mathrm{Re}\left\{\frac{1}{c_{j^*}} F1_j(z)\right\} + s_j \mathrm{Im}\left\{-\frac{1}{c_{j^*}} F1_j(z)\right\}\right] = \mathrm{Re}\left\{k\,e_{j^*} - \frac{1}{c_{j^*}} G1(z)\right\} \tag{III.62}$$

Interface
The first boundary condition expression is (III.21):

$$\mathrm{Re}\left\{\frac{\rho\Omega}{k} - \left(\frac{\rho\Omega}{k}\right)_c\right\} = (\rho - \rho_c)\mathrm{Im}\{z\}$$

Substitution of (III.45), and using:

$$(\rho - \rho_c)\,\mathrm{Im}\{z\} = \mathrm{Re}\{-i\,(\rho - \rho_c)z\}$$

yields:

$$\mathrm{Re}\left\{\frac{\rho}{k}\left[\sum_{j=1}^{n} q_j F0_j(z) + G0(z) + \Omega_0\right] + \right.$$
$$\left. - \left(\frac{\rho}{k}\left[\sum_{j=1}^{n} q_j F0_j(z) + G0(z) + \Omega_0\right]\right)_c\right\} = \mathrm{Re}\{-i(\rho - \rho_c)z\}$$

In rearranged form, this becomes:

$$\sum_{j=1}^{n}\left[r_j\,\mathrm{Re}\left\{\frac{\rho}{k}F0_j(z)\right\} + s_j\mathrm{Im}\left\{-\frac{\rho}{k}F0_j(z)\right\}\right] + \frac{\rho}{k}\Phi_0 +$$
$$+\left(\sum_{j=1}^{n}\left[r_j\mathrm{Re}\left\{-\frac{\rho}{k}F0_j(z)\right\} + s_j\mathrm{Im}\left\{\frac{\rho}{k}F0_j(z)\right\}\right] - \frac{\rho}{k}\Phi_0\right)_c =$$
$$= \mathrm{Re}\left\{-i(\rho - \rho_c)z - \frac{\rho}{k}G0(z) + \left(\frac{\rho}{k}G0(z)\right)_c\right\} \qquad \text{(III.63)}$$

If one of the fluids is stationary, the term

$$\mathrm{Re}\left\{\left(\frac{\rho\Omega}{k}\right)_c\right\} \text{ is constant:}$$

$$\mathrm{Re}\left\{\left(\frac{\rho\Omega}{k}\right)_c\right\} = \rho_c\phi_c$$

In that case this term is known, and can be placed in the right-hand side of the equation:

$$\sum_{j=1}^{n}\left[r_j\mathrm{Re}\left\{\frac{\rho}{k}F0_j(z)\right\} + s_j\mathrm{Im}\left\{-\frac{\rho}{k}F0_j(z)\right\}\right] + \frac{\rho}{k}\Phi_0 =$$
$$= \mathrm{Re}\left\{\rho_c\phi_c - i(\rho - \rho_c)z - \frac{\rho}{k}G0(z)\right\} \qquad \text{(III.64)}$$

The second boundary condition expression is (III.22):

$$\mathrm{Re}\left\{\frac{\rho}{k}\frac{d\Omega}{dz}\frac{\partial z}{\partial l} + \left(\frac{\rho}{k}\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right)_c\right\} = (\rho - \rho_c)\,\mathrm{Im}\left\{\frac{\partial z}{\partial l}\right\}$$

Substitution of (III.46), (III.51) and (III.52) yields:

$$\mathrm{Re}\left\{\frac{\rho}{k}\frac{1}{c_{j^*}}\left[\sum_{j=1}^{n} q_j F1_j(z) + G1(z)\right] + \right.$$

$$\left. + \left(\frac{\rho}{k}\frac{1}{c_{j^*}}\left[\sum_{j=1}^{n} q_j F1_j(z) + G1(z)\right]\right)_c\right\} = (\rho - \rho_c)e_{j^*}$$

In rearranged form this is:

$$\sum_{j=1}^{n}\left[r_j \mathrm{Re}\left\{\frac{\rho}{k}\frac{1}{c_{j^*}}F1_j(z)\right\} + s_j \mathrm{Im}\left\{-\frac{\rho}{k}\frac{1}{c_{j^*}}F1_j(z)\right\}\right] +$$

$$+ \left(\sum_{j=1}^{n}\left[r_j \mathrm{Re}\left\{\frac{\rho}{k}\frac{1}{c_{j^*}}F1_j(z)\right\} + s_j \mathrm{Im}\left\{-\frac{\rho}{k}\frac{1}{c_{j^*}}F1_j(z)\right\}\right]\right)_c =$$

$$= \mathrm{Re}\left\{(\rho - \rho_c)e_{j^*} - \frac{\rho}{k}\frac{1}{c_{j^*}}G1(z) + \left(\frac{\rho}{k}\frac{1}{c_{j^*}}G1(z)\right)_c\right\} \tag{III.65}$$

If one of the fluids is stationary, one term in (III.22) vanishes:

$$\mathrm{Re}\left\{\left(\frac{\rho}{k}\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right)_c\right\} = 0$$

In that case the second equation becomes:

$$\sum_{j=1}^{n}\left[r_j \mathrm{Re}\left\{\frac{\rho}{k}\frac{1}{c_{j^*}}F1_j(z)\right\} + s_j \mathrm{Im}\left\{-\frac{\rho}{k}\frac{1}{c_{j^*}}F1_j(z)\right\}\right] =$$

$$= \mathrm{Re}\left\{(\rho - \rho_c)\, e_{j^*} - \frac{\rho}{\mathrm{k}}\frac{1}{\mathrm{c}_{j^*}}G1(z)\right\} \tag{III.66}$$

The third boundary condition expression is (III.23):

$$\mathrm{Im}\left\{\frac{d\Omega}{dz}\frac{\partial z}{\partial l} + \left(\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right)_c\right\} = 0$$

The first term of the left-hand side of (III.23) is the same as the left-hand part of the first stream line condition. The second term has the same form, here for the adjoining sub-region. Using the first stream line equation (III.57) one finds:

$$\sum_{j=1}^{n}\left[r_j \mathrm{Im}\left\{\frac{1}{c_{j^*}}F1_j(z)\right\} + s_j \mathrm{Re}\left\{\frac{1}{c_{j^*}}F1_j(z)\right\}\right] +$$

$$+\left(\sum_{j=1}^{n}\left[r_j\,\mathrm{Im}\left\{\frac{1}{c_{j^*}}F1_j(z)\right\}+s_j\,\mathrm{Re}\left\{\frac{1}{c_{j^*}}F1_j(z)\right\}\right]\right)_c=$$

$$=\mathrm{Im}\left\{-\frac{1}{c_{j^*}}G1(z)-\left(\frac{1}{c_{j^*}}G1(z)\right)_c\right\} \tag{III.67}$$

The fourth boundary condition expression is (III.25):

$$\mathrm{Im}\left\{\frac{d\Omega}{dz}\frac{\partial^2 z}{\partial l^2}-\left(\frac{d\Omega}{dz}\frac{\partial^2 z}{\partial l^2}\right)_c+\frac{d^2\Omega}{dz^2}\left(\frac{\partial z}{\partial l}\right)^2-\left(\frac{d^2\Omega}{dz^2}\left(\frac{\partial z}{\partial l}\right)^2\right)_c\right\}=0$$

Substitution of $\frac{\partial^2 z}{\partial l^2}=0$ yields:

$$\mathrm{Im}\left\{\frac{d^2\Omega}{dz^2}\left(\frac{\partial z}{\partial l}\right)^2-\left(\frac{d^2\Omega}{dz^2}\left(\frac{\partial z}{\partial l}\right)^2\right)_c\right\}=0 \tag{III.68}$$

The first term of the left-hand side of (III.68) is the same as the left-hand part of the second stream line condition (III.58). The second term has the same form, here for the adjoining sub-region. Using the second stream line equation gives:

$$\sum_{j=1}^{n}\left[r_j\mathrm{Im}\left\{\frac{1}{c_{j^*}{}^2}F2_j(z)\right\}+s_j\mathrm{Re}\left\{\frac{1}{c_{j^*}{}^2}F2_j(z)\right\}\right]+$$

$$+\left(\sum_{j=1}^{n}\left[r_j\mathrm{Im}\left\{-\frac{1}{c_{j^*}{}^2}F2_j(z)\right\}+s_j\mathrm{Re}\left\{-\frac{1}{c_{j^*}{}^2}F2_j(z)\right\}\right]\right)_c=$$

$$=\mathrm{Im}\left\{-\frac{1}{c_{j^*}{}^2}G2(z)+\left(\frac{1}{c_{j^*}{}^2}G2(z)\right)_c\right\} \tag{III.69}$$

Inhomogeneity line with silt layer

The first boundary condition expression is (III.25):

$$\mathrm{Re}\left\{\frac{\Omega}{k}-\left(\frac{\Omega}{k}\right)_c-ic_s\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right\}=0$$

Substitution of (III.45), (III.46) and (III.51) yields:

$$\mathrm{Re}\left\{\frac{1}{k}\left(\sum_{j=1}^{n}q_jF0_j(z)+G0(z)+\Omega_0\right)-\left(\frac{1}{k}\left(\sum_{j=1}^{n}q_jF0_j(z)+G0(z)+\Omega_0\right)\right)_c+\right.$$

$$\left.-ic_s\frac{1}{c_{j^*}}\left(\sum_{j=1}^{n}q_jF1_j(z)+G1(z)\right)\right\}=0$$

In rearranged form, this is:

$$\sum_{j=1}^{n}\left[r_j\mathrm{Re}\left\{\frac{1}{k}H1_j(z)\right\}+s_j\mathrm{Im}\left\{-\frac{1}{k}H1_j(z)\right\}\right]+\frac{\Phi_0}{k}+$$
$$+\left(\sum_{j=1}^{n}\left[r_j\mathrm{Re}\left\{-\frac{1}{k}F0_j(z)\right\}+s_j\mathrm{Im}\left\{\frac{1}{k}F0_j(z)\right\}\right]-\frac{\Phi_0}{k}\right)_c=$$
$$=\mathrm{Re}\left\{-\frac{G0(z)}{k}+\left(\frac{G0(z)}{k}\right)_c+ic_s\frac{1}{c_{j^*}}G1(z)\right\} \tag{III.70}$$

where $H1_j(z)$ is given by (III.54).

The second boundary condition expression is (III.26):

$$\mathrm{Re}\left\{\frac{d\Omega}{dz}\left[\frac{1}{k}\frac{\partial z}{\partial l}-ic_s\frac{\partial^2 z}{\partial l^2}\right]+\left(\frac{1}{k}\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right)_c-\frac{d^2\Omega}{dz^2}\left[ic_s\left(\frac{\partial z}{\partial l}\right)^2\right]\right\}=0$$

Substitution of $\frac{\partial^2 z}{\partial l^2}=0$ yields:

$$\mathrm{Re}\left\{\frac{d\Omega}{dz}\frac{1}{k}\frac{\partial z}{\partial l}+\left(\frac{1}{k}\frac{d\Omega}{dz}\frac{\partial z}{\partial l}\right)_c-ic_s\frac{d^2\Omega}{dz^2}\left(\frac{\partial z}{\partial l}\right)^2\right\}=0$$

By substitution of (III.46), (III.47) and (III.51) there comes:

$$\mathrm{Re}\left\{\frac{1}{kc_{j^*}}\left(\sum_{j=1}^{n}q_jF1_j(z)+G1(z)\right)+\left(\frac{1}{kc_{j^*}}\left(\sum_{j=1}^{n}q_jF1_j(z)+G1(z)\right)\right)_c+\right.$$
$$\left.-i\frac{c_s}{{c_{j^*}}^2}\left(\sum_{j=1}^{n}q_jF2_j(z)+G2(z)\right)\right\}=0$$

Rearranged, this becomes:

$$\sum_{j=1}^{n}\left[r_j\mathrm{Re}\left\{\frac{1}{k}H2_j(z)\right\}+s_j\mathrm{Im}\left\{-\frac{1}{k}H2_j(z)\right\}\right]+$$
$$+\left(\sum_{j=1}^{n}\left[r_j\mathrm{Re}\left\{\frac{1}{kc_{j^*}}F1_j(z)\right\}+s_j\mathrm{Im}\left\{-\frac{1}{kc_{j^*}}F1_j(z)\right\}\right]\right)_c=$$
$$=\mathrm{Re}\left\{-\frac{1}{kc_{j^*}}G1(z)-\left(\frac{1}{kc_{j^*}}G1(z)\right)_c+i\frac{c_s}{{c_{j^*}}_2}G2(z)\right\} \tag{III.71}$$

where $H2_j(z)$ is given by (III.56).

The third and the fourth boundary condition expressions are (III.23) and (III.24). They are the same as for the interface, and so the corresponding equations are (III.67) and (III.69).

SUMMARY OF EXPRESSIONS AND EQUATIONS

Complex potential and derivatives:

$$\Omega(z) = \sum_{j=1}^{n} q_j F0_j(z) + G0(z) + \Omega_0 \tag{III.45}$$

$$\frac{d\Omega(z)}{dz} = \sum_{j=1}^{n} q_j F1_j(z) + G1(z) \tag{III.46}$$

$$\frac{d^2\Omega(z)}{dz^2} = \sum_{j=1}^{n} q_j F2_j(z) + G2(z) \tag{III.47}$$

Definitions:

$$\left.\begin{aligned}
F0_j(z) &= -\frac{c_j}{2\pi}\left\{(z - z_{2j})\ln[c_j(z - z_{2j})] - (z - z_{1j})\ln[c_j(z - z_{1j})]\right\} \\
F1_j(z) &= -\frac{c_j}{2\pi}\left\{\ln[c_j(z - z_{2j})] - \ln[c_j(z - z_{1j})]\right\} \\
F2_j(z) &= -\frac{c_j}{2\pi}\left\{\frac{1}{(z - z_{2j})} - \frac{1}{(z - z_{1j})}\right\} \\
G0(z) &= \sum_{j=1}^{m} \frac{Q_j}{2\pi} \ln(z - z_j) \\
G1(z) &= \sum_{j=1}^{m} \frac{Q_j}{2\pi(z - z_j)} \\
G2(z) &= \sum_{j=1}^{m} \frac{-Q_j}{2\pi(z - z_j)^2}
\end{aligned}\right\} \tag{III.44}$$

$$H1_j(z) = F0_j(z) - \frac{ikc_s}{c_{j^*}} F1_j(z) \tag{III.54}$$

$$H2_j(z) = \frac{1}{c_{j^*}} F1_j(z) - \frac{ikc_s}{c_{j^*}{}^2} F2_j(z) \tag{III.56}$$

$$c_j = \frac{|z_{2j} - z_{1j}|}{(z_{2j} - z_{1j})} \qquad \text{(III.34)}$$

$$c_{j^*} = \frac{|z_{2j^*} - z_{1j^*}|}{(z_{2j^*} - z_{1j^*})} \qquad \text{(III.51)}$$

$$e_{j^*} = \frac{\mathrm{Im}\{z_{2j^*} - z_{1j^*}\}}{|z_{2j^*} - z_{1j^*}|} \qquad \text{(III.52)}$$

Specific discharges:

$$\left.\begin{aligned} v_x &= -\,\mathrm{Re}\left\{\sum_{j=1}^{n} q_j F1_j(z) + G1(z)\right\} \\ v_y &= \mathrm{Im}\left\{\sum_{j=1}^{n} q_j F1_j(z) + G1(z)\right\} \end{aligned}\right\} \qquad \text{(III.48)}$$

Components of alteration of the position of a point of a phreatic line in a step in time:

$$\left.\begin{aligned} \Delta x &= -\,\mathrm{Re}\left\{\sum_{j=1}^{n} q_j F1_j(z) + G1(z)\right\}\frac{\Delta t}{\mu} \\ \Delta y &= \left[\mathrm{Im}\left\{\sum_{j=1}^{n} q_j F1_j(z) + G1(z)\right\} + N\right]\frac{\Delta t}{\mu} \end{aligned}\right\} \qquad \text{(III.49)}$$

Equations with respect to Ω_0

Potential:
One of the expressions (III.53), (III.59), (III.61), (III.63) or (III.70).

Stream function:

$$\sum_{j=1}^{n}\left[r_j \mathrm{Im}\{F0_j(z_0)\} + s_j \mathrm{Re}\{F0_j(z_0)\}\right] + \Psi_0 = \mathrm{Im}\{-G0(z_0)\} \qquad \text{(III.50)}$$

Boundary condition equations:

Equipotential line with silt layer:

$$\sum_{j=1}^{n}\left[r_j \mathrm{Re}\{H1_j(z)\} + s_j \mathrm{Im}\{-H1_j(z)\}\right] + \Phi_0 =$$

$$= \mathrm{Re}\left\{k\phi_1 - G0(z) + \frac{ikc_s}{c_{j^*}} G1(z)\right\} \tag{III.53}$$

$$\sum_{j=1}^{n}\left[r_j\mathrm{Re}\{H2_j(z)\} + s_j\mathrm{Im}\{-H2_j(z)\}\right] = \mathrm{Re}\left\{-\frac{1}{c_{j^*}} G1(z) + \frac{ikc_s}{c_{j^*}^2} G2(z)\right\} \tag{III.55}$$

Stream line boundary:

$$\sum_{j=1}^{n}\left[r_j\mathrm{Im}\left\{\frac{1}{c_{j^*}} F1_j(z)\right\} + s_j\mathrm{Re}\left\{\frac{1}{c_{j^*}} F1_j(z)\right\}\right] = \mathrm{Im}\left\{-\frac{1}{c_{j^*}} G1(z)\right\} \tag{III.57}$$

$$\sum_{j=1}^{n}\left[r_j\mathrm{Im}\left\{\frac{1}{c_{j^*}^2} F2_j(z)\right\} + s_j\mathrm{Re}\left\{\frac{1}{c_{j^*}^2} F2_j(z)\right\}\right] = \mathrm{Im}\left\{-\frac{1}{c_{j^*}^2} G2(z)\right\} \tag{III.58}$$

Seepage line with silt layer:

$$\sum_{j=1}^{n}\left[r_j\mathrm{Re}\{H1_j(z)\} + s_j\mathrm{Im}\{-H1_j(z)\}\right] + \Phi_0 =$$

$$= \mathrm{Re}\left\{-iz - G0(z) + \frac{ikc_s}{c_{j^*}} G1(z)\right\} \tag{III.59}$$

$$\sum_{j=1}^{n}\left[r_j\mathrm{Re}\{H2_j(z)\} + s_j\mathrm{Im}\{-H2_j(z)\}\right] =$$

$$= \mathrm{Re}\left\{k\, e_{j^*} - \frac{1}{c_{j^*}} G1(z) + \frac{ikc_s}{c_{j^*}^2} G2(z)\right\} \tag{III.60}$$

Phreatic line

$$\sum_{j=1}^{n}\left[r_j\mathrm{Re}\{F0_j(z)\} + s_j\mathrm{Im}\{-F0_j(z)\}\right] + \Phi_0 = \mathrm{Re}\{-ikz - G0(z)\} \tag{III.61}$$

$$\sum_{j=1}^{n}\left[r_j\mathrm{Re}\left\{\frac{1}{c_{j^*}} F1_j(z)\right\} + s_j\mathrm{Im}\left\{-\frac{1}{c_{j^*}} F1_j(z)\right\}\right] =$$

$$= \mathrm{Re}\left\{k\, e_{j^*} - \frac{1}{c_{j^*}} G1(z)\right\} \tag{III.62}$$

Interface:

$$\sum_{j=1}^{n}\left[r_j\mathrm{Re}\left\{\frac{\rho}{k}F0_j(z)\right\}+s_j\mathrm{Im}\left\{-\frac{\rho}{k}F0_j(z)\right\}\right]+\frac{\rho}{k}\Phi_0+$$

$$+\left(\sum_{j=1}^{n}\left[r_j\mathrm{Re}\left\{-\frac{\rho}{k}F0_j(z)\right\}+s_j\mathrm{Im}\left\{\frac{\rho}{k}F0_j(z)\right\}\right]-\frac{\rho}{k}\Phi_0\right)_c=$$

$$=\mathrm{Re}\left\{-i(\rho-\rho_c)\,z+-\frac{\rho G0(z)}{k}+\left(\frac{\rho G0(z)}{k}\right)_c\right\} \tag{III.63}$$

$$\sum_{j=1}^{n}\left[r_j\mathrm{Re}\left\{\frac{\rho}{k}\frac{1}{c_{j^*}}F1_j(z)\right\}+s_j\mathrm{Im}\left\{-\frac{\rho}{k}\frac{1}{c_{j^*}}F1_j(z)\right\}\right]+$$

$$+\left(\sum_{j=1}^{n}\left[r_j\mathrm{Re}\left\{\frac{\rho}{k}\frac{1}{c_{j^*}}F1_j(z)\right\}+s_j\mathrm{Im}\left\{-\frac{\rho}{k}\frac{1}{c_{j^*}}F1_j(z)\right\}\right]\right)_c=$$

$$=\mathrm{Re}\left\{(\rho-\rho_c)\,e_{j^*}-\frac{\rho}{k}\frac{1}{c_{j^*}}G1(z)+\left(\frac{\rho}{k}\frac{1}{c_{j^*}}G1(z)\right)_c\right\} \tag{III.65}$$

$$\sum_{j=1}^{n}\left[r_j\mathrm{Im}\left\{\frac{1}{c_{j^*}}F1_j(z)\right\}+s_j\mathrm{Re}\left\{\frac{1}{c_{j^*}}F1_j(z)\right\}\right]+$$

$$+\left(\sum_{j=1}^{n}\left[r_j\mathrm{Im}\left\{\frac{1}{c_{j^*}}F1_j(z)\right\}+s_j\mathrm{Re}\left\{\frac{1}{c_{j^*}}F1_j(z)\right\}\right]\right)_c=$$

$$=\mathrm{Im}\left\{-\frac{1}{c_{j^*}}G1(z)-\left(\frac{1}{c_{j^*}}G1(z)\right)_c\right\} \tag{III.67}$$

$$\sum_{j=1}^{n}\left[r_j\mathrm{Im}\left\{\frac{1}{c_{j^*}{}^2}F2_j(z)\right\}+s_j\mathrm{Re}\left\{\frac{1}{c_{j^*}{}^2}F2_j(z)\right\}\right]+$$

$$+\left(\sum_{j=1}^{n}\left[r_j\mathrm{Im}\left\{-\frac{1}{c_{j^*}{}^2}F2_j(z)\right\}+s_j\mathrm{Re}\left\{-\frac{1}{c_{j^*}{}^2}F2_j(z)\right\}\right]\right)_c=$$

$$=\mathrm{Im}\left\{-\frac{1}{c_{j^*}{}^2}G2(z)+\left(\frac{1}{c_{j^*}{}^2}G2(z)\right)_c\right\} \tag{III.69}$$

If one of the fluids is stationary (ρ_c and ϕ_c constant) then (III.63) becomes:

$$\sum_{j=1}^{n}\left[r_j\mathrm{Re}\left\{\frac{\rho}{k}F0_j(z)\right\}+s_j\mathrm{Im}\left\{-\frac{\rho}{k}F0_j(z)\right\}\right]+\frac{\rho}{k}\Phi_0=$$

$$=\mathrm{Re}\left\{\rho_c\phi_c-i(\rho-\rho_c)z-\frac{\rho}{k}G0(z)\right\} \tag{III.64}$$

and (III.65) becomes:

$$\sum_{j=1}^{n}\left[r_j\mathrm{Re}\left\{\frac{\rho}{k}\,\frac{1}{c_{j^*}}\,F1_j(z)\right\}+s_j\mathrm{Im}\left\{-\frac{\rho}{k}\,\frac{1}{c_{j^*}}\,F1_j(z)\right\}\right]=$$

$$=\mathrm{Re}\left\{(\rho-\rho_c)\,e_{j^*}-\frac{\rho}{k}\,\frac{1}{c_{j^*}}\,G1(z)\right\} \qquad \text{(III.66)}$$

Inhomogeneity line with silt layer:

$$\sum_{j=1}^{n}\left[r_j\mathrm{Re}\left\{\frac{1}{k}\,H1_j(z)\right\}+s_j\mathrm{Im}\left\{-\frac{1}{k}\,H1_j(z)\right\}\right]+\frac{\Phi_0}{k}+$$

$$+\left(\sum_{j=1}^{n}\left[r_j\mathrm{Re}\left\{-\frac{1}{k}\,F0_j(z)\right\}+s_j\mathrm{Im}\left\{\frac{1}{k}\,F0_j(z)\right\}\right]-\frac{\Phi_0}{k}\right)_c=$$

$$=\mathrm{Re}\left\{-\frac{G0(z)}{k}+\left(\frac{G0(z)}{k}\right)_c+i\,\frac{c_s}{c_{j^*}}\,G1(z)\right\} \qquad \text{(III.70)}$$

$$\sum_{j=1}^{n}\left[r_j\mathrm{Re}\left\{\frac{1}{k}\,H2_j(z)\right\}+s_j\mathrm{Im}\left\{-\frac{1}{k}\,H2_j(z)\right\}\right]+$$

$$+\left(\sum_{j=1}^{n}\left[r_j\mathrm{Re}\left\{\frac{1}{kc_{j^*}}\,F1_j(z)\right\}+s_j\mathrm{Im}\left\{-\frac{1}{kc_{j^*}}\,F1_j(z)\right\}\right]\right)_c=$$

$$=\mathrm{Re}\left\{-\frac{1}{kc_{j^*}}\,G1(z)-\left(\frac{1}{kc_{j^*}}\,G1(z)\right)_c+i\,\frac{c_s}{c_{j^*}^{\;2}}\,G2(z)\right\} \qquad \text{(III.71)}$$

$$\sum_{j=1}^{n}\left[r_j\mathrm{Im}\left\{\frac{1}{c_{j^*}}\,F1_j(z)\right\}+s_j\mathrm{Re}\left\{\frac{1}{c_{j^*}}\,F1_j(z)\right\}\right]+$$

$$+\left(\sum_{j=1}^{n}\left[r_j\mathrm{Im}\left\{\frac{1}{c_{j^*}}\,F1_j(z)\right\}+s_j\mathrm{Re}\left\{\frac{1}{c_{j^*}}\,F1_j(z)\right\}\right]\right)_c=$$

$$=\mathrm{Im}\left\{-\frac{1}{c_{j^*}}\,G1(z)-\left(\frac{1}{c_{j^*}}\,G1(z)\right)_c\right\} \qquad \text{(III.67)}$$

$$\sum_{j=1}^{n}\left[r_j\mathrm{Im}\left\{\frac{1}{c_{j^*}^{\;2}}\,F2(z)\right\}+s_j\mathrm{Re}\left\{\frac{1}{c_{j^*}^{\;2}}\,F2(z)\right\}\right]+$$

$$+\left(\sum_{j=1}^{n}\left[r_j\mathrm{Im}\left\{-\frac{1}{c_{j^*}^{\;2}}\,F2_j(z)\right\}+s_j\mathrm{Re}\left\{-\frac{1}{c_{j^*}^{\;2}}\,F2_j(z)\right\}\right]\right)_c=$$

$$=\mathrm{Im}\left\{-\frac{1}{c_{j^*}^{\;2}}\,G2(z)+\left(\frac{1}{c_{j^*}^{\;2}}\,G2(z)\right)_c\right\} \qquad \text{(III.69)}$$

17 Special Cases of Flow in a Half Plane

In the foregoing Chapter distributions of sources, sinks and vortices were used to generate flow patterns in regions that do not extend to infinity. In this Chapter it is shown that those distributions are also useful for two classes of half-plane flow, having the advantage that then it is not necessary to confine the region by an assumed fictive boundary. When the flow is generated in the whole half-plane, the influence of the local boundary conditions can be seen from the calculation results (generally this property is only found in analytical methods (see Chapters 5, 6, 7)). Using numerical methods, such as the finite element method and the finite difference method, always a schematised fictive boundary has to be taken into account to cut off a region of interest from the rest of the half plane.
Both classes discussed here deal with half-plane flow with each class being defined by special boundary conditions at the real axis. Of course, in principle also problems can be calculated that can be reduced to these classes by a conformal mapping technique. Regions that have a closed boundary in the form of a polygon can also be mapped upon a half plane; however, that is not relevant here as flow problems in that kind of regions can be solved by the method of the foregoing Chapters.

Two classes of half plane flow will now be defined. The half plane is assumed to be homogeneous and isotropic. The flow in the lower half-plane $y \leqslant 0$ is considered because this has a good connection with many problems in reality.
The two classes of half-plane flow are defined by:

Class I: On the real axis the following conditions hold (where $x_b > x_a$):

$$\left.\begin{array}{ll} x \geqslant x_b & \Phi = \Phi_1 \text{ (constant)} \\ x \leqslant x_a & \Phi = \Phi_2 \text{ (constant)} \end{array}\right\} \qquad \text{(III.72)}$$

Class II: On the real axis the following conditions hold (where $x_b > x_a$):

$$\left.\begin{array}{ll} x \geqslant x_b & \Psi = \Psi_1 \text{ (constant)} \\ x \leqslant x_a & \Psi = \Psi_2 \text{ (constant)} \end{array}\right\} \qquad \text{(III.73)}$$

For many practical problems one of the conditions (III.72) and (III.73) can be supposed to hold. Between the points x_a and x_b several boundary conditions may be met. (In practice there only the boundary types of stream line and equipotential line (with silt layer) can be used (in arbitrary number and combinations) because the boundary (the x-axis) is a straight line). For both classes the lower half-plane may include an arbitrary number of sources and sinks in its interior. Using the method of images (see Chapter 6), fictive sinks and sources are accounted in such a way that the relevant condition (III.72) or (III.73) remains satisfied.

First, some properties of the complex potential due to a distribution of sinks, sources or vortices over the real axis are shown:

A distribution of constant strength over the line segment x_1x_2 is given by (III.30):

$$\Omega(z) = -\frac{q}{2\pi}\{(z - x_2)\ln(z - x_2) - (z - x_1)\ln(z - x_1)\}$$

On the real axis Ω is given by:

$$\Omega(x) = -\frac{q}{2\pi}\{(x - x_2)\ln(x - x_2) - (x - x_1)\ln(x - x_1)\} \qquad \text{(III.74)}$$

let: $f(x) = (x - x_2)\ln(x - x_2) - (x - x_1)\ln(x - x_1)\}$

then (III.74) is: $\Omega(x) = -\dfrac{q}{2\pi}f(x)$

$$\text{or: } \Omega(x) = -\frac{q}{2\pi}\{\mathrm{Re}\{f(x)\} + i\,\mathrm{Im}\{f(x)\} \qquad \text{(III.75)}$$

The function $\mathrm{Im}\{f(x)\}$ has a special property at the x-axis:
Using the definition $\ln(z) = \ln|z| + i\arg(z)$, $\mathrm{Im}\{f(x)\}$ is given by:

$$\mathrm{Im}\{f(x)\} = (x - x_2)\arg(x - x_2) - (x - x_1)\arg(x - x_1)$$

For $x > x_2$	:	$\arg(x - x_1) = 0$
		$\arg(x - x_2) = 0$
then for $x > x_2$	:	$\mathrm{Im}\{f(x)\} = 0$
For $x < x_1$	:	$\arg(x - x_1) = -\pi$
		$\arg(x - x_2) = -\pi$
then for $x < x_1$	:	$\mathrm{Im}\{f(x)\} = \pi(x_2 - x_1)$
For $x_1 < x < x_2$	:	$\arg(x - x_1) = 0$
		$\arg(x - x_2) = -\pi$
then for $x_1 < x < x_2$	:	$\mathrm{Im}\{f(z)\} = -\pi(x - x_2)$

In figure III.10 the variation of $\mathrm{Im}\{f(x)\}$ at the x-axis is shown:

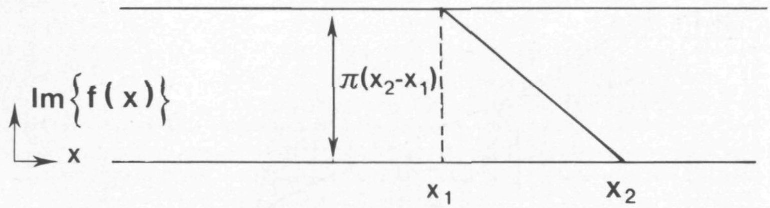

Figure III.10 Imaginary part of $f(x)$.

The real part of $f(x)$ does not have this special property. The function $\mathrm{Re}\{f(x)\}$ is given by:

$$\mathrm{Re}\{f(x)\} = (x - x_2)\ln|x - x_2| - (x - x_1)\ln|x - x_1|.$$

The special behaviour of $\mathrm{Im}\{f(x)\}$ is used for problems where the stream function Ψ of the potential Φ only varies between two points of the x-axis and has constant values outside these points, according to the defined Classes I and II. This will be shown now. When there is a distribution of sinks and sources only, q is real: $q = r + is = r$. Then from (III.75):

$$\Psi(x) = -\frac{r}{2\pi}\,\mathrm{Im}\{f(x)\}.$$

So for distributions of sinks and sources over a part of the real axis, the stream function only varies over that part. (This satisfies Class II). When there is a distribution of vortices only: $q \equiv r + i\,s = i\,s$ and it is seen from (III.75) that:

$$\Phi(x) = \frac{s}{2\pi}\,\mathrm{Im}\{f(x)\}.$$

So for distributions of vortices over a part of the real axis, the potential only varies over that part. (Thus it satisfies Class I).

By a combination of a number of these distributions (of only real or only imaginary strength) a flow pattern can be generated that satisfies (III.72) or (III.73), whereas between the points x_a and x_b an arbitrary variation of Φ or Ψ is approximated by a sequence of straight line segments. (Generally that arbitrary variation of Φ or Ψ is not known previously as it is the result of meeting the boundary conditions between x_a and x_b). Figure III.11 is a schematic representation. The figure shows (according to (III.72) or (III.73)) constant values of Φ or Ψ for $x \leqslant x_a$ and $x \geqslant x_b$, and at 5 line segments between x_a and x_b a linear variation of Φ or Ψ.

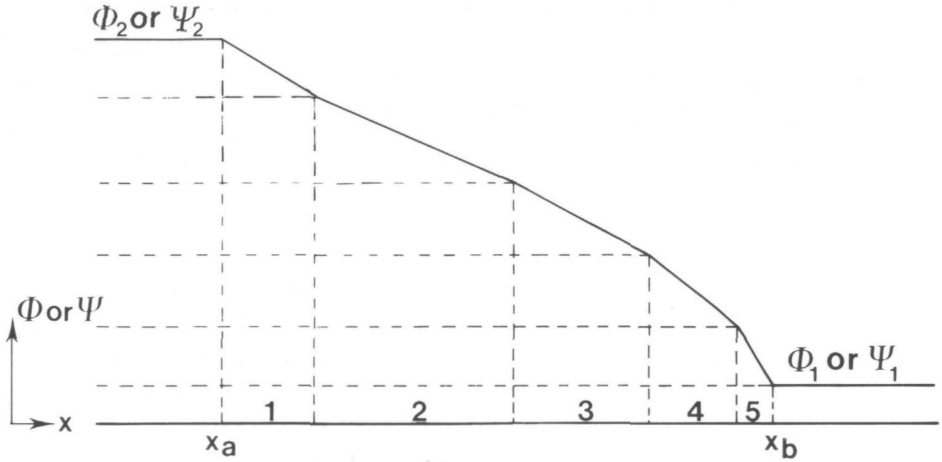

Figure III.11 Combination of distributions at the x-axis.

The complex potential due to a number of distributions of sinks and sources or vortices is found from (III.30):

$$\Omega(z) = \sum_{j=1}^{n} -\frac{q_j}{2\pi}\{(z - x_{2j})\ln(z - x_{2j}) - (z - x_{1j})\ln(z - x_{1j})\} + \Omega_0,$$

where q_j is real or imaginary.

Sources and sinks can be accounted by also using extra fictive sources and sinks in the upper half-plane so that the condition (III.72) or (III.73) remains valid. For Class I, where for $x \leqslant x_a$ and $x \geqslant x_b$ the potential is constant, fictive sources in the upper half-plane are accounted if there are sinks in the lower half-plane. For Class II, where for $x \leqslant x_a$ and $x \geqslant x_b$ the stream function is constant, fictive sinks in the upper half-plane are accounted when there are sinks in the lower half-plane. The expressions for the complex potential and its derivatives are given for both Classes by the following expressions:

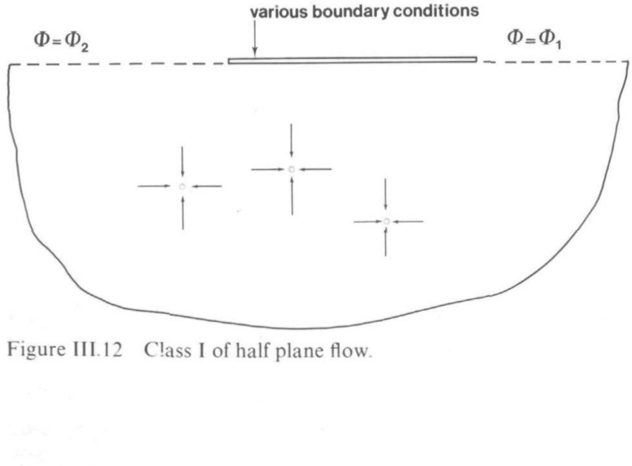

Figure III.12 Class I of half plane flow.

Class I: see figure III.12

$$\Omega(z) = \sum_{j=1}^{n} -\frac{is_j}{2\pi}\{(z-x_{2j})\ln(z-x_{2j})-(z-x_{1j})\ln(z-x_{1j})\} +$$

$$+\sum_{j=1}^{m}\frac{Q_j}{2\pi}\ln\frac{(z-z_j)}{(z-\bar{z}_j)} + \Omega_0$$

$$\frac{d\Omega(z)}{dz} = \sum_{j=1}^{n} -\frac{is_j}{2\pi}\{\ln(z-x_{2j})-\ln(z-x_{1j})\} +$$

$$+\sum_{j=1}^{n}\frac{Q_j}{2\pi}\left[\frac{1}{(z-z_j)}-\frac{1}{(z-\bar{z}_j)}\right]$$

$$\frac{d^2\Omega(z)}{dz^2} = \sum_{j=1}^{n} -\frac{is_j}{2\pi}\left\{\frac{1}{(z-x_{2j})}-\frac{1}{(z-x_{1j})}\right\} +$$

$$+\sum_{j=1}^{m}\frac{-Q_j}{2\pi}\left[\frac{1}{(z-z_j)^2}-\frac{1}{(z-\bar{z}_j)^2}\right]$$

Class II: see figure III.13

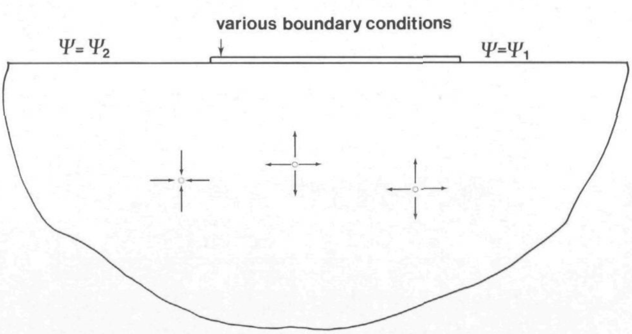

Figure III.13 Class II of half plane flow.

$$\Omega(z) = \sum_{j=1}^{n} -\frac{r_j}{2\pi}\{z-x_{2j})\ln(z-x_{2j})-(z-x_{1j})\ln(z-x_{1j})\} +$$

$$+\sum_{j=1}^{m}\frac{Q_j}{2\pi}\ln[(z-z_j)(z-\bar{z}_j)] + \Omega_0$$

$$\frac{d\Omega(z)}{dz} = \sum_{j=1}^{n} -\frac{r_j}{2\pi}\{\ln(z - x_{2j}) - \ln(z - x_{1j})\} +$$

$$+ \sum_{j=1}^{m} \frac{Q_j}{2\pi}\left[\frac{1}{(z - z_j)} + \frac{1}{(z - \bar{z}_j)}\right]$$

$$\frac{d^2\Omega(z)}{dz^2} = \sum_{j=1}^{n} -\frac{r_j}{2\pi}\left\{\frac{1}{(z - x_{2j})} - \frac{1}{(z - x_{1j})}\right\} +$$

$$+ \sum_{j=1}^{m} -\frac{Q_j}{2\pi}\left[\frac{1}{(z - z_j)^2} + \frac{1}{(z - \bar{z}_j)^2}\right]$$

A combination of the general solution with the boundary conditions between the points x_a and x_b at the x-axis leads to a set of linear equations, in which the unknowns are the distribution strengths. It is noted that here there is one degree of freedom per boundary segment. In the middle of each boundary segment the first of each pair of boundary conditions (see Chapter 14) is posed. It has already been mentioned that relevant boundary conditions at the boundary segments may be a stream line and an equipotential line with silt layer (in arbitrary number and combinations). The resulting equations can be derived directly from those of Chapter 16 by putting $r_j = 0$ for Class I problems and $s_j = 0$ for Class II problems:
The following survey gives the boundary condition equations and relevant expressions:

Definitions:

$$\left.\begin{aligned}
f0_j(z) &= -\frac{1}{2\pi}\{(z - x_{2j})\ln(z - x_{2j}) - (z - x_{1j})\ln(z - x_{1j})\} \\
f1_j(z) &= -\frac{1}{2\pi}\{\ln(z - x_{2j}) - \ln(z - x_{1j})\} \\
f2_j(z) &= -\frac{1}{2\pi}\left\{\frac{1}{(z - x_{2j})} - \frac{1}{(z - x_{1j})}\right\} \\
g0\mathrm{I}(z) &= \sum_{j=1}^{m} \frac{Q_j}{2\pi}\ln\left[\frac{z - z_j}{z - \bar{z}_j}\right] \\
g1\mathrm{I}(z) &= \sum_{j=1}^{m} \frac{Q_j}{2\pi}\left\{\frac{1}{(z - z_j)} - \frac{1}{(z - \bar{z}_j)}\right\} \\
g2\mathrm{I}(z) &= \sum_{j=1}^{m} \frac{Q_j}{2\pi}\left\{-\frac{1}{(z - z_j)^2} + \frac{1}{(z - \bar{z}_j)^2}\right\}
\end{aligned}\right\} \qquad \text{(III.76)}$$

$$\left.\begin{aligned}
g0\mathrm{II}(z) &= \sum_{j=1}^{m} \frac{Q_j}{2\pi} \ln[(z-z_j)(z-\bar{z}_j)] \\
g1\mathrm{II}(z) &= \sum_{j=1}^{m} \frac{Q_j}{2\pi} \left\{\frac{1}{(z-z_j)} + \frac{1}{(z-\bar{z}_j)}\right\} \\
g2\mathrm{II}(z) &= \sum_{j=1}^{m} \frac{Q_j}{2\pi} \left\{-\frac{1}{(z-z_j)^2} - \frac{1}{(z-\bar{z}_j)^2}\right\} \\
h1_j(z) &= f0_j(z) - ikc_s f1_j(z) \\
h2_j(z) &= f1_j(z) - ikc_s f2_j(z)
\end{aligned}\right\} \quad \text{(III.76)}$$

Complex potential and derivatives:

Class I:

$$\Omega(z) = \sum_{j=1}^{n} is_j f0_j(z) + g0\mathrm{I}(z) + \Omega_0 \quad \text{(III.77)}$$

$$\frac{d\Omega(z)}{dz} = \sum_{j=1}^{n} is_j f1_j(z) + g1\mathrm{I}(z) \quad \text{(III.78)}$$

$$\frac{d^2\Omega(z)}{dz^2} = \sum_{j=1}^{n} is_j f2_j(z) + g2\mathrm{I}(z) \quad \text{(III.79)}$$

Class II:

$$\Omega(z) = \sum_{j=1}^{n} r_j f0_j(z) + g0\mathrm{II}(z) + \Omega_0 \quad \text{(III.80)}$$

$$\frac{d\Omega(z)}{dz} = \sum_{j=1}^{n} r_j f1_j(z) + g1\mathrm{II}(z) \quad \text{(III.81)}$$

$$\frac{d^2\Omega(z)}{dz^2} = \sum_{j=1}^{n} r_j f2_j(z) + g2\mathrm{II}(z) \quad \text{(III.82)}$$

Specific discharges:

Class I:

$$\left.\begin{aligned}
v_x &= -\operatorname{Re}\left\{-\sum_{j=1}^{n} is_j f1_j(z) + g1\mathrm{I}(z)\right\} \\
v_y &= \operatorname{Im}\left\{\sum_{j=1}^{n} is_j f1_j(z) + g1\mathrm{I}(z)\right\}
\end{aligned}\right\} \quad \text{(III.83)}$$

Class II:

$$\left.\begin{aligned} v_x &= -\operatorname{Re}\left\{\sum_{j=1}^{n} r_j f1_j(z) + g1\mathrm{II}(z)\right\} \\ v_y &= \operatorname{Im}\left\{\sum_{j=1}^{n} r_j f1_j(z) + g1\mathrm{II}(z)\right\} \end{aligned}\right\} \quad \text{(III.84)}$$

Equations with respect to Ω_0

Potential:
Class I:

$$\sum_{j=1}^{n} s_j \operatorname{Im}\{-f0_j(z_0)\} + \Phi_0 = \operatorname{Re}\{k\phi_1 - g0\mathrm{I}(z_0)\} \quad \text{(III.85)}$$

Class II:

$$\sum_{j=1}^{n} r_j \operatorname{Re}\{-f0_j(z_0)\} + \Phi_0 = \operatorname{Re}\{k\phi_1 - g0\mathrm{II}(z_0)\} \quad \text{(III.86)}$$

Stream function:
Class I:

$$\sum_{j=1}^{n} s_j \operatorname{Re}\{f0_j(z_0)\} + \Psi_0 = \operatorname{Im}\{-g0\mathrm{I}(z_0)\} \quad \text{(III.87)}$$

Class II:

$$\sum_{j=1}^{n} r_j \operatorname{Im}\{f0_j(z_0)\} + \Psi_0 = \operatorname{Im}\{-g0\mathrm{II}(z_0)\} \quad \text{(III.88)}$$

Boundary condition equations:

Equipotential line with silt layer
Class I:

$$\sum_{j=1}^{n} s_j \operatorname{Im}\{-h1_j(z)\} + \Phi_0 = \operatorname{Re}\{k\phi_1 - g0\mathrm{I}(z) + ikc_s g1\mathrm{I}(z)\} \quad \text{(III.89)}$$

Class II:

$$\sum_{j=1}^{n} r_j \mathrm{Re}\{h1_j(z)\} + \Phi_0 = \mathrm{Re}\{k\phi_1 - g0\mathrm{II}(z) + ikc_s g1\mathrm{II}(z)\} \tag{III.90}$$

Stream line

Class I:

$$\sum_{j=1}^{n} s_j \mathrm{Re}\{f1_j(z)\} = \mathrm{Im}\{-g1\mathrm{I}(z)\} \tag{III.91}$$

Class II:

$$\sum_{j=1}^{n} r_j \mathrm{Im}\{f1_j(z)\} = \mathrm{Im}\{-g1\mathrm{II}(z)\} \tag{III.92}$$

18 Multi-Valued Character and Singularity of the used Functions

Multi-valued character

The general solution (II.35) and its derivative (III.36) contain logarithms of a complex variable. Such logarithms are defined by:

$$\ln(z) = \ln|z| + i\arg(z)$$

This function is multi-valued. In calculations generally a single-valued function is used that is defined by:

$$\left.\begin{array}{l} \ln(z) = \ln|z| + i\arg(z) \\ -\pi < \arg(z) \leqslant \pi \end{array}\right\} \quad \text{(III.93)}$$

This is the definition of the logarithm of a complex variable that is currently available in computers. From (III.93) it is seen that the imaginary part of $\ln(z)$ has a discontinuity 2π at ($x < 0$, $y = 0$). This discontinuity is encountered in the stream function due to the flow of a sink at the point z_p:

$$\Omega = \frac{Q}{2\pi}\ln(z - z_p)$$

or:

$$\Phi + i\Psi = \frac{Q}{2\pi}\ln|z - z_p| + i\frac{Q}{2\pi}\arg(z - z_p)$$

The argument discontinuity causes a stream function discontinuity at $\mathrm{Re}\{z - z_p\} < 0$, $\mathrm{Im}\{z - z_p\} = 0$, as shown in figure III.14:

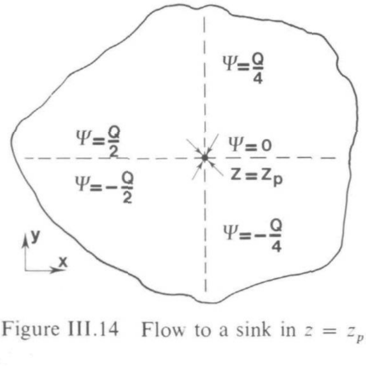

Figure III.14 Flow to a sink in $z = z_p$

The complex potential due to distributions of complex strength also involves logarithms, and therefore it has discontinuities. The expression for the complex potential due to a distribution of complex strength between the points z_1 and z_2 is (III.32):

$$\Omega = -\frac{qc}{2\pi}\{(z - z_2)\ln[c(z - z_2)] - (z - z_1)\ln[c(z - z_1)]\}$$

The argument discontinuity causes a discontinuity in the real and imaginary parts of Ω, and appears when the imaginary part of the complex logarithm is zero and the real part is negative.

For the first logarithm in (III.32) if:

$$\left.\begin{aligned} &\mathrm{Im}\{c(z - z_2)\} = 0 \\ &\mathrm{Re}\{c(z - z_2)\} < 0 \end{aligned}\right\} \qquad \text{(III.94)}$$

For the second logarithm in (III.32) if:

$$\left.\begin{aligned} &\mathrm{Im}\{c(z - z_1)\} = 0 \\ &\mathrm{Re}\{c(z - z_1)\} < 0 \end{aligned}\right\} \qquad \text{(III.95)}$$

This means that generally the complex potential according to (III.32) has discontinuities in the real and imaginary part, at the line through the points z_1 and z_2 over the part that is shown in figure III.15:

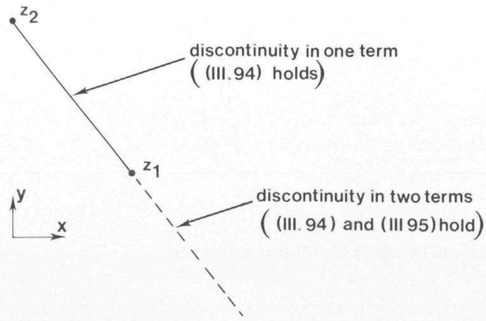

Figure III.15 Location of discontinuities.

Because the argument step is 2π, the discontinuity between z_1 and z_2 (see (III.32) and figure III.15)) is given by:

$$\Delta_1\Omega = \frac{qc}{2\pi}(z - z_2)i2\pi = iqc(z - z_2) \qquad \text{(III.96)}$$

At the extended part of z_2z_1 it is given by (see (III.32) and figure (III.15)):

$$\Delta_2\Omega = \frac{qc}{2\pi}\{(z - z_2)i2\pi - (z - z_1)i2\pi\} = iqc(z_1 - z_2) \qquad \text{(III.97)}$$

It is easily seen from (III.96), which is a linear expression, that the complex potential step $\Delta_1\Omega$ (between z_1 and z_2 has a linear variation going from $z = z_2$ (at $z = z_2$ it is zero) to $z = z_1$ (there it is $i\,q\,c\,(z_1 - z_2)$). In the extended part of z_2z_1, the step remains constant at this value, which is seen from (III.97).

Using the mentioned single-valued definition of the logarithm in some cases where the extended parts of the boundary segments are located in the flow region, the solution would have inadmissible discontinuities (see figure III.16). Therefore in the computer program mentioned in Chapter 20 the values of the complex logarithms are corrected so that the discontinuities are shifted to lines outside the flow region, see figure III.16.

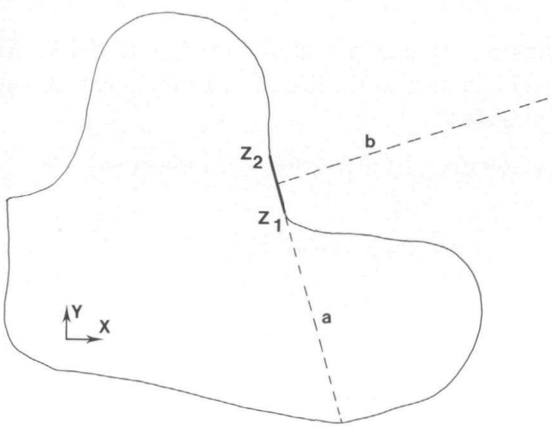

Figure III.16 Shift of discontinuities from line a to l line b.

It is noted that for multiply connected regions the discontinuities cannot be avoided in this way. In such cases the flow region is cut into two or more sub-regions in such a way that each sub-region is simply connected and the discontinuity of the mentioned function within the region can be avoided, see figure III.17.
Then at the cuts there is an inhomogeneity line where the properties of soil and fluid are the same at both sides.

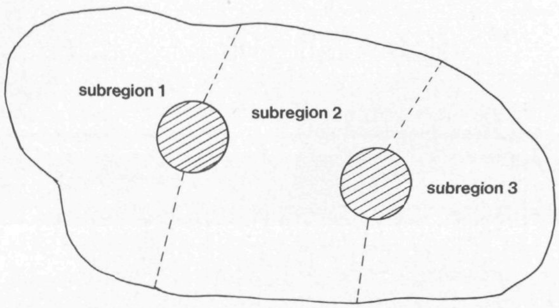

Figure III.17 Subdivision of multiply-connected regions.

Singularity

The general expression for the complex potential in the analytical function method is (III.35):

$$\Omega(z) = \sum_{j=1}^{n} -\frac{q_j c_j}{2\pi}\{(z - z_{2j})\ln[c_j(z - z_{2j})] - (z - z_{1j})\ln[c_j(z - z_{1j})]\} +$$
$$+ \sum_{j=1}^{m} \frac{Q_j}{2\pi}\ln(z - z_j) + \Omega_0$$

The function $\Omega(z)$ is singular at the sinks and the sources, but not at the end points of the boundary segments (z_{1j} and z_{2j}). This follows from the fact that the limit:

$$\lim_{z \to z_2} \{(z - z_2)\ln[c(z - z_2)]\}$$

is equal to zero. This can be shown simply by application of L'Hôpitals' rule:

$$\lim_{z \to z_2} \{(z - z_2)\ln[c(z - z_2)] = \lim_{z \to z_2} \left\{ \frac{\ln[c(z - z_2)]}{\dfrac{1}{(z - z_2)}} \right\} =$$

$$= \lim \frac{1}{(z - z_2)\dfrac{-1}{(z - z_2)^2}} = -\lim_{z \to z_2}\{z - z_2\} = 0$$

Of course, this holds for the point $z = z_1$ too. So the complex potential only has singularities at sinks and sources.

The derivatives of $\Omega(z)$ with respect to z, given by (III.36) and (III.37):

$$\frac{d\Omega(z)}{dz} = \sum_{j=1}^{n} - \frac{q_j c_j}{2\pi} \{\ln[c_j(z - z_{2j})] - \ln[c_j(z - z_{1j})]\} +$$

$$+ \sum_{j=1}^{m} \frac{Q_j}{2\pi(z - z_j)}$$

$$\frac{d^2\Omega(z)}{dz^2} = \sum_{j=1}^{n} - \frac{q_j c_j}{2\pi}\left[\frac{1}{(z - z_{2j})} - \frac{1}{(z - z_{1j})}\right] + \sum_{j=1}^{n} \frac{-Q_j}{2\pi(z - z_j)^2}$$

are singular at the points z_{1j}, z_{2j}, and z_j.

19 General Comparison with Other Methods

The following general comparison with other methods deals with aspects of pure two-dimensional flow because the analytical function method is restricted to two-dimensional flow.

The analytical function method has the properties that it is an approximative numerical method and it is rather general for two-dimensional flow.

In part II a review has been given of the most important existing solution methods. Only the finite difference method and the finite element method have these properties too, and in each of them, just like in the analytical function method, the solution is found by generating and solving a set of linear equations. However, there are some essential differences:

- There is a difference in theoretical background. In the finite difference method the solution is found using a discretisation of the differential equation, whereas in the finite element method it is found by using a variational principle. The analytical function method uses analytic complex functions.
- In the finite element method as well as the finite difference method the flow region is sub-divided into a large number of elements. Corresponding to this number of elements, the number of equations that is generated increases. In many cases where there are sharp alterations of the potential, such as may occur in the presence of sinks and sources, it may be necessary to increase the number of elements greatly to ensure accuracy. Then the number of equations becomes very large and, consequently, the required computer storage too. For the analytical function method this does not apply, because the number of equations is relatively small as the number of equations is defined by the number of boundary segments and, in addition, sinks and sources are simply accounted by extra terms in de general solution. The number of equations is proportional to the number of boundary segments. For the finite element method and the finite difference method the number of equations is proportional to the number of elements.
- For the finite element method and the finite difference method the array of coefficients is a diagonal band matrix. The array of coefficients for the analytical function method is complete in the case of one region and consists of blocs when there are more sub-regions; however, these arrays are generally smaller than the arrays of the finite element method and the finite difference method, because generally the number of boundary segments is relatively small.
- In the finite element method and the finite difference method the unknowns are the groundwater heads in a large number of points within the flow region from which discharges can be calculated. For the analytical function method the unknowns are

the complex distribution strengths. So for the calculation of groundwater heads in points of the region, values of an analytical function have to be evaluated. Then discharges are also known (because the analytical function contains the potential as well as the stream function).

– In the case of non-steady flow in the finite element method, the problem may be encountered that due to a sharp rise of a phreatic line the upper elements become very oblong. That may be disadvantageous for accuracy, because the basic assumptions about the variation of the groundwater head within an element might become a bad approximation for these upper elements. For such cases there has to be generated a new element grid or a part of it. Analogous problems may be encountered when there is a sharp fall in time of a phreatic line. These kinds of problems are not encountered in the analytical function method because it does not use an element discretisation of the region.

– When the coefficient of permeability varies considerably a sub-division in many sub-regions must be applied in the analytical function method. In such cases, the method becomes disadvantageous with respect to the finite element method and the finite difference method because then the number of sub-regions might become the same as the number of elements in the finite element method or the finite difference method. Then the computer time needed for calculations will be larger for the analytical function method because the unknowns in the set of equations are not groundwater heads as in the two other methods.

– The amount of input data for computer calculations is generally less for the analytical function method because a computer program based on that method needs only general information (number of sub-regions, coefficients of permeability, etc) and boundary information (position and properties). Consequently the amount of work for checking data is less. However, it is noted that when, e.g., a finite element program is used, for many problems a mesh generator can be used to reduce the amount of input work.

– For some classes of half-plane flow the analytical function method provides a solution for the whole half plane. For those classes it is not necessary to cut off the part of interest from the rest of the half plane. In the finite element method and the finite difference method such a cut always has to be made for half-plane flow problems.

– For the finite element method as well as the finite difference method the solution is approximative over the whole region. The analytical function method provides a solution that is exact within an approximative boundary. So the analytical function method has another character than the finite element method and the finite difference method. Because its solutions are exact within an approximative boundary, the method is semi-analytic.

20 Examples

To illustrate the analytical function method a computer program was made that is rather general for two-dimensional groundwater flow: It can be used for the calculation of steady and non-steady flow patterns in arbitrary shaped regions that may contain several fluids and inhomogeneities. The anisotropy may have arbitrary directions and magnitudes and there may be sources and sinks. The input of the program consists only of general information (number of sub-regions, coefficients of permeability, etc) and boundary information (position and properties). The computer program comprises a main programme, two subroutines and three small function subprograms. The main program has an overall controlling task and it provides the input and output. The subroutine *OPST* generates the equations, using the boundary condition formulation of the foregoing Chapters. The set of equations is solved by the subroutine *SIMQ*. That subroutine, or a corresponding one, is usually available on computers in a scientific subroutine package. The function sub-program CAF calculates some complex expressions that appear in many equations; the function sub-program CPF calculates the part $\Omega_1(z)$ of the complex potential that is due to the presence of sources and sinks; and the function sub-program COF evaluates the complex potential using the distribution strengths that are known from the solution of the set equations.

The output of the program consists of coordinates, groundwater heads, stream func-

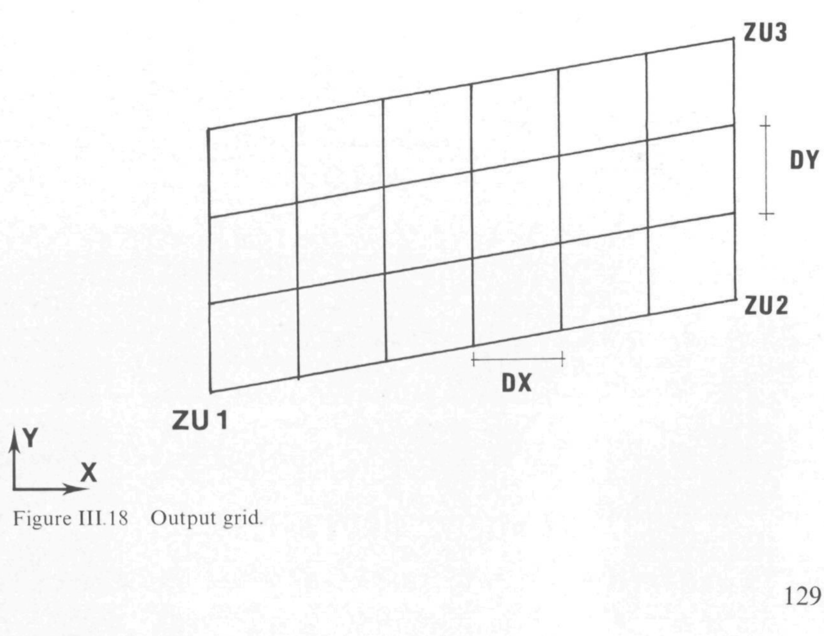

Figure III.18 Output grid.

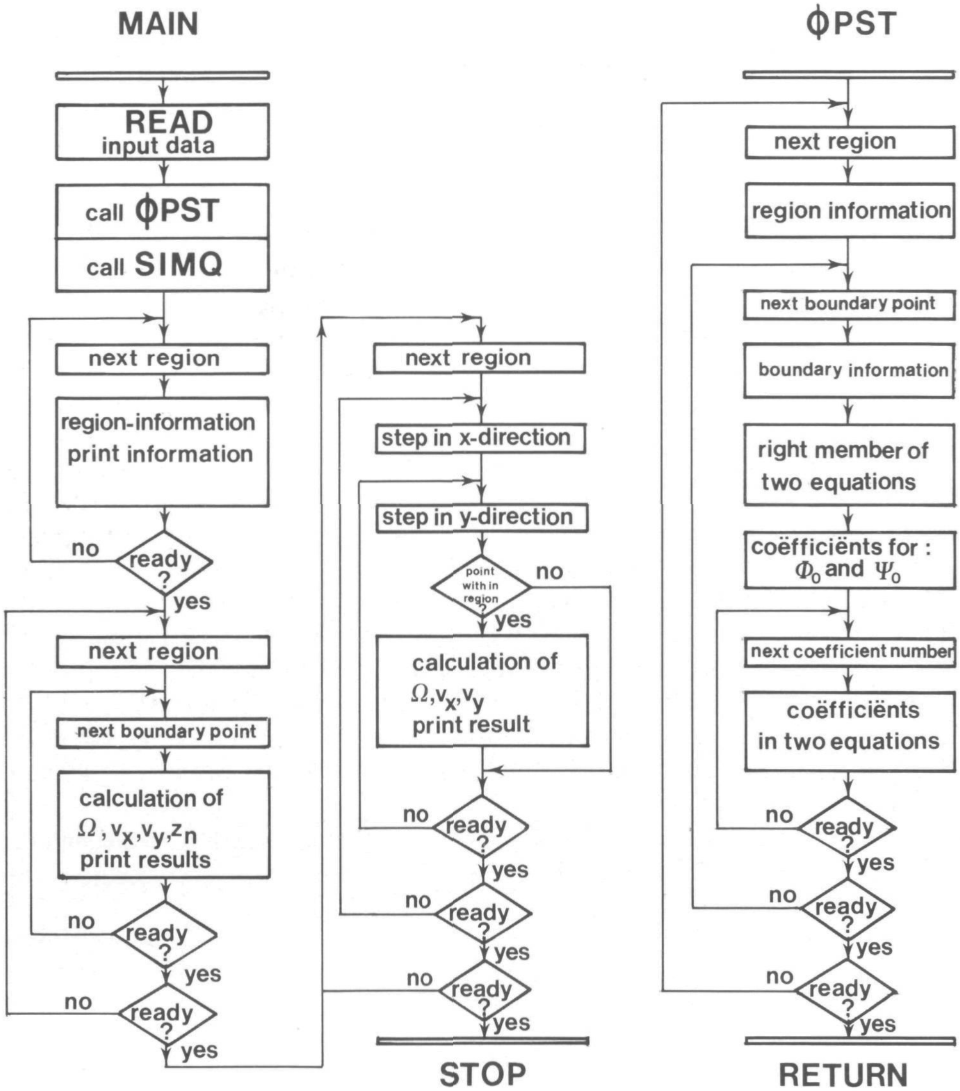

Figure III.19 Essential parts of computer program.

tion values, and x- and y-components of the specific discharge. These values are printed for the boundary points and one or more output grids, that are specified in the input data. These output grids are each defined by four complex parameters: starting point $ZU1$, end points $ZU2$ and $ZU3$, and the complex step parameter $(DX + i\,DY)$, as shown in figure III.18.

In addition, for points of the boundary the position is printed after one step in time. Before calculating the complex potential at a point of an output grid, it is first checked whether the point is within the region or not. If not, no calculation of the complex potential is carried out and the program passes to the next point of the output grid. This procedure has the advantage that output grids may be defined very roughly, so that all required information can be given to the program by a minimum of input data. Figure III.19 gives schematically the two relevant parts of the computer program: The MAIN program and the subroutine OPST. A listing of the computer program is given in Appendix 5. It is noted that for the computer program the terms $\frac{\partial^2 z}{\partial l^2}$ have been set to zero (see Chapter 16).

In the following some examples are discussed that illustrate the power of the calculation method outlined. The figures are roughly sketched by hand from numerical output data that were obtained using an early version of the computer program. In practise a plot program will be used together with the calculation program of Appendix 5 (The listed program conforms figure III.19). Then accurate plots can be produced automatically and the development using these programs is very easily done. If not defined otherwise, for all examples the following dimensions are used. Length: meters, time : days, mass : tons (1000 kg).

EXAMPLE 1

Steady flow in a dike; homogeneous isotropic soil

Figure III.20 shows the flow region. There is an impermeable revetment on the talus. The open part of the talus is clean; there is no silt layer, so it is an equipotential line. At some distance from the canal the flow is nearly horizontal; there a vertical equipotential line is used as boundary of the region. For this homogeneous isotropic soil the coefficient of permeability is 1 m/day and the groundwater head difference between the canal and the vertical equipotential line is 1 m. Figure III.20 shows the flow pattern. At the boundary the sub-division in boundary segments that was used is indicated.

From the groundwater heads at the talus it is seen that the outflow of groundwater is concentrated mainly at the upper zone of the open part of the talus.

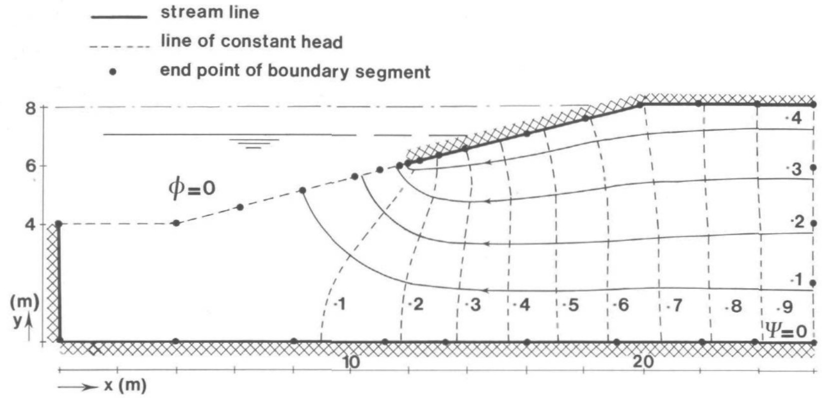

Figure III.20 Flow pattern in a dike.

EXAMPLE 2

Steady flow in a dike; homogeneous isotropic soil with a drain

In figure III.21 the same dike is shown as in the first example, but here there is a drain in the vicinity of the lower end of the revetment, possibly having been left behind from the building phase of the dike. It can now be used to reduce the overpressures at the revetment, for example, to make it possible for some repairs to be carried out on it. In another situation, it would be possible that the water flowing from the vertical line of constant head is polluted and the drain could be used to intercept the water in order to prevent it reaching the canal.

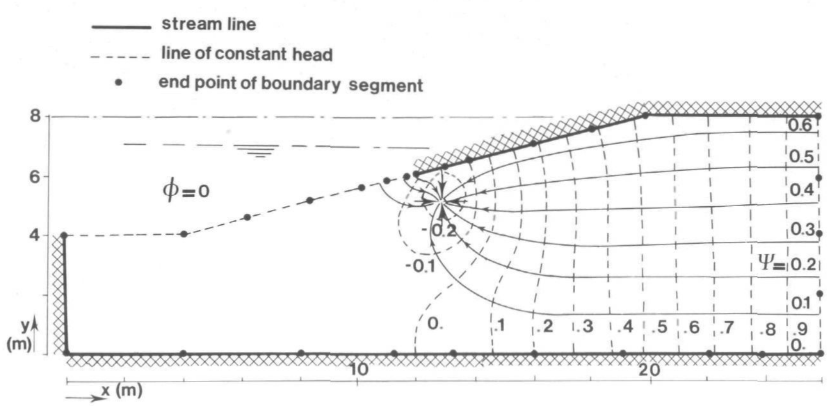

Figure III.21 Flow pattern in a dike with a sink.

The same boundary conditions are used as in the first example. The discharge of the

drain is 1.0 m^3/m'day (for a length of 500 m that is about 21 m^3/hr).
From figure III.21 it is seen that the polluted water is almost completely intercepted by the drain.

EXAMPLE 3

Steady flow in a dike; homogeneous anisotropic soil
Figure III.22 shows the same dike as in the first example, but in this case the soil is anisotropic. The direction of the maximum coefficient of permeability has an angle of 15° with the positive x-axis (that is, about parallel to the 1 : 4 talus). The ratio of the maximum to the minimum coefficient of permeability (the anisotropy factor) is 10.0.

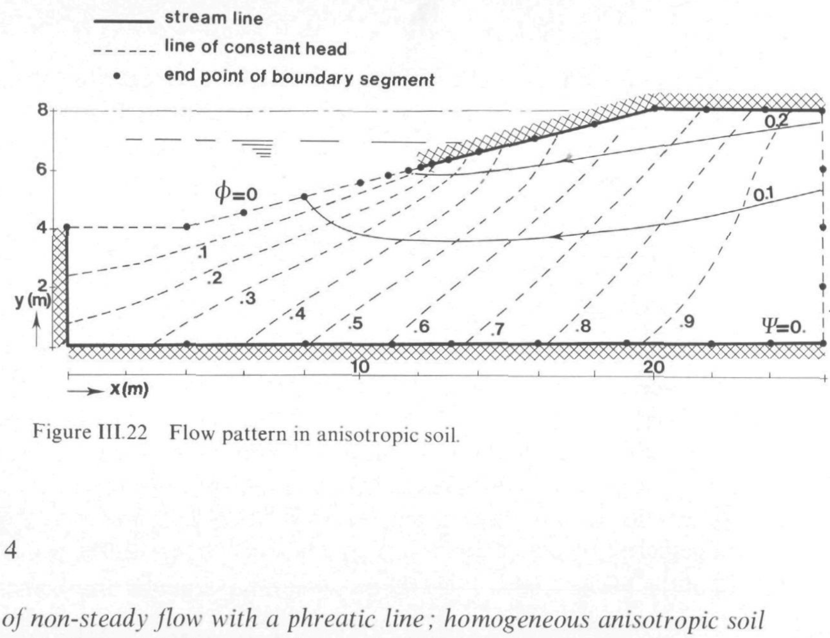

Figure III.22 Flow pattern in anisotropic soil.

EXAMPLE 4

Moment of non-steady flow with a phreatic line; homogeneous anisotropic soil
This example deals with the groundwater flow in a region that was rectangular in its initial state (see figure III.23). At the left, right and lower parts of the region the boundary is impermeable. The upper boundary is a phreatic line. In the region there is a sink and a source.
Figure III.23 shows the flow region and the flow pattern. The soil is anisotropic, the maximum coefficient of permeability is 1.4 m/day, the anisotropy factor is 2.0, and the direction of the maximum coefficient of permeability has an angle of 15° with the positive x-axis. The storage coefficient is 0.2.
This discharge of the sink is 2.5 m^3/m'day. The same discharge is infiltrated at the

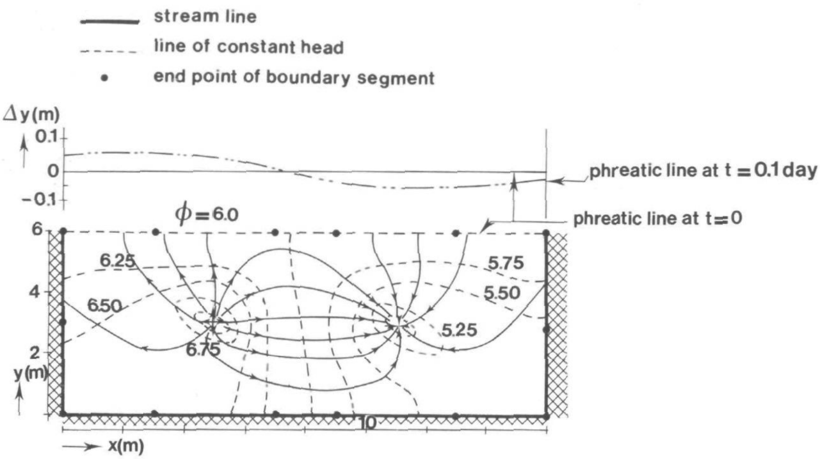

Figure III.23 Flow pattern in anisotropic soil with sink and source.

source. From the specific discharge distribution along the phreatic line the alteration of the position of the phreatic line in a step in time is found. After a period of 0.1 day, the phreatic line is raised about 6 cm above the source and the fall above the sink is also about 6 cm.

EXAMPLE 5

Moment of non-steady flow with an interface; homogeneous anisotropic soil with a sink and a source in the heavy fluid and a sink in the lighter fluid

Figure III.24 shows the flow region. There are two fluids. Left, right and lower boundaries are impermeable. The groundwater head at the upper boundary is constant.

The flow pattern is given in figure III.25. The soil is anisotropic, the maximum coefficient of permeability is 1.4 m/day, the anisotropy factor is 2.0, and the direction of the maximum coefficient of permeability has an angle of $-15°$ with the positive x-axis. The density of the lower fluid is 1,025 kg/m^3; that fluid contains a sink and a source, both with the same discharge (2.5 m^3/m'day). The density of the upper fluid is 1,000 kg/m^3; in this fluid there is a sink with a discharge of 3.5 m^3/m'day.

The upper boundary of the region is an equipotential line without silt layer.

It is seen from figure III.25 that there are groundwater head discontinuities at the interface, caused by the density difference between both fluids. The displacement of the interface in a step in time was calculated from the specific discharges at the interface. At the end of the step in time (2 days) the rise of the interface above the source is about 40 cm and the fall of the interface between the two sinks is also about 40 cm (η = 0.2). It is seen in figure III.25 that the upper sink mainly receives water originating from the upper equipotential line boundary.

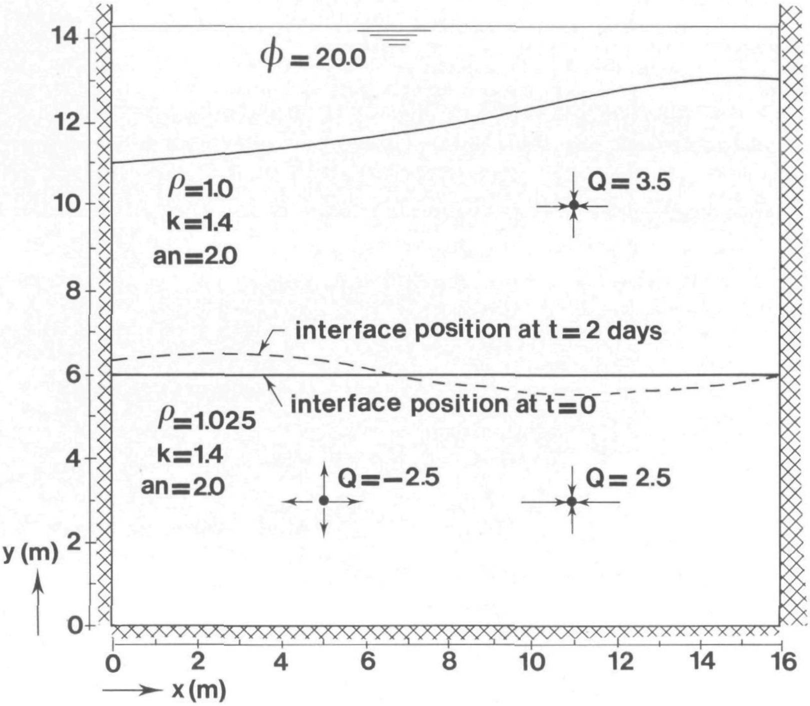

Figure III.24 Two fluids in anisotropic aquifer.

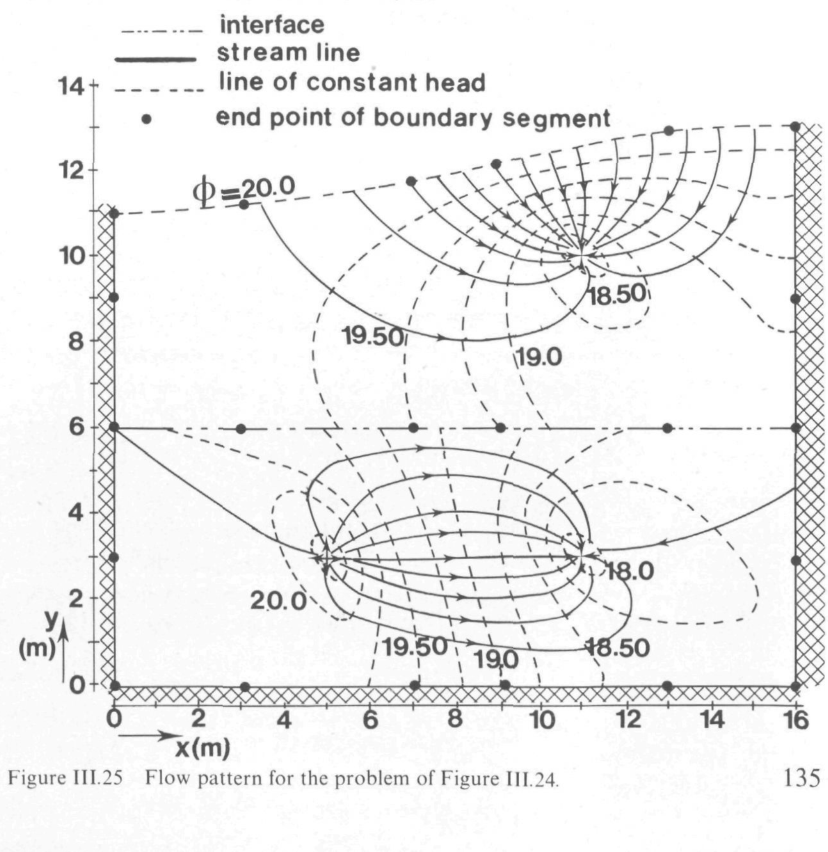

Figure III.25 Flow pattern for the problem of Figure III.24.

EXAMPLE 6

Moment of non-steady flow with an interface, a seepage line and a phreatic line; inhomogeneous anisotropic soil with a sink and a source in the heavy fluid and a sink in the lighter fluid

A fictive non-steady flow problem that involves together all boundary conditions that have been discussed in Chapter 14 is given in figure III.26. The flow region consists of two sand layers that are both anisotropic. The soil properties are different for both sand layers.

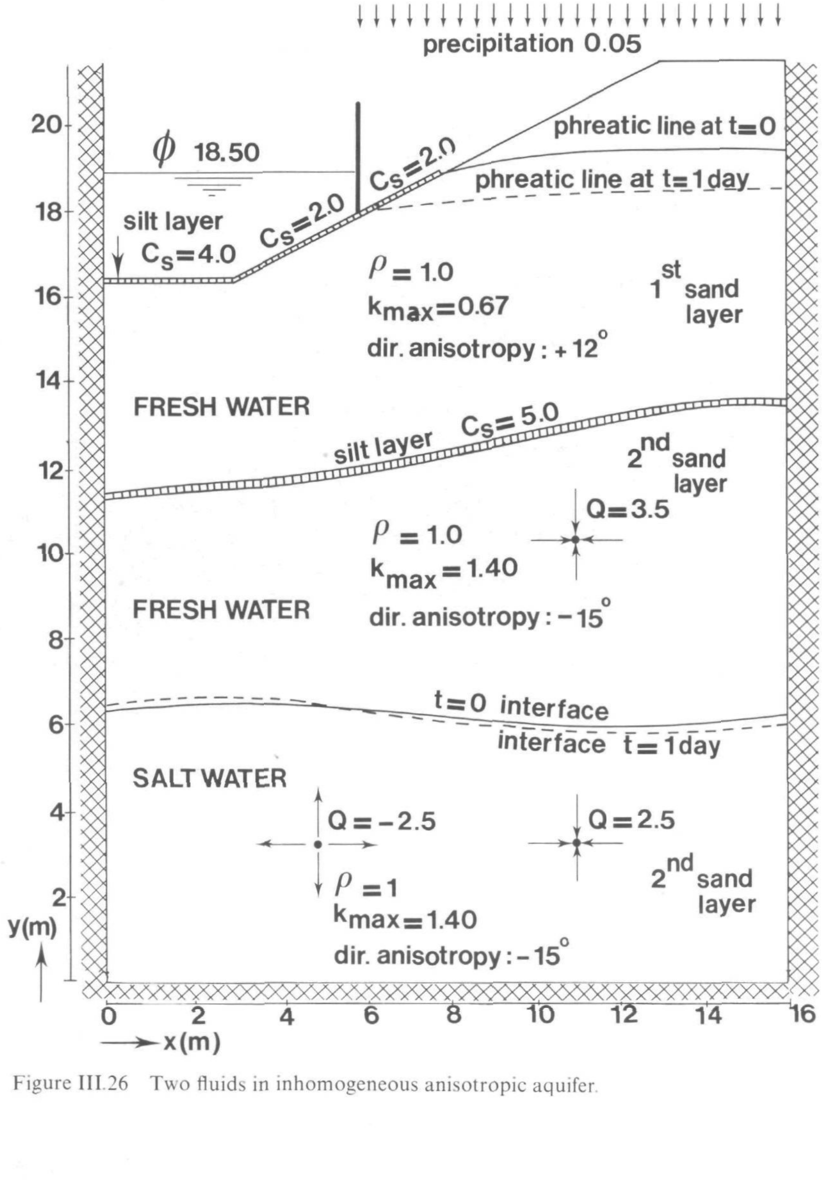

Figure III.26 Two fluids in inhomogeneous anisotropic aquifer.

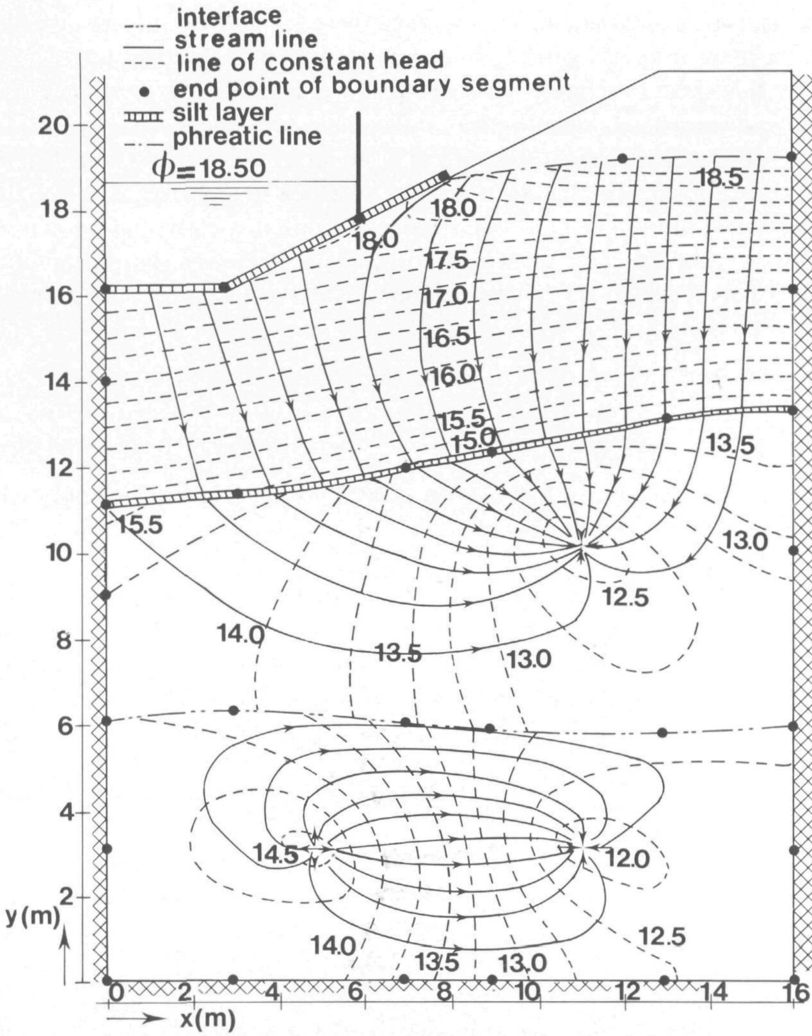

Figure III.27 Flow pattern for the problem of Figure III.26.

At the separation line between the sand layers there is a thin layer of clay. In the second sand layer there are two fluids with different densities. In the heavy fluid there are a sink and a source of equal discharge. In the lighter fluid there is a sink in the second sand layer. The upper boundary of the region involves a phreatic line with precipitation that ends at the talus of a canal. Perfect unimpeded streaming out of the soil solid is not possible because there is a silt layer on the talus.

This silt layer is present at the bottom of the canal too, but there it has a greater

resistance then at the talus. In the canal there is a fictive structure by which the water table in the canal is higher than the lowest point of the seepage line.
The flow pattern is given in figure III.27. It is clear from the figure that there are groundwater head discontinuities at the interface and at the clay and silt layers. From the specific discharges at the moving boundaries the displacements in a time interval were evaluated, and the positions after one step in time are given in figure III.26 by the dotted lines ($\mu = \eta = 0.2$). From the figure it is clear that there is only a small movement of the interface. On the other hand, there is a sharp drop of the phreatic line. This is caused by the abstraction of water at the upper sink in the second sand layer.

EXAMPLE 7

Moment of non-steady flow with two interfaces, a seepage line and a phreatic line; inhomogeneous anisotropic soil with a sink and a source in the salt water and a sink in the fresh water
The regional geometry here is almost the same as in the previous example, except for one essential difference. Instead of an impermeable lowest boundary here there is an interface with a third fluid. That fluid is assumed to have a constant groundwater head. The problem definition is given in figure III.28, and the flow pattern is given in figure III.29. The displacements of the moving boundaries in a time interval are indicated in figure III.28 by the dotted lines.
It is remarkable that now there is a sharp rise of a part of the lower interface and the whole upper interface.
These rises are the consequence of the abstraction of water at the upper sink in the second sand layer. Because there is a thin layer of clay between the sand layers, and now there is a possibility of water supply from underneath, this sink now receives considerably less water from above. Consequently the fall of the phreatic line is less than in Example 6.
As an illustration a part of the computer output is given in Appendix 3. The complex distribution strengths ($RCQ + i\,ICQ$) are given in the first pages, followed in later pages by the groundwater head (PHI), the stream function (PSI) and the components of the specific discharge (VX and VY). The parameters KO, FI and WR are not relevant in this context. This calculation was made using the sub-division of the boundary in the segments that are indicated in figure III.29. To illustrate the influence of the number of boundary segments, the calculation was repeated where the number of boundary segments was doubled. The corresponding part of the computer output is given in the next part of Appendix 3. It is seen that for this extreme case (high groundwater head gradients that are caused by sink and source), the differences between the results of the two calculations are comparatively small. Apparently the first choice of the number of boundary segments was reasonable.

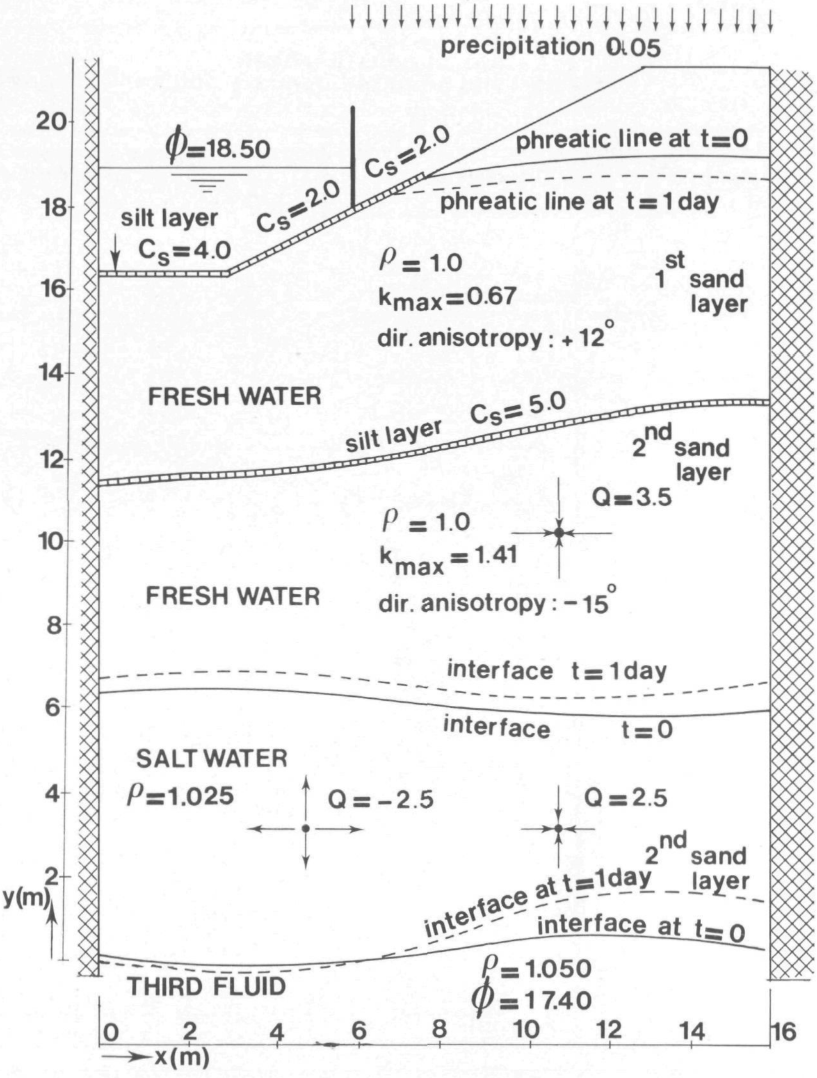

Figure III.28 Three fluids in inhomogeneous anisotropic aquifer.

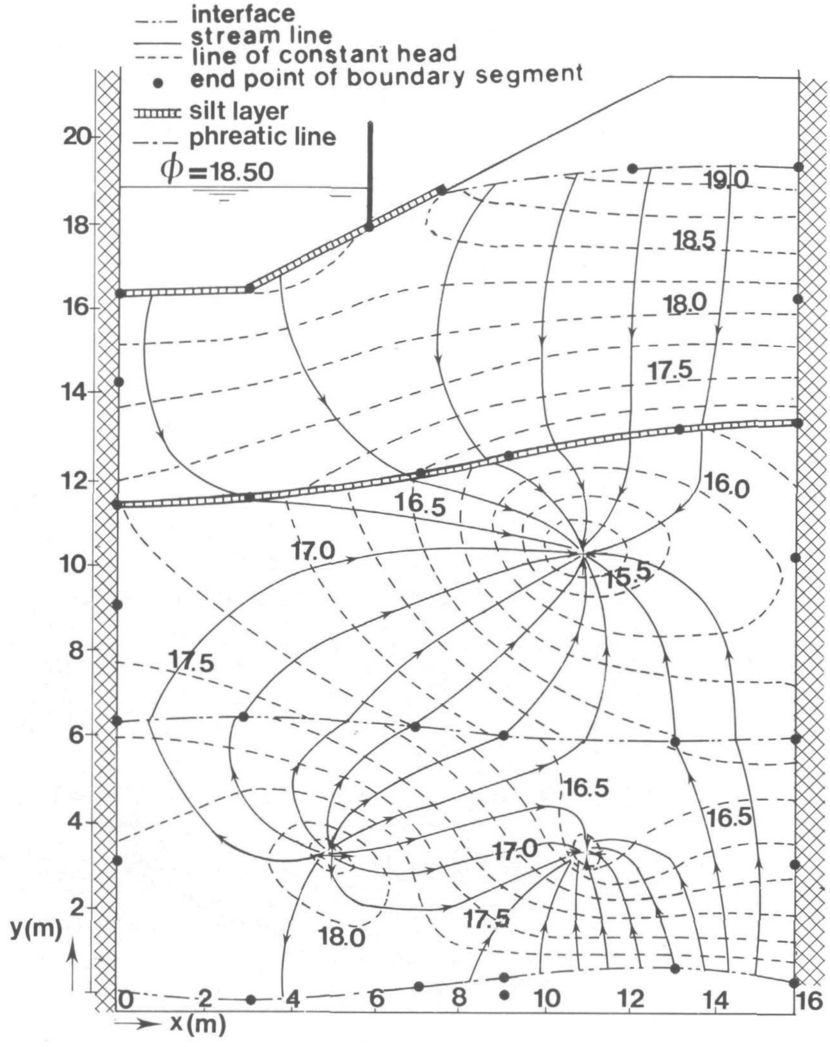

Figure III.29 Flow pattern for the problem of Figure III.28.

The behaviour in time of an interface; homogeneous isotropic soil

In figure III.30 the starting position of an interface between fresh and salt groundwater is given by the horizontal line indicated by $t \leqslant 0$. This example deals with groundwater flow in a polder aquifer. The upper boundary of the flow region consists of semi-pervious layers which locally have a smaller resistance ($c_s = 20$ days instead of $c_s = 100$ days). The sub-soil consists of sand, with a coefficient of permeability of 10 m/day. To a depth of 400 m the soil is homogeneous and isotropic; at that depth there is a very course layer (gravel or shells). As the supply of salt groundwater through that layer is quite possible, the horizontal line at 400 m depth is assumed to be an equipotential line. At the beginning ($t = 0$) the interface is horizontal at a depth of 200 m. This is a steady position if the groundwater head at the entire upper boundary is +4.0 m higher than the head in the salt groundwater (0.m).

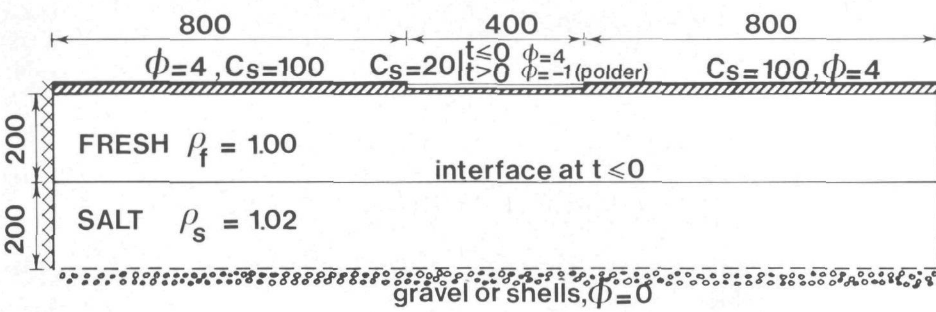

Figure III.30 Interface in polder aquifer.

At $t = 0$ a sudden fall of the groundwater head is brought about above the part of the upper boundary where $c_s = 20$ days (see figure III.30), (reclamation; the groundwater head is lowered from +4 to −1 m). The consequence of the reclamation is a flow of the fresh water as well as the salt water. Figure III.31 shows the position of the interface between fresh and salt groundwater as a function of time.

In the calculation only half the profile was used because the problem is symmetrical. From the figure it is clear that after some years there is salt water seepage into the new polder, but after about $3\frac{1}{2}$ years the steady state has practically been reached. The relatively high speed of the phenomenon was caused by the great groundwater head differences (5 m between the polder and its surroundings and 1 m between the polder and the groundwater head in the salt water at 400 m depth). To indicate the influence of the size of the step in time, the problem was calculated twice, where the time step size was respectively 0.4 and 0.2 year. Figure III.32 gives a comparison for the top of the interface and a point at 100 m from the top that does not reach the polder level. From the figure it is seen that there are differences, although these are relatively small. So the choice of a time step size of 0.4 year led to a reasonable result (for that time step size the polder received salt water seepage after 2.7 years instead of 2.9 years for the calculation with a time step size of 0.2 years).

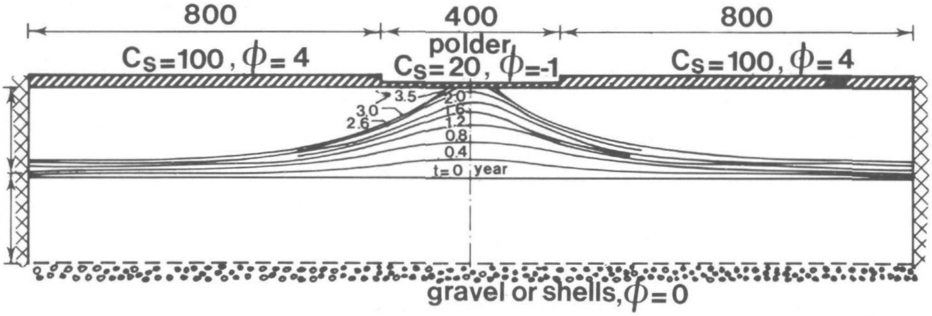

Figure III.31 Position of interface in time.

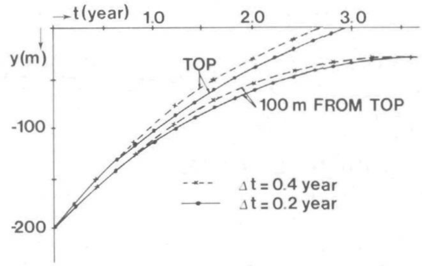

Figure III.32 Influence of time step size.

EXAMPLE 9

Simple half-plane flow

A simple example of half-plane flow is given in figure III.33:

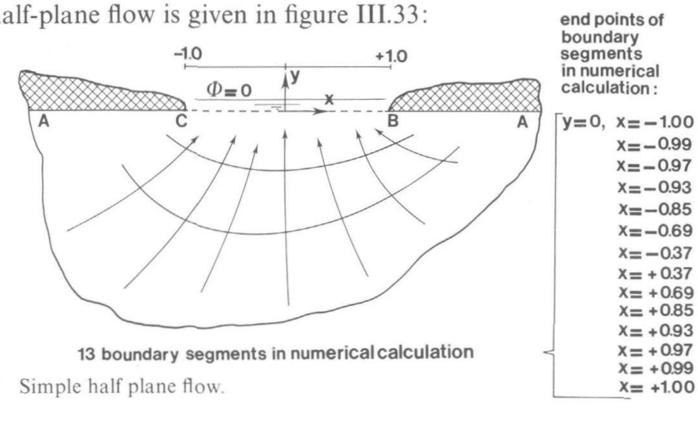

Figure III.33 Simple half plane flow.

The problem deals with the groundwater flow from the surroundings towards a low-situated polder. The analytic solution of the flow problem can simply be obtained by means of the method of Pavlovskii (conformal mapping, see Chapter 7). Figure III.34 shows the Ω-plane:

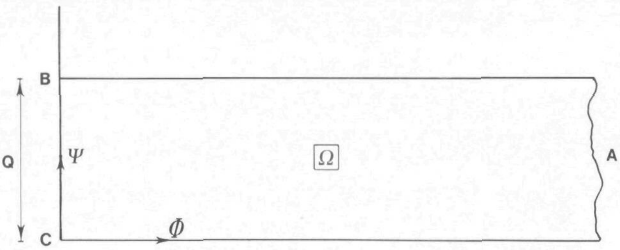

Figure III.34 Ω-plane.

Using the sine transform (see Appendix 2), the relationship between $z = x + iy$ and $\Omega = \Phi + i\Psi$ can easily be found. The result is:

$$\Omega = \frac{iQ}{\pi} \arccos(-z)$$

The flow was also calculated using the calculation method outlined in Chapter 17. A comparison of the results of both calculations is given in Appendix 4. From the tables of that Appendix it is seen that the sub-division in 13 boundary segments of the 'polder boundary' in the numerical calculation yields very accurate results. Including the reference equations, there were only 15 linear equations to be solved.

EXAMPLE 10

Half-plane flow

A somewhat more complicated problem is the groundwater flow from the surroundings towards three low-situated canals of which the water tables mutually differ. One of the canals has a bottom on which a silt layer is present. In the half plane there are two sinks of equal discharge, see figure III.35. The flow pattern is given in figure III.36. It is seen from the figure that the water abstracted from the right sink originates merely from the right canal and infinity. The left sink receives water from the middle canal and infinity. From the middle and the right canals water is abstracted, while the left canal receives water, which comes from the middle canal and infinity.

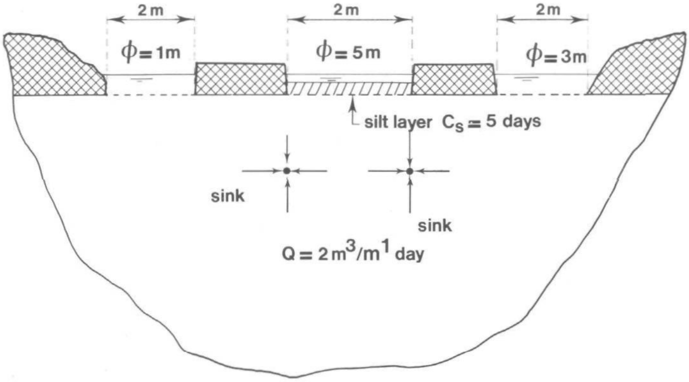

Figure III.35 Half plane flow.

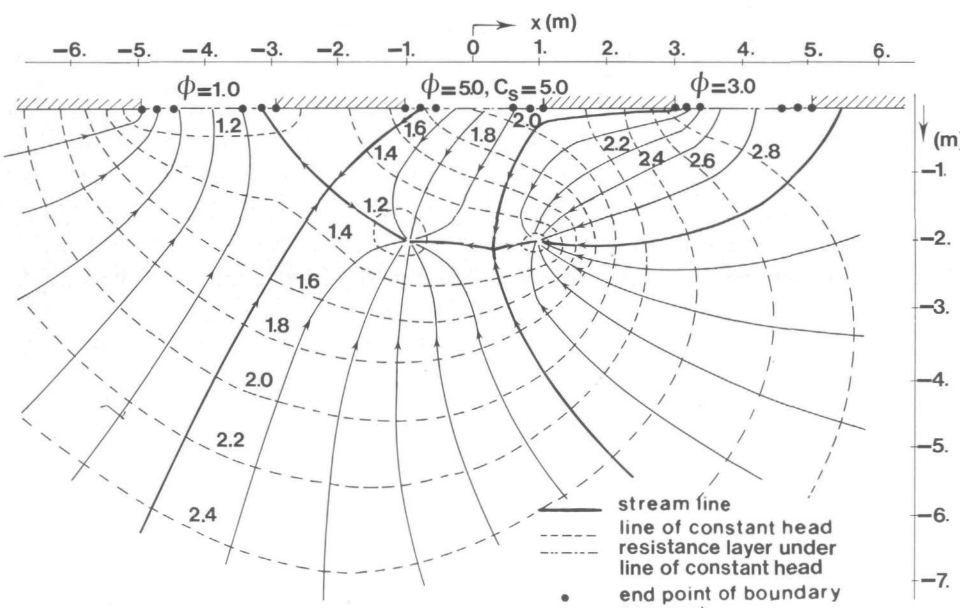

Figure III.36 Flow pattern for the problem of Figure III.35.

SAMENVATTING

Dit proefschrift over berekeningsmethoden voor tweedimensionale grondwaterstroming bestaat uit drie delen. In het eerste deel wordt de basistheorie vermeld. In het tweede deel wordt een overzicht gegeven van de belangrijkste bestaande oplossingsmethoden. In het derde deel wordt een numerieke rekentechniek beschreven die is gebaseerd op het gebruik van analytische functies voor het benaderen van tweedimensionale stromingsbeelden (analytische functie methode).

Een formulering van randvoorwaarden wordt gegeven waarin in de praktijk voorkomende randen zoals 'equipotentiaallijn met sliblaag', etc. voorkomen. Bij de berekening wordt het stromingsgebied onderverdeeld in deelgebieden die elk constante vloeistof- en grondeigenschappen hebben. De deelgebieden worden verbonden met behulp van aansluitvoorwaarden voor punten op de scheidingslijnen tussen de deelgebieden.

Twee klassen van stroming in een halfvlak kunnen ook worden berekend met deze methode met als voordeel dat het niet nodig is om op zekere afstand een schematische begrenzing aan te brengen.

Een computerprogramma werd geschreven dat gebaseerd is op de analytische functie methode. Dit programma kan worden gebruikt voor de berekening van beelden van stationaire en niet-stationaire stroming in gebieden van willekeurige vorm die putten en bronnen, meerdere vloeistoffen en anisotrope inhomogene grond mogen bevatten. De invoer van het computerprogramma bestaat louter uit algemene informatie (gegevens met betrekking tot doorlatendheid, anisotropie, dichtheid, etc.) en randinformatie (plaats en eigenschappen van de rand). De oplossingen die worden verkregen met behulp van de analytische functie methode zijn binnen een benaderde rand exact.

SUMMARY

This thesis on calculation methods for two-dimensional groundwater flow is subdivided into three Parts. In the first Part the basic theory is outlined. In the second Part a review is given of the most important existing solution methods. In the third Part a numerical calculation technique is developed that is based on the use of analytic functions for approximating two-dimensional flow patterns (analytical function method).

A boundary condition formulation has been given that involves practical boundaries like 'equipotential line with resistance', etc.

In the calculation a flow region is divided into sub-regions that all have constant properties of fluid and soil. The sub-regions are coupled by connecting conditions for points of the separation lines between the sub-regions.

Two classes of half-plane flow can also be calculated by this method, giving the advantage that for those problems no schematic boundary at some distance is necessary.

A computer program was written, based on the analytical function method. This program can be used for the calculation of steady and non-steady flow patterns in regions of arbitrary shape that may include sinks and sources, several fluids and soil that may be inhomogeneous and anisotropic. The input of the computer program consists only of general information (data with respect to permeability, anisotropy, density, etc.) and boundary information (position and properties of the boundary).

Solutions that have been found by the analytical function method are exact within an approximative boundary.

Appendix 1: Definitions with respect to Complex Denotations

Complex numbers
The complex number z is defined by:

$$z = x + i\,y$$

where x and y are real numbers and i is the so-called imaginary unit that has the following property:

$$i^2 = -1$$

The real numbers x and y are respectively the real and imaginary part of the complex number z. This is denoted by:

$$x = \mathrm{Re}\{z\}$$
$$y = \mathrm{Im}\{z\}$$

Complex functions
The complex function $f(z) = u(x, y) + i\,v(x, y)$ is analytic when u and v satisfy the so-called Cauchy-Riemann relationships:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

In the mathematical literature (e.g., Wylie, 1960) this can be found more comprehensively. From the Cauchy-Riemann expressions by differentiation one finds:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

Additions yields:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

In a similar way one finds:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

So the functions u and v that satisfy the Cauchy-Riemann relationships satisfy the Laplace differential equation. Functions that satisfy the Laplace differential equation are called harmonic functions, and when these satisfy the Cauchy-Riemann relationships they are called conjugate harmonic functions. So an analytic function $f(z) = u(x, y) + i\,v(x, y)$ consists of two conjugate harmonic functions. Conjugate harmonic functions (u and v) have the property that $u(x, y) = C_1$ (constant) and $v(x, y) = C_2$ (constant) are perpendicular lines.

Appendix 2: The Sine Transform

The sine transform has the property that a half-infinite strip is conformally mapped upon the upper half plane (see figure A.1) according to:

$$w = \sin(z)$$

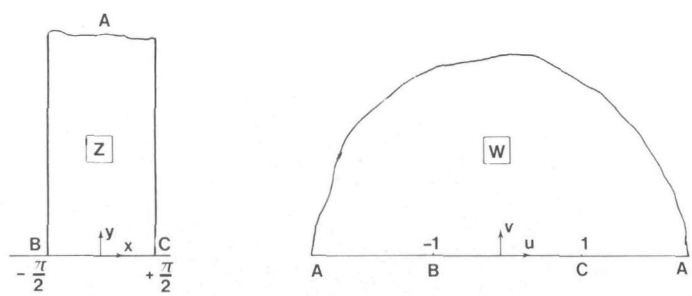

Figure A.1 Sine transform.

This is seen simply by:

$$w = \sin(x + iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy)$$

Using:

$$\begin{aligned}\cos(iy) &= \cosh(y)\\ \sin(iy) &= i\sinh(y)\end{aligned}$$

it follows: $w = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$,

$$\text{for } x = \pm\frac{\pi}{2} : w = \pm\cosh(y)$$

$$\text{so} : \operatorname{Im}\{w\} = 0$$

$$\text{for } y = 0 : w = \sin(x)$$

$$\text{so} : \operatorname{Im}\{w\} = 0.$$

For a point within the half-infinite strip:

$$-\frac{\pi}{2} < x < \frac{\pi}{2} \qquad \text{and so} : \cos(x) > 0$$

$$y > 0 \qquad \text{and so} : \sinh(y) > 0.$$

Then : $\mathrm{Im}\{w\} = \cos(x)\sinh(y) > 0.$

So a point of the half-infinite strip $\left(-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}, y \geqslant 0\right)$ is mapped upon a point of the upper half plane.

Appendix 3: Part of Computer Output of Example 7

Region 1 is the lower region; region 2 is the middle region and region 3 is the upper region (see figure III.28).

Region 1

X	Y	KO	FI	WR	RCQ	ICQ
.000	−.100	23	.000	.000	.482	.321
3.000	−.300	23	.000	.000	.483	.227
7.000	−.100	23	.000	.000	.448	.253
9.000	.200	23	.000	.000	.338	.134
13.000	.400	23	.000	.000	.393	−.312
16.000	.000	21	.000	.000	.523	−.060
16.000	3.000	21	.000	.000	.509	−.171
16.000	5.800	23	.000	.000	.472	.306
13.000	5.600	23	.000	.000	.531	.181
9.000	5.800	23	.000	.000	.538	.122
7.000	6.000	23	.000	.000	.492	.043
3.000	6.300	23	.000	.000	.508	−.061
.000	6.100	21	.000	.000	.412	.146
.000	3.000	21	.000	.000	.425	−.135
		Initial complex potential			6.933	−7.822

Region 2

X	Y	KO	FI	WR	RCQ	ICQ
.000	6.100	23	.000	.000	.389	.289
3.000	6.300	23	.000	.000	.377	−.003
7.000	6.000	23	.000	.000	.465	−.146
9.000	5.800	23	.000	.000	.524	−.128
13.000	5.600	23	.000	.000	.485	−.185
16.000	5.800	21	.000	.000	.587	.006
16.000	9.000	21	.000	.000	.600	−.033
16.000	13.000	31	.000	5.000	.550	.192
13.000	12.900	31	.000	5.000	.509	−.007
9.000	12.200	31	.000	5.000	.465	.101
7.000	11.700	31	.000	5.000	.384	−.034
3.000	11.200	31	.000	5.000	.376	−.209
.000	11.000	21	.000	.000	.585	.090
.000	9.000	21	.000	.000	.552	−0.45
		Initial complex potential			3.981	−6.148

Region 3

X	Y	KO	FI	WR	RCQ	ICQ
.000	11.000	31	.000	5.000	.281	.026
3.000	11.200	31	.000	5.000	.282	−.069
7.000	11.700	31	.000	5.000	.336	−.058
9.000	12.200	31	.000	5.000	.368	−.031
13.000	12.900	31	.000	5.000	.324	−.150
16.000	13.000	21	.000	.000	.347	.105
16.000	16.000	21	.000	.000	.343	−.038
16.000	19.000	22	.000	.000	.365	.064
12.000	19.000	22	.000	.000	.388	−.206
8.000	18.500	12	.000	2.000	.586	−.470
6.000	17.500	11	18.500	2.000	.075	−.272
3.000	16.000	11	18.500	4.000	−.066	−.255
.000	16.000	21	.000	.000	.225	−.092
.000	14.000	21	.000	.000	.213	−.139
		Initial complex potential			1.296	−.451

Region 1

X	Y	PHI	PSI	VX	VY
1.000	1.000	17.991	−.852	.008	−.063
3.000	1.000	18.020	−1.045	.001	−.077
5.000	1.000	18.036	−1.309	.041	−.106
7.000	1.000	17.918	−1.553	.095	−.048
9.000	1.000	17.742	−1.536	.061	.065
11.000	1.000	17.612	−1.192	−.010	.172
13.000	1.000	17.617	−.643	−.049	.192
15.000	1.000	17.615	−.197	−.031	.138
1.000	3.000	18.185	−.873	.011	−.021
3.000	3.000	18.277	−.916	−.105	−.009
5.000	3.000	22.064	−2.027	3838.164	−1028.405
7.000	3.000	17.925	−2.124	.279	.028
9.000	3.000	17.368	−2.008	.268	.054
11.000	3.000	13.249	−.714	−3838.036	1028.489
13.000	3.000	17.064	−.333	−.143	.087
15.000	3.000	17.199	−.089	−.045	.083
1.000	5.000	18.142	−3.382	.016	.024
3.000	5.000	18.092	−3.251	.015	.071
5.000	5.000	17.934	−.487	.102	.102
7.000	5.000	17.589	−2.57	.169	.056
9.000	5.000	17.204	−.162	.162	.012
11.000	5.000	16.957	−.171	.056	−.005
13.000	5.000	16.934	−.124	−.023	.037
15.000	5.000	16.974	.007	−.032	.046

Example 7:

The number of boundary segments is now doubled.

Region 1

X	Y	KO	FI	WR	RCQ	ICQ
.000	−.100	23	.000	.000	.747	.593
1.500	−.200	23	.000	.000	.792	.401
3.000	−.300	23	.000	.000	.733	.353
5.000	−.200	23	.000	.000	.752	.307
7.000	−.100	23	.000	.000	.730	.349
8.000	.050	23	.000	.000	.720	.315
9.000	.200	23	.000	.000	.681	.163
11.000	.300	23	.000	.000	.655	−.009
13.000	.400	23	.000	.000	.650	−.439
14.500	.200	23	.000	.000	.608	−.635
16.000	.000	21	.000	.000	.826	.083
16.000	1.500	21	.000	.000	.852	−.074
16.000	3.000	21	.000	.000	.850	−.196
16.000	4.400	21	.000	.000	.825	−.381
16.000	5.800	23	.000	.000	.727	.593
14.500	5.700	23	.000	.000	.758	.457
13.000	5.600	23	.000	.000	.814	.271
11.000	5.700	23	.000	.000	.815	.234
9.000	5.800	23	.000	.000	.843	.138
8.000	5.900	23	.000	.000	.842	.088
7.000	6.000	23	.000	.000	.809	.039
5.000	6.150	23	.000	.000	.801	−.089
3.000	6.300	23	.000	.000	.832	−.099
1.500	6.200	23	.000	.000	.825	−.262
.000	6.100	21	.000	.000	.743	.323
.000	4.550	21	.000	.000	.748	.105
.000	3.000	21	.000	.000	.752	−.101
.000	1.450	21	.000	.000	.741	−.308
		Initial complex potential			−.602	−11.851

Region 2

X	Y	KO	FI	WR	RCK	ICQ
.000	6.100	23	.000	.000	.520	.479
1.500	6.200	23	.000	.000	.538	.329
3.000	6.300	23	.000	.000	.560	.101
5.000	6.150	23	.000	.000	.565	−.053
7.000	6.000	23	.000	.000	.601	−.151
8.000	5.900	23	.000	.000	.608	−.194
9.000	5.800	23	.000	.000	.630	−.177
11.000	5.700	23	.000	.000	.617	−.171
13.000	5.600	23	.000	.000	.632	−.173
14.500	5.700	23	.000	.000	.624	−.312
16.000	5.800	21	.000	.000	.751	.131
16.000	7.400	21	.000	.000	.776	−.002
16.000	9.000	21	.000	.000	.777	−.058
16.000	11.000	21	.000	.000	.738	−.193
16.000	13.000	31	.000	5.000	.668	.409
14.500	12.950	31	.000	5.000	.683	.209
13.000	12.900	31	.000	5.000	.610	.142
11.000	12.550	31	.000	5.000	.648	.105
9.000	12.200	31	.000	5.000	.636	.163
8.000	11.950	31	.000	5.000	.630	.138
7.000	11.700	31	.000	5.000	.573	−.001
5.000	11.450	31	.000	5.000	.573	−.099
3.000	11.200	31	.000	5.000	.561	−.244
1.500	11.100	31	.000	5.000	.549	−.373
.000	11.000	21	.000	.000	.775	.186
.000	10.000	21	.000	.000	.783	.078
.000	9.000	21	.000	.000	.761	−.027
.000	7.550	21	.000	.000	.739	−.188
		Initial complex potential			−.196	−8.373

Region 3

X	Y	KO	FI	WR	RCQ	ICQ
.000	11.000	31	.000	5.000	.325	.071
1.500	11.100	31	.000	5.000	.333	−.008
3.000	11.200	31	.000	5.000	.328	−.052
5.000	11.450	31	.000	5.000	.337	−.109
7.000	11.700	31	.000	5.000	.367	−.082
8.000	11.950	31	.000	5.000	.378	−.079
9.000	12.200	31	.000	5.000	.388	−.090
11.000	12.550	31	.000	5.000	.373	−.080
13.000	12.900	31	.000	5.000	.338	−.170
14.500	12.950	31	.000	5.000	.327	−.226
16.000	13.000	21	.000	.000	.392	.153
16.000	14.500	21	.000	.000	.398	.065
16.000	16.000	21	.000	.000	.396	−.016
16.000	17.500	21	.000	.000	.385	−.120
16.000	19.000	22	.000	.000	.412	.133
14.000	19.000	22	.000	.000	.431	−.020
12.000	19.000	22	.000	.000	.449	−.186
10.000	18.750	22	.000	.000	.445	−.320
8.000	18.500	12	.000	2.000	.688	−.445
7.000	18.000	12	.000	2.000	.696	−.628
6.000	17.500	11	18.500	2.000	.030	−.433
4.500	16.750	11	18.500	2.000	.004	−.318
3.000	16.000	11	18.500	4.000	−.128	−.283
1.500	16.000	11	18.500	4.000	−.125	−.260
.000	16.000	21	.000	.000	.237	−.112
.000	15.000	21	.000	.000	.237	−.131
.000	14.000	21	.000	.000	.231	−.154
.000	12.500	21	.000	.000	.225	−.193
		Initial complex potential			.199	.061

Region 1

X	Y	PHI	PSI	VX	VY
1.000	1.000	17.968	−1.041	.011	−.056
3.000	1.000	17.993	−1.207	−.003	−.067
5.000	1.000	18.013	−1.443	.037	−.096
7.000	1.000	17.898	−1.659	.092	−.039
9.000	1.000	17.721	−1.616	.059	.072
11.000	1.000	17.596	−1.244	−.010	.186
13.000	1.000	17.589	−.678	−.053	.194
15.000	1.000	17.591	−.211	−.028	.142
1.000	3.000	18.110	−1.048	−.005	−.011
3.000	3.000	18.215	−1.064	−.110	.000
5.000	3.000	22.006	−2.150	3838.159	−1028.396
7.000	3.000	17.871	−2.221	.275	.037
9.000	3.000	17.317	−2.082	.265	.063
11.000	3.000	13.198	−.765	−3838.037	1028.497
13.000	3.000	17.012	−.363	−.143	.095
15.000	3.000	17.143	−.096	−.043	.092
1.000	5.000	18.044	−3.549	.014	.038
3.000	5.000	17.996	−3.387	.011	.082
5.000	5.000	17.846	−.596	.097	.110
7.000	5.000	17.504	−.343	.165	.065
9.000	5.000	17.126	−.225	.158	.018
11.000	5.000	16.880	−.218	.055	.001
13.000	5.000	16.853	−.152	−.023	.045
15.000	5.000	16.889	.003	−.030	.059

Appendix 4: Comparison of Exact and Approximative Solution of Example 9

$-x$	ϕ approximated	ϕ exact
0.	0.0000	0.
0.05	0.0001	
0.10	0.0002	
0.15	0.0004	
0.20	0.0006	
0.25	0.0003	
0.30	0.0003	
0.35	0.0003	
0.40	−0.0014	
0.45	−0.0014	
0.50	−0.0006	
0.55	0.0005	
0.60	0.0015	
0.65	0.0021	
0.70	−0.0005	
0.75	0.0006	
0.80	0.0010	
0.85	0.0003	
0.90	0.0006	
0.95	−0.0000	
1.00	0.0174	0.
1.05	0.1023	0.1002
1.10	0.1428	0.1412
1.15	0.1736	0.1722
1.20	0.1994	0.1981
1.25	0.2218	0.2206
1.30	0.2419	0.2408
1.35	0.2602	0.2591
1.40	0.2770	0.2760
1.45	0.2927	0.2917
1.50	0.3074	0.3063
1.55	0.3212	0.3202
1.60	0.3342	0.3333
1.65	0.3467	0.3457
1.70	0.3585	0.3575
1.75	0.3698	0.3689

$-x$	ϕ approximated	ϕ exact
1.80	0.3806	0.3797
1.85	0.3911	0.3901
1.90	0.4011	0.4002
1.95	0.4108	0.4099
2.00	0.4201	0.4192
2.05	0.4291	0.4282
2.10	0.4379	0.4370
2.15	0.4464	0.4455
2.20	0.4546	0.4537
2.25	0.4626	0.4617
2.30	0.4704	0.4695
2.35	0.4780	0.4771
2.40	0.4854	0.4845
2.45	0.4926	0.4917
2.50	0.4996	0.4987
2.55	0.5065	0.5056
2.60	0.5132	0.5123
2.65	0.5197	0.5189
2.70	0.5261	0.5253
2.75	0.5324	0.5316
2.80	0.5386	0.5377
2.85	0.5446	0.5437
2.90	0.5505	0.5496
2.95	0.5563	0.5554
3.00	0.5620	0.5611
3.05	0.5667	0.5667
3.10	0.5730	0.5721
3.15	0.5784	0.5775
3.20	0.5837	0.5828
3.25	0.5888	0.5830
3.30	0.5939	0.5931
3.35	0.5990	0.5981
3.40	0.6039	0.6031
3.45	0.6088	0.6079
3.50	0.6135	0.6127

$-x$	ϕ	
	approximated	exact
3.55	0.6182	0.6174
3.60	0.6229	0.6220
3.65	0.6275	0.6266
3.70	0.6320	0.6311
3.75	0.6364	0.6355
3.80	0.6408	0.6399
3.85	0.6451	0.6442
3.90	0.6493	0.6485
3.95	0.6535	0.6527
4.00	0.6576	0.6568
4.05	0.6617	0.6609
4.10	0.6658	0.6649
4.15	0.6697	0.6689
4.20	0.6737	0.6728
4.25	0.6775	0.6767
4.30	0.6814	0.6805
4.35	0.6851	0.6843
4.40	0.6889	0.6881
4.45	0.6926	0.6917
4.50	0.6962	0.6954
4.55	0.6998	0.6990
4.60	0.7034	0.7026
4.65	0.7069	0.7061
4.70	0.7104	0.7096
4.75	0.7139	0.7130

$-x$	ϕ	
	approximated	exact
4.80	0.7173	0.7164
4.85	0.7206	0.7198
4.90	0.7240	0.7231
4.95	0.7273	0.7264
5.00	0.7305	0.7297
5.05	0.7338	0.7329
5.10	0.7370	0.7361
5.15	0.7401	0.7393
5.20	0.7433	0.7424
5.25	0.7464	0.7455
5.30	0.7494	0.7486
5.35	0.7525	0.7517
5.40	0.7555	0.7547
5.45	0.7585	0.7577
5.50	0.7614	0.7606
5.55	0.7644	0.7635
5.60	0.7673	0.7664
5.65	0.7701	0.7693
5.70	0.7730	0.7722
5.75	0.7758	0.7750
5.80	0.7786	0.7778
5.85	0.7814	0.7806
5.90	0.7841	0.7833
5.95	0.7869	0.7860
6.00	0.7896	0.7887

$-x$	Ψ	
	approximated	exact
0.	0.5000	0.5000
0.05	0.4839	0.4841
0.10	0.4678	0.4681
0.15	0.4517	0.4521
0.20	0.4356	0.4359
0.25	0.4195	0.4196
0.30	0.4034	0.4030
0.35	0.3873	0.3862
0.40	0.3695	0.3690
0.45	0.3505	0.3514
0.50	0.3316	0.3333
0.55	0.3126	0.3146
0.60	0.2936	0.2952
0.65	0.2747	0.2748
0.70	0.2544	0.2532
0.75	0.2288	0.2301

$-x$	ψ	
	approximated	exact
0.80	0.2033	0.2048
0.85	0.1777	0.1766
0.90	0.1417	0.1436
0.95	0.0988	0.1011
1.00	0.0000	0.
1.05	0.0000	0.
1.10	0.0000	
1.15	0.0000	
1.20	0.0000	
1.25	0.0000	

Appendix 5: Listing of Computer Program

```
      IMPLICIT COMPLEX (C,Z)
      DIMENSION TEXT(54),AA(1)
      INTEGER NS3(5),NS4(5)
      COMMON /C01/ZPA(5,10),ZUA(5,5,4),ZOA(5),ZRA(5,100,1),DT,ROZ,FIZ,
     &FIA(6),GA(5,7),QPA(5,10),WRA(5,100,1),NG,POR(5),
     &NA(5,3,1),NGA(5,5,1),IFA(5,100,1),KRA(5,100,1),IRA(5,100,1),NFI
      COMMON /C02/BP(500),NV
      EQUIVALENCE (IND,AA(1))
      CANIF(CV,AN,AH)=CEXP((0.0,1.0)*AH)*CMPLX((REAL(CV)*SQRT(AN)),
     &AIMAG(CV))
C
C*    READ
C
      EPS=10.0**(-4)
      CI=(0.0,1.0)
      PI=3.1415926535
      READ(5,145)TEXT
      WRITE(6,146)TEXT
      WRITE(6,142)
      READ(5,101)NG,NFA,NFI,DT,ROZ,FIZ
      NS1=NG
      NG=IABS(NG)
      NS2=NFA
      NFA=IABS(NFA)
      IF(NFI.NE.0)READ(5,102)(FIA(I),I=1,NFI)
      DO 105 IG=1,NG
      READ(5,106)(NA(IG,J,1),J=1,3),(NGA(IG,J,1),J=1,NG)
      NS3(IG)=NA(IG,1,1)
      NA(IG,1,1)=IABS(NA(IG,1,1))
      READ(5,107)(GA(IG,J),J=1,4),POR(IG),ZOA(IG)
      NS4(IG)=1
      IF(GA(IG,1).LT.0.)NS4(IG)=-1
      GA(IG,1)=ABS(GA(IG,1))
      NP=NA(IG,2,1)
      IF(NP.GT.0)READ(5,108)(ZPA(IG,J),QPA(IG,J),J=1,NP)
      NU=NA(IG,3,1)
      IF(NU.EQ.0)GOTO 105
      READ(5,104)((ZUA(IG,J,I),I=1,4),J=1,NU)
  105 CONTINUE
      NFAM=NFA-1
      DO 109 IG=1,NG
      NR=NA(IG,1,1)
C

C     READ BOUNDARY AND INSERT EXTRA BOUNDARY SEGMENTS
C
      DO 113 IR=1,NR
      JT=0
      ISUBS=NFA*(IR-1)+1
      READ(5,110)ZRA(IG,ISUBS,1),KRA(IG,ISUBS,1),IFA(IG,ISUBS,1),
     &WRA(IG,ISUBS,1),IRA(IG,ISUBS,1)
      IF(IR.EQ.1)GOTO 113
      IF(IR.GT.1)ISUBM=ISUBS-NFA
      IF(NFA.EQ.1)GOTO 113
  140 CDZ=(ZRA(IG,ISUBS,1)-ZRA(IG,ISUBM,1))/NFA
      DO 119 J=1,NFAM
      ISUBMJ=ISUBM+J
      ZRA(IG,ISUBMJ,1)=ZRA(IG,ISUBM,1)+J*CDZ
      KRA(IG,ISUBMJ,1)=KRA(IG,ISUBM,1)
```

```
      IFA(IG,ISUBMJ,1)=IFA(IG,ISUBM,1)
      WRA(IG,ISUBMJ,1)=WRA(IG,ISUBM,1)
      IRA(IG,ISUBMJ,1)=IRA(IG,ISUBM,1)
  119 CONTINUE
      IF(.NOT.(IR.EQ.NR.AND.JT.EQ.0))GOTO 113
      ISUBM=NFA*(NR-1)+1
      ISUBS=1
      JT=1
      GOTO 140
  113 CONTINUE
      NA(IG,1,1)=NFA*NR
      IF(NFA.EQ.1)GOTO 109
      IGA=IG+100
      IGA=MOD(IGA,2)
      DO 141 IGB=1,NG
      IH=NGA(IG,IGB,1)
      IF(IH.EQ.0)GOTO 141
      IHC=NFA*(IH-1)+1
      IF(IGA.NE.0)IHC=IHC+NFA-1
      NGA(IG,IGB,1)=IHC
 141  CONTINUE
 109  CONTINUE
C


C     WRITE RESULT OF INPUT DATA (INCL. GENERATED EXTRA SEGMENTS)
C
      IF(NS1.GT.0)GOTO 138
      WRITE(6,101)NG,NFA,NFI,DT,ROZ,FIZ
      IF(NFI.NE.0)WRITE(6,102)(FIA(I),I=1,NFI)
      DO 138 IG=1,NG
      WRITE(6,106)(NA(IG,J,1),J=1,3),(NGA(IG,J,1),J=1,NG)
      WRITE(6,107)(GA(IG,J),J=1,4),POR(IG),ZOA(IG)
      NP=NA(IG,2,1)
      IF(NP.GT.0)WRITE(6,108)(ZPA(IG,J),QPA(IG,J),J=1,NP)
      NU=NA(IG,3,1)
      IF(NU.EQ.0)GOTO 138
      WRITE(6,104)((ZUA(IG,J,I),I=1,4),J=1,NU)
 138  CONTINUE
      NV=0
      DO 139 IG=1,NG
      GA(IG,4)=GA(IG,4)*PI/180.0
      NR=NA(IG,1,1)
      NV=NV+2*NR+2
      IF(NS1.LT.0)WRITE(6,110)(ZRA(IG,IR,1),KRA(IG,IR,1),
     &IFA(IG,IR,1),WRA(IG,IR,1),IRA(IG,IR,1),IR=1,NR)
  139 CONTINUE
      WRITE(6,142)
C*
      NV2=NV*NV
      CALL REQDA(AA(1),NV2,IER)
      IF(IER.NE.0)WRITE(6,901)IER
      IF(IER.NE.0)STOP
      CALL OPST(AA(IND))
C
      CALL SIMQ(AA(IND),BB,NV,KS)
C
C


C     WRITE REGION AND BOUNDARY DATA AND SOLUTION OF EQUATIONS
C
      IF(NS2.GT.0)GOTO 121
      DO 121 IG=1,NG
      NT=0
      NR=NA(IG,1,1)
      NP=NA(IG,2,1)
      AH=GA(IG,4)*180.0/PI
      IM1=1
      IF(IG.EQ.1)GOTO 127
```

```
      J=IG-1
      DO 128 I=1,J
      IM1=IM1+2*NA(I,1,1)+2
  128 CONTINUE
  127 WRITE(6,120)IG,GA(IG,1),GA(IG,2),GA(IG,3),AH,NP
      NT=NT+8
      IF(NP.EQ.0)GOTO 122
      WRITE(6,123)(ZPA(IG,J),QPA(IG,J),J=1,NP)
      WRITE(6,131)
      NT=NT+6
  122 WRITE(6,125)
      NT=NT+3
      DO 124 IR=1,NR
      IM=IM1+IR-1
      IFO=IFA(IG,IR,1)
      IF(IFO.EQ.0)FI=0.0
      IF(IFO.NE.0)FI=FIA(IFO)
      ISUBS=IM+NR
      WRITE(6,126)ZPA(IG,IR,1),KRA(IG,IR,1),FI,WRA(IG,IR,1),BB(IM),BB(
     $ISUBS)
      NT=NT+1
      IF(NT.EQ.50)WRITE(6,130)IG
      IF(NT.EQ.50)NT=0
  124 CONTINUE
      ISUBS=IM1+2*NR
      WRITE(6,129)BB(ISUBS),BB(ISUBS+1)
  121 CONTINUE
C
C     BOUNDARY OUTPUT
C
      DO 111 IG=1,NG
      NT=0
      NR=NA(IG,1,1)
      AN=GA(IG,3)
      AK=GA(IG,2)/SQRT(AN)
      AH=GA(IG,4)
      JA=0
      IF(NS4(IG).LT.0)GOTO 132
      DO 132 IR=1,NR
      IF(NS3(IG).LT.0)GOTO 144
      KO=KRA(IG,IR,1)
      IF(.NOT.(KO.EQ.22.OR.KO.EQ.23))GOTO 132
  144 IF(JA.EQ.0)WRITE(6,133)IG
      JA=1
      ZO=ZRA(IG,IR,1)
      IF(IR.LT.NR)ZP=ZRA(IG,IR+1,1)
      IF(IR.EQ.NR)ZP=ZRA(IG,1,1)
      Z=0.5*(ZO+ZP)
      CO=COF(IG,Z,0)
      CO=CMPLX((REAL(CO)/AK),AIMAG(CO))
      CV=COF(IG,Z,1)
      CV=CANIF(CV,AN,AH)
      CV=-CONJG(CV)
      ZN=Z+CV*DT/POR(IG)
      WRITE(6,134)Z,CO,CV,ZN
      NT=NT+1
      IF(NT.GE.25)WRITE(6,133)IG
      IF(NT.GE.25)NT=0
  132 CONTINUE
C
C     GRID OUTPUT
C
      NU=NA(IG,3,1)
      IF(NU.EQ.0)GOTO 111
      DO 116 J=1,NU
      NT=0
      WRITE(6,112)IG
```

```
      ZU1=ZUA(IG,J,1)
      ZU2=ZUA(IG,J,2)
      ZU3=ZUA(IG,J,3)
      ZU4=ZUA(IG,J,4)
      IF(CABS(ZU2-ZU1).GT.EPS)CR=(ZU2-ZU1)/CABS(ZU2-ZU1)
      IF(CABS(ZU2-ZU1).LT.EPS)CR=(ZU3-ZU2)/CABS(ZU3-ZU2)
      RECR=REAL(CR)
      AICR=AIMAG(CR)
      DX=REAL(ZU4)
      DY=AIMAG(ZU4)
      IF(ABS(RECR).LT.EPS)CST=CI*DY
      IF(ABS(RECR).GE.EPS)CST=CMPLX(DX,(DX*AICR/RECR))
      IF(ABS(DX).LT.EPS)NX=1
      IF(ABS(DY).LT.EPS)NY=1
      IF(ABS(DX).GT.EPS.AND.CABS(ZU2-ZU1).GE.EPS)
     &NX=((REAL(ZU2-ZU1)+EPS)/DX)+1
      IF(ABS(DX).GT.EPS.AND.CABS(ZU2-ZU1).LT.EPS)
     &NX=((REAL(ZU3-ZU2)+EPS)/DX)+1
      IF(ABS(DY).GT.EPS.AND.CABS(ZU3-ZU2).GT.EPS)
     &NY=((AIMAG(ZU3-ZU2)+EPS)/DY)+1
      IF(ABS(DY).GT.EPS.AND.CABS(ZU3-ZU2).LT.EPS)
     &NY=((AIMAG(ZU2-ZU1)+EPS)/DY)+1
      ZU=ZU1-CST-CI*DY
      DO 117 IY=1,NY
      ZU=ZU+CI*DY
      DO 118 IX=1,NX
      ZU=ZU+CST
      H=0.0
      DO 135 IR=1,NR
      ZO=ZRA(IG,IR,1)
      IF(IR.NE.NR)ZP=ZRA(IG,IR+1,1)
      IF(IR.EQ.NR)ZP=ZRA(IG,1,1)
      JA=0
      IF(CABS(ZU-ZO).LE.EPS)JA=1
      IF(JA.EQ.1)ZU=ZU+(ZP-ZO)/CABS(ZP-ZO)*2.0*EPS
      IF(CABS(ZU-ZP).LE.EPS)JA=2
      IF(JA.EQ.2)ZU=ZU+(ZO-ZP)/CABS(ZO-ZP)*2.0*EPS
      IF(ABS(AIMAG(CABS(ZP-ZO)/(ZP-ZO)*(ZU-ZO))).LT.EPS
     &.AND.CABS(ZU-ZP).LT.CABS(ZP-ZO)
     &.AND.CABS(ZU-ZO).LT.CABS(ZP-ZO))GOTO 143
      H=H+AIMAG(CLOG(CABS(ZO-ZU)/(ZO-ZU)*(ZP-ZU)))
  135 CONTINUE
      IF(ABS(H-2.0*PI).GT.0.01)GOTO 118
  143 CO=COF(IG,ZU,0)
      CO=CMPLX((REAL(CO)/AK),AIMAG(CO))
      CV=COF(IG,ZU,1)
      CV=CANIF(CV,AN,AH)
      CV=-CONJG(CV)
      NT=NT+1
      WRITE(6,114)ZU,CO,CV
      IF(JA.EQ.1)ZU=ZU-(ZP-ZO)/CABS(ZP-ZO)*2.0*EPS
      IF(JA.EQ.2)ZU=ZU-(ZO-ZP)/CABS(ZO-ZP)*2.0*EPS
      IF(NT.EQ.25)WRITE(6,112)IG
      IF(NT.EQ.25)NT=0
  118 CONTINUE
      ZU=ZU-NX*CST
  117 CONTINUE
  116 CONTINUE
  111 CONTINUE
C**
      STOP

  101 FORMAT(I5,2I9,F13.3,2F9.3)
  102 FORMAT(8F9.3)
  104 FORMAT(8F9.3)
  106 FORMAT(I5,7I9)
  107 FORMAT(5F9.3/2F9.3)
  108 FORMAT(3F9.3)
  110 FORMAT(2F9.3,I5,I9,F13.3,I5)
  112 FORMAT('1',5X,'REGION',I4///6X,'X',8X,'Y',11X,'PHI',6X,'PSI',9X,
     &'VX',7X,'VY'//)
```

```
  114 FORMAT(' ',2F9.3,3X,2F9.3,3X,2F9.3/)
  120 FORMAT('1',5X,'REGION',I4,7X,'RO =',F8.3/23X,'AK =',F8.3/23X,'AN =
     &',F8.3/23X,'AH =',F8.3/23X,'NP =',I4////)
  123 FORMAT(5X,'XP',7X,'YP',7X,'QP'//(3F9.3))
  131 FORMAT(///)
  125 FORMAT(5X,'X',8X,'Y',7X,'KO',8X,'FI',7X,'WR',10X,'RCQ',6X,'ICQ'//)
  126 FORMAT(2F9.3,3X,I3,3X,2F9.3,3X,2F9.3)
  130 FORMAT('1',5X,'REGION',I4,4X,'CONTINUED'//5X,'X',8X,'Y',7X,'KO',8X
     &,'FI',7X,'WR',9X,'RCQ',6X,'ICQ'//)
  129 FORMAT(/21X,'INITIAL COMPLEX POTENTIAL  ',2F9.3)
  133 FORMAT('1','REGION',I4///6X,'X',8X,'Y',11X,'PHI',6X,'PSI',12X,
     &'VX',7X,'VY',13X,'XN',7X,'YN'//)
  134 FORMAT(' ',2F9.3,3X,2F9.3,3X,3X,2F9.3,3X,3X,2F9.3/)
  142 FORMAT('1')
  145 FORMAT(18A4)
  146 FORMAT('1'///2(5X,74(1H*)/),3(5X,'*',72X,'*'/),
     &5X,'*',5X,'MOTGRO   :  MODEL  FOR  TWO-DIMENSIONAL  GROUNDWATER',
     &'  FLOW',9X,'*'/
     &5X,'*',17X,'BASED ON  ANALYTICAL  FUNCTION  METHOD',17X,'*'/
     &3(5X,'*',72X,'*'/),2(5X,74(1H*)/),2(5X,'*',72X,'*'/),
     &3(5X,'*',18A4,'*'/),2(5X,'*',72X,'*'/),5X,74(1H*))
  901 FORMAT(' IER=',I3,7X,'FOUT BIJ HET DYNAMISCH DECLAREREN'
     &/15X,'ZIE DOCUMENTATIE VAN REQDA'//)
      END
C
C
C


      SUBROUTINE OPST(AA)
      IMPLICIT COMPLEX (C,Z)
      LOGICAL LJ0
      DIMENSION AA(1)
      COMMON /C01/ZPA(5,10),ZUA(5,5,4),ZOA(5),ZRA(5,100,1),DT,ROZ,FIZ,
     &FIA(6),GA(5,7),QPA(5,10),WRA(5,100,1),NG,POR(5),
     &NA(5,3,1),NGA(5,5,1),IFA(5,100,1),KRA(5,100,1),IRA(5,100,1),NFI
      COMMON /C02/BB(500),NV
      CANF(Z,AN,AH)=CMPLX(REAL(Z*CEXP(-(0.0,1.0)*AH)),(SQRT(AN)*AIMAG(
     &Z*CEXP(-(0.0,1.0)*AH))))
      CANIF(CV,AN,AH)=CEXP((0.0,1.0)*AH)*CMPLX((REAL(CV)*SQRT(AN)),
     &AIMAG(CV))
      CI=(0.0,1.0)
      DO 101 I=1,NV
      BB(I)=0.0
      DO 102 J=1,NV
      ISUBS=(J-1)*NV+I
      AA(ISUBS)=0.0
  102 CONTINUE
  101 CONTINUE
C
      DO 103 IG=1,NG
C
C*    REGION
C
      J0=0
      IGB=IG
      NR=NA(IG,1,1)
      NRB=NR
      NP=NA(IG,2,1)
      RO=GA(IG,1)
      AK=GA(IG,2)/SQRT(GA(IG,3))
      AN=GA(IG,3)
      AH=GA(IG,4)
      IM1=1
      IF(IG.EQ.1)GOTO 118
      J=IG-1
      DO 104 I=1,J
      IM1=IM1+2*NA(I,1,1)+2
  104 CONTINUE
C*
```

```
      118 DO 105 IR=1,NR
C
C*        BOUNDARY-SECTION
C
          IM=IM1+2*(IR-1)
          IGC=IRA(IG,IR,1)
          IF(IGC.EQ.0)GOTO 106
          ROC=GA(IGC,1)
          AKC=GA(IGC,2)/SQRT(GA(IGC,3))
          ANC=GA(IGC,3)
          AHC=GA(IGC,4)
          NPC=NA(IGC,2,1)
      106 WR=WRA(IG,IR,1)
          IFO=IFA(IG,IR,1)
          IF(IFO.NE.0)FIO=FIA(IFO)
          KO=KRA(IG,IR,1)
          ZO=ZRA(IG,IR,1)
          IF(IR.LT.NR)ZP=ZRA(IG,IR+1,1)
          IF(IR.EQ.NR)ZP=ZRA(IG,1,1)
          Z=0.5*(ZO+ZP)
          CDZ=ZP-ZO
          DX=REAL(CDZ)
          DY=AIMAG(CDZ)
          DL=CABS(CDZ)
          DP=2.0*DX*DY
          DV=DX**2-DY**2
C*

C**       EQUATIONS
C
          J=1
C
C*        RIGHT-HAND AND OMEGA-ZERO
C
          LJ0=(KO.NE.21.AND.J0.EQ.0)
          IF(LJ0)ZF0=ZO+0.75*(ZP-ZO)
      107 FI1=0.0
          FI2=0.0
          IF(KO.EQ.11)FI1=RO*FIO
          IF(KO.EQ.12.OR.KO.EQ.22.OR.KO.EQ.23.AND.J*(IGB-IGC).GT.0)FI1=J*RO*
         &AIMAG(Z)
          IF(KO.EQ.23.AND.IGC.EQ.0)FI1=FI1+ROZ*(FIZ-AIMAG(Z))
          IF(KO.EQ.12.OR.KO.EQ.22.OR.KO.EQ.23.AND.J*(IGB-IGC).GT.0)FI2=J*RO*
         &AIMAG(ZP-ZO)/CABS(ZP-ZO)
          IF(KO.EQ.23.AND.IGC.EQ.0)FI2=FI2-ROZ*AIMAG(ZP-ZO)/CABS(ZP-ZO)
          AL1=0.0
          AL2=0.0
          AL3=0.0
          AL4=0.0
          IF(.NOT.(KO.EQ.21.OR.J*(IGB-IGC).LT.0))AL1=J*RO/AK
          IF(KO.EQ.11.OR.KO.EQ.12.OR.KO.EQ.31.AND.J*(IGB-IGC).GT.0)AL2=((J+1
         &)/2)*RO*WR
          IF(KO.EQ.21)AL2=1.0
          IF(J*(IGB-IGC).LT.0)AL2=J*J
          AL3=J*AL1
          AL4=AL2
          IF(J*(IGB-IGC).LT.0)AL4=J
          CP0=CPF(IGB,Z,0)
          CP1=CPF(IGB,Z,1)
          CP2=CPF(IGB,Z,2)
          CP1=CANIF(CP1,AN,AH)
          CP2=CANIF(CP2,AN,AH)
          IF(.NOT.LJ0)GOTO 121
          CP00=CPF(IGB,ZF0,0)
          CP10=CPF(IGB,ZF0,1)
          CP10=CANIF(CP10,AN,AH)
      121 BB(IM)=BB(IM)+FI1-AL1*REAL(CP0)-AL2/DL*(DY*REAL(CP1)+DX*AIMAG(CP1)
         &)
          BB(IM+1)=BB(IM+1)+FI2-AL3/DL*(DX*REAL(CP1)-
         &DY*AIMAG(CP1))-AL4/(DL**2)*(DP*REAL(CP2)+DV*AIMAG(CP2))
```

```
      IF(LJ0.AND.J.EQ.1)IM0=IM1+2*NRB
      IF(LJ0)BB(IM0)=BB(IM0)+FI1-AL1*REAL(CP00)-AL2/DL*(DY*REAL(CP10)+
     &DX*AIMAG(CP10))
      IF(LJ0)ISUBS=(IM1+2*NRB-1)*NV+IM0
      IF(LJ0)AA(ISUBS)=AL1
      ISUBS=(IM1+2*NRB-1)*NV+IM
      AA(ISUBS)=AL1
      IF(.NOT.(J.EQ.1.AND.IR.EQ.NR))GOTO 110
      Z0=Z0A(IG)
      BB(IM+3)=-AIMAG(CPF(IG,Z0,0))
      ISUBS=(IM1+2*NR)*NV+IM+3
      AA(ISUBS)=1.0
  110 CONTINUE
C*


      DO 112 IRB=1,NRB
      ZOV=ZRA(IGB,IRB,1)
      IF(IRB.LT.NRB)ZPV=ZRA(IGB,IRB+1,1)
      IF(IRB.EQ.NRB)ZPV=ZRA(IGB,1,1)
C
C*    LEFT-HAND
C
      CA0=CAF(Z,ZOV,ZPV,AN,AH,0)
      CA1=CAF(Z,ZOV,ZPV,AN,AH,1)
      CA2=CAF(Z,ZOV,ZPV,AN,AH,2)
      CA1R=CANIF(CA1,AN,AH)
      CA1S=CANIF((CI*CA1),AN,AH)
      CA2R=CANIF(CA2,AN,AH)
      CA2S=CANIF((CI*CA2),AN,AH)
      IF(.NOT.LJ0)GOTO 122
      CA00=CAF(ZF0,ZOV,ZPV,AN,AH,0)
      CA10=CAF(ZF0,ZOV,ZPV,AN,AH,1)
      CA10R=CANIF(CA10,AN,AH)
      CA10S=CANIF((CI*CA10),AN,AH)
  122 ISUBS=(IM1+IRB-2)*NV+IM
      AA(ISUBS)=AL1*REAL(CA0)+AL2/DL*(DY*REAL(CA1R)+
     &DX*AIMAG(CA1R))
      ISUBS=(IM1+IRB-2+NRB)*NV+IM
      AA(ISUBS)=AL1*REAL(CI*CA0)+AL2/DL*(DY*REAL(CA1S)+
     &DX*AIMAG(CA1S))
      ISUBS=(IM1+IRB-2)*NV+IM+1
      AA(ISUBS)=AL3/DL*(DX*REAL(CA1R)-DY*AIMAG(CA1R))+AL4/
     &(DL**2)*(DP*REAL(CA2R)+DV*AIMAG(CA2R))
      ISUBS=(IM1+IRB-2+NRB)*NV+IM+1
      AA(ISUBS)=AL3/DL*(DX*REAL(CA1S)-DY*AIMAG(CA1S))+
     &AL4/(DL**2)*(DP*REAL(CA2S)+DV*AIMAG(CA2S))
      IF(.NOT.LJ0)GOTO 123
      ISUBS=(IM1+IRB-2)*NV+IM0
      AA(ISUBS)=AL1*REAL(CA00)+AL2/DL*(DY*REAL(CA10R)+
     &DX*AIMAG(CA10R))
      ISUBS=(IM1+IRB-2+NRB)*NV+IM0
      AA(ISUBS)=AL1*REAL(CI*CA00)+AL2/DL*(DY*REAL(CA10S)+
     &DX*AIMAG(CA10S))
  123 IF(.NOT.(J.EQ.1.AND.IR.EQ.NR))GOTO 113
      CA0=CAF(Z0,ZOV,ZPV,AN,AH,0)
      ISUBS=(IM1+IRB-2)*NV+IM+3
      AA(ISUBS)=AIMAG(CA0)
      ISUBS=(IM1+IRB-2+NR)*NV+IM+3
      AA(ISUBS)=AIMAG(CI*CA0)
  113 CONTINUE
C*


  112 CONTINUE
      IF(J.EQ.1)GOTO 114
      J=1
      GOTO 115
  114 IF(.NOT.(KO.EQ.31.OR.KO.EQ.23.AND.IGC.NE.0))GOTO 116
      J=-1
  115 IH=IGB
```

```
      IGB=IGC
      IGC=IH
      RH=RO
      RO=ROC
      ROC=RH
      RH=AK
      AK=AKC
      AKC=RH
      RH=AN
      AN=ANC
      ANC=RH
      RH=AH
      AH=AHC
      AHC=RH
      IH=NP
      NP=NPC
      NPC=IH
      NRB=NA(IGB,1,1)
      CDZ=-(ZP-ZO)
      DX=REAL(CDZ)
      DY=AIMAG(CDZ)
      DL=CABS(CDZ)
      DP=2.0*DX*DY
      DV=DX**2-DY**2
      IM1=1
      IF(IGB.EQ.1)GOTO 120
      JH=IGB-1
      DO 117 I=1,JH
      IM1=IM1+2*NA(I,1,1)+2
  117 CONTINUE
  120 IF(J.NE.1)GOTO 107
  116 IF(LJ0)J0=1
C**
  105 CONTINUE
  103 CONTINUE
      RETURN
      END
C
C
C


      COMPLEX FUNCTION CAF(Z,ZOV,ZPV,AN,AH,J)
      IMPLICIT COMPLEX (C,Z)
      CANF(Z,AN,AH)=CMPLX(REAL(Z*CEXP(-(0.0,1.0)*AH)),(SQRT(AN)*AIMAG(
     &Z*CEXP(-(0.0,1.0)*AH))))
      EPS=10.0**(-4)
      PI2=2.0*3.1415926535
      CI=(0.0,1.0)
      N1=0
      N2=0
      ZOW=CANF(ZOV,AN,AH)
      ZPW=CANF(ZPV,AN,AH)
      ZW=CANF(Z,AN,AH)
      IF(J.NE.0.AND.(CABS(ZW-ZOW).LE.EPS.OR.CABS(ZW-ZPW).LE.EPS))ZW=ZW+
     &EPS
      C=CABS(ZPW-ZOW)/(ZPW-ZOW)
      CZ1=C*(ZW-ZOW)
      CZ2=C*(ZW-ZPW)
      D=CABS(ZPW-ZOW)*0.5
      IF(AIMAG(CZ1).LT.EPS.AND.(ABS(REAL(CZ1))-D).LT.EPS)CZ1=CZ1+CI*EPS
      IF(AIMAG(CZ2).LT.EPS.AND.(ABS(REAL(CZ2))-D).LT.EPS)CZ2=CZ2+CI*EPS
      IF(AIMAG(CZ1).LT.0.0.AND.REAL(CZ1).LT.D)N1=1
      IF(AIMAG(CZ2).LT.0.0.AND.REAL(CZ2).LT.(-D))N2=1
      IF(J.NE.0)GOTO 101
      IF(CABS(ZW-ZOW).LE.EPS)CAF=-CZ2*CLOG(CZ2)-N2*PI2*CI*CZ2
      IF(CABS(ZW-ZPW).LE.EPS)CAF=CZ1*CLOG(CZ1)+N1*PI2*CI*CZ1
      IF(CABS(ZW-ZOW).GT.EPS.AND.CABS(ZW-ZPW).GT.EPS)CAF=-CZ2*CLOG(CZ2)+
     &CZ1*CLOG(CZ1)+PI2*CI*(N1*CZ1-N2*CZ2)
      GOTO 102
  101 IF(J.EQ.1)CAF=-C*(CLOG(CZ2)-CLOG(CZ1))+PI2*CI*C*(N1-N2)
```

```
      IF(J.EQ.2)CAF=-C*(1.0/(ZW-ZPW)-1.0/(ZW-ZOW))
  102 CAF=CAF/PI2
      RETURN
      END
C
C
C
```

```
      COMPLEX FUNCTION CPF(IG,Z,J)
      IMPLICIT COMPLEX (C,Z)
      COMMON /C01/ZPA(5,10),ZUA(5,5,4),ZOA(5),ZRA(5,100,1),DT,ROZ,FIZ,
     &FIA(6),GA(5,7),QPA(5,10),WRA(5,100,1),NG,POR(5),
     &NA(5,3,1),NGA(5,5,1),IFA(5,100,1),KRA(5,100,1),IRA(5,100,1),NFI
      CANF(Z,AN,AH)=CMPLX(REAL(Z*CEXP(-(0.0,1.0)*AH)),(SQRT(AN)*AIMAG(
     &Z*CEXP(-(0.0,1.0)*AH))))
      PI2=2.0*3.1415926535
      EPS=10.0**(-4)
      CPF=(0.0,0.0)
      NP=NA(IG,2,1)
      IF(NP.EQ.0)GOTO 101
      AN=GA(IG,3)
      AH=GA(IG,4)
      ZW=CANF(Z,AN,AH)
      DO 102 I=1,NP
      QP=QPA(IG,I)
      ZP=ZPA(IG,I)
      ZP=CANF(ZP,AN,AH)
      IF(CABS(ZW-ZP).LE.EPS)ZW=ZW+EPS
      IF(J.EQ.0)CPF=CPF+QP/PI2*CLOG(ZW-ZP)
      IF(J.EQ.1)CPF=CPF+QP/(PI2*(ZW-ZP))
      IF(J.EQ.2)CPF=CPF-QP/(PI2*(ZW-ZP)**2)
  102 CONTINUE
  101 RETURN
      END
C
C
C
```

```
      COMPLEX FUNCTION COF(IG,Z,J1)
      IMPLICIT COMPLEX (C,Z)
      COMMON /C01/ZPA(5,10),ZUA(5,5,4),ZOA(5),ZRA(5,100,1),DT,ROZ,FIZ,
     &FIA(6),GA(5,7),QPA(5,10),WRA(5,100,1),NG,POR(5),
     &NA(5,3,1),NGA(5,5,1),IFA(5,100,1),KRA(5,100,1),IRA(5,100,1),NFI
      COMMON /C02/BB(500),NV
      CI=(0.0,1.0)
      AN=GA(IG,3)
      AH=GA(IG,4)
      NR=NA(IG,1,1)
      IM1=1
      IF(IG.EQ.1)GOTO 103
      J=IG-1
      DO 101 I=1,J
      IM1=IM1+2*NA(I,1,1)+2
  101 CONTINUE
  103 COF=(0.0,0.0)
      DO 102 IR=1,NR
      ZOV=ZRA(IG,IR,1)
      IF(IR.LT.NR)ZPV=ZRA(IG,IR+1,1)
      IF(IR.EQ.NR)ZPV=ZRA(IG,1,1)
      ISUBS1=IM1+IR-1
      ISUBS2=IM1+IR+NR-1
      CQ=CMPLX(BB(ISUBS1),BB(ISUBS2))
      COF=COF+CQ*CAF(Z,ZOV,ZPV,AN,AH,J1)
  102 CONTINUE
      ISUBS=IM1+2*NR
      COF=COF+(1-J1)*CMPLX(BB(ISUBS),BB(ISUBS+1))+CPF(IG,Z,J1)
      RETURN
      END
```

Principal Notations

$\Omega = \Phi + i\Psi$	complex potential (L^2t^{-1})
$\Phi = k\phi$	potential (L^2t^{-1})
Ψ	stream function (L^2t^{-1})
ϕ	groundwater head (L)
$z = x + iy$	complex variable (L)
$\bar{z} = x - iy$	complex variable (L)
x, y	cartesian coordinates (L)
v_x, v_y	components of specific discharge (Lt^{-1})
Δt	step in time (t)
k	coefficient of permeability (Lt^{-1})
μ	storage coefficient (dimensionless)
η	effective porosity (dimensionless)
ρ	density (ML^{-2})
p	pressure ($ML^{-1}t^{-2}$)
Q	discharge of source or sink (L^2t^{-1})
$q_j = r_j + is_j$	complex variable (Lt^{-1})
r_j, s_j	distribution strength of sources (or sinks), resp. vortices (Lt^{-1})
l	boundary variable (L)
$\frac{\partial z}{\partial l}, \frac{\partial^2 z}{\partial l^2}$	complex variable for boundary segment (dimensionless)
c_s	boundary resistance (t)
Δx, Δy	components of displacement of a point of moving boundary in a step in time (L)
N	precipitation (Lt^{-1})
c	subscript that refers to an adjoining sub-region
m	number of sources and sinks
n	number of boundary segments
γ_j	complex constant
$F_j(z)$	analytic function
$\mathrm{Re}\{\ \}$	real part
$\mathrm{Im}\{\ \}$	imaginary part

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Rijkswaterstaat, Delft University of Technology, Delft Hydraulics Laboratory, 1976

No. 27. *Asphalt Revetment of Dyke Slopes*
Committee on the Compaction of Asphalt Revetments of Dyke Slopes, 1977

* out of print

Stellingen

I

De door Verruyt in 1968 gepubliceerde exacte oplossing van het probleem van Badon Ghijben kan worden uitgebreid tot een oplossing waarin de aanwezigheid van nuttige neerslag op de freatische lijn is verwerkt.

Verruijt, A., 1968. A note on the Ghijben-Herzberg formula. (Bull. Int. Ass. Sci Hydrol., XIII: 4–12.)
Van der Veer, P., 1977. Analytical solution for steady interface flow in a coastal aquifer involving a phreatic surface with precipitation (J. Hydrol., 34: 1–11).

II

Bij een geschikte keuze van twee constanten in de op een ééndimensionale beschouwing gebaseerde formule voor de ligging van de grenslijn tussen zoet en zout grondwater in het probleem van Badon Ghijben, is deze formule gelijk aan die welke volgt uit de exacte oplossing van het tweedimensionale probleem.

Van Dam, J. C., 1976. Fresh Water – Salt Water Relationships (college geohydrologie TH-Delft, afd. Civiele Techniek).
Van der Veer, P., 1977. Analytical solution for steady interface flow in a coastal aquifer involving a phreatic surface with precipitation (J. Hydrol., 34: 1–11).

III

De sinds 1929 bekende, door Vreedenburgh gevonden, exacte oplossing voor de grondwaterstroming naar een drainage, waarvan in 1959 door Glover een gewijzigde vorm werd gepubliceerd die de stroming van zoet grondwater boven stilstaand zout grondwater beschrijft, kan worden uitgebreid tot een vorm die een gelijktijdige stroming van zoet en zout grondwater beschrijft.

De Vos, H. C. P., 1929. Enige beschouwingen omtrent de verweekingslijn in aarden dammen. (De Waterstaatsingenieur, 17, 335–354).
Glover, R. E., 1959. The pattern of fresh water flow in a coastal aquifer. (J. Geophys. Res., 69. no. 8, 457–459.)
Van der Veer, P., 1977. The pattern of fresh and salt groundwater flow in a coastal aquifer (Delft Progr. Rep., 2, 137–142).

IV

De oplossing voor een gelijktijdige stroming van zoet en zout grondwater kan niet met behulp van de hodograafmethode worden gevonden; de exacte oplossing voor het geval waarbij behalve het grensvlak ook een freatische lijn aanwezig is, dus het probleem van Badon Ghijben uitgebreid met de stroming van zout grondwater, kan echter toch worden gegeven.

Van der Veer, P., 1978. Analytical solution for a two-fluid flow in a coastal aquifer involving a phreatic surface with precipitation (J. Hydrol., 34: 271–278).

V

In een bijzonder geval van grondwaterstroming met een freatische lijn en een grenslijn tussen zoet en (al dan niet stromend) zout grondwater is zowel de freatische lijn als de grenslijn een rechte lijn; de grenslijn is ook een rechte als er in plaats van de freatische lijn een rechte kwellijn aanwezig is.

Van der Veer, P., 1977. Analytical solution for steady interface flow in a coastal aquifer involving a phreatic surface with precipitation (J. Hydrol., 34: 1–11).
Van der Veer, P., 1978. Analytical solution for a two-fluid flow in a coastal aquifer involving a phreatic surface with precipitation (J. Hydrol., 34: 271–278).

VI

Door de definiering van een geschikte variabele kunnen analytische oplossingen worden gevonden voor problemen van tweedimensionale grondwaterstroming in gebieden waarvan de rand bestaat uit een horizontale semi-doorlatende lijn en (eventueel) verticale stroom- en/of potentiaallijnen. In deze gebieden mogen putten en bronnen voorkomen.

Van der Veer, P., 1978. Exact solutions for two-dimensional groundwater flow problems involving a semi-pervious boundary (J. Hydrol., 37: 159–168).

VII

De in de vorige stelling bedoelde variabele is niet geschikt voor het berekenen van de grondwaterstroming tussen twee horizontale semi-doorlatende lagen, ook al zijn de weerstandseigenschappen van die lagen gelijk.

VIII

Naast het verschil tussen de hoogteligging van de freatische lijn tegen een gesloten taludbekleding en de buitenwaterstand tegen het talud kan het karakter van de grondwaterstroming, (stationair of niet-stationair), een grote invloed hebben op de overdrukken onder die taludbekleding.

Van der Veer, P., 1976. Overdrukken onder gesloten dijkbekledingen. Geohydrologische aspecten van de waterbouwkunde 1 (Pt-b, 31, nr. 9, 547–550).

IX

Het *zoeken naar* de analytische oplossing van een stromingsprobleem geeft veelal meer inzicht dan het *vinden van* een oplossing met behulp van een numeriek model.

X

Het verdient aanbeveling om de actieve muziekbeoefening op scholen sterk te bevorderen.

XI

De Nederlandse Spoorwegen verdienen een compliment van de minister van Wetenschappen omdat het sinds de invoering van zgn. werkcoupé's in treinen voor forenzen mogelijk is om wetenschappelijk onderzoek te doen tijdens treinreizen in het woon-werkverkeer.

XII

Het vervangen van auto's door trapauto's kan een gunstige invloed hebben op de verkeersveiligheid, het energieverbruik en de volksgezondheid.

Stellingen behorende bij het proefschrift **Calculation Methods for Two-dimensional Groundwater Flow** van P. van der Veer, Delft, 1978.