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On computing order quantities for perishable inventory control with non-stationary demand*

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Abstract We study the global optimal solution for a planning problem of inventory control of perishable products and non-stationary demand.

Keywords: Inventory control, Perishable products

1. Introduction

The basis of our study is a SP model published in [3] for a practical production planning problem over a finite horizon of T periods of a perishable product with a fixed shelf life of J periods. The demand is uncertain and non-stationary such that one produces to stock. To keep waste due to out-dating low, one issues the oldest product first, i.e. FIFO issuance. Literature provides many ways to deal with perishable products, order policies and backlogging, e.g. [5, 1]. The model we investigate aims to guarantee an upper bound for the expected demand that cannot be fulfilled for every period.

The solution for such a model is a so-called order policy. Given the inventory situation $\mathbf{I}$ at the beginning of period moment t, an order policy should advice the decision maker on the order quantity Q_t. For the decision maker, simple rules are preferred. We consider a policy with a list of order periods Y with order quantities Q_t.

2. Stochastic Programming Model

The stochastic demand implies that the model has random inventory variables $\boldsymbol{I}_{jt}$ apart from the initial fixed levels I_{j0}. In the notation, $P(.)$ denotes a probability to express the chance constraints and $E(.)$ is the expected value operator for the expected costs. Moreover, we use $x^+ = max\{x, 0\}$. A formal description of the SP model from [2] is given.

Indices

t period index, $t = 1, \ldots, T$, with T the time horizon

j age index, $j = 1, \ldots, J$, with J the fixed shelf life

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Data

- $\boldsymbol{d}_t$ Normally distributed demand with expectation $\mu_t > 0$ and variance $(cv \times \mu_t)^2$ where cv is a given coefficient of variation
- k fixed ordering cost, $k > 0$
- c unit procurement cost, $c > 0$
- h unit inventory cost, $h > 0$
- w unit disposal cost, is negative when having a salvage value, $w > -c$
- β service level, $0 < \beta < 1$

Variables

- $Q_t \geq 0$ ordered and delivered quantity at the beginning of period t
- $Y_t \in \{0, 1\}$ setup of order
- $\boldsymbol{I}_{jt}$ Inventory of age j at end of period t, initial inventory fixed $I_{j0} = 0$, $\boldsymbol{I}_{jt} \geq 0$ for $j = 1, \ldots, J$

The total expected costs over the finite horizon is to be minimized.

$$f(Q) = \sum_{t=1}^{T} \left(C(Q_t) + E \left(h \sum_{j=1}^{J-1} \boldsymbol{I}_{jt} + w \boldsymbol{I}_{Jt} \right) \right), \tag{1}$$

where procurement cost is given by the function

$$C(x) = k + cx, \text{ if } x > 0, \text{ and } C(0) = 0. \tag{2}$$

The FIFO dynamics of inventory of items of different age j starts by defining waste

$$\boldsymbol{I}_{Jt} = (\boldsymbol{I}_{J-1,t-1} - d_t)^+, \ t = 1, \ldots, T, \tag{3}$$

followed by the inventory of other ages that still can be used in the next period:

$$\boldsymbol{I}_{jt} = \left(\boldsymbol{I}_{j-1,t-1} - (\boldsymbol{d}_t - \sum_{i=j}^{J-1} \boldsymbol{I}_{i,t-1})^+ \right)^+, \ t = 1, \ldots, T, \ j = 2, \ldots, J-1. \tag{4}$$

and finally the incoming and freshest products, with $j = 1$:

$$\boldsymbol{I}_{1t} = \left(Q_t - (\boldsymbol{d}_t - \sum_{j=1}^{J-1} \boldsymbol{I}_{j,t-1})^+ \right)^+, \ t = 1, \ldots, T. \tag{5}$$

Lost sales for period t is defined by

$$\boldsymbol{X}_t = \left(\boldsymbol{d}_t - \sum_{j=1}^{J-1} \boldsymbol{I}_{j,t-1} - Q_t \right)^+. \tag{6}$$

The service level constraint for every period is

$$E(X_t) \leq (1 - \beta)\mu_t, \ t = 1, \ldots, T \tag{7}$$

Notice that the incoming products are the freshest product, $j = 1$. We consider a simple order policy, where the decision maker is provided a list of order periods Y_t and order quantities Q_t where $Y_t = 0$ implies $Q_t = 0$. This can be considered an MINLP problem to derive what are the optimal values of the (continuous) order quantities Q_t and the corresponding optimal (integer) order timing Y.

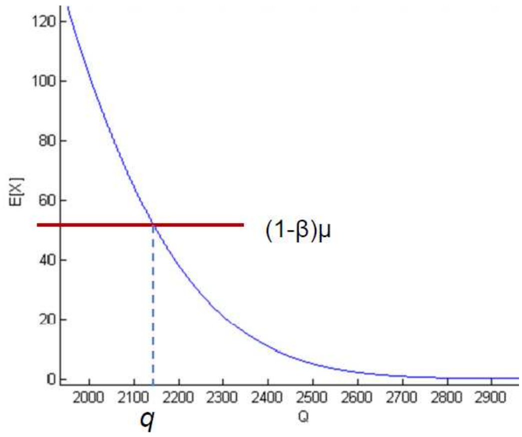

Figure 1: One period loss function for $\boldsymbol{d} \sim N(1950, 0.25 \cdot 1950)$ and corresponding basic order quantity q.

3. Replenishment cycles and basic order quantities

We study several theoretical properties of the order quantities Q and the list of order periods Y. We first focus on the concept of replenishment cycles and determine in which cases a so-called basic order quantity defines the optimal order quantity in Section 3.2.

3.1 Feasible replenishment cycles

Literature on inventory control e.g. [5] applies the concept of a replenishment cycle, i.e. the length of the period R for which the order of size Q is meant. For stationary demand, the replenishment cycle is fixed, but for non-stationary demand the optimal replenishment cycle R_t may depend on the period.

Definition 1. *Given list of order periods $Y \in \{0,1\}^T$ and $N = \sum_{t=1}^{T} Y_t$. The order timing vector $A(Y) \in \mathbb{N}^N$ gives the order moments $A_i < A_{i+1}$ such that $Y_{A_i} = 1$.*

Definition 2. *Given list of order periods $Y \in \{0,1\}^T$ and $N = \sum_{t=1}^{T} Y_t$ Replenishment cycle $R_i(Y) = A_{i+1} - A_i,\ i = 1, \ldots, N-1$ and $R_N = T - R_n + 1$.*

Notice that for the perishable case with a shelf life J, to fulfil the service level constraint, practically the replenishment cycle can not be larger than the shelf life J; so $R_i \leq J$.

Lemma 3. *Let Y be an order timing vector of the SP model, i.e. $Y_t = 0 \Rightarrow Q_t = 0$. Y provides an infeasible solution of the SP model, if it contains more than $J-1$ consecutive zeros.*

This means that a feasible order timing vector Y does not contain a consecutive series with more than $J-1$ zeros.

3.2 Basic order quantities

Consider a replenishment cycle of one period $R = 1$, zero inventory and the order quantity q that minimizes the cost function such that the service level constraint (7) is fulfilled. The expected lost sales $L(q)$ is

$$L(q) = E(\boldsymbol{d} - q)^+ = \int_q^\infty (x-q) f(x) dx \qquad (8)$$

where f is the density function of $\boldsymbol{d}$. L is known as the loss function.

The cost function is monotonously increasing in the order quantity Q, so in order to minimize it we need to find q such that $L(q) = (1-\beta)\mu$ as illustrated in Figure 1. Since demand is normally distributed, the solution has to be calculated numerically. Here there are several ways to proceed. One can use the derivative of loss function $L'(q) = \int_{-\infty}^{q} f(x)dx - 1 = F(q) - 1$ to approximate q using *Newton Raphson*. For the described model, the determination of q, only has to be done once.

Lemma 4. *Let $\boldsymbol{d} \sim N(\mu_1, cv \times \mu)$ and φ be the pdf and Φ the cdf of the standard normal distribution. The solution of $L_d(q) = (1-\beta)\mu$ fulfils $q = \mu(1+cv\times\hat{q})$ where $\hat{q}$ solves $\varphi(\hat{q}) - (1-\Phi(\hat{q}))\,\hat{q} = \frac{1-\beta}{cv}$.*

Proof. Using the results in [4] for $\boldsymbol{d} \sim N(\mu, cv \times \mu)$, the loss function can be expressed as

$$L_d(q) = cv \times \mu \left(\varphi\left(\frac{q-\mu}{cv \times \mu}\right) - \left(1 - \Phi\left(\frac{q-\mu}{cv \times \mu}\right)\right) \frac{q-\mu}{cv \times \mu} \right). \tag{9}$$

The equation $L(q) = (1-\beta)\mu$ substituting $q = \mu(1 + cv \times \hat{q})$ implies

$$\varphi\left(\frac{q-\mu}{cv \times \mu}\right) - \left(1 - \Phi\left(\frac{q-\mu}{cv \times \mu}\right)\right) \frac{q-\mu}{cv \times \mu} = \varphi(\hat{q}) - (1-\Phi(\hat{q}))\hat{q} = \frac{1-\beta}{cv}. \tag{10}$$

□

The basic order quantity $\overline{Q}_{1t} = \mu_t(1 + cv \times \hat{q})$ provides an upper bound on the order quantity Q_t if $R_t = 1$, because inventory may be available. The basic order quantities for longer replenishment cycles are far more complicated; $R_t = 2$ implies

$$E\left(\boldsymbol{d}_{t+1} - (\overline{Q}_{2t} - \boldsymbol{d}_t)^+)\right)^+ = (1-\beta)\mu_{t+1}$$

and $R_t = 3$ implies

$$E\left((\boldsymbol{d}_{t+2} - ((\boldsymbol{d}_{t+1} - (\overline{Q}_{3t} - \boldsymbol{d}_t)^+)^+)\right)^+ = (1-\beta)\mu_{t+2},$$

where we also have to take the constraint $\overline{Q}_{1t} \le \overline{Q}_{2t} \le \overline{Q}_{3t}$ into account. These basic order quantities can only be found by simulation.

4. Conclusions

An MINLP model has been presented to determine order quantities for a perishable product inventory control problem. So far, basic order quantities can be determined that provide a feasible policy of the model. The next question is how given this starting policy to find the optimal order quantities and order timing for the problem.

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