551.521.13

MEDEDELINGEN VAN DE LANDBOUWHOGESCHOOL TE WAGENINGEN, NEDERLAND 60 (15), 1-6 (1960)

ON THE SOLAR CONSTANT AND THE ENERGY DISTRIBUTION OF THE SOLAR RADIATION

by

P. J. BRUIJN

Laboratorium voor Natuur- en Weerkunde van de Landbouwhogeschool, Wageningen, Nederland.

(Received 29.10.'60)

SYNOPSIS

Theoretical expressions of the solar constant and of the energy distribution of solar radiation on the basis of blackbody radiation are derived. Theoretical and empirical energy distribution are compared.

INTRODUCTION

The solar constant Z is of vital importance for agricultural physics and for the study of both terrestrial and extra-terrestrial applications of solar radiation as an energy source.

The constant Z is defined as the energy flux density of the solar radiation, outside the atmosphere, in the vicinity of the earth at mean solar distance (cal. $cm^{-2}min^{-1}$ or erg. $cm^{-2}sec^{-1}$).

An attempt to a quantitative explanation of its magnitude and of the distribution of the energy in solar radiation involves consideration of two factors:

1. The composition of the (thermal) radiation emitted by the sun's "surface".
2. The spatial dimensions of the system *sun → receptor*.

1. SPECTRAL RADIANT EMITTANCE H^+ OF THE SUN

To find an approximation of the thermal radiation emitted by the sun we consider a model in which the sun has been replaced by a spherical blackbody with a diameter equal to the sun's optical diameter.

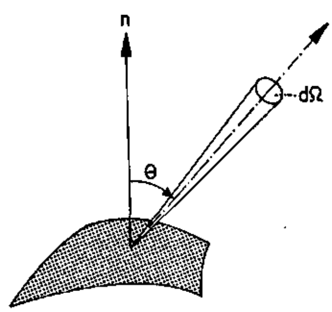

FIG. 1. Infinitesimal solid angle $d\Omega$ including angle θ with normal n at surface in radiating point.

Using Planck's radiation function*

$$J = (hc^2/\lambda^5)\ [\exp\,(hc/\lambda kT) - 1]^{-1} \quad (\text{erg.cm}^{-3}\text{sec}^{-1}) \tag{1}$$

we can write the radiant emittance in the wave-length interval between λ and $\lambda + d\lambda$ of a surface element dS at temperature T (°K) in an infinitesimal solid angle $d\Omega$ deviating by an angle θ from the normal direction (Fig. 1)

$$Hd\lambda = J \cos\theta\ d\Omega dS d\lambda \quad (\text{erg.sec}^{-1}) \tag{2}$$

for *linearly polarised* radiation and

$$H^{+}d\lambda = 2J \cos\theta d\Omega dS d\lambda \quad (\text{erg.sec}^{-1}) \tag{3}$$

for *non-polarised* radiation.

Groups of constants occurring in (1) are often combined into "radiation constants" c_1 and c_2, but care must be taken here as the meaning of c_1 may differ appreciably with different authors! Following Jahnke and Emde [1] one has

$$c_1 = hc^2 = 0.589 \times 10^{-5} \quad (\text{erg.cm}^2\text{sec}^{-1}) \tag{4}$$

and

$$c_2 = hc/k = 1.43 \quad (\text{cm.°K}) \tag{5}$$

Substitution in (1) yields

$$J = (c_1/\lambda^5)\ [\exp\,(c_2/\lambda T) - 1]^{-1} \quad (\text{erg.cm}^{-3}\text{sec}^{-1}) \tag{6}$$

These authors introduce auxiliary quantities x and y

$$x = T/c_2 = T/1.43 \tag{7}$$

$$y = J\,T^{-5}.10^6/0.980 \tag{8}$$

and present a table for y against x. ([1], addenda, page 46)**.

It should be stressed, however, that the function J of Jahnke and Emde concerns *linearly polarised* radiation, though this has not been clearly stated in the table. For non-polarised radiation a factor 2 must be added to the values for J obtained with this table!

2. Spatial dimensions of the system

To compute the solar irradiancy at normal incidence outside the atmosphere of the earth per square centimeter, the distance of the sun and the spatial extension of the radiating body must be taken into account (vide Fig. 2).

As the solar distance is about a hundred times the sun's (optical) diameter the following approximations are introduced.

I. The visible part of the sun covers a hemisphere.

II. The distance of *any* point of this hemisphere to the receptor equals the mean solar distance.

III. All the rays are parallel to the radius vector joining the receptor with the sun's centre.

Fig. 2 shows the hemisphere facing the receptor. The rays, emanating from a point with spherical polar coordinates Φ and θ and heating the receptor are with regard to simplification III considered to include collectively the same angle θ with the normal n at the radiator's surface in this point.

The well known expression for the surface element dS of a sphere with radius R, corresponding to simultaneous variations $d\Phi$ of the azimuth Φ and $d\theta$ of the polar angle (collatitude) θ, reads (vide Fig. 3)

* Notation of Jahnke-Emde, [1].

** It can be shown that the pairs (x, y) are solutions of the equation $[\exp\,(1/x) - 1]\,x^5y = 1$, hence, the table of y against x is universal, i.e. not explicitly depending on either λ or T, [1].

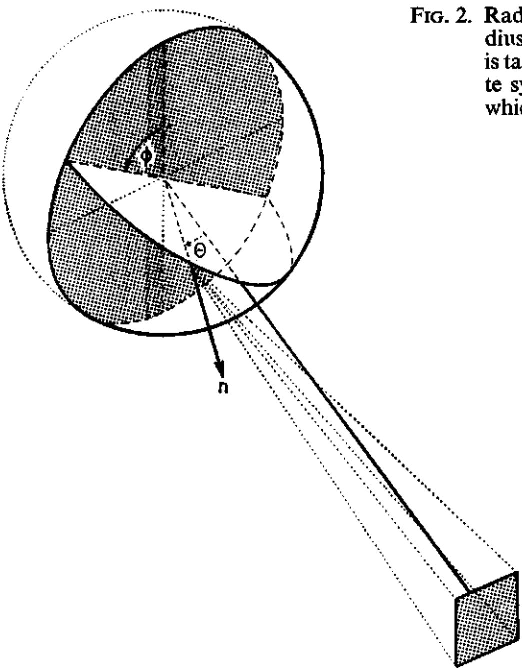

FIG. 2. Radiating hemisphere and receptor. Radiusvector from centre of sphere to receptor is taken as polar axis of spherical coordinate system. Receptor is parallel to plane in which azimuth Φ is measured.

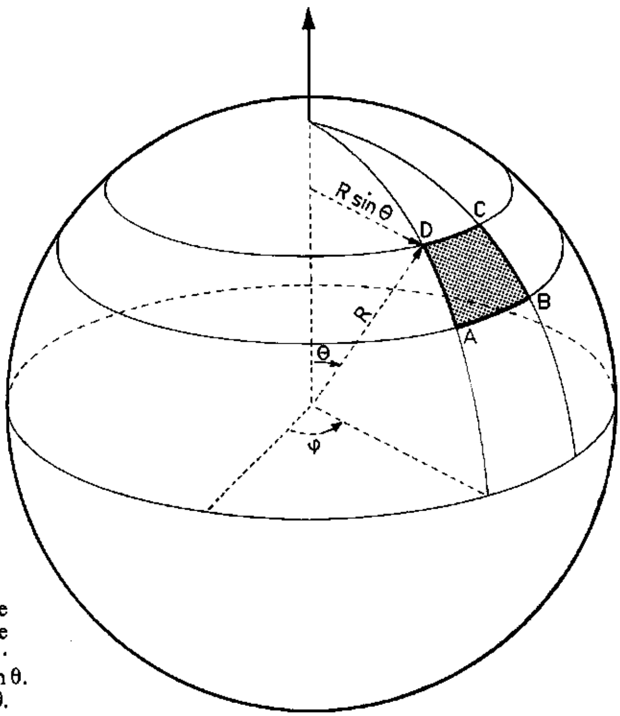

FIG. 3. Spherical coordinate system and surface element $dS = AD \cdot DC = Rd\theta \cdot \mathrm{R} \sin\theta$. $d\Phi = \mathbf{R}^2 \sin\theta d\Phi d\theta$.

$$dS = R^2 \sin \theta d\Phi d\theta \tag{9}$$

If m is the mean solar distance, then the receptor covers a fraction $1/4\pi m^2$ of the area of a sphere with its centre in the radiating point under consideration (Fig. 2). Hence the rays towards the square centimeter fill a small solid angle

$$d\Omega = (1/4\pi m^2).\ \ 4\pi = 1/m^2 \text{ steradians} \tag{10}$$

Substitution of (9) and (10) turns (3) into

$$H^+ d\lambda = 2J \cos\theta \cdot (1/m^2) \cdot R^2 \sin\theta d\Phi d\theta d\lambda \quad (\text{erg.sec}^{-1}) \tag{11}$$

Integration of (11) over the radiating hemisphere yields

$$\begin{aligned} E^+ d\lambda &= \int_{\Phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} H^+ d\lambda d\Phi d\theta = \\ &= (2JR^2 d\lambda/m^2) \int_{\Phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \cos\theta \sin\theta \, d\Phi d\theta \end{aligned} \tag{12}$$

or

$$E^+ d\lambda = (2\pi J R^2 d\lambda)/m^2 \quad (\text{erg.sec}^{-1}) \tag{13}$$

Here $E^+ d\lambda$ stands for the energy (erg) received per sec from non-polarised "solar" radiation of wave-length between λ and $\lambda + d\lambda$ by the receptor (spectral energy flux density).

3. The effective temperature of the sun

The temperature of the imaginary blackbody cannot be predicted theoretically but must be based on experiment. T must be selected such, that the theoretical value of the solar constant Z agrees with the observational one. The temperature, thus defined, is termed the effective temperature of the sun T_{eff}.

4. The theoretical solar constant Z and the evaluation of T_{eff}

The total energy received by the receptor per sec is obtained by integration of (13) over all the wave-lengthes

$$Z = \int_{\lambda=0}^{\infty} E^+ d\lambda = (2\pi R^2/m^2) \int_{\lambda=0}^{\infty} J d\lambda \quad (T = T_{\text{eff}}) \tag{14}$$

Introducing the substitution $z = c_2/\lambda T_{\text{eff}}$, one has

$$Z = (2\pi R^2/m^2)\,(c_1 T_{\text{eff}}^4/c_2^4) \int_{z=0}^{\infty} z^3 \left[\exp(z) - 1\right]^{-1} dz \tag{15}$$

The evaluation of the latter integral can be found in many textbooks on physics, for instance [4], page 544. The result is

$$\int_{z=0}^{\infty} z^3 \left[\exp(z) - 1\right]^{-1} dz = \pi^4/15 = 6.4950 \tag{16}$$

Hence the theoretical value for the solar constant Z yields

$$Z = (2\pi R^2/m^2)\,(c_1 T_{\text{eff}}^4/c_2^4)\,(\pi^4/15) = 1.287 \times 10^{-9} \cdot T_{\text{eff}}^4 \,(\text{erg.sec}^{-1}\text{cm}^{-2}) \tag{17}$$

Equating (17) to the observational value of 1.94 $\text{cal.cm}^{-2}\text{min}^{-1} = 1.35 \times 10^6\ \text{erg.cm}^{-2}\text{sec}^{-1}$ yields

$$T_{\text{eff}} = 5740\ {}^\circ\text{K} \tag{18}$$

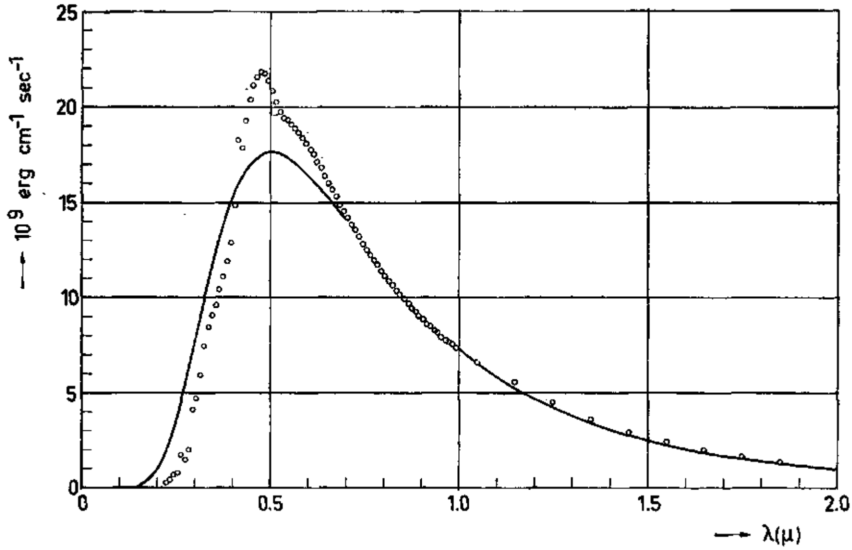

FIG. 4. Theoretical and empirical energy distribution. Full drawn curve presents theoretical value of E^+, experimental values are represented by dots.

5. COMPARISON OF THEORETICAL AND EMPIRICAL ENERGY DISTRIBUTION

If we substitute the value (18) into formula (13) we find a quantitative expression of the spectral energy flux density E^+ of the solar radiation in the vicinity of the earth. In Fig. 5 the function E^+ against λ is compared with the experimental values given by the Smithsonian Tables. For $\lambda > 0.7\ \mu$ the agreement is excellent. Near the maximum of the curve, however, deviations up to 25% occur. The experimental values show a maximum for $\lambda = 0.475\ \mu$, whereas the curve E^+ has a maximum for $\lambda = 0.54\ \mu$. A temperature $T = 6100\,°K$ of the blackbody would shift the maximum of E^+ to $\lambda = 0.475\ \mu$. This temperature, however, would raise the numerical value of the solar constant to 2.5 cal $cm^{-2}min^{-1}$.

ACKNOWLEDGEMENT

I thank Prof. Dr. W. R. VAN WIJK for suggesting the problem.
The author is research associate of the "Stichting voor Fundamenteel Onderzoek der Materie" (F.O.M.), financially supported by the „Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek" (Z.W.O.).

LIST OF SYMBOLS

[ ] indicates reference for numerical value cited, ( ) indicates number of explaining formula.

c	$= 2.9985 \times 10^{10}$	cm. sec^{-1}	velocity of light [1]
c_1	$= 0.589 \times 10^{-5}$	erg.cm^2sec^{-1}	first radiation constant [1]
c_2	$= 1.43$	cm °K	second radiation constant [1]
E^+	=	erg.$cm^{-1}sec^{-1}$	spectral energy flux density (12)

h	=	0.655×10^{-27}	erg.sec	Planck's constant [1]
H	=		$erg.cm^{-1}sec^{-1}$	spectral radiant emittance (lin. pol.) (2)
H^+	=		$erg.cm^{-1}sec^{-1}$	spectral radiant emittance (non-pol.) (3)
k	=	1.372×10^{-16}	$erg. (°K)^{-1}$	Boltzmann's constant [1]
J	=		$erg.cm^{-3}sec^{-1}$	Planck's radiation function [1]
m	=	1.4967×10^{13}	cm	mean solar distance [3]
R	=	6.9635×10^{10}	cm	solar radius [3]
dS	=		cm^2	surface element (9)
T	=		°K	absolute temperature
T_{eff}	=		°K	effective temperature of sun
x	=			auxiliary variable (7)
y	=			auxiliary variable (8)
z	=			integration variable
Z	=	1.94	$cal.cm^{-2}min^{-1}$	solar constant [2]
Φ	=		rad	azimuth of point at radiating hemisphere
θ	=		rad	polar angle of point at radiating hemisphere
λ	=		cm	wavelength
$d\Omega$	=		sterad	solid angle element

References

1. E. Jahnke and F. Emde, Tables of functions, 4th edition, New York, 1945.
2. Smithsonian Meteorological Tables, 6th edition, Washington, 1951.
3. Landolt-Börnstein, Astronomie und Geophysik, Band III, Berlin, 1952.
4. G. Joos, Lehrbuch der theoretischen Physik, 8. Auflage, Leipzig, 1954.