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A nomogram for vertical capillary flow

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The fundamental equation

The movement of water in the capillary zone may be upward as well as downward. It is commonly accepted that this flow may be described by an equation which is valid for diffusion as well as moisture flow

$$v = -k \frac{d\varphi}{dz}$$

v = velocity of flow
φ = flow potential
z = length in direction of flow

For the upward flow this assumption will be valid. For the downward flow it is not certain whether not, in case cracks are present, a flow through these cracks may occur, which satisfies other laws of moisture flow. Where no cracks and fissures are present in the soil, the assumption of a linear equation of flow will hold.

The Poiseuille equation for granular material

The capillary flow is governed by a coefficient for unsaturated permeability. The formula of Poiseuille describes moisture flow in straight cylindrical capillaries of uniform radius as:

$$q = \frac{r^2}{8\eta} \pi r^2 \frac{d\varphi}{dz}$$

The mean velocity v_c of this flow may be obtained by dividing the quantity of flow q by the cross-sectional area of the capillary πr^2, therefore

$$v_c = \frac{r^2}{8\eta} \frac{d\varphi}{dz}$$

η = viscosity of the fluid
r = radius of pore

This formula is valid for straight cylindrical capillaries of uniform size.

The shape of the capillary system in soils however does not resemble a system of straight uniform tubes. It differs with respect to the variation in pore size, the complicated network and the winding pathway.

The difference between uniformity and distributed pore size is met by a distribution constant. The difference between unlinked parallel tubes and a network of tubes is met by a ramification constant. The difference between straight tubes and the winding tubes is met by two tortuosity constants.

The influence of pore size distribution

Mathematical considerations as to the influence of the differences in pore size reveal that to the exponent 2 of r in the last equation has to be added + b for p = 1 and -b for p = 0, where p and b are parameters of the desorption function:

$$b(A - \log \Psi) = p \log v - (1 - p) \log (P - v)$$

Ψ = moisture tension or negative pressure
v = moisture content
P = max. moisture content
A, b, p = constants

The influence of the ramified pore system

The ramified shape of the network of capillaries and voids is accounted for by multiplying the added value of b with a reduction factor β which is of the order of 0,5 for granular material.

The influence of tortuosity

The tortuosity of the flow path is accounted for by multiplying dz by an elongation factor t. There are further indications in litterature that the exponent of r may be increased by a value τ, accounting for the increase in resistance due to the size of the bends of the winding flow path. The flow resistance k of the diffusion equation may therefore be presented by

$$k = \frac{r^{2+-\beta b+\tau}}{8\eta t} = \alpha r^{n}$$

Here the conductivity factor k stands for the capillary velocity v_c at a pressure gradient $\frac{d\Psi}{dz}$ equal to unity.

In this formula the r has the meaning of the radius of the largest pore which actually conveys water.

The importance of the air entry point

The conductivity of the soil is obtained by integrating the quantity of flow over the whole range of pores present in the soil. For unsaturated soils the lower limit of integration is zero. The upper limit is the value of r of the largest pore containing water. Saturated conductivity or permeability requires integration between the limits of the smallest pore, which is again zero, and the largest pore present in the soil of which the size has a certain, often unknown value.

Integration is performed over the pressure gradient expressed in dΨ and dz. Now the diameter r is a function of Ψ and the formula has to be expressed in Ψ. This poses the question what the value of Ψ is belonging to the value of r used as upper limit of integration.

In saturated soil it is the Ψ value corresponding with the largest radius present in the soil. It is however not $\Psi = 0$ corresponding with $r = \infty$. In unsaturated soils it is the radius, belonging to the moisture stress in the soil. In both cases the value of Ψ may be calculated according the formula:

$$\Psi = \frac{0.15}{r}$$

For the saturated permeability δ the formula reads:

$$\delta = \alpha\, r^{n}_{max.} \text{ or } \alpha = \delta / r^{n}_{max.}$$

From this follows for the unsaturated permeability:

$$k_{unsat.} = \delta \left(\frac{r}{r_{max.}} \right)^{n}$$

In order to integrate the formula over $d\Psi$ the expression in r is reduced to an expression in Ψ and reads:

$$k_{unsat.\Psi} = \delta\left(\frac{\Psi_{min}}{\Psi}\right)^n$$

This value of k is inserted into the diffusion equation.

Comparison with the Gardner expression

Criticism on the formula $k_{unsat.} = \frac{\alpha}{\Psi^n}$ has been passed and objection is made that for $\Psi = 0$ the value of $k_{sat.}$ becomes infinite. It was suggested to counter this result by assuming a formula (GARDNER):

$$k_{unsat.} = \frac{\alpha}{\Psi^n + a}$$

which for $\Psi = 0$ gives $k_{sat.} = \alpha / a$. This suggestion passes over at the fundamental fact that the permeability is primarily a function of the pore radius and not of the tension.

Assuming a value of Ψ equal to zero which seems reasonable, in fact very unreasonably means assuming maximum pore diameter of infinite size. The integration limit, however, should be $r_{max.}$ or $\Psi_{min.}$ and taking this into consideration, complications of our formula for k with an extra constant may be avoided.

The equation of capillary flow

The pressure gradient deserves some attention. The water flows under influence of the gradient due to the degree of unsaturatedness and the influence of gravity, By upward flow the two gradients partly neutralize each other, by downward flow the two gradients add up. The gradient due to gravity is, in case water is the moving liquid, equal to unity.

The differential equation

Now we may write:

Upward flow

$$v_c = \delta\Psi^n_{min.}\frac{1}{\Psi^n}\left\{\frac{d\Psi}{dz} - 1\right\}$$

$$dz = \frac{d\Psi}{\frac{v_c}{\delta\Psi^n_{min.}}\Psi^n + 1}$$

Downward flow

$$v_c = \delta\Psi^n_{min.}\frac{1}{\Psi^n}\left\{\frac{d\Psi}{dz} + 1\right\}$$

$$dz = \frac{d\Psi}{\frac{v_c}{\delta\Psi^n_{min.}}\Psi^n - 1}$$

The simplest solution, obtained for n = 2, reads:

$$\Psi = \sqrt{\frac{\delta}{v_c}}\ tg\left(\sqrt{\frac{v_c}{\delta}}\ z\right) \qquad \Psi = \sqrt{\frac{\delta}{v_c}}\ tgh\left(\sqrt{\frac{v_c}{\delta}}\ z\right) = \sqrt{\frac{\delta}{v_c}}\left(\frac{1-e^{-2\sqrt{\frac{v_c}{\delta}}\ z}}{1+e^{-2\sqrt{\frac{v_c}{\delta}}\ z}}\right)$$

Indications for the construction of a nomogram.

The integration of these equations, simple in case the value of n is equal to 2, becomes more complicated for other values of n. The integrations were solved and the relation between Ψ and z computed. The solution is obtained by equating n to p/q with p and q whole numbers. Now calculation is done with new variables

$$\left(\frac{v_c}{\delta \Psi^n_{min.}}\ \Psi^{p/q}\right)^{1/p} = S \quad \text{and} \quad \left(\frac{v_c}{\delta \Psi^n_{min.}}\ z^{p/q}\right)^{1/p} = T$$

A nomogram of the equations for various values of n could be made by drawing up a number of curves for the relation

$$S = f\ T$$

as the main nomogram and by constructing a network with which the values of S and T can be computed separately. In this nomogram the value of $\delta\Psi^n_{min.}$ is contracted to d. In cases where $\Psi_{min.}$ is not known its value best may be assumed to 0,1 or log $\Psi_{min.}$ = 1,0. (see loose nomogram sheets added at the end of the paper)

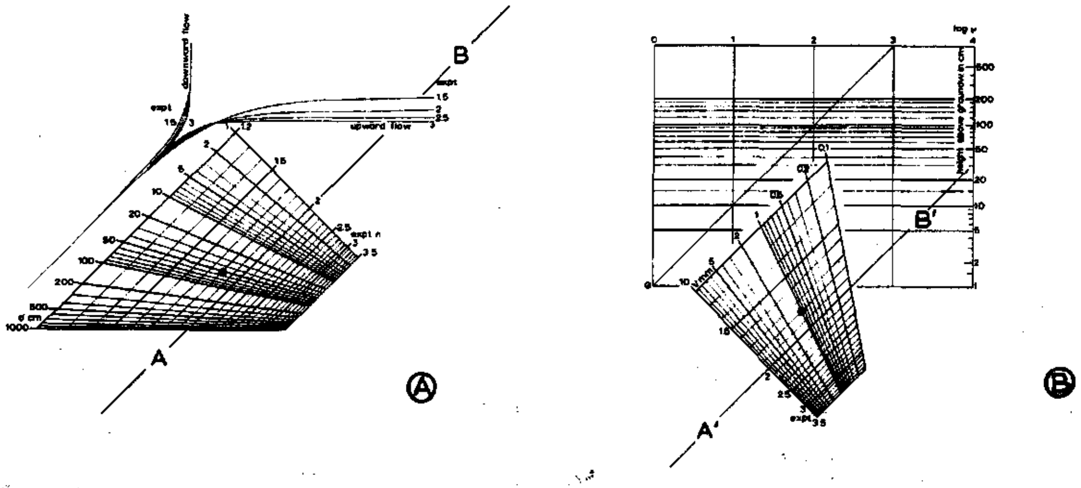
How to use the nomogram

The nomogram consists of two sheets A and B which should be laid upon each other in such a way that the curves on sheet A for upward and downward movement are brought in the correct position with respect to the coordinate system of sheet B for log Ψ - the horizontal coordinates - and log z - the vertical coordinate axis. The coordinate axis for log z is plotted on a logarithmic scale, but is numerated with the values of z. Here z indicates the height of the point of observation in the profile above the groundwater table, for which the log Ψ of the solution of the equation found with the nomogram is valid.

The two sheets take up the proper position with respect to each other when the oblique lines AB and A'B' coincide. But then a shift along the line is still possible. The exact position is found as follows:

A value of the exponent n is selected, say 1,8. Also a value of $\delta\Psi^{n}_{min.}$ = = d is selected, say 0,50 m. On sheet A this point is marked.

Then a value for the capillary velocity is chosen, say 1 mm. a day. On sheet B the point for v_c = 1 mm. and the exponent n = 1,8 is also marked.



Now the lines AB and A'B' are shifted along each other till the two points coincide. In this position the curved lines of sheet A and the coordinate system of sheet B match and the values for log Ψ and z belonging together may be read by using the curves for the exponent 1,8 which is visually interpolated between the lines given.

In the example chosen we read the following solution, using the two loose sheets of the nomogram added to this paper:

Upward flow

log Ψ =	0,5	1,0	1,5	2,0	2,5	3,0	3,5	4,0
z cm. =	3	10	23	40	48	52	53	54

Downward flow

z cm. =	1	5	10	30	50	70	100	150
log Ψ =	0	0,66	0,95	1,34	1,42	1,46	1,48	1,48

The question what the situation is at 70 cm. at log Ψ = 3 above the water table may be read by shifting the sheets to a position where the interpolated curve for n = 1,8 intersects the log Ψ = 3 and the z = 0,70 m. coordinate lines. In the network the lines for d = 0,50 and n = 1,8 intersect on the line for v_c = 0,58 mm. At this height of 70 cm. and with the suction value selected, the upward capillary flow reduces from 1 mm. to 0,58 mm. a day.

With the aid of the desorption curve these results for Ψ may be converted to moisture contents.

Considerations for the choice of the constants

Under practical circumstances no accurate information generally will be available as to the value of $\Psi_{min.}$ and of the exponent n. It may be assumed that values for the constants A, b and p of the desorption curve are obtainable from laboratory determinations or correlation with profile descriptions. The value of $\Psi_{min.}$, β and τ will have to be estimated in an other way. Consideration of soil texture and soil structure may be helpful..

Considerations as to $\Psi_{min.}$

As approximation may be used that the largest pore diameter will be of the order of 20% of the diameter of the largest particles which are still present in such a quantity that they have an influence on the pore space distribution. A particle size might be selected which is exceeded by only 20% of the particles with respect to volume. The granular analyses may provide these data. This consideration is only valid for sandy soils. In clay soils the water primarily flows through cracks.

Consideration as to β

The ramification factor will be of the order of 0,5 in soils, where the soil moisture flows through the network of pores between the soil particles. This is the case by sandy soils. In densely packed soils with tetrahedral particle arrangement the ramification factor will be somewhat higher than in loosely packed soils, with cubical arrangement. In the latter case more branching off is possible. In clay soils, where the water flows through cracks the flow path is not very ramified and the value of β may be nearer to unity.

Considerations as to t and τ

The tortuosity factor t, accounting for the lengthening of the flow path, disappears by bringing the saturated permeability into the formula. The factor τ will depend on the type of flow path, using cracks or pores in granular material. A certain correlation between the ramification factor β and the tortuosity factor τ may be assumed. For $\beta = 1$ the value of τ will therefore be zero, for $\beta = 0{,}5$ this value will decrease to 0,5. This may be described by $\tau = 1 - \beta$. If the flow path is straight there is no branching off.

Considerations as to the distribution of pore sizes

The pore space distribution, as described by the desorption curve influences the exponent in such a way, that - as earlier is stated in this paper -, by the value of $p = 0$ for n the value $2 - \beta b + \tau$ matches. For $p = 1$ this value becomes $2 + \beta b + \tau$. Linear interpolation for intermediate values of p may be done by using

$$n = 2 + \beta(2p - 1)\ b + \tau$$

what gives the same results for $p = 0$ or 1.

Final estimate of the exponent

By inserting in this formula the expected correlation between β and τ, just mentioned, the exponent might be given as

$$n = 2 + \beta(2p-1)b + (1-\beta)$$

or

$$n = 3 + \beta\left\{(2p-1)b - 1\right\}$$

Because b and p follow from the analyses of constants for the desorption curve, the problem how to use of the nomogram, simplifies to the problem of estimating $\Psi_{min.}$ and β. The value of β may be chosen upon the impression of the soil structure being fissured or of a structureless granular type. Also information on the permeability in horizontal or vertical direction may support this impression of soil structure in case of straight line flow in orientated granular material.

It will be clear that all these estimates may prove to be inaccurate and a direct laboratory determination of $\Psi_{min.}$ and n will make these considerations superfluous. The determination of the unsaturated conductivity is however difficult and the aim of the paper is to make capillary problems accessable to the project engineer and the agricultural advisor often lackin laboratory support in the field.

Considerations as to the errors of the estimates

The flow equation is based on the value of Ψ, $\Psi_{min.}$ and n according

$$\gamma = \left(\frac{\Psi}{\Psi_{min.}}\right)^n$$

γ = proportionality factor

The procentual error due to incorrect estimates of $\Psi_{min.}$ and n now is

$$\frac{d\gamma}{\gamma} = \log \frac{\Psi}{\Psi_{min.}} dn - \frac{n}{\Psi_{min.}} d\Psi_{min.}$$

dn = error of n

$d\Psi_{min.}$ = error of $\Psi_{min.}$

On first sight it will be clear that the largest influence on the accuracy is exerted by $d\Psi_{min.}$ The logarithm will have a value of 1 or 2 and $n/\Psi_{min.}$ will be - though variable - not much smaller than 0,2. An equal influence of the errors of n and $\Psi_{min.}$ would be obtained for dn equal to 10% of $d\Psi_{min.}$. We suppose that the error of $\Psi_{min.}$ will often be twice as much This shows that any activity to increase the accuracy should at first concentrate on a more accurate description of the desorption curve at its saturated lower end by assessing the maximum pore size. This is of more importance than assessing the exponent n by the rather difficult determination of the unsaturated permeability.

An example may give an indication of the magnitude of the error.

	$r_{max.}$	$\Psi_{min.}$	n	Ψ
assumed value in cm.	0,015	10	2	100
actual value in cm.	0,030	5	2,5	
magnitude of error		$d_{min.} = -5$	dn = 0,5	

$$\frac{d\gamma}{\gamma} = \log \frac{100}{10} \times (+\,0{,}5) - \frac{2}{10} \times (-5)$$

$$= +\,1{,}0 \times 0{,}5 + 1{,}0$$

$$= +\,1{,}5$$

The error should be calculated with respects to that value - assumed or observed - that yields a positive error $d\gamma$

In the equation:

$$\frac{v_c}{\partial}\left(\frac{\Psi}{\Psi_{min.}}\right)^n = \frac{v_c}{\partial}\gamma$$

the error of γ is equal to:

$$d\gamma = 1{,}5\,\gamma$$

In case the value of z in the nomogram is given, a value of 1,5 times γ will be compensated by a value of $v_c/1{,}5$. A result for v_c is obtained which is only 67% of what it in reality should be.

Considerations as to the propagation of the errors and some tentative calculutions will be valuable to make to justified use of the nomogram.