NOTA 1028 II december 1977

Instituut voor Cultuurtechniek en Waterhuishouding
Wageningen

TRANSFORMATION OF STORM MODELS

CAUSED BY STOCHASTIC COMPONENTS

dr. Ph.Th. Stol

C O N T E N T S

1. INTRODUCTION

Storm models can be used to mathematically describe the process of the occurrence of storms in a certain area and the process of measuring rainfall amounts. Such models can be deterministic or stochastic.

Adding stochastic components to deterministic models means that rainfall amounts become larger or smaller due to a probabilistic mechanism. The most commonly used condition in such case is that the statistical characteristics of stochastic components do not depend on the order of magnitude of the rainfall amount.

This means that, apart from systematic errors, *on the average* nothing will change. However, large negative values of the stochastic component obtained by simulation and added to small rainfall amounts can cause negative values. Used as generated rainfall time series one would replace such negative values by zero's. This means that automatically the percentage of zero's will change and so does the expectation and the variance of the time series of rainfall amounts.

In this report the effect of such changes to mathematical rainfall models will be investigated and it will be tried to find mathematical expressions that describe the transformed models.

2. MATHEMATICAL INTRODUCTION

The problem stated in the introduction reads in mathematical terms as follows.

Given a two-dimensional storm function

$$h = f(x)$$

where h = rainfall amount

x = co-ordinate in the storm

defined by the following conditions

$$h \geqq 0$$

$$h = f(0) = 0$$

$$h = f(\tfrac{1}{2}B) = H$$

where B = storm diameter or storm width

H = maximum rainfall amount at the center of the storm

We assume storms to be symmetrical about the center and so, for our present purpose, the complete storm function is defined by

$$h = f(x) \begin{cases} {}^{1}f(x), & 0 \leqq x \leqq \tfrac{1}{2}B \\ {}^{2}f(x), & \tfrac{1}{2}B < x \leqq B \end{cases} \tag{2.1}$$

where

$$ {}^{2}f(x) = {}^{1}f(B-x) \tag{2.2}$$

Without ambiguity we understand by $f(x)$ the first half of the storm function since, because of the symmetry, transformations applied to ${}^{1}f(x)$ can analogously be applied to ${}^{2}f(x)$, using Eq.(2.2). In the present study we therefore shall confine ourselves to

$$h = f(x) = {}^{1}f(x)$$

Particular values of h measured, say, at $x = a$ are denoted by $h_a = f(a)$, $0 \leqq a \leqq \tfrac{1}{2}B$, and analogously for h_b.

3. ADDING STOCHASTIC COMPONENTS

Now we consider the case that stochastic components are added to rainfall amounts h. In real life this could mean that we take into account random fluctuations of storm intensities, measuring and exposure errors, etc. Influences of these components will be called random fluctuations and no further distinction between their origin will be made.

Random fluctuations are superposed to h = f(x) according to (see STOL, 1977a p.14)

$$\underline{h}_a = f(a) + \underline{\varepsilon}_a \,, \quad \underline{\varepsilon}_a = \tau\underline{\chi} \qquad (3.1)$$

$$\underline{h}_b = f(b) + \underline{\varepsilon}_b \,, \quad \underline{\varepsilon}_b = \tau\underline{\chi} \qquad (3.2)$$

where

$$E(\underline{\varepsilon}_a) = E(\underline{\varepsilon}_b) = \eta = 0, \quad \text{all a and b}$$

and

$$E(\underline{\varepsilon}_a^2) = E(\underline{\varepsilon}_b^2) = \tau^2 \,, \quad \text{all a and b}$$

and $\underline{\chi}$ is assumed to be normally distributed with expectation 0 and variance 1.

To complete this model we define

$$\mathrm{Cov}(\underline{\varepsilon}_a \; \underline{\varepsilon}_b) = \Theta\tau^2$$

where Θ obviously is the correlation between random fluctuations occurring in the storm at x = a and x = b respectively.

In conclusion, and defining μ_a and σ_a

$$\mu_a = E(\underline{h}_a) = f(a) \quad \text{and} \quad \mu_b = E(\underline{h}_b) = f(b) \qquad (3.3)$$

$$\sigma_a^2 = E\,\{\underline{h}_a - f(a)\}^2 = E(\underline{\varepsilon}_a^2) = \tau^2$$

$$\sigma_a^2 = E\,\{\underline{h}_b - f(b)\}^2 = E(\underline{\varepsilon}_b^2) = \tau^2$$

$$\text{Cov}\ \{\underline{h}_a - f(a)\}\ \{\underline{h}_b - f(b)\} = \text{Cov}\ (\underline{\varepsilon}_a\ \underline{\varepsilon}_b) = \Theta\tau^2$$

The symbols a and b, rather than the running variable x, are to be considered points at which rainfall amounts are measured in the storm.

4. TRUNCATED RAINFALL DISTRIBUTIONS

In equations (3.1) and (3.2) we assume that $\underline{\chi}$ is truncated such that $\underline{h} \geqq 0$, since rainfall amounts can not be negative.

In order to be able to apply truncated normal distributions to rainfall amounts, basic theory is given in Appendix 1, where the distribution, its mathematical expectation and its variance are derived. Definitions of special symbols used in truncating distributions are defined in Appendix 1 as well.

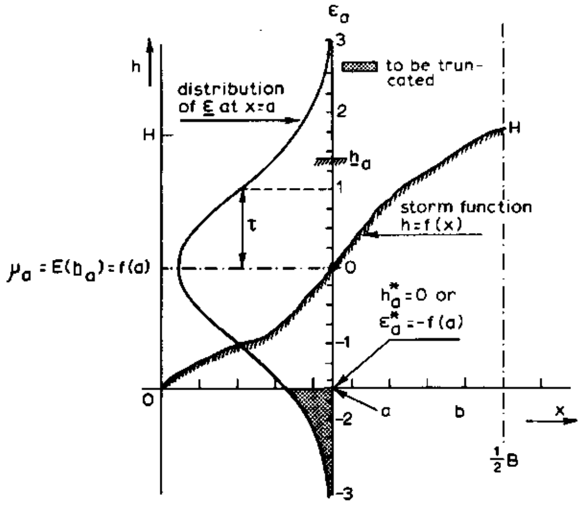
To obtain non-negative rainfall amounts, $\underline{\chi}$ has to be truncated, given x = a, at the lower point of truncation, viz.

$$h_a^* = 0, \quad 0 \leqq a \leqq \tfrac{1}{2}B \tag{4.1}$$

or considering the distribution of $\underline{\varepsilon}$, at the lower point of truncation

$$\varepsilon_a^* = -\ f(a) \tag{4.2}$$

in virtue of (3.1). The relationship between the Equations (4.1) and (4.2) is depicted in Figure 1.



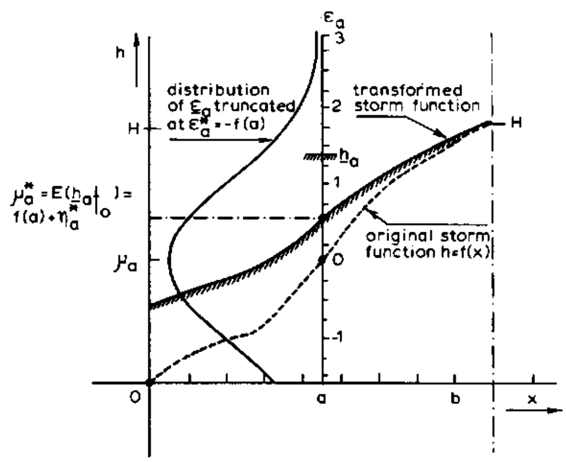


Fig. 1a. Distribution of random fluctuations $\underline{\varepsilon}$ about the value of the storm function at x = a. Except for difference in size of the truncated area, the distribution is the same for all x that satisfies $0 \leqq x \leqq \frac{1}{2}B$. Realization of a rainfall amount is illustrated by $\underline{h}_a = f(a) + \underline{\varepsilon}_a$. The standarized point of truncation for $\underline{\varepsilon}$ is $z_a^* = -f(a)/\tau$

Fig. 1b. Truncated distribution of random fluctuations $\underline{\varepsilon}$ about the value of the storm function at x = a. The distribution is n o t the same for different values of x, the transformed storm function approaches the original function for increasing values of h. The discrepancy equals η_a^* which depends on the location in the storm

The degree of truncation from below then is

$$P(\underline{\varepsilon}_a \leqq \varepsilon_a^*) = N\{-f(a)\} = N(\varepsilon_a^*)$$

or, standarized

$$P(\underline{\varepsilon}_a \leqq \varepsilon_a^*) = \Phi\{\frac{-f(a)}{\tau}\} = \Phi(\frac{\varepsilon_a^*}{\tau})$$

According to results obtained in Appendix 1, Section 4, we now have

$$\eta_a^* = E(\underline{\varepsilon}_a \nmid_{\varepsilon_a^*}) = \frac{\phi(\varepsilon_a^*/\tau)}{1 - \Phi(\varepsilon_a^*/\tau)} \tau > 0$$

and so

$$\mu_a^* = E\ \{\underline{h}_a \nmid_0\} = f(a) + \eta_a^*$$

In an analogous way the variance ${\sigma_a^*}^2$ can be obtained.

Parameter values of the distribution of h under truncation from below can be obtained from Table B, in Appendix 1. We have to interpret the headings of the columns in the following way (see Table B, in Appendix 1, lower headings).

Column 1: The standarized lower point of truncation $z_a^* = - f(a)/\tau$

Column 2: The degree of truncation from below viz. $P(\underline{z} \leqq z_a^*)$ or $P(\underline{\varepsilon}_a \leqq \varepsilon_a^*)$ or $P(\underline{h}_a \leqq 0)$, in percentages

Column 3: Values of the expression $\frac{\mu_a^* - f(a)}{\tau}$ from which μ_a^* [being $E\ \{\underline{h}_a \nmid_0\}$] can be solved by a linear transformation

Column 4: Values of the expression $\frac{\sigma_a^*}{\tau}$ from which σ_a^*, the standard deviation of the truncated distribution of h, can be solved by multiplication

The expectation of $\underline{h}$ thus depends on the location in the storm. Again using the continuous variable x we can express this fact by the following formula

$$\mu_x^*(0) = f(x) + \frac{\phi\{-f(x)/\tau\}}{1 - \Phi\{-f(x)/\tau\}}\,\tau \qquad (4.3)$$

where the expectation without truncation reads

$$\mu_x = f(x)$$

5. CORRELATION BETWEEN RANDOM FLUCTUATIONS

Mostly it is assumed that random fluctuations are not correlated amongst themselves. In Fig. 1 let a be a first point of interest and b be a second one then the correlation Θ between random errors is

$$\Theta(\underline{\varepsilon} \text{ at a}, \underline{\varepsilon} \text{ at b}) = 0 \qquad \text{(see Fig. 2a)}$$

A special case, however, arises when we take Θ = 1. This means that

$$\underline{\varepsilon}_a = \underline{\varepsilon}_b = \dots, \qquad \text{for all points in } [0, \tfrac{1}{2}B]$$

and a random constant value is added to the storm function, according to

$$\left.\begin{aligned} \underline{h}_a &= f(a) + \underline{\varepsilon} \\ \underline{h}_b &= f(b) + \underline{\varepsilon} \end{aligned}\right\} \text{for the same storm} \qquad \text{(see Fig. 2b)}$$

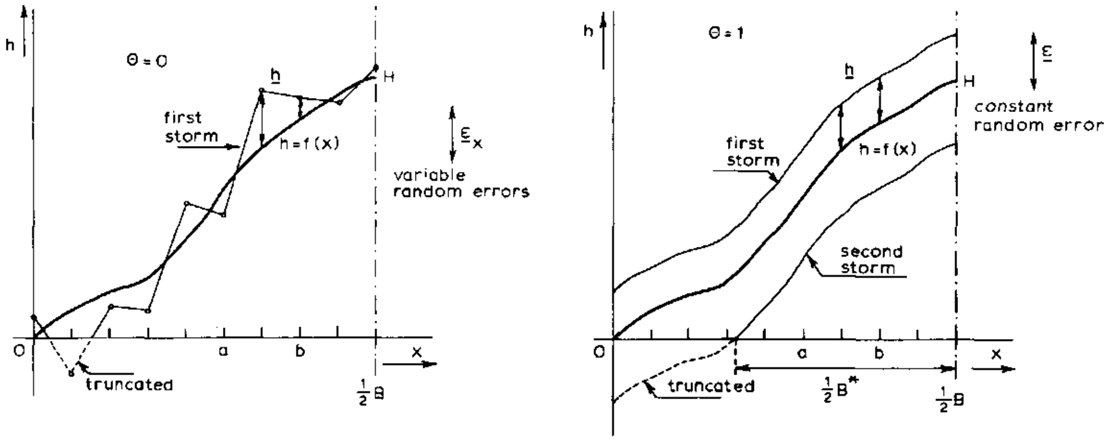


Fig. 2a and 2b. Two models for adding random errors to storm functions
For Θ = 0 random errors are variable from point to point
For Θ = 1 random errors are constant per storm

From Fig. 2 it is clear that the two cases behave differently. Both cases are briefly commented.

$\boxed{\Theta = 0}$ Fig. 2a

All points in the storm operate independently. Negative values need not necessarily occur in adjacent points. In each point a truncated normal distribution is in action with properties

$$\underline{\varepsilon}_x \geq - f(x) \;, \quad \text{parameters} \quad 0 \text{ and } \tau$$

The storm width B remains constant although zero's may occur, especially near the edges: x = 0 and x = ½B.

$\boxed{\Theta = 1}$ Fig. 2b

All points in the storm operate in the same way. Negative values occur in adjacent points. For the whole storm (or if one wishes: in only one point) a truncated normal distribution is in action with properties

$$\underline{\varepsilon} \geq - f(x), \quad \text{parameters} \quad 0 \text{ and } \tau$$

The storm width B does not remain constant. If zero's occur they act if the storm width were variable (Fig. 2b, second storm with reduced width B*).

At the present state of art this complication is not solved and therefore will not be paid attention to in this report.

In this last case, however, storms behave also if it were the maximum height H that is variable. There is one case in this class of storm functions that can be treated with the present models, namely the rectangular storm type (STOL, 1977c). Since rainfall magnitudes then are constant in each storm random fluctuations that cause negative values do so for the entire storm: cancelling negative values means that that particular storm is cancelled as a whole. This situation is sketched in Fig. 3: storm 3 and 6.

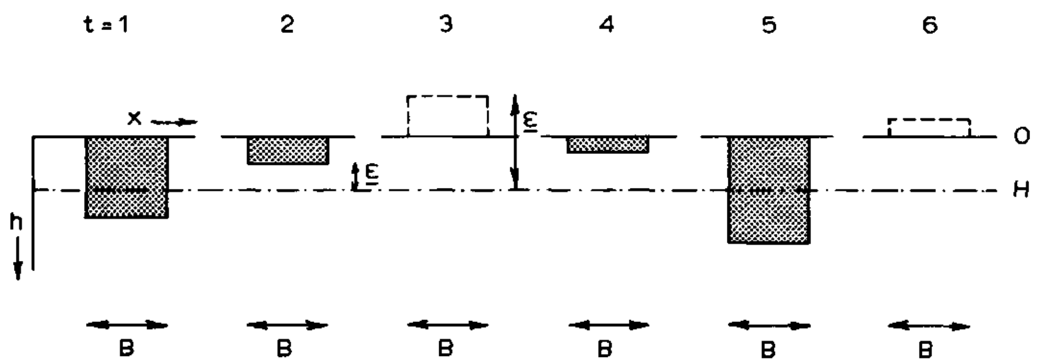


Fig. 3. Example of 6 rectangular storms affected by random fluctuations $\underline{\varepsilon}$. Great negative values of $\underline{\varepsilon}$ cause h < 0. Cancelling these storms (no. 3 and 6) means adding dry days to the time series

However, cancelling a complete storm means adding a dry day to the time series. In this case we will find an increase of the percentage of (completely) dry days (STOL, 1977b).

The degree to which the percentage of dry days is extended, depends on the degree of truncation. Let, for short, the degree of truncation $\Phi(\frac{-H}{\tau})$ be P* under truncation, P* say, becomes with $\Theta = 1$ for the rectangular storm type

$$p^* = p + (1-p)P^*$$

Values of P^* are given in percentages in Table B, Appendix 1, column 2.

A start has been made to solve this particular case analytically. No further details are given here.

6. TRANSFORMED STORM FUNCTIONS

The storm profile as expressed by the storm function in Eq. (2.1) changes under the influence of random fluctuations. O n t h e a v e r a g e the profile will show up as being

$$h = f(x) + \frac{\frac{1}{\sqrt{2\pi}} \exp \frac{-1}{2} \{-f(x)/\tau\}^2}{1 - \Phi\{-f(x)/\tau\}} \tau \tag{6.1}$$

as a result of Eq. (4.3).

Correlation functions derived for particular expressions of the storm function $f(x)$ have now to be developed taking into account the second term in (6.1).

However, since the fraction contains in the denominator the integral Φ of the normal distribution, it is not possible to write Eq. (6.1) with elementary functions. The entire expression (6.1) has to be approximated by more simple functions. Because of the structure we will use exponentials, and, in particular, the exponential storm function.

The fundamental idea of the construction of transformed storm models can be illustrated with the following example.

Given the storm function, see Fig. 4,

$$h = f(x)$$

Values of h can be considered to be the mean value about which random fluctuations are superposed, with standard deviation $\tau = 5$, say.

So at x = a, for $0 \leqq a \leqq \frac{1}{2}B$, we have

$$\underline{h}_a = f(x) + \underline{\varepsilon}_a$$

and according to Eq. (3.3)

$$\mu_a = E(\underline{h}_a) = f(a)$$

However, since negative values are truncated, the mean value will increase. The new values of the mean under truncation viz. $E(\underline{h}_a|_0)$, or $\mu_a^*(0)$, can be obtained from Table B in Appendix 1. To prepare the elaboration, values of h = f(x) to be understood the mean value μ_x are collected in Table 1, columns 1 and 2.

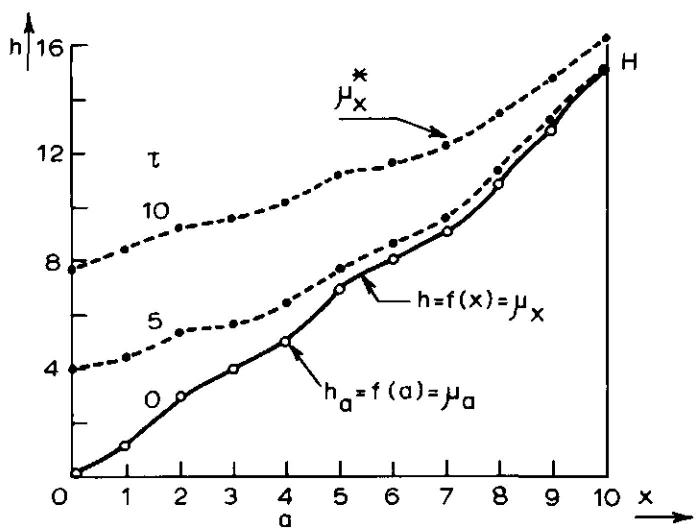


Fig. 4. Example of a storm function h = f(x) and the function obtained as the locus of mean values (expectations) after truncation

Values of μ_a are expressed in standard units by dividing them by τ. Table B in Appendix 1 is entered with the negative value $z_a = -\mu_a/\tau$ and the standarized expectation under truncation is returned (Table 1, columns 3 and 4). Next the inverse linear transformation is applied (Table 1, columns 5 and 6). Finally the expectation under truncation, μ_a^*, is plotted in Fig. 4. For comparison the function under truncation for τ = 10 is plotted as well. The original function (no random errors) is valid for τ = 0.

Table 1. Determination of the mathematical expectation in a storm model as given in Fig. 4 under truncation of random fluctuations with standard deviation $\tau = 5$

Preparation		See Table B of Appendix 1		Solution of μ_a^*		
a	f(a)	$\frac{-f(a)}{\tau}$	$\frac{\mu_a^* - f(a)}{\tau} =$	$\mu_a^* - f(a) =$	$\mu_a^* =$	$\sigma_a^* =$
1	2	3	4	5	6	7
0	0	0	0.7979	3.9895	3.99	3.01
1	1	-0.2	0.6751	3.3755	4.38	3.20
2	3	-0.6	0.4591	2.2955	5.30	3.58
3	4	-0.8	0.3676	1.8380	5.84	3.78
4	5	-1.0	0.2876	1.4380	6.43	3.97
5	7	-1.4	0.1629	0.8145	7.81	4.32
6	8	-1.6	0.1174	0.5870	8.59	4.47
7	9	-1.8	0.0819	0.4095	9.41	4.60
8	11	-2.2	0.0360	0.1800	11.18	4.79
9	13	-2.6	0.0136	0.0680	13.07	4.90
10	15	-3.0	0.0044	0.0220	15.02	4.97

7. DISCUSSION

From the foregoing result we observe that small values of h are influenced much more by truncation then large values. However, this is true in a relative sense only, since the elaboration depends on $-f(x)/\tau$.

The transformed function, the dotted line in Fig. 4, is a function of x again, with - among others - τ as a parameter. This function reads, according to Eq. 6.1

$$\mu_x^*(0) = f(x) + \frac{\frac{1}{\sqrt{2\pi}} \exp \frac{-1}{2} \{-f(x)/\tau\}^2}{1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-f(x)/\tau} \exp(-\tfrac{1}{2}z^2)dz} \tau \qquad (7.1)$$

This is not a simple function of x and cannot be expressed in elementary functions. The main properties of these functions are:

- if f(x) is large, relative to τ, the numerator in the second term becomes small, and the denominator tends towards 1 and so

$$\mu_x^* \simeq f(x) \qquad \text{if } f(x) \text{ large (near center)}$$

- if f(x) is small, relative to τ, and close to zero, the numerator approaches $\frac{1}{\sqrt{2\pi}}$ and the denominator tends towards $\frac{1}{2}$ and so

$$\mu_x^* \simeq f(x) + \frac{2\tau}{\sqrt{2\pi}}, \qquad \text{if } f(x) \text{ small (near edges)}$$

$$\simeq f(x) + 0.8\tau \qquad \text{(more exact 0.7979)}$$

In our case the general shape of the transformed storm function will be: close to the original function in the neighbourhood of the center, increasing deviations with a maximum of 0.7979τ near the edges.

The type of formula for μ_x^* and the shape of the dotted lines in Fig. 4 suggest that we can try an exponential storm function to approximate the complicated expression in Eq. (7.1) and to analytically describe the situation under presence of random errors. This will be done for the triangular and the exponential storm function.

The problem that remains now is to determine the values of the parameters of that exponential storm function that describes the triangular and exponential storm functions, in which random fluctuations are present, best.

8. WORKED EXAMPLES

8.1. The influence of the size of random fluctuations

Before giving details of examples we first consider the influence of the size of random errors to the correlation function.

In Fig. 5 the correlation function is given for a triangular storm with characteristics: storm width B = 0.5 (dimensionless, expressed in units of area size L), maximum storm value H = 20 (cm say) at the center of the storm. Random fluctuations are chosen to have zero expectation, and standard deviation $\tau = 0, 2, 5, 10, 20$ and 100 (cm), or dimensionless $H/\tau = \infty, 10, 4, 2, 1, 0.20$.

The first graph shows the situation $\tau = 0$, compare with STOL (1977d, Fig. 4). The graphs illustrate the mean correlation for various interstation distances, obtained by simulation (dots), and the interstation correlation function calculated from analytically derived formulas (full line). For small values of τ the theory describes the simulated results well. From $\tau \geqq 5$ ($\frac{H}{\tau} \leqq 4$) discrepancies become apparent. They can be explained as follows.

Random errors cause fluctuations about the mean value $h = f(x)$. When these fluctuations are large, algebraically obtained rainfall amounts can become negative. Cancelling these negative values mean that the expectation of $\underline{h}$ increases and so the analytically derived formula for triangular storms with H = 20 is not valid any more (Fig. 2). Even the storm model does not hold any longer since the increase of the mean value depends on x, the location in the storm (Fig. 4).

Although the value of the standard deviation $\tau = 100$ is rather irrealistic compared with the value of the maximum rainfall amount H = 20, we will use this illustrative example to modify the storm function to make it fit the points in Fig. 5 (last case) again.

However, to systematically treat the three important storm functions, in the examples we will start, after having made some general remarks, with the rectangular storm type.

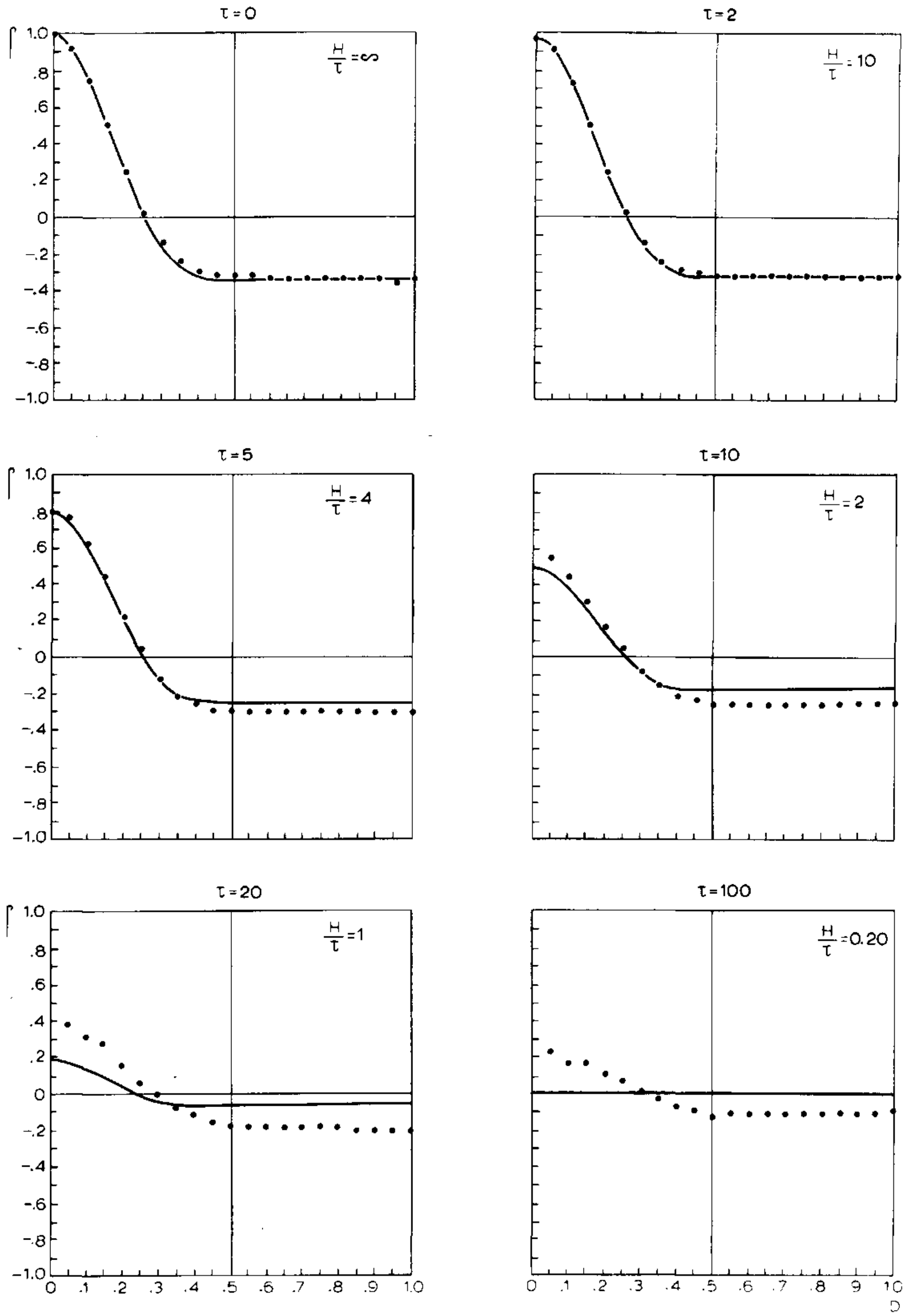


Fig. 5. The simulated (dots) and theoritical (full line) correlation function for a triangular storm with increasing values for the standard deviation τ of random errors

8.2. General remarks about examples

After the theory has been fully discussed, examples are briefly commented. The essential storm characteristics are H, the maximum rainfall at the center of the storm, and τ, the standard deviation of the random fluctuations. The storm diameter B is taken B = 0.5. The correlation between random fluctuations at different points is taken $\Theta = 0$, and the fraction of dry days $p = 0$. Interstation distance D and storm diameter B are expressed in units of area length L, and so are dimensionless with $0 \leqq D \leqq 1$.

The following items are discussed and illustrated by graphs.

a) definition of the storm function (STOL, 1977c).
Only the left part of the function needs further concern.

b) transformation of the storm function to account for additional zero's caused by algebraically obtained negative precipitation amounts. Determination of storm characteristics H and τ.

c) determination of further parameters of the transformed storm function using 11 points in the storm, viz. $\frac{a}{\frac{1}{2}B} = 0(0.1)\ 1$

d) graphical representation of simulated correlations at various interstation distances D, represented by .... Distances taken are D = 0(0.05) 1. The simulated values are obtained as a mean of two series of 1000 storms, in which negative values are replaced by zeros. They are referred to as *simulated values*. Storm characteristics used are H and τ.

e) general formulas of the correlation function for the storm type under discussion, after STOL (1977b). The correlation functions are defined as

$\rho_{I}(D)$ for $0 \leqq D < \frac{1}{2}B$

$\rho_{II}(D)$ for $\frac{1}{2}B \leqq D \leqq B$

$\rho_{III}(D)$ for $B < D \leqq 1$

Properties and graphical representation of the theoretical correlation function without transformation of the storm function with parameter values as used under d) are given. Characteristics B, O and p, are given values as mentioned in the Introduction to this sub-section.

f) graphical representation of the correlation function with transformed storm functions with combinations of values for the storm characteristics, as follows (see Table 2)

Table 2. Combination of values of storm characteristics and symbols with which the correlation function based on them are represented in the Figures

storm maximum	standard deviation	curve number	curve type	(H,τ)
H	τ	1	———————	(-,-)
H*	τ	2	———————	(*,-)
H	τ*	3	-.-.-.-	(-,*)
H*	τ*	4	--------	(*,*)

*values valid under truncation

g) conclusion

It must be noted that if the negative rainfall amounts obtained by simulation are n o t made zero, which is an option in the computer program, the empirically found correlations are in accordance with the untransformed theoretical correlation function. This means that the analytical solution gives algebraically correct results.

9. THE RECTANGULAR STORM TYPE

9.1. Definition

The rectangular storm type is defined as follows

$$h = {}^{1}f(x) = H, \qquad 0 \leqq x \leqq \tfrac{1}{2}B$$

$$h = {}^{2}f(x) = H, \qquad \tfrac{1}{2}B < x \leqq B$$

So we pay attention to the storm function

$$\boxed{h = f(x) = H, \qquad 0 \leqq x \leqq \tfrac{1}{2}B}$$

where H is constant and also represents the maximum amount in the center.

With uncorrelated random fluctuations the model reads

$$\underline{h}_a = f(a) + \underline{\varepsilon}_a, \qquad (\underline{\varepsilon}_a = \tau\underline{\chi})$$

$$= H + \underline{\varepsilon}_a$$

We choose $H = 10$, $E(\underline{\varepsilon}_a) = 0$ for all a, and $\tau^2 = E(\underline{\varepsilon}_a)^2 = 2500$ and so $\tau = 50$. With these values the correlation function has been evaluated.

9.2. Transformation

Truncation of the normal distribution of $\underline{h}_a$ is at $h = 0$. Truncation of the normal distribution of $\underline{\varepsilon}_a$ is at $-H$, which in standard units reads $z_a^* = -H/\tau$, numerically this equals -0.20.

From column 3 of Table B in Appendix 1 we read

$$E(\underline{z}_a \dagger_{-0.20}) = 0.6751 = \frac{\mu_a^* - H}{\tau}$$

and so

$$\mu_a^* = 50 \times 0.6751 + 10$$

$$= 43.755 \qquad \text{for all } a: \ 0 \leqq a \leqq \tfrac{1}{2}B$$

The standard deviation after truncation becomes

$$\sigma_a^* = 50 \times 0.6397$$

$$= 31.985$$

The transformed storm function now reads

$$h = H^* = 43.755 \qquad 0 \leqq x \leqq \tfrac{1}{2}B$$

9.3. Parameters

No further parameters need to be determined.

9.4. Simulated values

Simulated values of the correlation coëfficient after applying zero's are collected for several values of the interstation distance D in Fig. 6 and are represented by dots (...).

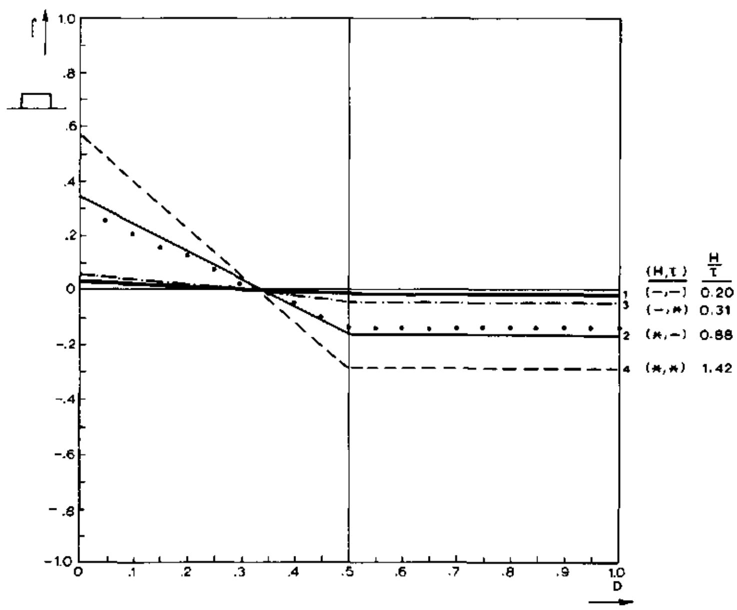


Fig. 6. Illustration of elaborations with the rectangular storm type. Explanation in text

9.5. The correlation function

The general formula of the correlation function in the present case reads, and can be written

$$\rho_{I,II} = 1 - \frac{1+B}{B} \cdot \frac{B + D(H/\tau)^2}{(1+B) + (H/\tau)^2}$$

$$\rho_{III} = 1 - (1+B) \cdot \frac{1 + (H/\tau)^2}{(1+B) + (H/\tau)^2}$$

from which it is easily verified that

$$\rho_{I,II} = 0 \quad \text{for} \quad D = \frac{B}{1+B}\text{, independent of H and } \tau$$

All curves with $B = 0.5$ intersect at $(D, \rho) = (\frac{1}{3}, 0)$. Allowing $\tau \to \infty$ produces $\frac{H}{\tau} \to 0$ and the correlation function for large values of τ with respect to H, becomes a horizontal straight line, since then

$$\rho_{I,II} \to 0 \quad \text{and} \quad \rho_{III} \to 0 \tag{9.1}$$

For $H = 10$ and $\tau = 50$, the employed values, we almost have the situation given by Eq. (9.1). See curve 1 in Fig. 6.

9.6. Correlation function for transformed storm functions

According to the scheme given in Table 2, combinations of values of storm characteristics are given in Fig. 6, curve 1, 2, 3 and 4, respectively.

9.7. Conclusion

After transformation the correlation function with (H^*, τ) gives the best result in approximating the correlation coefficients obtained by simulation. The transformed storm model with (H^*, τ^*)(curve 4) apparently has too low a standard deviation of random fluctuations to be considered an adequate approximation to the simulated values.

10. THE TRIANGULAR STORM TYPE

10.1. Definition

The triangular storm type is defined as follows

$$h = {}^{1}f(x) = \frac{2H}{B}x, \qquad 0 \leqq x \leqq \tfrac{1}{2}B$$

$$h = {}^{2}f(x) = 2H - \frac{2H}{B}x, \qquad \tfrac{1}{2}B < x \leqq B$$

So we pay attention to the storm function

$$\boxed{h = f(x) = \frac{2H}{B}x, \qquad 0 \leqq x \leqq \tfrac{1}{2}B}$$

where H is the maximum rainfall at the center and B is the storm width.

With uncorrelated random fluctuations the model reads

$$\underline{h}_a = \frac{2H}{B}a + \underline{\varepsilon}_a, \qquad (\underline{\varepsilon}_a = \tau\underline{\chi})$$

We choose H = 20, to obtain the same mean storm value as in Section 9 (STOL, 1977e), $E(\underline{\varepsilon}_a) = 0$ for all a, and $\tau^2 = E(\underline{\varepsilon}_a)^2 = 10\ 000$ and so $\tau = 100$. With these values the correlation function has been evaluated.

10.2. Transformation

Truncation of the normal distribution of $\underline{h}_a$ is at h = 0. Truncation of the normal distribution of $\underline{\varepsilon}_a$ is at -2Ha/B, which in standard units reads -2Ha/Bτ, numerically this equals $-2 \times 20a/0.5 \times 100 = -\frac{4}{5}a$, where a is taken a = 0(0.05 B) 0.5 B.

Special values occur at a = 0 and a = ½B giving a = 0 and a = 0.25 giving for the standardized point of truncation

$$z^*_0 = 0 \quad \text{and} \quad z^*_{0.25} = -0.20$$

From Table B, Appendix I, we read

$$E(\underline{z}_a \dagger_0) = 0.7979 \quad \text{and} \quad E(\underline{z}_a \dagger_{-0.20}) = 0.6751$$

and so, since H = 20 and τ = 100, μ_a^* becomes

$$E(\underline{h}_a \dagger_0 \; a = 0) = 79.79 \quad \text{and} \quad E(\underline{h}_a \dagger_0 \; a = \tfrac{1}{2}B) = 87.51$$

The standard deviation after truncation becomes

$$\sigma_a^* = 60.28 \quad (a=0) \quad \text{and} \quad \sigma_a^* = 63.97 \quad (a=\tfrac{1}{2}B)$$

respectively. From these last values an average of σ^*= 62 is constant, has been employed for numerical elaborations.

Since $\mu_a^* = E(\underline{h}_a \dagger_0)$ depends on a, the storm function is not linear anymore. This is illustrated in Fig. 7.

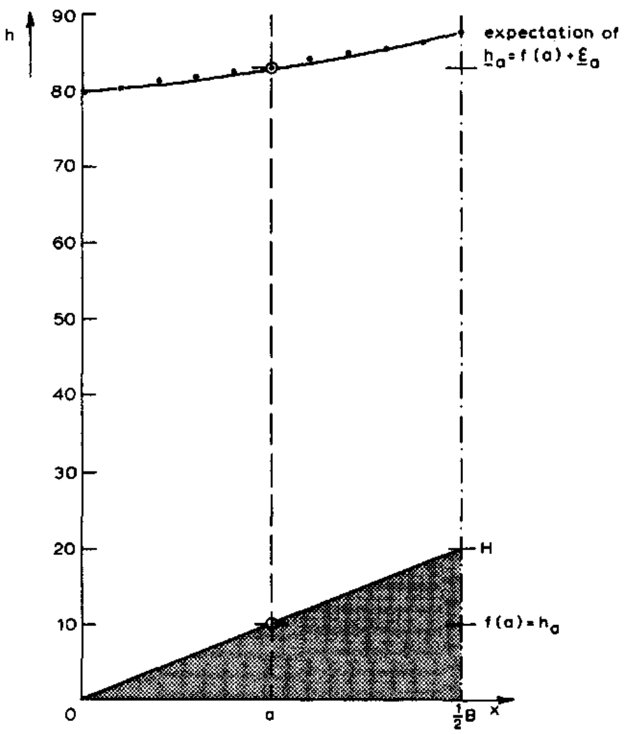


Fig. 7. The triangular storm function and its transformed function. Dots: calculated expectations according Table B of Appendix I; curve: approximation of calculated expectations by an exponential function

The structure of Equation (7.1) and the shape of the locus of expectations suggested an exponential function to approximate the obtained values. Use was therefore made of the exponential storm type with

$$h = f(x) = H^* e^{2b(x - \frac{1}{2}B)}, \qquad 0 \leqq x \leqq \tfrac{1}{2}B$$

in which a further parameter, b, occurs. The storm characteristic H is taken $H^* = 87.51$.

10.3. Parameters

It remains to determine the parameter b such that the exponential function fits the points in Fig. 7 best. This problem was solved as a simple case of a least squares problem. However, it was treated iteratively as a nonlinear problem without taking logarithms (STOL, 1975).

Numerical results are obtained by a computerprogram written by MAASSEN (1977a).

The starting value for the iterative process was obtained by

$$b_o = \frac{1}{B} \ln \frac{h^*_{\frac{1}{2}B}}{h^*_o}$$

which gives

$$b_o = \frac{1}{0.5} \ln \frac{87.51}{79.79}$$

$$= 2 \ln 1.0967$$

$$= 2 \times 0.0923$$

$$= 0.1846$$

After 3 iteration cycles the following result was obtained (Table 3). Discrepancies between values to be used and their approximation are small (less than 0.03). The final value of b then is $b = 0.18523$.

It must be noted, however, that for smaller values of τ the fit is less accurate. See Appendix 2.

Table 3. Numerical results of the fit of the exponential storm function to data obtained by simulation with a triangular storm function (Fig. 7)

a	h = f(a)	$z^*_a = \frac{-f(a)}{\tau}$	$E(\underline{z}_a)$	$E(\underline{h}_a)$	Exponential approximation
.0	0	0	.79788	79.79	79.77
.025	2	-.02	.78520	80.52	80.51
.050	4	-.04	.77260	81.26	81.26
.075	6	-.06	.76008	82.01	82.01
.100	8	-.08	.74766	82.77	82.78
.125	10	-.10	.73533	83.53	83.55
.150	12	-.12	.72309	84.31	84.32
.175	14	-.14	.71095	85.09	85.11
.200	16	-.16	.69889	85.89	85.90
.225	18	-.18	.68693	86.69	86.70
.250	20	-.20	.67507	87.51	87.51

Mean rainfall in storm before transformation 10.00
" " " " after " 83.58

10.4. Simulated values

Simulated values of the correlation coefficient after applying zero's are given in Fig. 8 and are represented by dots (...).

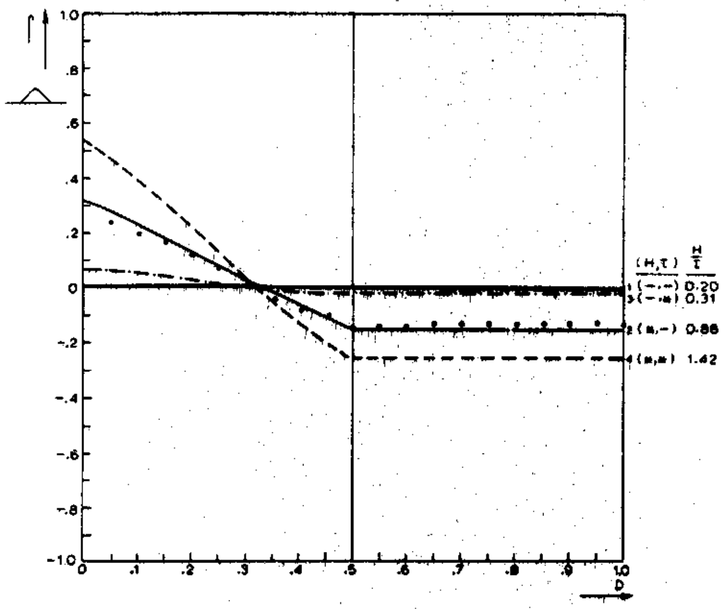


Fig. 8. Illustration of elaborations with the triangular storm type. Explanation in text

10.5. The correlation function

The general formula of the correlation function in the present case reads

$$\rho_I = 1 - \frac{12(1+B)}{B^3} \cdot \frac{2H^2(B-D)\,D^2 + B^3\tau^3}{(1+B)(H^2+12\tau^2) + 3H^2}$$

$$\rho_{II} = 1 - \frac{4(1+B)}{B^3} \cdot \frac{H^2\{B^3 - 2(B-D)^3\} + 3B^3\tau^2}{(1+B)(H^2+12\tau^2) + 3H^2}$$

$$\rho_{III} = 1 - 4(1+B) \cdot \frac{H^2 + 3\tau^2}{(1+B)(H^2+12\tau^2) + 3H^2}$$

from which it easily can be verified that $\rho_I = 0$ if

$$\frac{B^3(4+B)}{12(1+B)} = (B-D)\,D^2,$$ independent of H and τ.

For B = 0.5 we have D = 0,25, so all curves with B = 0.5 intersect at $(D, \rho) = (\frac{1}{4}, 0)$. Dividing through by τ^2 and allowing

$\tau \to \infty$ produces $\frac{H}{\tau} \to 0$ and the correlation function for large values of τ with respect to H, becomes a horizontal straight line, since then

$$\rho_I \to 0, \quad \rho_{II} \to 0 \quad \text{and} \quad \rho_{III} \to 0 \tag{10.1}$$

For H = 20 and τ = 100, the employed values, we almost have the situation given by Eq. (10.1). See curve 1 in Fig. 8.

10.6. Correlation function for transformed storm functions

According to the scheme given in Table 2, combinations of values of storm characteristics are given in Fig. 8, curve 1, 2, 3 and 4, respectively.

10.7. Conclusion

After transformation the correlation function with (H*, τ) as characteristics gives the best result in approximating the correlation coefficients obtained by simulation. The transformed storm model with (H*, τ*) (curve 4) apparently has too low a standard deviation of random fluctuations to be considered an adequate approximation to the simulated data.

11. THE EXPONENTIAL STORM TYPE

11.1. Definition

The exponential storm type is defined as follows

$$h = {}^1f(x) = He^{2b(x-\frac{1}{2}B)}, \quad 0 \leq x \leq \tfrac{1}{2}B$$

$$h = {}^2f(x) = He^{2b(\frac{1}{2}B-x)}, \quad \tfrac{1}{2}B < x \leq B$$

So we pay attention to the storm function

$$h = f(x) = He^{2b(x-\frac{1}{2}B)}, \quad 0 \leqq x \leqq \tfrac{1}{2}B$$

where H is the maximum and b is a further parameter.

With uncorrelated random fluctuations the model reads

$$\underline{h}_a = He^{2b(a-\frac{1}{2}B)} + \underline{\varepsilon}_a, \quad (\underline{\varepsilon}_a = \tau\underline{\chi})$$

We choose H = 20, to obtain the same maximum storm value as in Section 10, $E(\underline{\varepsilon}_a) = 0$ for all a, and $\tau^2 = E(\underline{\varepsilon}_a)^2 = 10\,000$ and so $\tau = 100$.

The parameter b has been chosen such that the mean rainfall amount in the storm equals 10, as was the case for the rectangular and the triangular storm type. The appropriate value then is b = 3.187 (STOL, 1977e).

With these values the correlation function has been evaluated.

11.2. Transformation

Truncation of the normal distribution of $\underline{h}_a$ is at $h = 0$. Truncation of the normal distribution of $\underline{\varepsilon}_a$ is at $-H\exp\{2b(a-\frac{1}{2}B)\}$, which has to be devided by τ to obtain standard units. Numerical results are

$$z_a^* = \frac{-20\exp\{2b(a-0.25)\}}{100} \qquad (b = 3.187)$$

where a is taken a = 0(0.05 B) 0.5 B.

Special values occur at a = 0 and a = $\frac{1}{2}$B, so a = 0 and a = 0.25, giving $h_0 = 4.1$ and $h_{0.25} = 20$ which yields for the standardized point of truncation

$$z_0^* = -0.041 \quad \text{and} \quad z_{0.25}^* = -0.20$$

From Table B, Appendix 1, we read

$$E(\underline{z}_a \dagger_{-0.041}) = 0.7722 \quad \text{and} \quad E(\underline{z}_a \dagger_{-0.20}) = 0.6751$$

and so, since H = 20 and τ = 100, μ_a^* becomes

$$E(\underline{h}_a \dagger_0\ a = 0) = 81.28 \quad \text{and} \quad E(\underline{h}_a \dagger_0\ a = \tfrac{1}{2}B) = 87.51$$

The standard deviation after truncation becomes

$$\sigma_a^* \simeq 60.28 \quad (a = 0) \quad \text{and} \quad \sigma_a^* \simeq 63.97 \quad (a = \tfrac{1}{2}B)$$

respectively. From these last values an average of σ^*= 62 is constant, has been employed for numerical elaborations. Since $\mu_a^* = E(\underline{h}_a \dagger_0)$ depends on a, the storm function cannot be obtained by a simple shift in vertical direction of the original storm function. This is illustrated in Fig. 9.

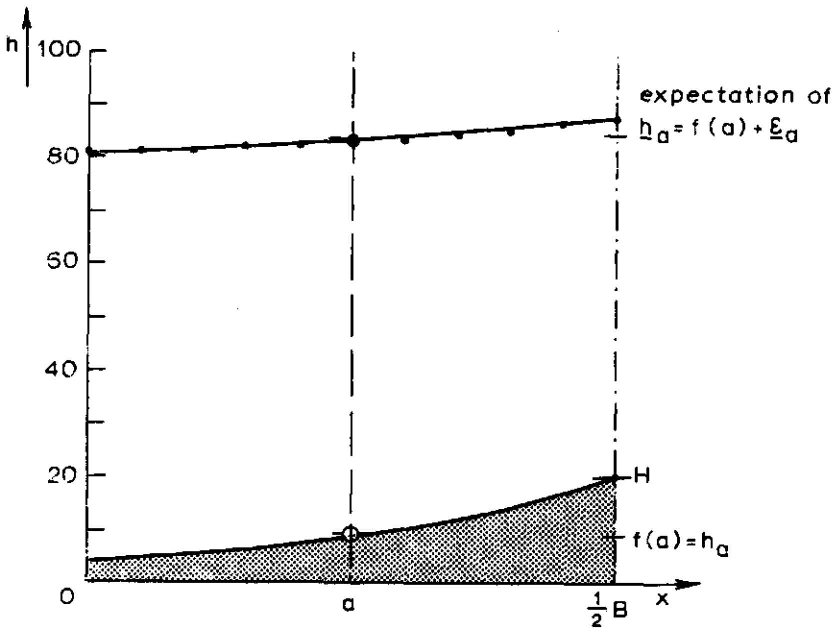


Fig. 9. The exponential storm function and its transformed function. Dots: calculated expectations according Table 2 of Appendix 1, curve: approximation of calculated expectations by an exponential function

The structure of Eq.(7.1) and the shape of the locus of expectations suggested an exponential function to approximate the obtained values. Use was therefore made of the exponential storm type with

$$h = f(x) = H^{*} e^{2b(x-\frac{1}{2}B)}, \quad 0 \leq x \leq \tfrac{1}{2}B$$

in which a further parameter, b, occurs. The storm characteristic H is taken H* = 87.51.

11.3. P a r a m e t e r s

It remains to determine the parameter b such that the exponential function fits the points in Fig. 9 best. The same procedure as the one described in the former Section has been applied.

The starting value this time was

$$b_0 = \frac{1}{0.5} \ln \frac{87.51}{81.28} = 0.1476$$

After 3 iteration cycles the following result was obtained (Table 4). Discrepancies between values to be used and their approximation are small but greater than in the former case (less than 0.9). The final value of b then is b = 0.17155.

Table 4. Numerical results of the fit of the exponential storm function to data obtained by simulation with an exponential storm function (Fig. 9)

a	h = f(a)	$z^*_a = \frac{-f(a)}{\tau}$	$E(\underline{z}_a)$	$E(\underline{h}_a)$	Exponential approximation
.0	4.064	-0.041	.77219	81.28	80.31
.025	4.766	-0.048	.76780	81.55	81.01
.050	5.590	-0.056	.76264	81.85	81.70
.075	6.555	-0.066	.75663	82.22	82.41
.100	7.688	-0.077	.74960	82.65	83.12
.125	9.016	-0.090	.74139	83.15	83.83
.150	10.573	-0.106	.73181	83.75	84.56
.175	12.400	-0.124	.72066	84.47	85.28
.200	14.542	-0.145	.70767	85.31	86.02
.225	17.054	-0.171	.69258	86.31	86.76
.250	20.000	-0.200	.67507	87.51	87.51

P.T.O.

Mean rainfall in storm before transformation = 10

" " " " after " = 85.64

11.4. Simulated values

Simulated values of the correlation coefficient after applying zero's are given in Fig. 10 and are represented by dots (...).

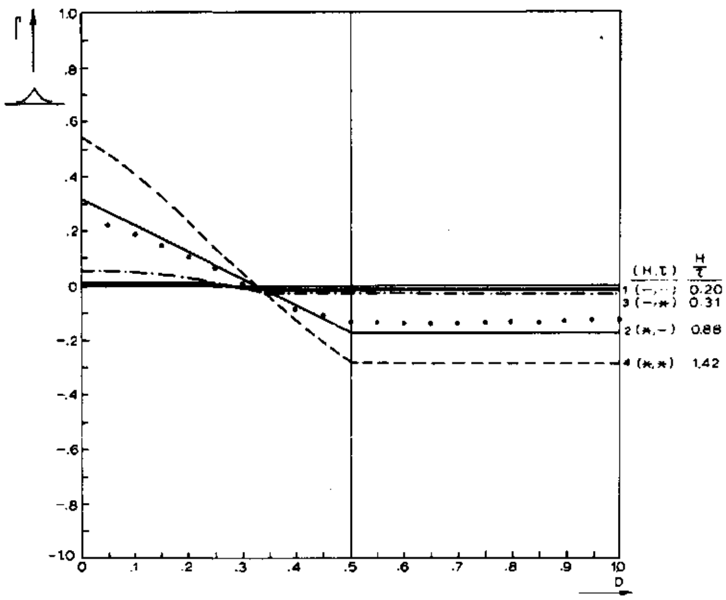


Fig. 10. Illustration of elaborations with the exponential storm type. Explanation in text

11.5. The correlation function

The correlation function for the exponential storm type is even more complicated than the one for the triangular storm type. (Section 10.5), it will not be given here explicitely but the reader is referred to to STOL (1977b).

The properties of this correlation function, however, are analogous to those of the former functions discussed in this report. Compare Fig. 10 with Fig. 8.

11.6. Correlation function for transformed storm functions

According to the scheme given in Table 2, combinations of values of storm characteristics are given in Fig. 10, curve 1, 2, 3 and 4, respectively.

11.7. Conclusion

After transformation the correlation function with (H^*, τ) as characteristics gives the best result in approximating the correlation coefficients obtained by simulation. The transformed storm model with (H^*, τ^*) (curve 4) apparently has too low a standard deviation of random fluctuations to be considered an adequate approximation to the simulated data.

12. CONCLUDING REMARKS

1) In all examples values of the storm characteristics H and τ are chosen such that they meet theorytical conditions. Values used are collected in next summary (Table 5):

Table 5. Summary of combinations of used values for storm characteristics H and τ

No.	Characteristics	Rectangular type		Triangular and exponential type		$\frac{H}{\tau}$
1	H, τ	10	50	20	100	0.20
3	H, τ^*	10	31	20	62	0.31
2	H^*, τ	44	50	88	100	0.88
4	H^*, τ^*	44	31	88	62	1.42

Although in Figures 6, 8 and 10 these values are used, no. 2 (H^* under truncation, τ without truncation) giving the best results in all three cases, it is the ratio $\frac{H}{\tau}$ that really matters. This is the reason that in all three cases about the same results are obtained since the ratio's are the same. It still is to explain why the unaltered value of τ gives the best results. A still better approximation could be obtained with a slightly lower value of $\frac{H}{\tau}$ but it is not clear whether τ should be taken less than 88 (but greater than 20).

These alternative choices are not supported by the theory developed thus far.

2) All storm models employed behave in the same way. Introducing large random fluctuations does give correlation functions that are much alike. To demonstrate this the simulated values of the correlation coefficient are collected in Fig. 11. Here the conclusion is that the random fluctuations, actually the H/τ ratio, determine in the present study the shape of the correlation function.

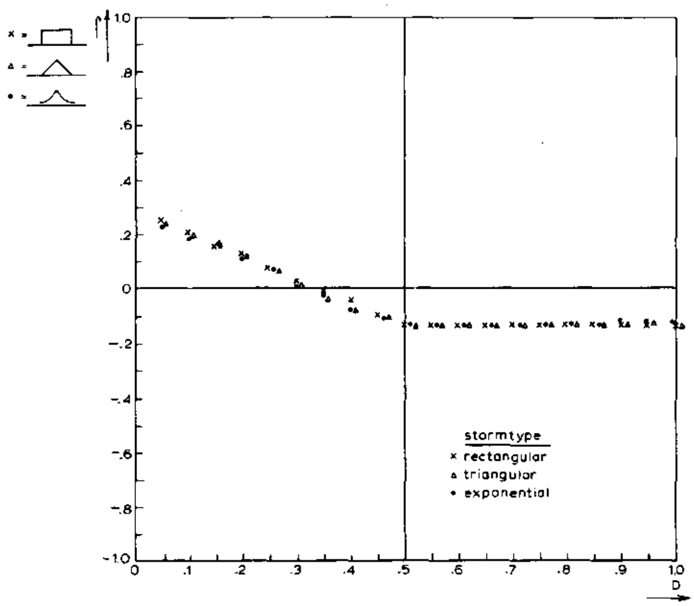


Fig. 11. Values of simulated correlation coefficients have been collected from Figs. 6, 8 and 10. The shape of the simulated correlation function appears to be the same

3) Finally it should be remembered that an irrealistic value of τ had been used. This to enlarge discrepancies between simulated and analytically derived values. Other combinations of values, applying stormwidth B as a parameter as well, could be used to enlarge insight in the problem stated.

REFERENCES

GRAF, U., H.J. HENNING, K. STANGE, 1966. Formeln und Tabellen der mathematischen Statistiek. 2. Auflage. Berlin/Heidelberg/ New York. Springer-Verlag pp 362

HALD, A., 1967. Statistical theory with engineering applications. 7th printing. John Wiley & Sons, Inc. New York, London, Sydney, pp. 783

HASTINGS, C., 1955. Approximations for digital computers. Princeton University Press. Princeton, New Jersey. pp. 201

JOHNSON, N.L. and S. KOTZ., 1970. Distributions in statistics: Continuous univariate distributions-1. Houghton Mifflin Series in Statistics. John Wiley & Sons, Inc. New York. pp. 300

MAASSEN, J.R., 1977a. Computerprogram BUIEN (SHOWERS) to optimize the parameter in the exponential storm type. FORTRAN IV, extended

_____ 1977b. Computer program ERROR for numerically approximated normal distributions and error-functions according to Hastings. FORTRAN IV, extended

MOOD, A.M. and F.A. GRAYBILL, 1963. Introduction to the theory of statistics. Int. student ed. 2nd ed. McGraw-Hill Book Company, Inc., New York, London, pp. 443

STOL, PH.TH., 1975. A contribution to nonlineair parameter optimization. Agric. Research Report 835, pp. 197. Pudoc, Wageningen

_____ 1977a. Principles underlying an analytic model for the process of measuring rainfall amounts and the determination of inter-station correlations. Nota I.C.W. 992. pp. 37

_____ 1977b. The fraction of dry days as a parameter in analytic rainfall interstation correlation functions. Nota I.C.W. 1002, pp. 23

_____ 1977c. Solution of integrals necessary to determine rainfall interstation correlation functions. Nota I.C.W. 993, pp. 34

_____ 1977d. Rainfall interstation correlation functions: an analytic approach. in print

_____ 1977e. Rain storm models and the relationship between their parameters. Nota I.C.W. 1001. pp. 31

TRUNCATED NORMAL DISTRIBUTIONS

This Appendix is meant to give some theory on trucated normal distributions that will be used in connection with storm functions. Symbols introduced in this Appendix do not match those in the Report especially the use of the constants a and b is different. Symbols commonly used in statistics are employed here. All symbols are defined in the text.

1. General

Let $\underline{\chi}$ be normally distributed with expectation 0 and variance 1, to be written $\underline{\chi}$ (0, 1). Then

$$\underline{x} = \mu + \sigma\underline{\chi}$$

is normally distributed with expectation μ and variance σ^2, or is $\underline{x}$ (μ, σ^2). Following MOOD and GRAYBILL (1963)'s notation and writing densities with the aid of differentials, so employing probability elements (HALD, 1967, p. 93) we define the density n(x) of x by

$$n(x)\ dx = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

and the cumulative distribution by

$$N(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt$$

Some particular values are given in Table A.

Table A. Particular values of the normal distribution and its density

x	n(x)	N(x)
$-\infty$	0	0
μ	$1/\sigma\sqrt{2\pi}$	1/2
$+\infty$	0	1

Now we assume that, given the normal distribution, values below a given value x = a do not occur and that values greater than a given value x = b do not occur either. Thus the distribution of $\underline{x}$ is assumed to be defined on the interval

$$a \leqq x \leqq b$$

and the distribution of $\underline{\chi}$ consequently on the interval

$$\frac{a - \mu}{\sigma} \leqq \chi \leqq \frac{b - \mu}{\sigma}$$

or, by definition, in standard units

$$\alpha \leqq \chi \leqq \beta$$

This produces a so-called two sided truncated normal distribution.

We employ the following notational convention. In connection with random variables:

$\underline{x} \nmid_{a}^{b}$ = 'under truncation', lower point being a, upper point being b

$\underline{x} \nmid_{a} = \underline{x} \nmid_{a}^{\infty}$ = under truncation from below

$\underline{x} \nmid_{0}$ = under truncation from below: only positive values and zero's can occur

In connection with variables and parameters an asterisk (*) is used. We define its use as follows:

$x^* = a$ lower point of truncation for the random variable $\underline{x}$ which is truncated at $x = a$ from below

$\mu^*(a)$ mathematical expectation of the random variable $\underline{x}$, when the distribution of $\underline{x}$ is truncated at $x = a$ from below

Without ambiguity the argument in the last definition can sometimes be dropped.

Since the two-sided truncated distribution is defined on the interval $[a,b]$ only, probabilities have to be expressed in fractions of the total probability mass valid for the distribution.
So

$$P(\underline{x} < x \,\big|_a^b) = N(x \,\big|_a^b) = \frac{N(x) - N(a)}{N(b) - N(a)}$$

which is illustrated in Fig. 1.

The density of this distribution is obtained by differentiating this expressing with respect to x which yields

$$\frac{dP}{dx} = n(x \,\big|_a^b) = \frac{\frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2\}}{N(b) - N(a)}$$

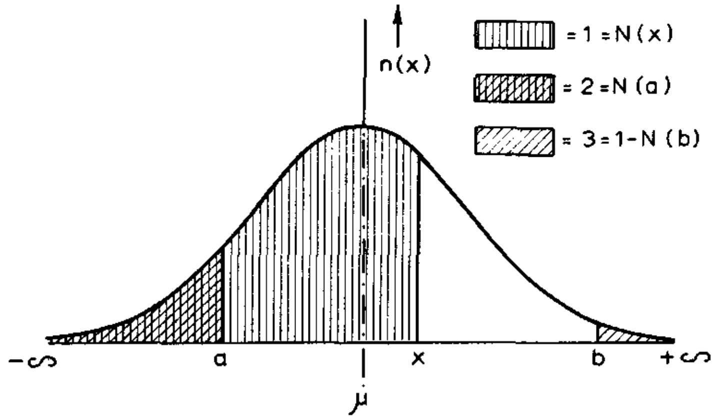


Fig. 1. Example of a normal distribution supposed to be truncated at a and b respectively. Legend: 1 = N(x) = area under the curve from - ∞ to x; 2 = N(a) = area under the curve from - ∞ to a: the degree of truncation from below; 3 = 1 - N(b) = area under the curve from b to + ∞: the degree of truncation from above

In this case a is called the lower truncation point and b is called the upper truncation point, while N(a) stands for the degree of truncation from below and 1 - N(b) for the degree of truncation from above.

2. T h e m a t h e m a t i c a l e x p e c t a t i o n

The mathematical expectation of $\underline{x}$ and functions of $\underline{x}$ can be obtained from the density. So we have

$$E(\underline{x} \overset{b}{\underset{a}{\dagger}}) = \frac{\frac{1}{\sigma\sqrt{2\pi}} \int_a^b x \, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx}{N(b) - N(a)} \qquad (1)$$

The integral in the numerator can be worked out as follows:

$$\sigma \int_a^b \left(\frac{x-\mu}{\sigma} + \frac{\mu}{\sigma}\right) \exp \left\{-\tfrac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} dx$$

$$= \sigma^2 \int_a^b \frac{x-\mu}{\sigma} \exp\{-\tfrac{1}{2}(\tfrac{x-\mu}{\sigma})^2\}\, d(\tfrac{x-\mu}{\sigma}) + \mu\{N(b) - N(a)\}\, \sigma\sqrt{2\pi}$$

$$= -\sigma^2 \int_a^b \exp\{-\tfrac{1}{2}(\tfrac{x-\mu}{\sigma})^2\}\, d\{-\tfrac{1}{2}(\tfrac{x-\mu}{\sigma})^2\} + \mu\{N(b) - N(a)\}\, \sigma\sqrt{2\pi}$$

$$= -\sigma^2\{\exp -\tfrac{1}{2}(\tfrac{x-\mu}{\sigma})^2\} \Big|_a^b + \mu\{N(b) - N(a)\}\, \sigma\sqrt{2\pi}$$

which by means of symbols introduced before becomes

$$= -\sigma^2\ \{n(b) - n(a)\}\ \sigma\sqrt{2\pi} + \mu\ \{N(b) - N(a)\}\ \sigma\sqrt{2\pi}$$

Inserted in Eq. (1) the mathematical expectation becomes

$$E(\underline{x} \mathop{+}_a^b) = \frac{\mu\{N(b) - N(a)\} - \sigma^2\{n(b) - n(a)\}}{N(b)-N(a)}$$

and finally, see also Mood and Graybill (1963, p. 138):

$$\boxed{E(\underline{x} \mathop{+}_a^b) = \mu + \frac{n(a) - n(b)}{N(b) - N(a)}\,\sigma^2}$$

Now we will express a and b in standard units and define the parameters α and β as follows

$$z = \frac{x-\mu}{\sigma} \qquad \text{so} \qquad x = \mu + \sigma z$$

$$x = a \;\rightarrow\; z = \frac{a-\mu}{\sigma} = \alpha \quad \text{and} \quad a = \mu + \alpha\sigma$$

$$x = b \;\rightarrow\; z = \frac{b-\mu}{\sigma} = \beta \quad \text{and} \quad b = \mu + \beta\sigma$$

$$x = t \;\rightarrow\; z = \frac{t-\mu}{\sigma} \qquad \text{and} \quad t = \mu + \sigma z$$

with differentials $dx = \sigma dz$

and $dt = \sigma dz$

This gives

$$n(a) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\alpha^2}$$

$$n(b) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\beta^2}$$

and

$$N(a) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\mu+\alpha\sigma} \exp \{-\tfrac{1}{2}(\frac{t-\mu}{\sigma})^2 \ dt$$

The upper boundary for t, viz. $t = a$, is written $a = \mu + \alpha\sigma$. Consequently $z = \frac{x - \mu}{\sigma}$ is satisfied for $x = a$ with the corresponding value $z = \alpha$ and we can write

$$N(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} \exp \{-\tfrac{1}{2}z^2\} \ dz$$

and analogously

$$N(b) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\beta} \exp \{-\tfrac{1}{2}z^2\} \ dz$$

Define $\Phi(z)$ to represent the cumulative s t a n d a r i z e d normal distribution, and $\phi(z)$ to represent its density so

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} \ dt$$

and

$$\phi(z)dz = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \ dz$$

then we have

$$n(a) = \frac{1}{\sigma}\,\phi(\alpha) = \frac{1}{\sigma}\,\phi(\frac{a-\mu}{\sigma})$$

$$n(b) = \frac{1}{\sigma}\,\phi(\beta) = \frac{1}{\sigma}\,\phi(\frac{b-\mu}{\sigma})$$

$$N(a) = \Phi(\alpha) = \Phi(\frac{a-\mu}{\sigma})$$

$$N(b) = \Phi(\beta) = \Phi(\frac{b-\mu}{\sigma})$$

and so we can express the required expectation with standarized normal distribution functions by

$$E(\underline{x} \mathop{+}\limits_{a}^{b}) = \mu + \frac{\phi(\frac{a-\mu}{\sigma}) - \phi(\frac{b-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}\,\sigma$$

as given by JOHNSON and KOTZ (1970, p 81).

3. The variance

Truncating normal distributions means that the scattering of individual points about the mean becomes less. The variance of the truncated normal distribution therefore is smaller than the variance of the original distribution. The general formula for the variance of truncated normal distributions can be derived along lines given in Section 2. The result as given by JOHNSON and KOTZ (1970) reads

$$Var(\underline{x} \mathop{+}\limits_{a}^{b}) = \left[1 + \frac{(\frac{a-\mu}{\sigma})\,\phi(\frac{a-\mu}{\sigma}) - (\frac{b-\mu}{\sigma})\,\phi(\frac{b-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})} - \left\{ \frac{\phi(\frac{a-\mu}{\sigma}) - \phi(\frac{b-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})} \right\}^2 \right] \sigma^2$$

4. S p e c i a l   c a s e s

Next we assume that $\underline{x}$ is truncated only from the left and so $b \to +\infty$ giving

$$\phi(\frac{b-\mu}{\sigma}) \to 0$$

$$\Phi(\frac{b-\mu}{\sigma}) \to 1$$

This means that, using our notational convention

$$E(\underline{x} \underset{a}{\mid}) = \mu + \frac{\phi(\frac{a-\mu}{\sigma})}{1-\Phi(\frac{a-\mu}{\sigma})}\sigma = \mu^*(a) \qquad (2)$$

the variance being

$$Var(\underline{x} \underset{a}{\mid}) = \left[1 + \frac{a-\mu}{\sigma} \frac{\phi(\frac{a-\mu}{\sigma})}{1-\Phi(\frac{a-\mu}{\sigma})} - \left\{\frac{\phi(\frac{a-\mu}{\sigma})}{1-\Phi(\frac{a-\mu}{\sigma})}\right\}^2\right]\sigma^2 = \sigma^{*2}(a)$$

so without arguments,

$$\sigma^{*2} = \left[1 + \frac{a-\mu}{\sigma}(\frac{\mu^*-\mu}{\sigma}) - (\frac{\mu^*-\mu}{\sigma})^2\right]\sigma^2 \qquad (3)$$

or

$$\sigma^{*2} = \sigma^2 + (a-\mu)(\mu^*-\mu) - (\mu^*-\mu)^2$$

If $a = \mu$ we have a further special case, namely (see Table A),

$$\left.\begin{aligned} \mu^*(\mu) &= \mu + \frac{1/\sqrt{2\pi}}{1/2} = \mu + 0.7978\ \sigma \\ \sigma^{*2}(\mu) &= \sigma^2 - (0.7978\ \sigma)^2 = 0.6028\ \sigma \end{aligned}\right\} \text{ for } \underline{\chi}\ (\mu,\sigma^2)$$

If we have $\mu = 0$ and $\sigma = 1$ the final most simple result reads

$$\left.\begin{aligned} \mu^*(0) &= 0.7978 \\ \sigma^{*2}(0) &= 0.6028 \end{aligned}\right\} \text{ for } \underline{\chi}\ (\mu,\sigma^2) = \underline{\chi}\ (0,\ 1)$$

which is valid for standard normal distributions.

5. Special case for rainfall

Applied to rainfall amounts we will consider normal distributions that are truncated at zero to avoid the occurrence of negative precipitation values. So a = 0 and we have from Eq. (2)

$$\mu^*(0) = \mu + \frac{\phi(\frac{-\mu}{\sigma})}{1 - \Phi(\frac{-\mu}{\sigma})}\sigma \qquad (4)$$

and from Eq. (3) after adding terms

$$\sigma^{*2}(0) = \left[1 - \frac{\mu^*}{\sigma} \cdot \frac{\mu^*-\mu}{\sigma}\right]\sigma^2 \qquad (5)$$

Since the integral Φ of the normal distribution cannot be expressed in elementary functions values of it can be obtained only by numerical integration. In our case, however, values are obtained by approximating formulas given by HASTINGS (1955) that furnish enough accurate decimals for our practical purposes. Values are calculated in a computer program developped by MAASSEN (1977b). Results are given in Table B.

In Table B the following values are tabulated.

Column 1: The standardized lower point of truncation, $z^* = \alpha = -\mu/\sigma$, (which corresponds with $x^* = \mu$) ranging -3.0(0.2)3.0

Column 2: The degree of truncation from below viz. $P(\underline{z} \leqq \alpha)$ or $P(\underline{x} \leqq \mu)$, in percentages.

Column 3: Values of the expression $\frac{\mu^*-\mu}{\sigma}$ according Eq. (4)

Column 4: Values of the expression $\frac{\sigma^*}{\sigma}$ according Eq.(5) by taking the square root

Application of Table B is discussed in the main text of this report. The meaning of the 'heading at the bottom' shall be explained in Section 4.

Table B. Characteristic values of the standard normal distribution which is truncated from below

1	2	3	4
point of truncation	degree of truncation	expectation	standard deviation
$\alpha = -\mu/\sigma$	$P(\underline{z} \leq \alpha)$	$\frac{\mu^{*}-\mu}{\sigma} =$	$\frac{\sigma^{*}}{\sigma} =$
∞	100	∞	0
3.0	99.87	3.2829	0.2667
2.8	99.74	3.0978	0.2784
2.6	99.53	2.9140	0.2913
2.4	99.18	2.7319	0.3056
2.2	98.61	2.5515	0.3211
2.0	97.72	2.3732	0.3380
1.8	96.41	2.1973	0.3563
1.6	94.52	2.0241	0.3762
1.4	91.92	1.8541	0.3977
1.2	88.49	1.6876	0.4210
1.0	84.13	1.5251	0.4462
0.8	78.81	1.3674	0.4734
0.6	72.57	1.2150	0.5027
0.4	65.54	1.0688	0.5341
0.2	57.93	0.9294	0.5675
0.0	50.00	0.7979	0.6028
- 0.2	42.07	0.6751	0.6397
- 0.4	34.46	0.5619	0.6779
- 0.6	27.43	0.4591	0.7167
- 0.8	21.19	0.3676	0.7555
- 1.0	15.87	0.2876	0.7935
- 1.2	11.51	0.2194	0.8298
- 1.4	8.08	0.1629	0.8634
- 1.6	5.48	0.1174	0.8936
- 1.8	3.59	0.0819	0.9197
- 2.0	2.28	0.0552	0.9415
- 2.2	1.39	0.0360	0.9589
- 2.4	0.82	0.0226	0.9723
- 2.6	0.47	0.0136	0.9820
- 2.8	0.26	0.0079	0.9888
- 3.0	0.13	0.0044	0.9933
- ∞	0	0	1
$z_a^{*} = \frac{-f(a)}{\tau}$	$P(\underline{z} \leq z_a^{*})$	$\frac{\mu_a^{*}-f(a)}{\tau} =$	$\frac{\sigma_a^{*}}{\tau} =$

The values given in columns 1, 3 and 4 are plotted in a graph to visualize the relevant relationships between point of truncation and the mathematical expectation and standard deviation for standard normal distributions (Fig. 2).

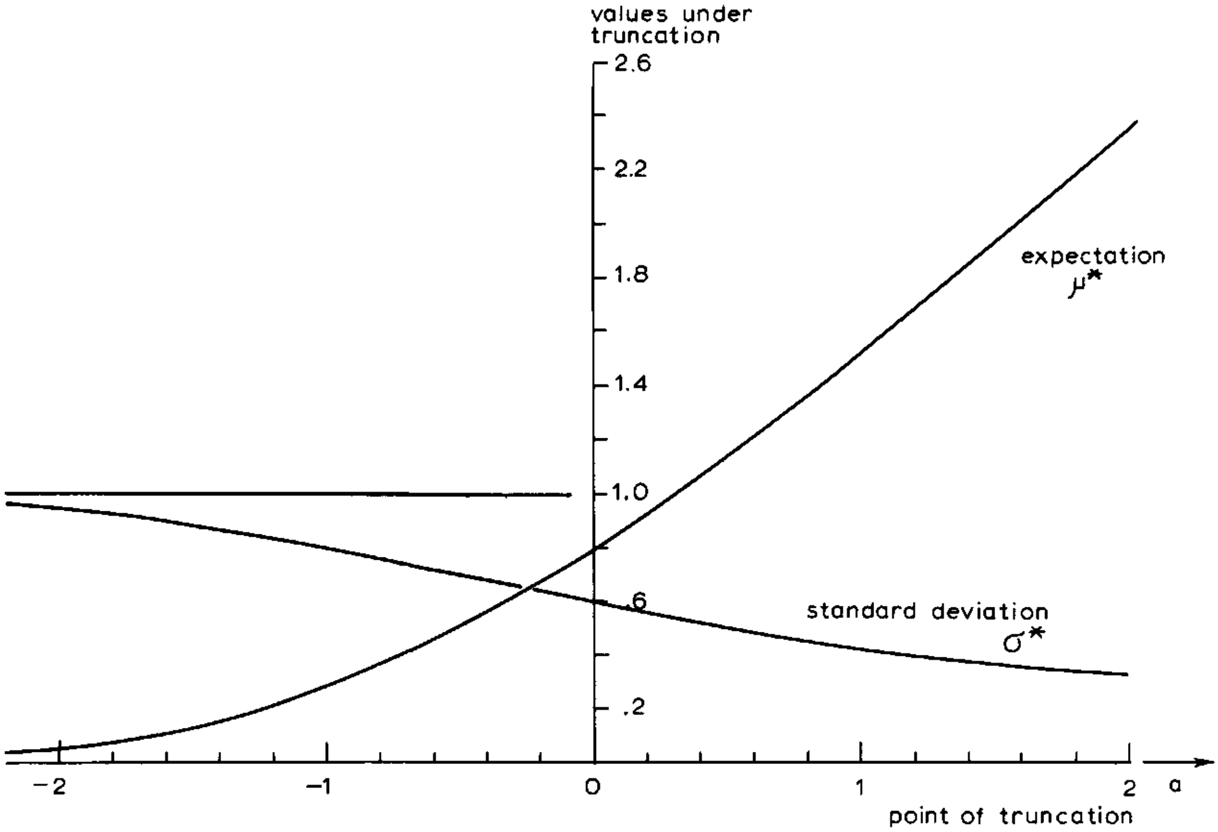


Fig. 2. Curves for the expectation (μ*) and standard deviation (σ*) at increasing points of truncation from below for a standard normal distribution (see also Table B)

It is seen that with increasing degree of truncation from below the mean value increases while the standard deviation decreases.

THE FIT OF THE TRIANGULAR STORM TYPE BY AN EXPONENTIAL STORM FUNCTION WHEN τ IS SMALL

In Section 10.3 the fit of the triangular storm type with random fluctuations, by an exponential storm function was treated. Storm characteristics were H = 20 and τ = 100. The fit was reasonable. (Fig. 7 and Table 3).

If the value of τ is much smaller, τ = 5, say, then the influence of random fluctuations near the center of the storm is limited. Namely in this neighbourhood $-h/\tau$ is approximately $-20/5 = -4$ and the degree of truncation is small. This causes the shape of the storm function to remain straight. (See Fig. 3). Discrepancies from the straight line only occur for values x <0.25 B. The exponential fit to the straight line is poor and less accurate as the one demonstrated in Section 10.3 where τ = 100.

The results for H = 20, τ = 5 are given in Table C and Fig. 3.

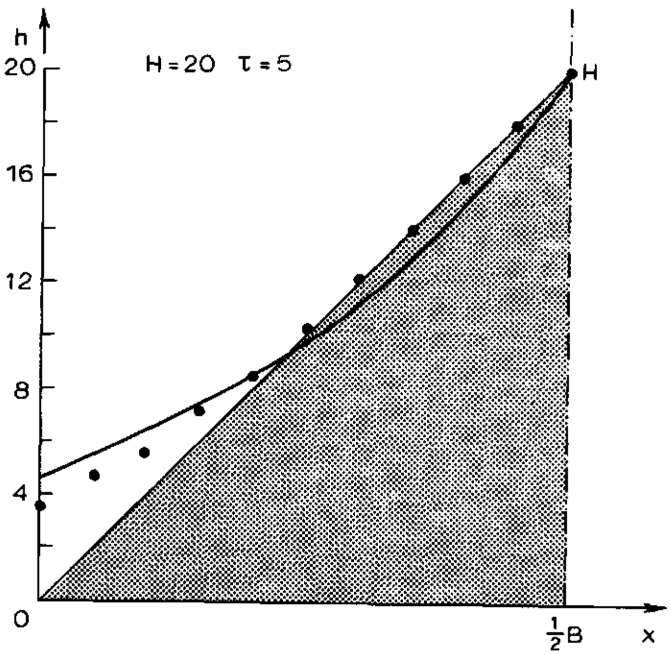


Fig. 3. The triangular storm function and its transformed function for H = 20 and τ = 5
Dots: calculated expectations according Table B of Appendix 1;
Curve: approximation of calculated expectations by an exponential function

Table C. Numerical results of the fit of the exponential storm function to data obtained by simulation with a triangular storm function (Fig. 3) where H = 20 and τ = 5

a	h=f(a)	$z^*_a = \frac{-f(a)}{\tau}$	$E(\underline{z}_a)$	$E(\underline{h}_a)$	Exponential approximation
.0	0	0	.79788	3.99	4.87
.025	2	- .4	.56188	4.81	5.60
.050	4	- .8	.36756	5.84	6.46
.075	6	-1.2	.21943	7.10	7.44
.100	8	-1.6	.11735	8.59	8.56
.125	10	-2.0	.05525	10.28	9.86
.150	12	-2.4	.02258	12.11	11.36
.175	14	-2.8	.00794	14.04	13.09
.200	16	-3.2	.00239	16.01	15.08
.225	18	-3.6	.00061	18.00	17.36
.250	20	-4.0	.00013	20.00	20.00

Mean rainfall in storm before transformation 10.00
" " " " after " 10.71

The starting value of b was b_0 = 3.2242

after 5 iterations the value b = 2.8271 was found. Discrepancies between values to be used and their approximation now amounts .88 or less. This is much greater than the value of 0.03 which was found in Section 10.3.