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ANALYTICAL DESCRIPTION OF STEADY FLOW ABOVE A SHALLOW WATER TABLE WITH WATER UPTAKE BY ROOTS USING DIFFERENT HYDRAULIC CONDUCTIVITY FUNCTIONS

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CONTENTS

I. INTRODUCTION*

One of the critical assumptions in modelling water movement in soils under unsaturated conditions is the one relating hydraulic conductivity K to pressure head Ψ. A survey by RAATS and GARDNER (1974) lists six empirical relationships that have been used.

The choice of the K(Ψ) relationship is not the only problem, however, applying data from laboratory experiments to field sites can also be of concern. It is recognized that soil in its undisturbed state has different properties than disturbed soil samples (WESSELING and WIT, 1966). Even when taking undisturbed samples, the variation of soil properties within a small region of what appears to be homogeneous soils, may be such that problems will arise.
This is called spatial variability and is addressed by WARRICK et al. (1977a, 1977b), NIELSEN et al. and MULLEN and PARASHER at the Symposium on International Drainage in Field Soils (see EGS abstracts, 1978).

A natural question that arises is: "How sensitive are the results from a mathematical model of flow in unsaturated soil to changes in the K(Ψ) relationship?"

In this paper two types of functions are used: exponential variation of K with Ψ (see e.g. GARDNER, 1958) and a variation according to a power law (WIND, 1955; WESSELING, 1957). The soil used is a heavy clay soil for which J. BOUMA** did measurements with a so-called crust method (for the wet range) and a dry hot air method (for the

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dry range). For details see BOUMA (1977). The flow cases studied are taken from the investigations of FEDDES (1971) at the groundwater level experimental field Geestmerambacht and concern a red cabbage crop grown on a sticky clay soil in the presence of a shallow groundwater table.

Specific sink functions will be taken and comparisons will be made between pressure head ($\log \psi = pF$) profiles as typical in the field.

In general one can state that the sensitivity of a mathematical model to changes in hydraulic conductivity is not simply an academic question. The answer to this question should provide some assistance in determining how much consideration should be given to the variability of soil types.

Analytic solutions are usually restricted to specific, mostly simplified flow cases. However, they provide exact answers for the situation investigated and can thus be used to check complex numerical schemes, for which it is difficult to detect errors.

II. HYDRAULIC CONDUCTIVITY AS AN EXPONENTIAL FUNCTION OF PRESSURE HEAD

a. General considerations

Darcy's law states that the velocity of water moving in unsaturated soil is proportional to the gradient of the total head, i.e.

$$v = - K\nabla H \tag{1}$$

The constant of proportionality, K, is called the unsaturated hydraulic conductivity and is usually taken to be a function of the pressure head Ψ($H = \Psi - z$ if z is positive downward into the soil).

If we now apply the principle of conservation of mass in the horizontal region between the water table and the soil surface we obtain

$$\frac{\partial v}{\partial z} = - S \tag{2}$$

for an equilibrium situation where S is the volume of water used by the roots per unit volume of soil per unit time. Substituting eq. (1) into eq. (2) gives for strictly vertical flow

$$\frac{d}{dz}\left[K \frac{dH}{dz}\right] = \frac{d}{dz}\left[K(\frac{d\Psi}{dz} - 1)\right] = S \tag{3}$$

We now assume that the unsaturated hydraulic conductivity changes as

$$K = K_o \exp(\alpha\Psi) \tag{4}$$

and define a new independent variable, ϕ, by

$$\phi = \int_{-\infty}^{\Psi} K(\Psi)\, d\Psi = K/\alpha \tag{5}$$

This transformation has been used by GARDNER (1958) and others to obtain solutions to problems in soil physics. It is also known as

Kirchhoff's transformation (see IRMAY (1966)). If we differentiate the first part of eq. (5) and substitute the result into eq. (3), the final form of our differential equation becomes

$$\frac{d}{dz}\left[\frac{d\phi}{dz} - \alpha\phi\right] = S \quad , \qquad 0 < z < L \tag{6}$$

For a specified sink function, S, we seek solutions of eq. (6) subject to specified boundary conditions at z = 0 and z = L. If a surface flux is specified at z = 0, we need

$$- K \frac{dH}{dz} = - \frac{d\phi}{dz} + \alpha\phi = v_o \qquad \text{at} \qquad z = 0 \tag{7}$$

The second boundary condition specifies the value of pressure head of the water table. In terms of ϕ we can say

$$\phi = \phi_o \qquad \text{at} \qquad z = L \tag{8}$$

b. Sink functions depending only on z

If S = S(z), eq. (6) may be directly integrated and yields

$$\frac{d\phi}{dz} - \alpha\phi = \int_0^z S(z')\, dz' - v_o \tag{9}$$

(note that we have already satisfied eq. (7)). If we rewrite the left hand side of eq. (9) as

$$\frac{d\phi}{dz} - \alpha\phi = e^{\alpha z} \frac{d}{dz}(e^{-\alpha z}\ \phi)$$

we can integrate eq. (9) and obtain

$$e^{-\alpha z}\ \phi = - \int_z^L \left[\int_0^{z''} S(z')\, dz' - v_o\right] e^{-\alpha z''}\, dz'' + C \tag{10}$$

The constant C in eq. (10) must be chosen as $\phi_o\ e^{-\alpha L}$ in order to satisfy the boundary condition at the water table (eq. (8)).

Carrying out the only integration possible for a general S puts eq. (10) in the form

$$\phi(z) = v_o/\alpha + (\phi_o - v_o/\alpha)\, e^{-\alpha(L-z)} - e^{\alpha z} \int_z^L e^{-\alpha z''} \int_0^{z''} S(z')\, dz' dz'' \tag{11}$$

For any desired water uptake pattern as a function of z, the integrals in eq. (11) may be evaluated and ϕ(also Ψ) is determined. Among the examples of WARRICK (1974) is one where S equals the constant a. This gives

$$\phi(z) = v_o/\alpha + \left[\alpha\phi_o - v_o + a(L + 1/\alpha)\right] \alpha^{-1}\, e^{\alpha(z-L)} - a(\alpha z + 1)/\alpha^2 \tag{12}$$

c. Sink functions given implicity in terms of $\exp(\alpha\Psi)$

1. $S = a_1 + a\phi$

Consider now the solution of eq. (6) subject to boundary conditions (egs. (7) and (8)) when the sink function, S, is not specified a priori as a function of depth, but is allowed to change as the pressure head changes. As long as we avoid saturated conditions, we can account for increased water uptake by plant roots under wet conditions and decreased uptake for dry conditions. LOMEN and WARRICK (1976) have given solutions for five different sink functions over a deep water table. These solutions become quite lengthy and will not be repeated here. However, a misprint in the expression for S_3 of Table 1 of LOMEN and WARRICK (1976) should be corrected where D is actually given by

$$D = (\alpha - \sqrt{a}\, e^{-\alpha z_1})\, e^{-\sqrt{a}/\alpha} \exp(\sqrt{a}\, e^{-\alpha z_1}/\alpha) + (\alpha + \sqrt{a}\, e^{-\alpha z_1})\, e^{\sqrt{a}/\alpha} \exp(-\sqrt{a}\, e^{-\alpha z_1}/\alpha) \tag{13}$$

Unfortunately these five functions typify irrigated soils where most of the water uptake by roots occurs near the soil surface. In much of the low lying areas of the Netherlands a shallow water table causes the water uptake pattern to have a different distribution. FEDDES (1971) reports a lack of root activity near the soil surface as well as close to the water table. These facts will be included in the sink functions used in the remainder of this report.

The first example assumes that the sink function is linearly related to the matrix flux potential ϕ. Thus we write

$$S = \begin{cases} 0 & 0 < z < z_1 \\ a_1 + a\,\phi & z_1 < z < z_2 \\ 0 & z_2 < z < L \end{cases} \qquad (14)$$

In the central layer ($z_1 < z < z_2$) we have the differential equation

$$\frac{d^2\phi}{dz^2} - \alpha\frac{d\phi}{dz} - a\phi - a_1 = 0 \qquad (15)$$

First notice that if we add a_1/a to a solution of

$$\frac{d^2\phi}{dz^2} - \alpha\frac{d\phi}{dz} - a\phi = 0 \quad , \qquad (16)$$

we have the solution to eq. (15). Eq. (16) is a linear differential equation which can be solved by assuming a solution in the form of an exponential and determining the constant from eq. (16) by the quadratic formula. Solutions of eq. (14) for $0 < z < z_1$ and $z_2 < z < L$ can be obtained using the techniques of Section II.b. A solution of eq. (14) over the entire range of z may be written as

$$\phi = \begin{cases} v_o/\alpha + A\,e^{\alpha z} & 0 < z < z_1 \\ B\,e^{mz} + C\,e^{nz} - a_1/a & z_1 < z < z_2 \\ \phi_o + D(e^{\alpha z} - e^{\alpha L}) & z_2 < z < L \end{cases} \qquad (17)$$

where $m = (\alpha - \sqrt{\alpha^2 + 4a})/2$, $n = (\alpha + \sqrt{\alpha^2 + 4a})/2$.

This function satisfies the two boundary conditions (eqs.(7) and (8)) automatically. The four arbitrary constants (A, B, C, D) remaining in eq. (17) are determined by demanding that ϕ and its derivative be continuous at $z = z_1$ and z_2. This is equivalent to having the pressure head, Ψ, and the flux continuous. This will result in solving four equations in four unknowns, namely

$$\begin{aligned}
- Ae^{\alpha z_1} + Be^{mz_1} + Ce^{nz_1} &= v_o/\alpha + a_1/a \\
- \alpha Ae^{\alpha z_1} + mBe^{mz_1} + nCe^{nz_1} &= 0 \\
Be^{mz_2} + Ce^{nz_2} + D(e^{\alpha L} - e^{\alpha z_2}) &= \phi_o + a_1/a \\
mBe^{mz_2} + nCe^{nz_2} - \alpha De^{\alpha z_2} &= 0
\end{aligned} \qquad (18)$$

The solution of eq. (18) is

$$\begin{aligned}
A &= e^{-\alpha z_1}\left[m(v_o + \alpha a_1/a) + (n^2 - m^2)\, e^{nz_1} C\right]/(n\alpha) \\
B &= \left[(v_o + \alpha a_1/a)\, e^{-mz_1} - me^{(n-m)\, z_1} C\right]/n \\
D &= e^{-\alpha z_2}\left[m(v_o + \alpha a_1/a)\, e^{m(z_2-z_1)} + n^2\, e^{nz_2} - m^2\, e^{nz_1+m(z_2-z_1)} C\right]/(n\alpha) \\
C &= \frac{\alpha\phi_o + \alpha a_1/a - (v_o + \alpha a_1/a)\,(1 + m/ne^{\alpha(L-z_2)})\, e^{m(z_2-z_1)}}{e^{nz_2}(m + ne^{\alpha(L-z_2)}) - me^{nz_1+m(z_2-z_1)}(1 + m/ne^{\alpha(L-z_2)})}
\end{aligned} \qquad (19)$$

Any time there is a lot of algebra involved in obtaining a solution, it is reassuring to have special cases to check against. In this situation we have three such check points:

- If we let $z_1 \to 0$ and $z_2 \to L$ we should have the same solutions as S_6 in LOMEN and WARRICK (1976). Taking these limits achieves this agreement if we make the associations $B \to A_1$, $C \to B_1$, $a_1 \to a_o$ and $L \to z_1$.

- If we let $z_2 = L \to \infty$ and $z_1 \to 0$ we obtain S_1 of LOMEN and WARRICK (1976). In this case note that we can write (if $z_2 = L$ and $z_1 = 0$) C as

$$C = \frac{e^{-mL}\, n[\phi_o + a_1/a] - (v_o + \alpha a_1/a)}{ne^{(n-m)\,L} - m}$$

before taking limits to get

$$\lim_{L \to \infty} C = 0$$

and

$$\lim_{L \to \infty} B = (v_o + \alpha a_1/a)/n$$

This gives the proper agreement by noting $a_1 = - a\phi_\infty$.

- The final check case is to let $z_1 = 0$, $z_2 = L$ and take the limit as $a \to 0$. This will prove that the resulting ϕ agrees with that of eq. (12) which was derived for a constant uptake function. Since a appears in the denominator of several expressions in the solution much manipulation is required along with the use of L'Hopitals Rule. No details will be given as the computation is long and messy, however, the final conclusion is that the expressions agree.

A computer program was written in Fortran to evaluate the function in eq. (17) and the constants in eq. (19). The listing appears as Fig. 1 with sample output as Fig. 2. The effects of a_1 and a on the sink function can be observed in Fig. 3.

```
      REAL L
C                               SINK = SM(Z-Z1)/(ZM-Z1)EXP((ZM-Z)/(ZM-Z1))
C                         PROGRAM SENK(INPUT,OUTPUT)
      V0 = 0.
      N = 0
    9 READ(9,8,END=90) SM,B,Z1,ZM,Z2,L,HL,SN,SA
    8 FORMAT (2E10.3,7F8.3)
    6 FORMAT (F10.1,7E10.3)
      PRINT 30
      PRINT 1
      PRINT 2,SM,B,Z1,ZM,Z2,L,HL,SN,SA
    2 FORMAT (2E10.3,7F8.2)
    1 FORMAT (/1H ,5X,2HSM,8X,1HB,7X,2HZ1,6X,2HZM,6X,2HZ2,6X,1HL,7X,2HHL
     *,7X,1HN,12H    SMALL A/)
   30 FORMAT (1H1)
      E = 2.7182818
      EZ2 = EXP((ZM-Z2)/(ZM-Z1))
      C1 = -V0
      C2 = -V0 + SM*(ZM-Z1)*E
      C3 = C2 - SM*(Z2 + ZM - 2.*Z1)*EZ2
      C4 = 0.
      D4 = 0.
      D3 = SA*(ABS(HL))**(1.-SN)/(SN-1.)-C3*L
      D2 = D3 + (C3-C2)*Z2-SM*(ZM-Z1)*(Z2+2.*ZM-3.*Z1)*EZ2
      D1 = D2 + SM*(ZM-Z1)*(2.*ZM-Z1)*E
      TU = SM*((ZM-Z1)*E-(Z2+ZM-2.*Z1)*EZ2)
      PRINT 50, TU
      PRINT 3
    3 FORMAT (/1H ,5X,2HC1,8X,2HC2,8X,2HC3,7X,2HC4,8X,2HD1,8X,2HD2,9X,2H
     *D3,8X,2HD4,11H     UPTAKE/)
      PRINT 4,C1,C2,C3,C4,D1,D2,D3,D4,TU
    4 FORMAT (9E10.3)
   50 FORMAT (1X,E10.3)
      PRINT 5
      DO 80 J=1,21
      Z = 5.*(J-1)
      EZ = EXP((ZM-Z)/(ZM-Z1))
      IF(Z.GT.Z1) GO TO 10
      HH = C1*Z + D1
      UP = 0.
      SINK = 0.
      FLU = C1
      GO TO 12
   10 IF(Z.GT.Z2) GO TO 11
      HH = SM*(ZM-Z1)*(Z + 2.*ZM-3.*Z1)*EZ + C2*Z + D2
      UP = SM*((ZM-Z1)*E - (Z+ZM-2.*Z1)*EZ)/TU
      SINK = SM*(Z-Z1)*EZ/(ZM-Z1)
      FLU = -SM*(Z + ZM-2.*Z1)*EZ + C2
      GO TO 12
   11 HH = C3 *Z + D3
      UP = 1.
      SINK = 0.
      FLU = C3
   12 H= -(ABS((1.-SN)*HH/SA))**(1./(1.-SN))
      FLUX = -1.*FLU
      PSI = H + Z
      PF = ALOG10(ABS(PSI))
      DHDZ = FLU*(-1.*H)**SN/SA
    5 FORMAT(/1H ,6X,1HZ,4X,15HTOTAL HEAD  PSI,5X,16HPERCENT UP  SINK,8X
     *,2HHH,6X,4HFLUX,6X,4H PF /)
      PRINT 6,Z,H,PSI,UP,SINK,HH,FLUX,PF
   80 CONTINUE
      N = N + 1
```

Fig. 1. Program listing for S of eq. (14)

KO	ALPHA	VO	PHIZERO	Z1	Z2	L	A	A1
42.500	.082	0.000	515.800	10.000	56.000	92.000	.0001	0.0001

AA	BB	CC	DD	PHIO-DEAL	T. UPTAKE
.258E+00	.846E-02	.252E+00	.273E+00	-.381E+00	.313E-01

Z	PHI	HEAD	UPTAKE	P.UPTAKE	SINK
0.	.258E+00	-.927E+02	.294E-02	.100E+01	0.
.500E+01	.389E+00	-.877E+02	.294E-02	.100E+01	0.
.100E+02	.587E+00	-.827E+02	.294E-02	.100E+01	0.
.150E+02	.885E+00	-.777E+02	.363E-03	.116E-01	.885E-04
.200E+02	.134E+01	-.727E+02	.911E-03	.291E-01	.134E-03
.250E+02	.202E+01	-.676E+02	.174E-02	.556E-01	.202E-03
.300E+02	.306E+01	-.626E+02	.299E-02	.957E-01	.306E-03
.350E+02	.464E+01	-.575E+02	.489E-02	.156E+00	.464E-03
.400E+02	.703E+01	-.524E+02	.777E-02	.248E+00	.703E-03
.450E+02	.106E+02	-.474E+02	.121E-01	.388E+00	.106E-02
.500E+02	.161E+02	-.423E+02	.187E-01	.599E+00	.161E-02
.550E+02	.245E+02	-.372E+02	.287E-01	.918E+00	.245E-02
.600E+02	.370E+02	-.322E+02	.287E-01	.100E+01	0.
.650E+02	.560E+02	-.271E+02	.287E-01	.100E+01	0.
.700E+02	.846E+02	-.221E+02	.287E-01	.100E+01	0.
.750E+02	.128E+03	-.171E+02	.287E-01	.100E+01	0.
.800E+02	.193E+03	-.121E+02	.287E-01	.100E+01	0.
.850E+02	.290E+03	-.707E+01	.287E-01	.100E+01	0.
.900E+02	.438E+03	-.206E+01	.287E-01	.100E+01	0.
.950E+02	.660E+03	.294E+01	.287E-01	.100E+01	0.
.100E+03	.994E+03	.795E+01	.287E-01	.100E+01	0.

Fig. 2. Sample output of the listing in Fig. 1

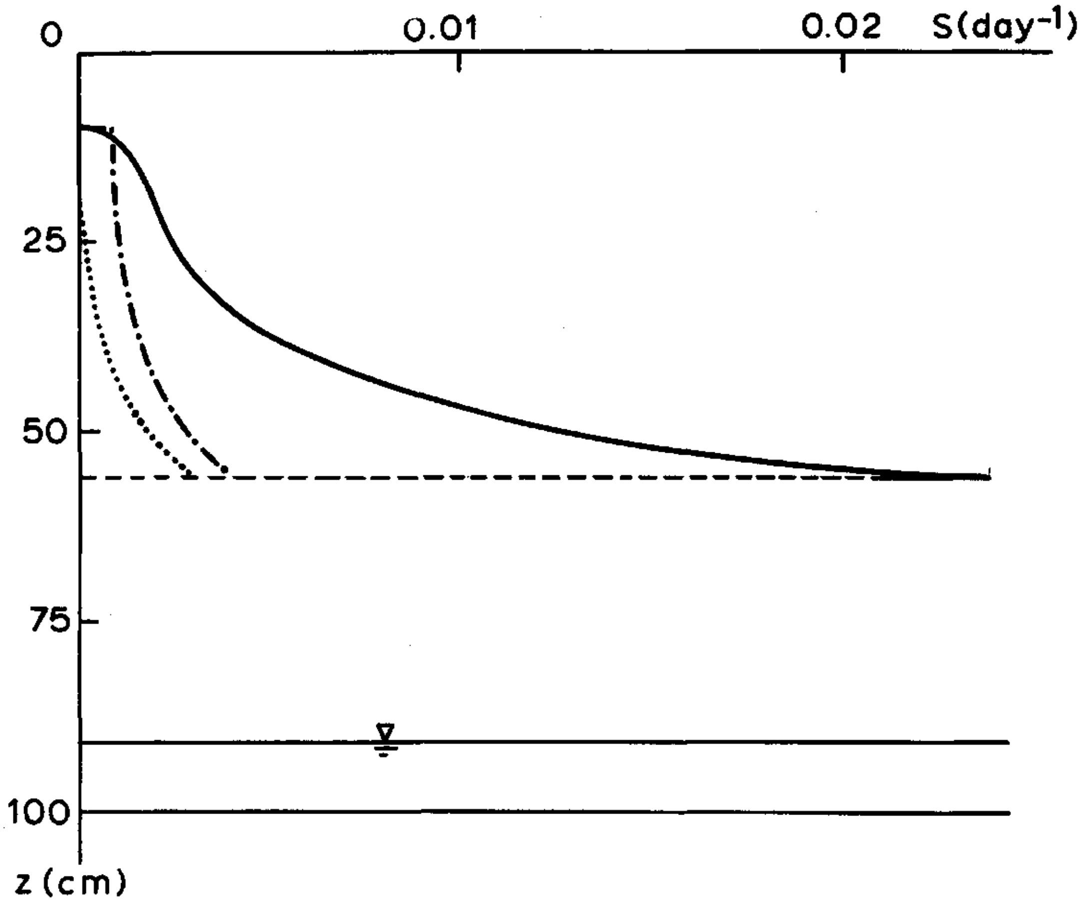


Fig. 3. Water uptake patterns for an implicit sink function (eq. (14))

The top curve in Fig. 3 has $a = 10^{-3}$, $a_1 = 10^{-4}$, the middle curve $a = 10^{-4}$, $a_1 = 10^{-3}$ and the bottom curve $a = 10^{-4}$, $a_1 = 10^{-4}$. Notice that changing a_1 from 10^{-4} to 10^{-3} (from the bottom curve to the middle one) almost has the exact effect of shifting the curve by the amount of increase and retaining the same shape. (Recall a_1 is the constant term in the sink function). However, a similar increase in a causes the shape of the curve, as well as the extent, to be drastically altered.

c.2. $S = a(\phi - b) \exp(-2\alpha z)$

A second form of the sink function was used to incorporate an explicit depth dependence in addition to the dependence on ϕ. Specifically we consider the form

$$S = \begin{cases} 0 & 0 < z < z_1 \\ ae^{-2\alpha z}(\phi - b) & z_1 < z < z_2 \\ S_M & z_2 < z < z_3 \\ 0 & z_3 < z < L \end{cases} \tag{20}$$

The solution of eq. (6) with this sink function is

$$\phi = \begin{cases} A_1 e^{\alpha z} + v_o/\alpha & 0 < z < z_1 \\ b + A_2 \exp(\sqrt{a}e^{-\alpha z}/\alpha) + B_2 \exp(-\sqrt{a}e^{-\alpha z}/\alpha) & z_1 < z < z_2 \\ -S_M z/\alpha + A_3 e^{\alpha z} + B_3 & z_2 < z < z_3 \\ \phi_o + A_4(e^{\alpha z} - e^{\alpha L}) & z_3 < z < L \end{cases} \tag{21}$$

Notice that the boundary conditions imposed by eqs. (7) and (8) have already been satisfied by the solution in eq. (21). The six remaining arbitrary constants may be evaluated in the manner illustrated in the previous section by requiring ϕ and its derivative to be continuous at $z = z_1$, z_2 and z_3. The resulting values for A_1, A_2, B_2, A_3, B_3 and A_4 are summarized at page 13.

d, F u r t h e r   c o n s i d e r a t i o n s

FEDDES and ZARADNY (1977) consider a sink function which depends upon Ψ as

$$S = \begin{cases} 0 & \Psi_1 < \Psi < 0 = \Psi_o \\ S_M & \Psi_2 < \Psi < \Psi_1 \\ S_M(\Psi - \Psi_3)/(\Psi_2 - \Psi_3) & \Psi_3 < \Psi < \Psi_2 \\ 0 & \Psi < \Psi_3 \end{cases} \tag{22}$$

Constants for eq. (21)

$$A_1 = \left[A_2 \exp(\sqrt{ae^{-\alpha z_1}}/\alpha) + B_2 \exp(-\sqrt{ae^{-\alpha z_1}}/\alpha) + b - v_o/\alpha\right] e^{-\alpha z_1}$$

$$A_2 = [a_{22}c_1 - a_{12}c_2]/D \qquad D = a_{11}a_{22} - a_{12}a_{21}$$

$$B_2 = [a_{11}c_2 - a_{21}c_1]/D$$

$$A_3 = \left[\{-A_2 \exp(\sqrt{ae^{-\alpha z_2}}/\alpha) + B_2 \exp(-\sqrt{ae^{-\alpha z_2}}/\alpha)\} \sqrt{ae^{-\alpha z_2}} + S_M/\alpha\right] e^{-\alpha z_2}/\alpha$$

$$B_3 = \phi_o + S_M z_3/\alpha + S_M(e^{\alpha(L-z_3)} - 1)/\alpha^2 - e^{\alpha L} A_3$$

$$A_4 = A_3 - S_M e^{-\alpha z_3}/\alpha^2$$

$$a_{11} = (\sqrt{ae^{-\alpha z_1}} + \alpha) \exp(\sqrt{ae^{-\alpha z_1}}/\alpha) ,$$

$$a_{12} = (\alpha - \sqrt{ae^{-\alpha z_1}}) \exp(-\sqrt{ae^{-\alpha z_1}}/\alpha)$$

$$a_{21} = (\alpha + \sqrt{ae^{-\alpha z_2}}(1 - e^{\alpha(L-z_2)}) \exp(\sqrt{ae^{-\alpha z_2}}/\alpha)$$

$$a_{22} = (\alpha - \sqrt{ae^{-\alpha z_2}}(1 - e^{\alpha(L-z_2)}) \exp(-\sqrt{ae^{-\alpha z_2}}/\alpha)$$

$$c_1 = v_o - \alpha b$$

$$c_2 = \alpha(\phi_o - b) + S_M(z_3 - z_2) + S_M(e^{\alpha(L-z_3)} - e^{\alpha(L-z_2)})/\alpha$$

The region $\Psi_1 < \Psi < 0$ is near the water table and below the root zone (at least in the absence of infiltration from the surface), $\Psi_2 < \Psi < \Psi_1$ denotes the zone where maximum water uptake occurs while for $\Psi_3 < \Psi < \Psi_2$ uptake decreases until the "wilting" point is reached at Ψ_3.

The relationship between Ψ and the matrix flux potential, ϕ, is

$\phi = K_o/\alpha \exp(\alpha\Psi)$, so if $\phi_i = \phi_o \exp(\alpha\Psi_i)$, $i = 0, 1, 2, 3$ we can directly transform eq. (22) into

$$S = \begin{cases} 0 & \phi_1 < \phi < \phi_0 \\ S_M & \phi_2 < \phi < \phi_1 \\ S_M(\ln \phi - \ln \phi_3)/(\ln \phi_2 - \ln \phi_3) & \phi_3 < \phi < \phi_2 \\ 0 & 0 < \phi < \phi_3 \end{cases} \qquad (23)$$

Several attempts were made to solve the differential equation in the interval $\phi_3 < \phi < \phi_2$ but with limited success. The change of variables $Y = \ln \phi$ will change

$$\phi''(z) - \alpha\phi'(z) = A + B \ln \phi(z) \qquad (24)$$

to

$$Y''(z) + (Y'(z))^2 - \alpha Y'(z) + (A + BY(z))\, e^{-Y(z)} = 0 \qquad (25)$$

Now change the dependent variable from Y to p and the independent variable from z to Y by the relationship

$$\frac{dY}{dz} = p \quad \text{so} \quad z = \int \frac{dY}{p(Y)}$$

This results in an Abel equation of the second kind (see KAMKE (1956) Chapter 1, 4, 11A)

$$pp' + p^2 - \alpha p - (A + BY)\, e^{-Y} = 0, \; p = p(Y) \qquad (26)$$

We can obtain an Abel equation of the first kind (4.10) KAMKE (1956) by letting $p = 1/u(Y)$, namely

$$u' + (A + BY)\, e^{-Y} u^3 + \alpha u^2 - u = 0 \qquad (27)$$

KAMKE lists further transformations to be carried out but I sincerely doubt that evaluating all the integrals required to find ϕ again will result in a tractable expression. No more attempts were made along this path of endeavor.

A different possibility is to expand ln ϕ from eq. (23) in a Taylor series and obtain

$$\phi'' - \alpha\phi' = A + B\phi$$

We notice that for no infiltration and a water table, ϕ will have small positive values near the soil surface and take its maximum value at the water table. Thus an approximation would be

$$\phi'' - \alpha\phi' = \begin{cases} 0 & 0 < z < z_1 \\ A + B\phi & z_1 < z < z_2 \\ S_M & z_2 < z < z_3 \\ 0 & z_3 < z < L \end{cases} \qquad (28)$$

which is the same as eq. (14) if $z_2 = z_3$. The solution of this system can be obtained very easily in the same manner of Section II.c.

III. HYDRAULIC CONDUCTIVITY AS A POWER FUNCTION OF PRESSURE HEAD

In the next analysis the form of hydraulic conductivity function is chosen as a power function of pressure head. WIND (1955) and WESSELING (1957) both used the relationship

$$K = a(-\Psi)^{-n} \quad , \quad n > 0 \tag{29}$$

for conditions away from saturation. Recent experiments with the dry hot air method by BOUMA (1978) down to Ψ values of -10^5 cm also indicate that eq. (29) might be reasonable for heavy clay soils from the rather wet to the rather dry range. Thus it seems appropriate to seek solutions of the basic differential equation (3) for this situation. If we have $S = S(z)$, eq. (3) may be integrated to obtain

$$a(-\Psi)^{-n}\left[\frac{d\Psi}{dz} - 1\right] = \int_0^z S(z')\,dz' + C \tag{30}$$

or

$$a\,\frac{d\Psi}{dz} - (-\Psi)^n\left[\int_0^z S(z')\,dz' + C\right] = a \tag{31}$$

For general values of n this seems difficult to solve, even for simple functions of S. Note even though $n = 1$ makes the equation linear, the solution is still not simple, so further efforts were not expended along this direction. However, it was noted that solutions were readily available if K was a power function of the t o t a l head! This assumption will be made in the following section.

IV. HYDRAULIC CONDUCTIVITY AS A POWER FUNCTION OF TOTAL HEAD

a. General sink functions

In the previous section we noted that some researchers have assumed $K = a(-\Psi)^{-n}$. Since $\Psi = H + z$ there is little difference between Ψ and H for small values of z or large values of H. For many problems of interest this is the case so we take the hydraulic conductivity of the form

$$K = a(-H)^{-n} \tag{32}$$

(We note that GARDNER (1958) also used a modified form of (29) as $K = a((-\Psi)^n + C)^{-1}$ to have a finite value of K at $\Psi = 0$).

Now we seek solutions of eq. (6) written as

$$\frac{d}{dz}\left[a(-H)^{-n}\frac{dH}{dz}\right] = S \tag{33}$$

If S is only a function of z we can integrate directly and obtain

$$a(-H)^{-n}\frac{dH}{dz} = \int S(z')\,dz' + c_1 \tag{34}$$

This term is also integrable as

$$-\frac{a(-H)^{-n+1}}{-n+1} = \int_0^z\int_0^{z''} S(z')\,dz'dz'' + c_1 z + c_2$$

and may be solved for H yielding

$$H = -\left[\left(\frac{n-1}{a}\right)\left(\int_0^z\int_0^{z''} S(z')\,dz'dz'' + c_1 z + c_2\right)\right]^{1/(1-n)} \tag{35}$$

Solutions for H for specific sink functions S(z) will be obtained and illustrated in the next section.

If the sink function is allowed to depend only upon H, $S = S(H)$, the differential equation in question, (33), can be written as

$$H''(z) - nH^{-1}[H'(z)]^2 - (-H)^n S(H)/a = 0 \tag{36}$$

If one defines a new dependent variable by

$$p(H) = \frac{dH}{dz} \quad , \quad \text{i.e. } z = \int \frac{dH}{p(H)} \tag{37}$$

one obtains a Bernoulli equation

$$p' - nH^{-1} p = \left[\frac{(-H)^n S(H)}{a}\right] p^{-1} \tag{38}$$

The standard way of solving this equation is by letting

$$Y = p^2$$

yielding

$$Y'(H) - 2nH^{-1}Y = \frac{2(-H)^n S(H)}{a} \tag{39}$$

This is a linear differential equation with an integrating factor $\exp(\int(-2nH^{-1})\,dH) = H^{-2n}$ so we can write eq. (39) as

$$\frac{d}{dH}[H^{-2n}Y] = \frac{2(-H)^{-n}S(H)}{a} \tag{40}$$

or

$$Y(H) = H^{2n}\left[2\int(-H)^{-n}S(H)/a\,dH + C\right] \tag{41}$$

For example if

$$S(H) = b(-H)^m + B\ , \tag{42}$$

$$Y(H) = 2[b(-H)^{m+n+1}/(n-m-1) + B(-H)^{n+1}/(n-1)]/a + C(-H)^{2n} \tag{43}$$

Notice that while Y(H) is completely determined, $p = Y^{1/2}$ and z must be determined from the integration in eq. (37).

For special cases this might not be so hopeless. In particular if $m = n = 1.5$ and $B = 0$, using DWIGHT (1961) 129.9, we obtain

$$H = \frac{2b/a}{C^2(D + z)^2/4 - C} \tag{44}$$

where D is an arbitrary constant of integration.
If we choose C and D to satisfy conditions of no flux at the soil surface and $H = H_L$ at $z = L$ we obtain $D = 0$ and
$C = 2(1 + \sqrt{1 + 2bL^2/aH_L})/L^2$.

If $m = n = 1$ the solution can also be developed.

In practice the slope of K(Ψ) line will probably not be some "nice" number so we will stop this approach and return to having $S = S(z)$ only.

b. Sink functions determined by connected straight lines

The model of the sink function of FEDDES and ZARADNY (1977) which depends on the value of the pressure head was mentioned before (see eq. (22) in Section II.d). One of the outputs of their simulation model is the change of this sink function with depth. Many of these predicted sink functions can be quite closely approximated by straight lines. For this reason a prescribed sink function was chosen as illustrated in Fig. 4 and given by (45)

$$S(z) = \begin{cases} 0 & 0 < z < z_1 \\ Az + B & z_1 < z < z_2 \\ Az_2 + B & z_2 < z < z_3 \\ 0 & z_3 < z < L \end{cases} \tag{45}$$

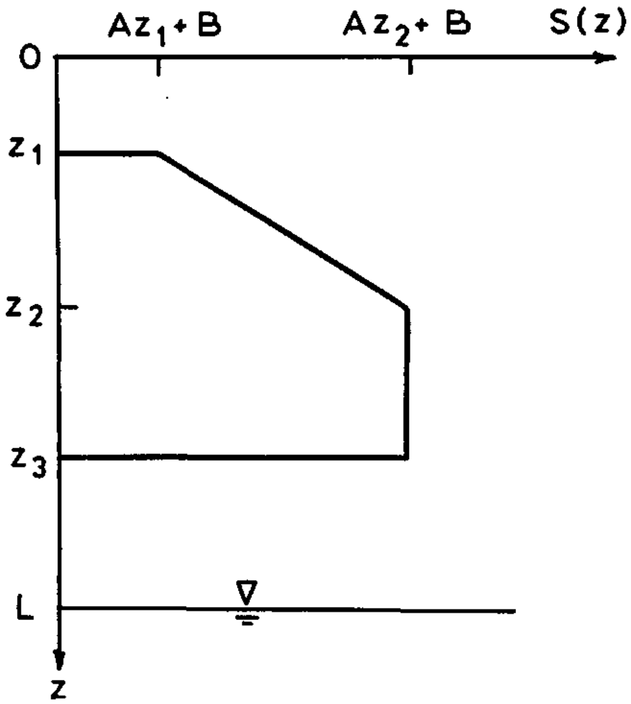


Fig. 4. Sink with connected straight lines

The jump at z_1 can be avoided by choosing $Az_1 + B = 0$. The differential equation we must solve is from (33)

$$\frac{d}{dz}[K \frac{dH}{dz}] = S(z) \tag{46}$$

subject to the boundary condition $- K \frac{dH}{dz} = v_o$ at $z = 0$, and to $H = H_L$ for $z = L$ (see eqs. (7) and (8)). Even though the general form of the solution of eq. (46) has been given by eq. (35), it is instructive to rework each step. These intermediate results are useful in their own right. If we integrate eq. (46) once with S(z) given by eq. (45) we obtain

$$K \frac{dH}{dz} = \begin{cases} C_1 & 0 < z < z_1 \\ Az^2/2 + Bz + C_2 & z_1 < z < z_2 \\ (Az_2 + B)\, z + C_3 & z_2 < z < z_3 \\ C_4 & z_3 < z < z_L \end{cases} \tag{47}$$

Now the flux is given by $- K \frac{dH}{dz}$, so if we satisfy the flux boundary condition at the surface and assume the flux is continuous, we obtain

$$
\begin{aligned}
C_1 &= - v_o \\
C_2 &= - v_o - Az_1^2/2 - Bz_1 \\
C_3 &= C_2 - Az_2^2/2 \\
C_4 &= C_3 + (Az_2 + B)\, z_3
\end{aligned}
\tag{48}
$$

Since $K = a(-H)^{-n}$ the left hand side of eq. (47) is equal to the derivative of $\frac{a(-H)^{-n+1}}{n-1}$, so one more integration yields

$$
HH = \frac{a(-H)^{-n+1}}{n-1} = \begin{cases} C_1 z + D_1 & 0 < z < z_1 \\ Az^3/6 + Bz^2/2 + C_2 z + D_2 & z_1 < z < z_2 \\ (Az_2 + B)\, z^2/2 + C_3 z + D_3 & z_2 < z < z_3 \\ C_4 z + D_4 & z_3 < z < L \end{cases}
\tag{49}
$$

In realistic situations $a > 0$, $n > 1$ and $H < 0$ so the quantity HH should be positive. This quantity is printed out in the output of the computer program for evaluating the pressure head as a function of depth. Negative values will occur when the specified water uptake by plant roots is greater than can be supplied from the groundwater table or surface flux.

To determine the constants D_i, $i = 1, 2, 3, 4$ we specify H at $z = L$ and then require that H, and therefore HH, be continuous throughout $0 < z < L$. This gives

$$
\begin{aligned}
D_4 &= aH_L^{1-n}/(n-1) - C_4 L \\
D_3 &= D_4 + (Az_2 + B)\, z_3^2/2 \\
D_2 &= D_4 + (Az_2 + B)\, z_3^2/2 - Az_2^3/6 \\
D_1 &= D_2 + Az_1^3/6 + (C_2 - C_1)\, z_1 + Bz_1^2/2
\end{aligned}
\tag{50}
$$

The values of a and n strongly affect the value of D_4 which in turn occurs in D_1, D_2 and D_3. Thus not only can the lack of water (surface flux or water table) give rise to inappropriate values of H, (i.e. imaginary) but also possible combinations of a and n. The mathematical requirement that H be real, implies that each of the expressions on the right hand side of eq. (49) must be greater than or equal to zero. For the case of no infiltration $v_o = 0$ the soil will be driest near the surface and this condition gives $D_1 > 0$.

From eq. (50) this mathematical inequality can be written in terms of a and n, but it is a non-linear inequality that cannot be solved exactly.

From eq. (49) we see that

$$H = -[(n - 1)\ HH/a]^{1/(1-n)} \tag{51}$$

To avoid needless delays and messages in the printout for HH < 0, the absolute value of HH is used in this calculation. The values of HH should always be checked on the computer printout to make sure that HH > 0.

The last calculation we need to make will determine the water uptake by plant roots as a function of depth.This is a straight forward calculation from eq. (45) and yields

$$\text{Uptake} = \int_0^z S(z')\,dz' = \begin{cases} 0 & 0 < z < z_1 \\ A(z^2 - z_1^2)/2 + B(z - z_1) & z_1 < z < z_2 \\ A(z_2 z - (z_2^2 + z_1^2)/2) + B(z - z_1) & z_2 < z < z_3 \\ A(z_2 z_3 - (z_2^2 + z_1^2)/2) + B(z_3 - z_1) & z_3 < z < L \end{cases} \tag{52}$$

The various functions occurring for the sink consisting of straight lines are summarized in Table 1.

The listing of the FORTRAN computer program that evaluates those functions is given in Fig. 5. The output of a typical run is given in Fig. 6.

Table 1. Results for a sink with connected straight lines

$S(z)$	$K \frac{dH}{dz}$	HH	z
0	C_1	$C_1 z + D_1$	$0 < z < z_1$
$Az + B$	$Az^2/2 + Bz + C_2$	$Az^3/6 + Bz^2/2 + C_2 z + D_2$	$z_1 < z < z_2$
$Az_2 + B$	$(Az_2 + B)\ z + C_3$	$(Az_2 + B)\ z^2/2 + C_3 z + D_3$	$z_2 < z < z_3$
0	C_4	$C_4 z + D_4$	$z_3 < z < L$

$C_1 = - v_o$	$D_4 = aH_L^{1-n}/(n - 1) - C_4 L$
$C_2 = - v_o - Az_1^2/2 - Bz_1$	$D_3 = D_4 + (Az_2 + B)\ z_3^2/2$
$C_3 = C_2 - Az_2^2/2$	$D_2 = D_4 + (Az_2 + B)\ z_3^2/2 - Az_2^3/6$
$C_4 = C_3 + (Az_2 + B)\ z_3$	$D_1 = D_2 + Az_1^3/6 + (C_2 - C_1)\ z_1 + Bz_1^2/2$

```
      REAL L
      OPEN 9,"@DATA"
      OPEN 10,"@LIST",ATT="P"
      V0 = 0.
      N = 0
C                              SINK WITH TWO STRAIGHT LINES(Z1,Z2,Z3,L)
    9 READ (9,8,END=90) A,B,Z1,Z2,Z3,L,HL,SN,SA
      WRITE (10,30)
      WRITE (10,2) A,B,Z1,Z2,Z3,L,HL,SN,SA
      C1 = -V0
      C2 = -A*Z1*Z1*.5 - V0 - B*Z1
      C3 = C2 - .5*A*Z2*Z2
      C4 = C3 + (A*Z2 + B)*Z3
      D4 = SA*(ABS(HL))**(1.-SN)/(SN-1.)-C4*L
      D3 = D4 + .5*(A*Z2 + B)*Z3*Z3
      D2 = D4 + .5*(A*Z2 + B)*Z3*Z3 - A*Z2**3/6.
      D1 = D2 + A*Z1**3/6. + (C2 - C1)*Z1 + B*Z1*Z1*.5
      TU = A*(Z2*Z3 - .5*(Z2*Z2 + Z1*Z1)) + B*(Z3-Z1)
      WRITE (10,4) C1,C2,C3,C4,D1,D2,D3,D4,TU
      WRITE (10,5)
      DO 80 J=1,21
      Z = 5.*(J-1)
      IF(Z.GT.Z1) GO TO 10
      HH = C1*Z + D1
      UP = 0.
      SINK = 0.
      FLU = C1
      GO TO 12
   10 IF(Z.GT.Z2) GO TO 11
      HH = (A*Z/6. + .5*B)*Z*Z + C2*Z + D2
      UP = (A*.5*(Z*Z-Z1*Z1) + B*(Z-Z1))/TU
      SINK = A*Z + B
      FLU = (A*Z*.5 + B)*Z + C2
      GO TO 12
   11 IF(Z.GT.Z3) GO TO 13
      HH = (A*Z2 +B)*Z*Z*.5 + C3*Z + D3
      UP = (A*(Z2*Z-.5*(Z2*Z2 + Z1*Z1)) + B*(Z-Z1))/TU
      SINK = A*Z2 + B
      FLU = (A*Z2 + B)*Z + C3
      GO TO 12
   13 HH = C4*Z + D4
      UP = 1.
      SINK = 0.
      FLU = C4
   12 H = -(ABS((1.-SN)*HH/SA))**(1./(1.-SN))
      FLUX = -1.*FLU
      PSI = H + Z
      PF = ALOG10(ABS(PSI))
      DHDZ = FLU*(-1.*H)**SN/SA
      WRITE (10,6) Z,H,PSI,UP,SINK,HH,FLUX,PF
   80 CONTINUE
      N = N + 1
      GO TO 9
   90 STOP
C
    2 FORMAT (/1H ,5X,1HA,9X,1HB,8X,2HZ1,6X,2HZ2,6X,2HZ3,6X,1HL,7X,2HHL,
     *7X,1HN,3X,7HSMALL A//1H ,2E10.3,7F8.2)
    4 FORMAT (/1H ,5X,2HC1,8X,2HC2,8X,2HC3,7X,2HC4,8X,2HD1,8X,2HD2,9X,
     *2HD3,//1H ,7E10.3//1H ,5X,2HD4,7X,6HUPTAKE//1H ,2E10.3/)
    5 FORMAT(/1H ,6X,1HZ,4X,15HTOTAL HEAD  PSI,5X,16HPERCENT UP  SINK,8X
     *,2HHH,6X,4HFLUX,6X,4H PF /)
    6 FORMAT (F10.1,7E10.3)
    8 FORMAT (2E10.3,7F8.3)
```

Fig. 5. Program listing for S of eq. (45)

A	B	Z1	Z2	Z3	L	HL	N	SMALL A
0.447E-03	-0.223E-02	5.00	20.00	30.00	35.00	35.00	1.20	15.00

C1	C2	C3	C4	D1	D2	D3
0.[illegible]E+00	0.553E-02	-0.833E-01	0.117E+01	0.233E+02	0.233E+02	0.237E+02

D4	UPTAKE
0.2[illegible]E+02	0.117E+00

Z	TOTAL HEAD	PSI	PERCENT UP	SINK	JH	FLUX	PF
0.0	-0.346E+03	-0.346E+03	0.000E+00	0.000E+00	0.233E+02	0.000E+00	0.254E+01
5.0	-0.346E+03	-0.34[illegible]E+03	0.000E+00	0.000E+00	0.233E+02	0.000E+00	0.253E+01
10.0	-0.345E+03	-0.335E+03	0.477E-01	0.224E-02	0.233E+02	-0.560E-02	0.253E+01
15.0	-0.340E+03	-0.325E+03	0.191E+00	0.447E-02	0.234E+02	-0.224E-01	0.251E+01
20.0	-0.328E+03	-0.303E+03	0.427E+00	0.671E-02	0.235E+02	-0.503E-01	0.249E+01
25.0	-0.305E+03	-0.280E+03	0.714E+00	0.671E-02	0.239E+02	-0.839E-01	0.245E+01
30.0	-0.275E+03	-0.245E+03	0.100E+01	0.671E-02	0.244E+02	-0.117E+00	0.239E+01
35.0	-0.244E+03	-0.209E+03	0.100E+01	0.000E+00	0.250E+02	-0.117E+00	0.232E+01
40.0	-0.217E+03	-0.177E+03	0.100E+01	0.000E+00	0.256E+02	-0.117E+00	0.225E+01
45.0	-0.194E+03	-0.149E+03	0.100E+01	0.000E+00	0.261E+02	-0.117E+00	0.217E+01
50.0	-0.174E+03	-0.124E+03	0.100E+01	0.000E+00	0.267E+02	-0.117E+00	0.209E+01
55.0	-0.156E+03	-0.101E+03	0.100E+01	0.000E+00	0.273E+02	-0.117E+00	0.200E+01
60.0	-0.140E+03	-0.801E+02	0.100E+01	0.000E+00	0.279E+02	-0.117E+00	0.190E+01
65.0	-0.126E+03	-0.613E+02	0.100E+01	0.000E+00	0.285E+02	-0.117E+00	0.179E+01
70.0	-0.114E+03	-0.447E+02	0.100E+01	0.000E+00	0.291E+02	-0.117E+00	0.164E+01
75.0	-0.103E+03	-0.232E+02	0.100E+01	0.000E+00	0.297E+02	-0.117E+00	0.145E+01
80.0	-0.936E+02	-0.135E+02	0.100E+01	0.000E+00	0.303E+02	-0.117E+00	0.113E+01
85.0	-0.850E+02	-0.115E-14	0.100E+01	0.000E+00	0.308E+02	-0.117E+00	-0.498E+01
90.0	-0.774E+02	0.124E+02	0.100E+01	0.000E+00	0.314E+02	-0.117E+00	0.110E+01
95.0	-0.705E+02	0.249E+02	0.100E+01	0.000E+00	0.320E+02	-0.117E+00	0.139E+01
100.0	-0.644E+02	0.354E+02	0.100E+01	0.000E+00	0.326E+02	-0.117E+00	0.155E+01

Fig. 6. Sample output of the listing in Fig. 5

Examination of water extraction patterns as determined from measured data of FEDDES (1971) shows that roots do not stop as abruptly near the water table as given by the models of Fig. 3. A simple modification of this would be to use a slanted line instead of a horizontal one at z_3. Such a sink function is given in Fig. 7. Since the solution of the resulting system is so similar to that just concluded, we omit all details and simply present the results in Table 2. The only expression lacking is that of uptake, but that is identical with $K \frac{dH}{dz}$ if we set $v_0 = 0$ in C_1, C_2, C_3, C_4 and C_5.

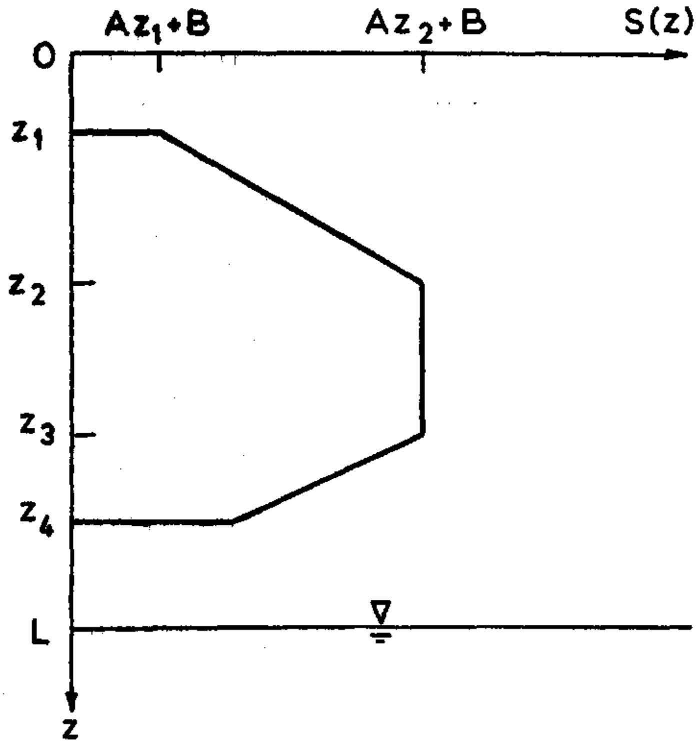


Fig. 7. Polygonal sink function

c. P a r a b o l i c s i n k f u n c t i o n s

One of the disadvantages of the 'straight-line' models discussed under b is the amount of information to be specified. For example in Fig. 7, four depths, z_1, z_2, z_3 and z_4, two slopes, A and A*, as well

Table 2. Results for a polygonal sink function

S(z)	$K \frac{dH}{dz}$	HH	z
0	C_1	$C_1 z + D_1$	$0 < z < z_1$
$Az + B$	$Az^2/2 + Bz + C_2$	$Az^3/6 + Bz^2/2 + C_2 z + D_2$	$z_1 < z < z_2$
$Az_2 + B = S_m$	$S_m z + C_3$	$S_m z^2/2 + C_3 z + D_3$	$z_2 < z < z_3$
$S_m + A^*(z-z_3)$	$S_m z + A^*(z-z_3)^2/2 + C_4$	$S_m z^2/2 + A^*(z-z_3)^3/6 + C_4 z + D_4$	$z_3 < z < z_4$
0	C_5	$C_5 z + D_5$	$z_4 < z < L$

$C_1 = -v_o$	$D_5 = aH_L^{1-n}/(n-1) - C_5 L$
$C_2 = -v_o - Az_1^2/2 - Bz_1$	$D_4 = D_5 + z_4(C_5-C_4) - S_m z_4^2/2 - A^*(z_4-z_3)^3/6$
$C_3 = C_2 - Az_2^2/2$	$D_3 = D_4$
$C_4 = C_3$	$D_2 = D_3 - Az_2^3/6$
$C_5 = C_3 + S_m z_4 + A^*(z_4-z_3)^2/2$	$D_1 = D_2 - Az_1^3/3 - Bz_1^2/2$

as S_m may all be independently assigned. While this is fine for constructing a model to describe known results, it is not so good if the model is to be used as a predictive tool. To construct a sink function which uses only the minimum and maximum values of the root zone as well as S_m, the maximum value of the uptake function, we consider a parabola as given in Fig. 8. The intercepts of the parabola are taken at z_1 en z_2 and the maximum value is taken as S_m. Since the integration is straightforward and similar to the previous examples, the results are summarized in Table 3. The only expression lacking is the uptake which again is the same as the expressions for $K \frac{dH}{dz}$ if $v_o = 0$ in C_1, C_2 and C_3. A listing of the FORTRAN program written to evaluate the expressions in Table 3 is shown in Fig. 9. In the input data, z_m refers to the place where S has its maximum value. This is needed for the exponential function shown in Section IV.d, but is overridden in this program by an instruction giving z_m as the arithmetic mean of z_1 and z_2. The inclusion of z_m in the input allows the same input and output to be used for these two different sink functions.

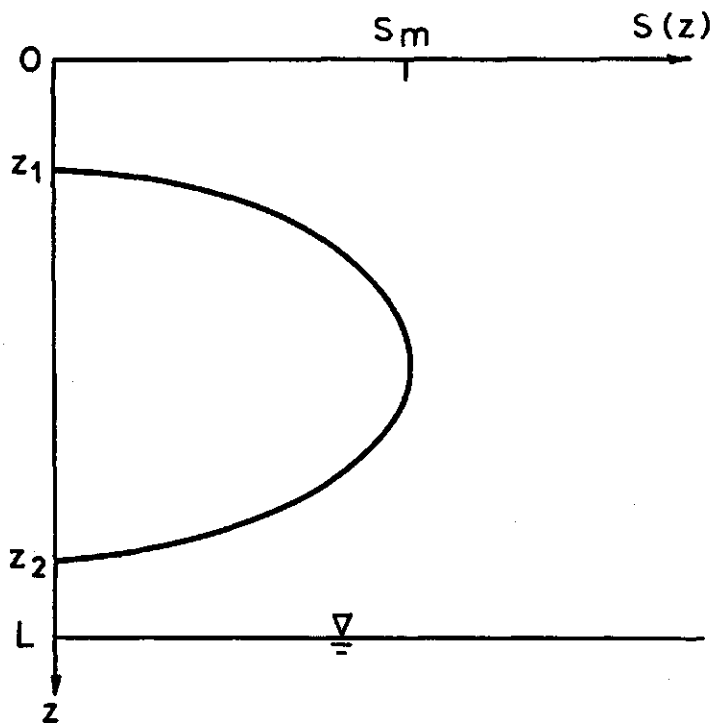


Fig. 8. Parabolic sink function

Table 3. Results for a parabolic sink function

S(z)	$K\frac{dH}{dz}$	HH	z
0	C_1	$C_1 z + D_1$	$0 < z < z_1$
$Q(z-z_1)(z-z_2)$	$Q(z-z_1)^2(2z+z_1-3z_2)/6 + C_2$	$Q(z-z_1)^3(z+z_1-2z_2)/12 + C_2 z + D$	$z_1 < z < z_2$
0	C_3	$C_3 z + D_3$	$z_2 < z < L$

$$Q = -4S_m(z_2-z_1)^{-2}$$

$C_1 = -v_o$ $\qquad$ $D_3 = aH_L^{1-n}/(n-1) - C_3 L$

$C_2 = C_1$ $\qquad$ $D_2 = D_3 + S_m(z_2^2-z_1^2)/3$

$C_3 = C_1 + 2S_m(z_2-z_1)/3$ $\qquad$ $D_1 = D_2$

```
      REAL L
C                                    PARABOLIC SINK PROFILE
      V0 = 0.
      N = 0
      OPEN 5,"@DATA",ATT="B"
    9 READ(5,8,END=90) SM,B,Z1,ZM,Z2,L,HL,SN,SA
    8 FORMAT (2E10.3,7F8.3)
    6 FORMAT (F10.1,7E10.3)
      PRINT 30
      PRINT 1
      Q = -4.*SM/(Z2-Z1)**2
      ZM = (Z1 + Z2)/2.
      PRINT 2,SM,B,Z1,ZM,Z2,L,HL,SN,SA
    2 FORMAT (2E10.3,7F8.2)
    1 FORMAT (/1H ,5X,2HSM,9X,1HB,7X,2HZ1,6X,2HZM,6X,2HZ2,6X,1HL,7X,2HHL
     *,7X,1HN,12H      SMALL A/)
   30 FORMAT (1H1)
      C1 = -V0
      C2 = C1
      C3 = C2 + 2.*SM*(Z2-Z1)/3.
      C4 = 0.
      D3 = SA*(ABS(HL))**(1.-SN)/(SN-1.)-C3*L
      D2 = D3 + SM*(Z2-Z1)*(Z2+Z1)/3.
      D1 = D2
      TU = 2.*SM*(Z2-Z1)/3.
      PRINT 50,TU
      PRINT 3
    3 FORMAT (/1H ,5X,2HC1,8X,2HC2,8X,2HC3,7X,2HC4,8X,2HD1,8X,2HD2,9X,2H
     *D3,8X,2H Q)
      PRINT 4,C1,C2,C3,C4,D1,D2,D3,Q
   50 FORMAT (1X,E10.3)
    4 FORMAT (9E10.3)
      PRINT 5
      DO 80 J=1,21
      Z = 5.*(J-1)
      IF (Z.GT.Z1) GO TO 10
      HH = C1*Z + D1
      UP = 0.
      SINK = 0.
      FLU = C1
      GO TO 12
   10 IF(Z.GT.Z2) GO TO 11
      HH = Q*(Z-Z1)**3*(Z+Z1-2.*Z2)/12. + C2*Z + D2
      UP = Q*(Z-Z1)**2*(2.*Z+Z1-3.*Z2)/6./TU
      SINK = Q*(Z-Z1)*(Z-Z2)
      FLU = UP*TU + C2
      GO TO 12
   11 HH = C3*Z + D3
      UP = 1.
      SINK = 0.
      FLU = C3
   12 H = -(ABS((1.-SN)*HH/SA))**(1./(1.-SN))
      FLUX = -1.*FLU
      PSI = H + Z
      PF = ALOG10(ABS(PSI))
      DHDZ = FLU*(-1.*H)**SN/SA
    5 FORMAT(/1H ,6X,1HZ,4X,15HTOTAL HEAD  PSI,5X,16HPERCENT UP  SINK,8X
     *,2HHH,6X,4HFLUX,6X,4H PF /)
      PRINT 6,Z,H,PSI,UP,SINK,HH,FLUX,PF
   80 CONTINUE
      N = N + 1
      GO TO 9
   90 STOP
```

Fig. 9. Program listing for parabolic sink function

d. Exponential sink function

This type of sink function has the advantage that only four constants are needed to describe it. Besides the z_1, z_2 and S_m needed before, the value of z, z_m, where S_m is achieved must also be specified. A diagram showing such a sink function is given in Fig. 10 with the resulting solution given in Table 4. The FORTRAN listing of the computer program used to evaluate the various interesting terms is given in Fig. 11.

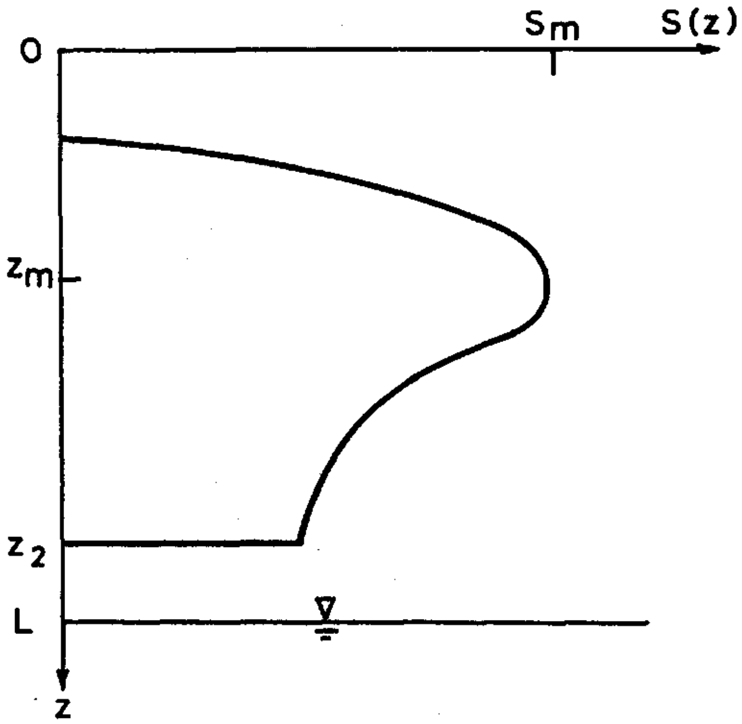


Fig. 10. Exponential sink function

e. Examples

In this section we give examples to show the effect the two parameters a and n (occurring in the unsaturated hydraulic conductivity function) have on the pressure head distribution in the soil. The values of a and n are typical of those given in BOUMA et al. (1979) for heavy clay.

Table 4. Results for an exponential sink function

S(z)	$K \frac{dH}{dz}$	HH	z
0	C_1	$C_1 z + D_1$	$0 < z < z_1$
$S_m(z-z_1)/(z_m-z_1)E$	$-S_m(z+z_m-2z_1)E + C_2$	$S_m(z_m-z_1)(z+2z_m-3z_1)E + C_2 z + D_2$	$z_1 < z < z_2$
0	C_3	$C_3 z + D_3$	$z_2 < z < L$

$$E = \exp\,[(z_m-z)(z_m-z_1)]$$

$C_1 = -v_0$

$C_2 = C_1 + S_m(z_m-z_1)e$

$C_3 = C_2 - S_m(z_2+z_m-2z_1)\exp[(z_m-z_2)/(z_m-z_1)]$

$D_3 = aH_L^{1-n}/(n-1) - C_3 L$

$D_2 = D_3 + (C_3-C_2)z_2 - S_m(z_m-z_1)(z_2+2z_m-3z_1)\exp[(z_m-z_2)/(z_m-z_1)]$

$D_1 = D_2 + S_m(z_m-z_1)(2z_m-z_1)e$

```
      PROGRAM SINK(INPUT,OUTPUT)
      REAL KO,MZ21
      CALL CONNEC(5LINPUT)
      N = 0
   15 READ 10,KO,ALP,VO,PHO,Z1,Z2,RL,A,A1
   10 FORMAT(9F8.4)
      RM = .5*(ALP - SQRT(ALP*ALP + 4.*A))
      RN = ALP - RM
      B1 = ALP*(PHO + A1/A)
      B2 = VO + ALP*A1/A
      AMZ21 = EXP(RM*(Z2-Z1))
      EMZ21 = EXP(-RM*(Z2-Z1))
      MZ21 = RM*(Z2-Z1)
      ENZ1 = EXP(RN*Z1)
      ENZ2 = EXP(RN*Z2)
      EALZ = EXP(ALP*(RL-Z2))
      RMDN = RM/RN
      BTM = EXP(RN*Z2-MZ21)*(RM+RN*EALZ)-RM*ENZ1*(1.+RMDN*EALZ)
      CC = (B1*EMZ21 - B2*(1. + RMDN*EALZ))/BTM
      DD = (RMDN*B2*EXP(MZ21-ALP*Z2)+(RN*EXP((RN-ALP)*Z2)
     * -RM*RMDN*ENZ1*EXP(MZ21-ALP*Z2))*CC)/ALP
      BB = B2/RN*EXP(-RM*Z1) - RMDN*EXP((RN-RM)*Z1)*CC
      AA = (RMDN*B2 + ALP*(1.-RMDN)*ENZ1*CC)/ALP*EXP(-ALP*Z1)
      DL = PHO - DD*EXP(ALP*RL)
      TU = A*CC*ENZ1*(EXP(RN*(Z2-Z1))-AMZ21)/RN + B2*(1. - AMZ21)
      PRINT 70
   70 FORMAT(1H1)
      PRINT 20
      PRINT 60,KO,ALP,VO,PHO,Z1,Z2,RL,A,A1
      PRINT 30
      PRINT 40,AA,BB,CC,DD,DL,TU
   60 FORMAT(9F10.3)
   20 FORMAT(/1H ,3X,2HKO,8X,5HALPHA,5X,2HVO,6X,7HPHIZERO,6X,2HZ1,8X,2HZ
     *2,9X,1HL,9X,1HA,8X,2HA1/)
   30 FORMAT(/1H ,3X,2HAA,9X,2HBB,8X,2HCC,9X,2HDD,3X,9HPHIO-DEAL,1X,9HT.
     * UPTAKE/)
   40 FORMAT(6E10.3)
      PRINT 50
      DO 1 J = 1,21
      Z =  5.*(J-1)
      IF(Z.GT.Z1) GO TO 11
      PHI = VO/ALP+AA*EXP(ALP*Z)
      U = 0.
      SK = 0.
      GO TO 13
   11 IF(Z.GT.Z2) GO TO 12
      PHI = BB*EXP(RM*Z) + CC*EXP(RN*Z) - A1/A
      SK = A1 + A*PHI
      U = B2*(1.-EXP(RM*(Z-Z1)))+A*CC*ENZ1*(EXP(RN*(Z-Z1))-EXP(RM*(Z-Z1
     *)))/RN
      PU = U/TU
      GO TO 13
   12 PHI = DL + DD*EXP(ALP*Z)
      SK = 0.
      U = 0.
      PU = 1.
   13 RH = ALOG(ALP*PHI/KO)/ALP
      PRINT 40,Z,PHI,RH,U,PU,SK
   50 FORMAT(/1H ,4X,1HZ,8X,3HPHI,6X,4HHEAD,5X,6HUPTAKE,3X,9H P.UPTAKE,3
     *X,4HSINK/)
    1 CONTINUE
      N = N + 1
      IF(N.LT.4) GO TO 15
```

Fig. 11. Program listing for exponential sink function

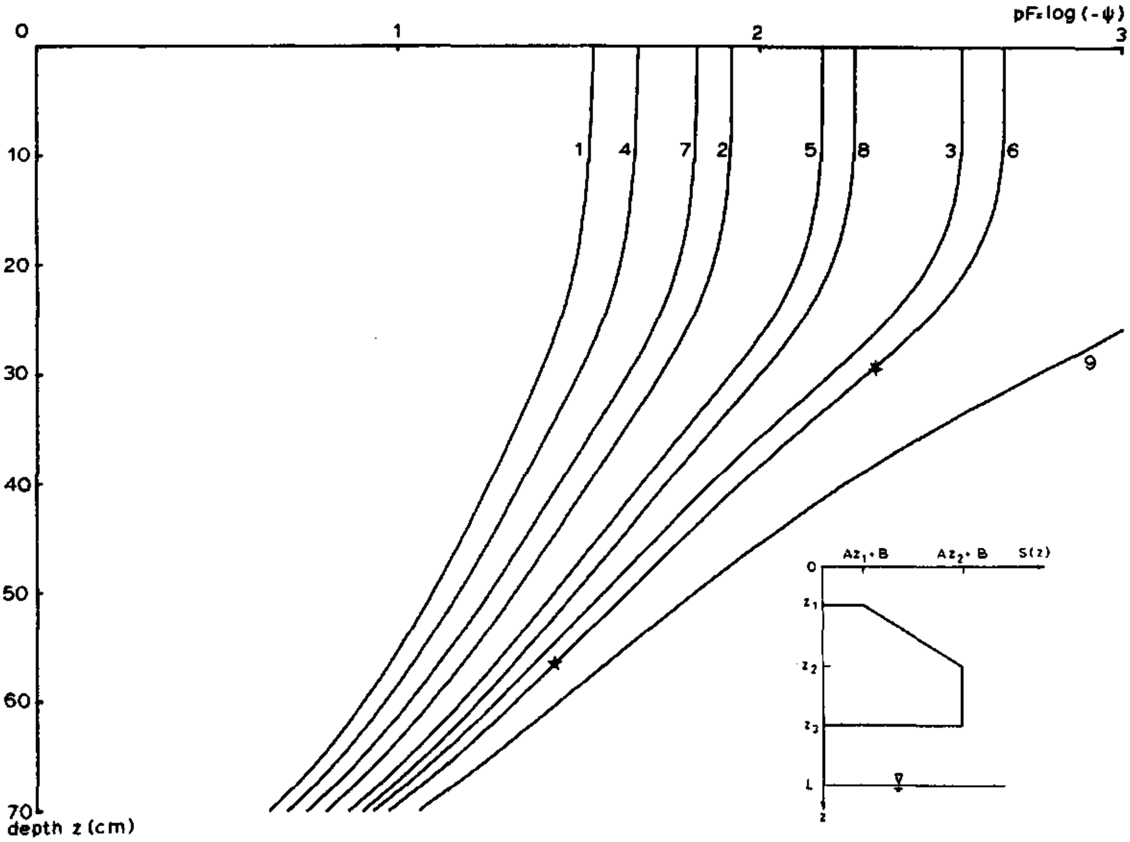
We will use the 3 different sink functions of Figs. 4, 8 and 10 to model water uptake by plant roots. All the water used by the roots is assumed to come from the water table and the flux across the soil surface is taken to be zero. In all examples in this section we plot pF = log(- Ψ) versus the depth beneath the soil surface z ($0 \leqslant z \leqslant 70$). The lower limit of 70 was chosen simply for plotting convenience. We choose a root pattern consistent with that of day 185 in Fig. 44 of FEDDES et al. (1978). The input parameters are listed directly below each figure.

In the Figs. 12 and 13 the input parameters of the 2nd - 8th column apply to the inserted Fig. 4. The 9th - 10th column contain parameters of the unsaturated hydraulic conductivity function. The numbers on the curves are associated with the various cases. The difference between Fig. 12 and 13 is that Fig. 12 uses a = 8 for the cases 3, 6 and 9, while Fig. 13 uses a = 9 for the same threee corresponding numbers. It is obvious from these two figures that increasing the value of n, or decreasing the value of a causes an increase in the value of pF. Also decreasing the value of a can cause a slight 'bending back' of the curves in the middle. To state this mathematically consider curve 6 in Fig. 12.

The concavity between z = 0 and the first * and between the second* and z = 70 is to the left while between the two *'s the curve bends to the right. It should be noted that the maximum pF for curve 9 is 4.45.

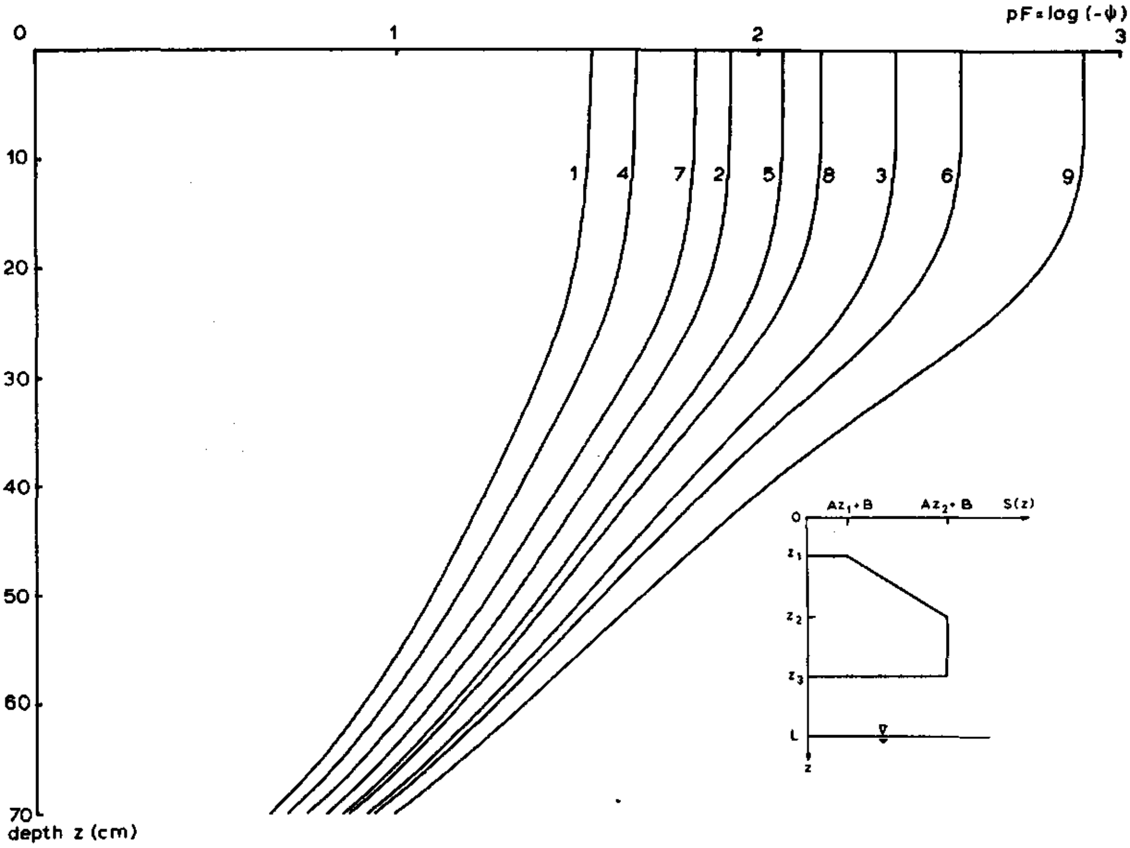
In Figs. 14 and 15 a parabolic water uptake pattern is used consistent with Fig. 8. The parameters in the 2nd - 6th column again apply to the inserted Fig. 8. The shapes of the curves are very similar to those for the straight line sink given before. Notice that changing the value of a from 8 to 9 for curve 6 in Figs. 14 and 15 results in pF-values of 3.82 and 3.48 respectively.

The column on the bottom of Fig. 16 are as in the Figs. 14 and 15 with the addition of a column (3) to denote where the exponential sink obtains its maximum value. Figs. 15 and 16 have comparable values of a and n. The large increase in pF between the two curves is because of the different uptake pattern between the parabola and the exponential. As is indicated in the Figs. 8 and 10, for identical



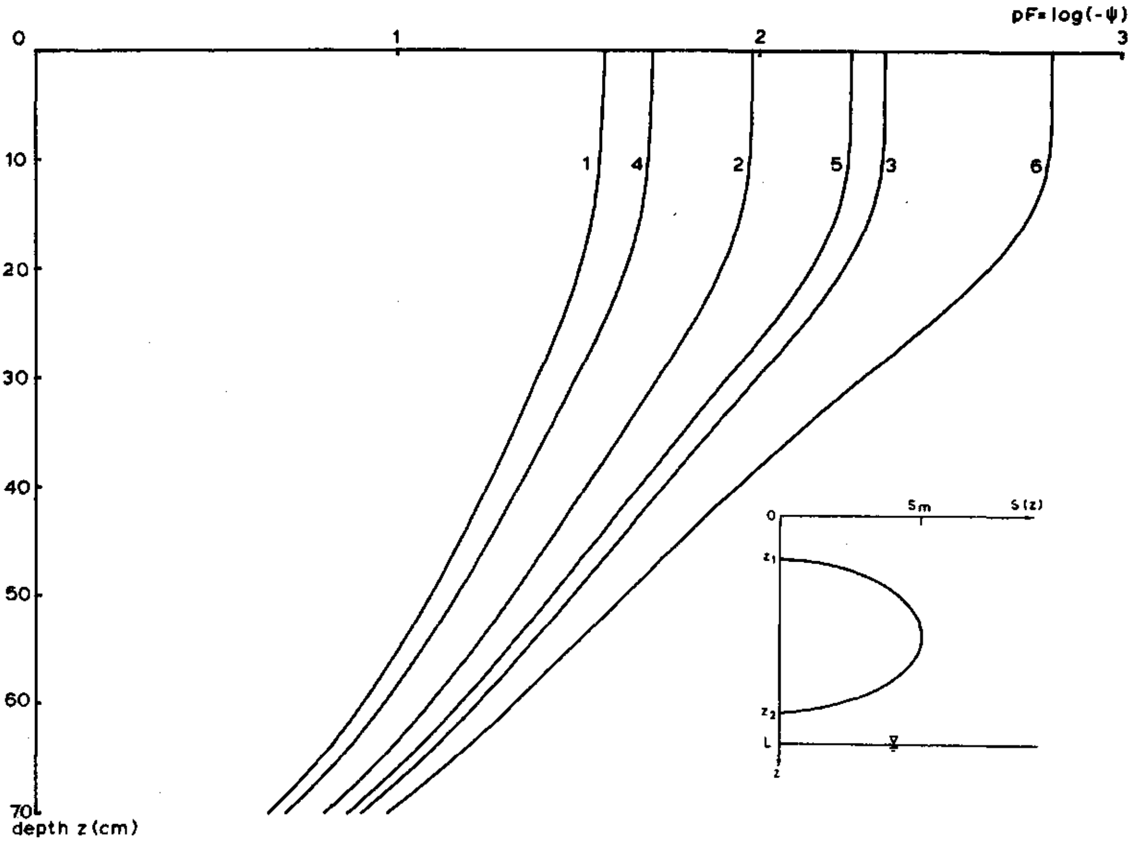
No.	A	B	z_1	z_2	z_3	L	h_L	$K = a(-H)^{-n}$ n	a
1	.447E-03	-2.233E-03	5.	20.	30	85.	85.	1.2	15
2	.447E-03	-2.233E-03	5.	20.	30.	85.	85.	1.2	10
3	.447E-03	-2.233E-03	5.	20.	30.	85.	85.	1.2	8.
4	.447E-03	-2.233E-03	5.	20.	30.	85.	85.	1.23	15
5	.447E-03	-2.233E-03	5.	20.	30.	85.	85.	1.23	10
6	.447E-03	-2.233E-03	5.	20.	30.	85.	85.	1.23	8.
7	.447E-03	-2.233E-03	5.	20.	30.	85.	85.	1.26	15
8	.447E-03	-2.233E-03	5.	20.	30.	85.	85.	1.26	10
9	.447E-03	-2.233E-03	5.	20.	30.	85.	85.	1.26	8.

Fig. 12. Plots of pF versus depth for 9 different cases using a sink term with connected straight lines. A listing of the various input parameters applied is given above



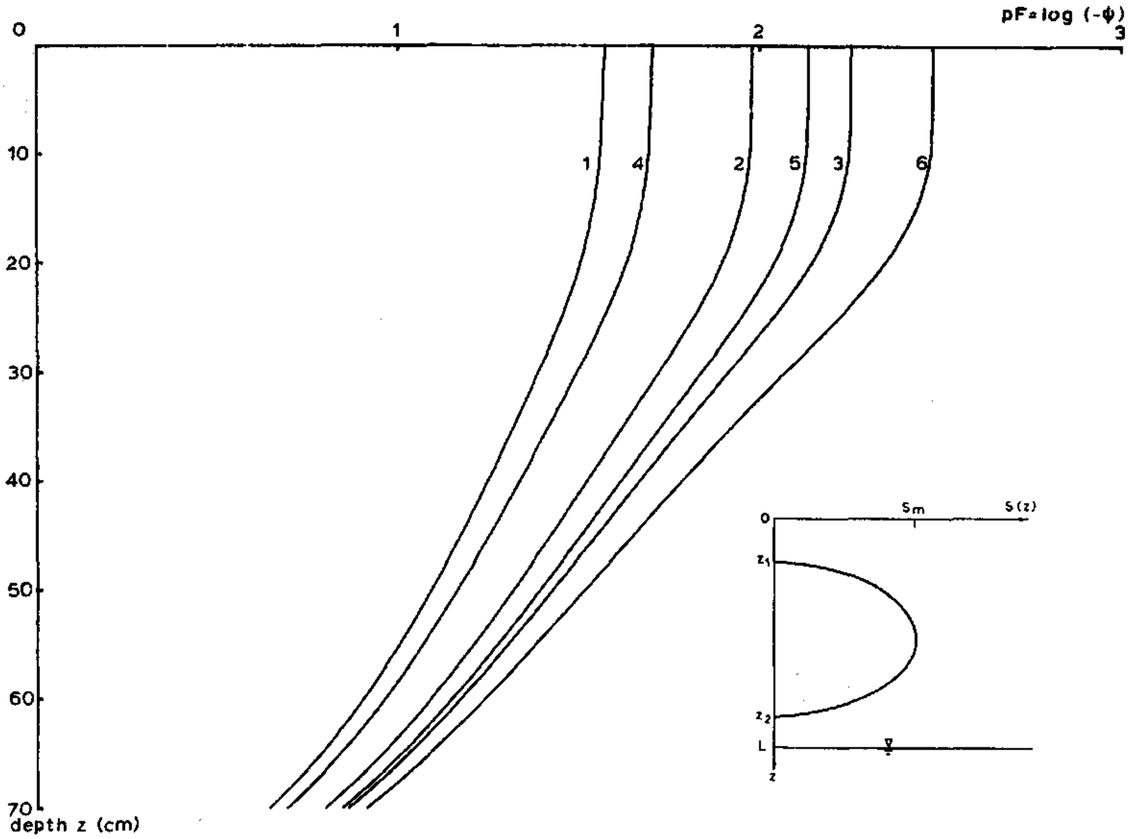
No.	A	B	z_1	z_2	z_3	L	h_L	$K = a(-H)^{-n}$ n	a
1	.447E-03	-2.233E-03	5.	20.	30.	85.	85.	1.2	15
2	.447E-03	-2.233E-03	5.	20.	30.	85.	85.	1.2	10
3	.447E-03	-2.233E-03	5.	20.	30.	85.	85	1.2	9.
4	.447E-03	-2.233E-03	5.	20.	30.	85.	85.	1.23	15
5	.447E-03	-2.233E-03	5.	20.	30.	85.	85.	1.23	10
6	.447E-03	-2.233E-03	5.	20.	30.	85.	85.	1.23	9.
7	.447E-03	-2.233E-03	5.	20.	30.	85.	85.	1.26	15
8	.447E-03	-2.233E-03	5.	20.	30.	85.	85.	1.26	10
9	.447E-03	-2.233E-03	5.	20.	30.	85.	85.	1.26	9.

Fig. 13. As Fig. 12, but with different values of parameter a used for the cases 3, 6 and 9.



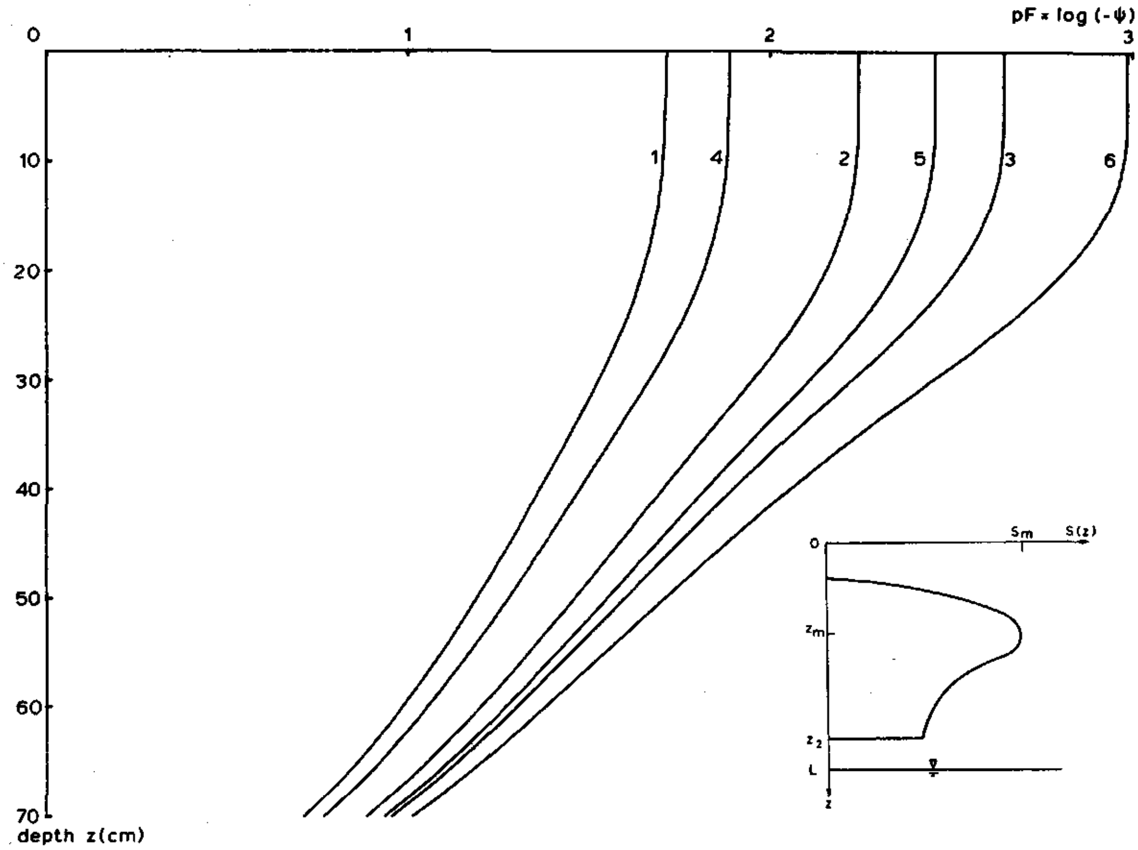
No.	S_m	z_1	z_2	L	h_L	$K = a(-H)^{-n}$	
						n	a
1	.701E-02	5.	30.	85.	85.	1.2	15
2	.701E-02	5.	30.	85.	85.	1.2	10
3	.701E-02	5.	30.	85.	85.	1.2	8.
4	.701E-02	5.	30.	85.	85.	1.23	15
5	.701E-02	5.	30.	85.	85.	1.23	10
6	.701E-02	5.	30.	85.	85.	1.23	8.

Fig. 14. Plots of pF versus depth for 6 different cases using a parabolic sink function. A listing of the various input parameters applied is giving above



No.	S_m	z_1	z_2	L	h_L	$K = a(-H)^{-n}$	
						n	a
1	.701E-02	5.	30.	85.	85.	1.2	15
2	.701E-02	5.	30.	85.	85.	1.2	10
3	.701E-02	5.	30.	85.	85.	1.2	9.
4	.701E-02	5.	30.	85.	85.	1.23	15
5	.701E-02	5.	30.	85.	85.	1.23	10
6	.701E-02	5.	30.	85.	85.	1.23	9.

Fig. 15. As Fig. 14 but with different values of parameter a for the cases 3 and 6



No.	S_m	z_1	z_m	z_2	L	h_L	$K = a(-H)^{-n}$	
							n	a
1	.701E-02	5.	20.	30.	85.	85.	1.2	15
2	.701E-02	5.	20.	30.	85.	85.	1.2	10
3	.701E-02	5.	20.	30.	85.	85.	1.2	9.
4	.701E-02	5.	20.	30.	85.	85.	1.23	15
5	.701E-02	5.	20.	30.	85.	85.	1.23	10
6	.701E-02	5.	20.	30.	85.	85.	1.23	9.

Fig. 16. Plots of pF versus depth for 6 different cases using an exponential sink function. A listing of the various input parameters applied is given above

values of S_m, z_1 and rooting depths, the area under the parabola is greater than the area under the exponential. This means that with less water extraction, the profiles for the exponential sink will be much wetter than for a parabolic sink. If one desires the same total plant water uptake for the two cases and identical root location, then the maximum value (S_m) for the sink function for the exponential must be increased over that of the parabola (Exact expressions for the total uptake for the three sink functions are listed as TU in the computer listings in the Figs. 5, 9 and 11).

V. SUMMARY AND CONCLUSIONS

We have considered two types of hydraulic conductivity pressure head relationships and given analytical solution for one-dimensional flow with various types of functions describing water uptake. A surface flux was prescribed to allow for rainfall or irrigation and a shallow water table was assumed. Mathematically the pressure head was prescribed at a specific depth so deep water tables can also be described by these solutions with appropriate choices for this pressure head.

For the case of $K = K_o \exp(\alpha\Psi)$ the sink function can be given explicitly in terms of depth, or explicitly in terms of the pressure head and two arbitrary parameters. (The latter formulation is the linear Taylor series expansion of the $K(\Psi)$ function of FEDDES et al., 1978 over the dry range of Ψ). The FORTRAN listing is given and an example shows the sensitivity of the uptake pattern to these two parameters. For this type of model, the uptake pattern and moisture profile are outputs of the model, with the two empirical parameters, surface flux, water table depth etc. being the inputs.

Solutions for a $K(\Psi) = a(-\Psi)^{-n}$ can be obtained only for special values of n, i.e. 1 and 1.5. However, for $K(\Psi) = a(-\Psi + z)^{-n} = a(-H)^{-n}$ the resulting moisture profiles may be easily obtained. Analytical expressions are obtained for the resulting moisture profiles when the sink function $S(z)$ is given explicitly terms of depth. The three types of patterns for $S(z)$ are straight line, parabolic and exponential.

Plots of pF versus z curves are given for each of these three functions and six or twelve combinations of a and n. To use these analytical results, the exact uptake pattern must be described as an input to the system with the moisture profile being the output.

As mentioned in the body of this nota, care must be made in the choice of a and n values in the hydraulic conductivity function. Certain choices will not allow for enough water movement from the water table to meet the demand of the plant roots. This in turn gives rise to nonsensical values of pF. When this occurs, HH will be negative and all results should be ignored.

BOUMA et al (1979) note a wide range of a and n values for heavy clays. These values greatly depend on the range of Ψ over which the least squares fit is taken. For best results, one should use a and n values which are obtained by a best fit over the exact range of Ψ one is dealing with. In other words if one is operating in the dry range, one should obtain a and n from a best fit over that range. As an example consider that the values of a and n change from 5.44 and 1.14 to 7.83 and 1.228 respectively by simply ignoring all data with Ψ greater than -100!

The analytical expressions developed in this nota may generally be used to test complex numerical schemes for which it is difficult to detect errors.

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