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RAIN STORM MODELS AND THE RELATIONSHIP BETWEEN THEIR PARAMETERS

dr. Ph.Th. Stol

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C O N T E N T S

1. INTRODUCTION

Rainfall interstation correlation functions can be obtained with the aid of analytic rainfall or storm models (STOL, 1977a; 1977b; 1977c).

Correlation functions for various storm models, although completely different concerning their mathematical structure, do not differ so much in shape when plotted in a graph.

If we want to mutually compare the graphical representation of different correlation functions, this can be done best by comparable storm models.

Since alternative storm models have different mathematical formulas, comparison should be based on equallity of parameters like storm diameter, mean rainfall amount, storm maximum or total storm volume.

In this report we will discuss some useful stormmodels and the relationship between the parameters of three models already used to obtain the correlation function analytically.

2. REVIEW

Storm models can be used in analytic hydrological research.

Since the storm model is the starting point for further elaborations, their mathematical definition must be simple to be sure that the required analytic treatment can be fulfilled.

Even strongly simplified models may lead to rather complicated mathematical results when deriving the correlation function. This means that the search is for simple models that should be, however, reasonable realistic. For this reason we first will discuss models that have been suggested in the literature.

In most of the following formulas the variables are consistent and have the following meaning:

h = rainfall depth

x,y = coordinates to define locations in the storm

H = maximum rainfall depth in the center of the storm

B = storm width, storm diameter

r = storm radius measured from the center to an isohyetal

A = area, enclosed by isohyetals

S = total storm volume

a,b = further storm parameters

$\underline{\varepsilon}$ = random component

Occasionally some other variables are used. They are defined in the text.

Random variables will be underlined. The following expression means:

$\underline{\varepsilon} = 0$, the random variable takes the value 0 (in this special case)

In this Section the origin of the co-ordinate system is located at the center of the storm.

COURT (1961) has given a review on area-depth rainfall formulas. He refers, among others, to Frühling, who in 1894 proposed a parabolic equation

$$h = H(1 - \sqrt{\frac{x}{B}}) \qquad (1)$$

where B was taken 12 km. Most of the other functions referred to are based on areas enclosed by isohyetals and try to give average values over that area. Special mention must be made of his reference to Light from whom a logarithmic curve is discussed, suggested in a 1947 research. COURT quotes Light's conclusion that in '... a single-celled rainfall pattern with concentric circles as isohyets, ...

rainfall decreases logarithmically with distance from the storm center'.

The general structure of this suggestion reads

$$h = a - b \ln A, \quad A > 50 \text{ mi}^2$$

which was scaled accordingly to avoid unrealistic results.

Court himself suggested the use of a Gaussian type formula namely

$$h(x,y) = H\, e^{(-a^2x^2-b^2y^2)} \tag{2}$$

Finally Court concludes that 'the several formulas discussed ... indicate that short-duration storms tend to have steeper precipitation gradients than those of longer duration and larger area'. He also mentions some conditions to be met by storm functions namely: 'Any realistic representation of the variation of rainfall amount with distance from the storm center should be smooth at the center. This means that the first derivative of the function should be zero when evaluated at the center' ... 'At the other extreme, an asymptotic approach to zero rainfall with increasing distance seems desirable'. In conclusion he adds: 'The Gaussian formula, in addition, has some probabilistic justification, and may be suitable as an area-depth formula'.

McCULLOCH, 1961, in an article on statistical assessment of rainfall, gives examples for Africa from which he concludes: '... that it is unreasonable to expect satisfactory results on the assumption of a statistical model of the type

$$\underline{h}_i = H + \underline{\varepsilon}_i \tag{3}$$

where $\underline{h}_i$ is the rainfall measured at a given point in the area, H is the true rainfall of the area and $\underline{\varepsilon}_i$ is the deviation from the mean of any particular observation $\underline{h}_i$.'...

'Unless the area being considered is very small or the rainfall is widespread, 'cold-front' type rainfall, it is unrealistic to propose a general mean;...'

The above mentioned model will be called the 'rectangular' type. It lacks, however, the definition of the storm size (or storm diameter) and so it is not a complete model.

On the basis of his experience McCulloch proceeds with saying that '.... in these circumstances, there is no option but to propose a sampling model of the type:

$$\underline{h}_i = H + ax + by + \underline{\varepsilon}_i \qquad (4)$$

where a and b may be considered as pure numbers in the first instance and x and y are rectangular co-ordinates of distance in two dimensions'.

The constants a and b need be chosen such that 'the pattern is one of a heavy deluge over a small area with rainfall decreasing perhaps to zero at a relatively short distance from the center of the storm.'

Using only two dimensions the foregoing statement refers to a triangular storm type. However, like model (3), the expression given by (4) lacks the storm diameter as a parameter. So, a (and b) cannot be considered as 'pure numbers' but should be chosen such that at the boudary of the storm $\underline{h}_i = 0$, apart from random fluctuations.

McCulloch finally mentions: 'In particular cases it may be necessary to postulate quadratic, cubic or even more complicated dependence on the distance co-ordinates.' However, no further suggestions are given.

On the other hand EPSTEIN (1966) considered circular precipitation cells in a study on point and area precipitation probabilities. Here, indeed, the storm diameter was given as a parameter. It was assumed that each cell covered an area A so having a diameter 2r obtained from the radius r given by

$$r = \sqrt{\frac{A}{\pi}} \qquad (5)$$

No further details on the distribution of rainfall intensities within each call were given since the main objective was to derive probabilities of any amount of precipitation.

This was done by FOGEL and DUCKSTEIN (1969) who used circular patterns with a Gaussian-type rainfall distribution to obtain rainfall frequencies.

BOYER (1957) refers to a study of thunderstorm rainfall made by the U.S. Weather Bureau and the Corps of Engineers in 1947. An indication was obtained '... that for such storms the precipitation rate h along an isohyetal is an exponential function of distance from the storm center ...'. It appeared to hold for much larger storms as well. The formula reads

$$h = H\, e^{-ax} \tag{6}$$

where

- h = precipitation along any isohyetal
- H = maximum at the eye of the storm
- a = a coefficient of distribution
- x = distance from the storm center to the isohyetal, measured along an axis of the storm

This formula is the basic form for the exponential storm type.

In an investigation on the sensitivity of peak catchment discharge to the characteristic spatial variability of convenctive and cyclonic storm rainfall, EAGLESON (1967) used the model given by (6) to represent 'great cyclonic storms'.

For convective storms Eagleson refers to WOOLHISER and SCHWALEN (1959) who fitted the average areal rainfall distribution with a storm-centered function, where radial symmetry and a circular area is assumed. From this function, which is essentially linear, Eagleson derives the relationship

$$\frac{P_T(r)}{P_T(0)} = 1 - 0.72\,\frac{r}{r_o} \tag{7}$$

where

- r = storm radius (distance from center)
- P_T = total storm depth
- r_o = storm correlation radius defined by $\frac{\Phi_P(r_o)}{\Phi_{P(0)}} = 0.5$
- Φ = energy density spectrum

In his analysis r_o was found to be 1.73 $P_T(0)$ giving

$$P_T(r) = P_T(0).\{1 - 0.42\ r\} \tag{8}$$

which is a linearly decreasing function with maximum rainfall depth $P_T(0)$ at r = 0 and storm diameter r = 2.38 giving $P_T(2.38) = 0$.

Although the functions used by Eagleson are linear (8) and exponential (6) functions, the symbols used are explained in his article by a three-dimensional bell-shaped storm rainfall pattern (Eagleson, 1967, fig. 2) which, however, was not used.

This was done by HUTCHINSON (1970) who used the bell-shaped form as a model rainstorm. The shape was defined as a circular storm. Of 172 actual storms the shape appeared to be '... somewhat irregular, but about 80% were more circular than elliptical...'.

The rainstorms were thus given by

$$i = I \exp \left[\frac{-18\{(x-\bar{x})^2 + (y-\bar{y})^2\}}{d^2} \right] \tag{9}$$

where

I = maximum intensity

x,y = general co-ordinates

$\bar{x},\bar{y}$ = co-ordinates of the center of the storm

d = diameter

i = intensity at any point (x,y)

This model was used in the following way:

I and d are supposed to be able to be represented by two parameter log normal distributions, with

for I: μ = 0.095 in. σ = 1.9 in.

for d: μ = 17 miles σ = 5.38 miles

$\bar{x},\bar{y}$ were obtained from a rectangular (0, 1)-distribution and scaled accordingly to suit an area of 500 miles x 500 miles

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This model was used by Hutchinson to obtain interstation correlations on basis of storm simulations. Although a storm model was used, no analytic solutions had been pursued.

Finally mention must be made of the study by RODRIGUEZ-ITURBE and MEJIA (1974) who applied a formula for '... the areal extension of convective storms...', viz.

$$P_t(r) = P_t(0) \exp(-\pi r^2 t) \qquad (10)$$

where t is a dispersion parameter given by

$$t = 0.27\ e^{-0.67}\ P_t(0) \qquad (11)$$

Here

$P_t(0)$ = total depth in inches at the storm center
πr^2 = area at distance r from the center

and so rainfall depth is expressed as a function of circular areas surrounding the center, and radial symmetry is assumed.

The formulas are due to FOGEL and DUCKSTEIN (1969). They write: 'When storms were selected for analysis, only those of less than two hours duration were chosen. Thunderstorms often consist of a group of three or more cells adjacent to each other'. So their formula may be regarded as a function for single storms.

3. PRINCIPLES OF STORM FUNCTIONS

Rainfall patterns are often very irregular concerning the isohyetal plots. For analytic models more regular patterns need be used to make all required elaborations possible.

Examples given in the literature only give elementary functions. For our purpose (STOL, 1977a, 1977c) the following mathematical treatment must be possible.

Given a storm diameter B, rainfall amounts h depending on the stormcoordinate x, and a storm maximum H, we define a storm in a two dimensional model by

$$h = {}^1f(x), \quad 0 \leq x < \tfrac{1}{2}B \quad \text{(left of center)}$$

$$h = {}^2f(x), \quad \tfrac{1}{2}B \leq x \leq B \quad \text{(right of center)}$$

Symmetry about the center $x = \frac{1}{2}B$ is assumed; ${}^1f(x)$ is a monotonic increasing, ${}^2f(x)$ a monotonic decreasing function.

Particular values are

$$h = {}^1f(0) = 0, \quad h = {}^2f(B) = 0$$

$$h = {}^1f(\tfrac{1}{2}B) = {}^2f(\tfrac{1}{2}B) = H$$

The assumption of symmetry tells that

$${}^1f(x) = {}^2f(B-x)$$

The total storm volume S is given by

$$S = 2\int_0^{\frac{1}{2}B} {}^1f(x)\,dx$$

If we choose a point $\underline{a}$ at random on B, uniformly distributed, then the probability P, based on intervals, that the corresponding rainfall depth $h_a = h(a)$ is not exceeded, is given by

$$P(h_{\underline{a}} \leq h_a) = \frac{2a}{B} \tag{12}$$

(See Fig. 1)

Expressed in the variable h, this becomes, defining ${}^1f^{-1}(h)$ to be the inverse function of ${}^1f(x)$,

$$P(h_{\underline{a}} \leq h_a) = \frac{2}{B}\,{}^1f^{-1}(h_a) \tag{13}$$

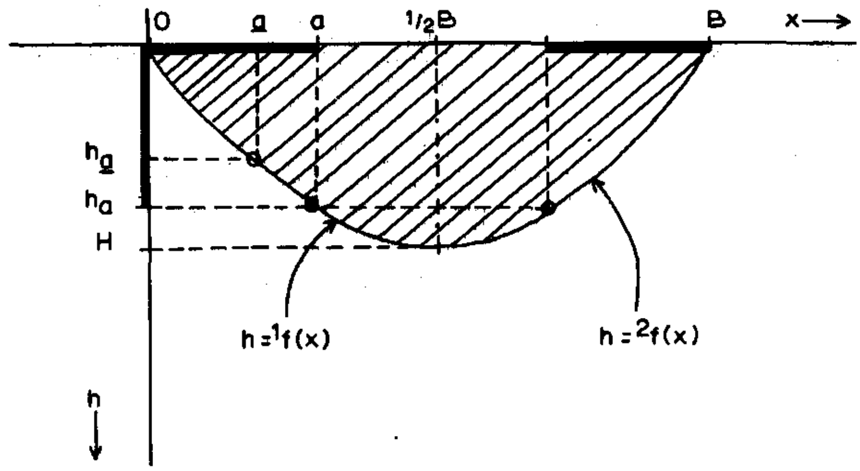

Fig. 1. Schematic illustration of the probability, based on intervals, that $h_{\underline{a}}$ is less than h_a. The heavy bars have the same length when the storm function is symmetric

The density of this function is, dropping the subscript a and defining i_o,

$$\frac{dP(h)}{dh} = \frac{2}{B}\,\frac{d}{dh}\,{}^{1}f^{-1}(h) = \frac{2}{B}\,i_o \tag{14}$$

To obtain statistical parameters the following elaborations must be possible.

For the expectation μ,

$$\mu = \frac{2}{B}\int_0^H h\,\frac{d}{dh}\,{}^{1}f^{-1}(h)\,dh \tag{15}$$

For the variance σ^2,

$$\sigma^2 + \mu^2 = \frac{2}{B}\int_0^H h^2\,\frac{d}{dh}\,{}^{1}f^{-1}(h)\,dh \tag{16}$$

In stead of the statistical parameter μ we can calculate the mean value $\bar{h}$ as follows

The total storm volume S reads

$$S = 2\int_0^{\frac{1}{2}B} {}^{1}f(x)\,dx$$

Since the base of this volume equals B, the mean heigth of the storm is

$$\bar{h} = \frac{S}{B}$$

or

$$\bar{h} = \frac{2}{B} \int_{x=0}^{x=\frac{1}{2}B} {}^{1}f(x).dx$$

Since ${}^{1}f(x) = h$ and $x = {}^{1}f^{-1}(h)$ we may write the last integral in terms of h, viz.

$$\bar{h} = \frac{2}{B} \int_{{}^{1}f^{-1}(h)=0}^{{}^{1}f^{-1}(h)=\frac{1}{2}B} h.d\ {}^{1}f^{-1}(h)$$

which equals

$$= \frac{2}{B} \int_{h=0}^{h=H} h.\frac{d\,{}^{1}f^{-1}(h)}{dh}\ dh$$

so

$$\bar{h} = \mu \qquad (17)$$

and the mean rainfall depth $\bar{h}$ over the total storm diameter B equals the mathematical expectation μ for randomly chosen points in the storm.

Comparing parameters of different storms it obviously does not matter whether the mean value or the mathematical expectation is used.

The covariance (Fig. 2) between two points h_a and h_b at distance D gives rise to even more complicated formulas (STOL, 1977c). However, if (15) and (16) can not be solved the analytic approach already breaks down here. These integrals are used as a test case to decide

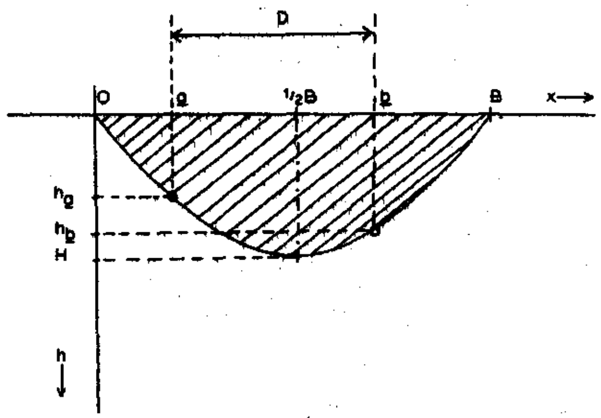

Fig. 2. Schematic illustration of the relationship between two randomly chosen points connected by their distance D. The covariance between $h_{\underline{a}}$ and $h_{\underline{b}}$ depends on D

whether or not it makes sense to pay attention to the storm model by which they are produced.

4. DISCUSSION

From the foregoing review it follows that there are four main types that need further concern. They are, including remarks given by the respective authors,

a. The rectangular storm type

- smooth at the center
- no approach to zero
- valid for longer durations and larger areas
- valid for 'cold-front' wide spread rainfall or very small areas

b. The triangular storm type

- not smooth at the center
- reaches zero at finite distance
- valid for short-duration storms
- heavy deluge over a small area
- convective storms

c. The exponential storm type

- Not smooth at the center
- Asymptotic approach to zero
- Valid for short-duration storms
- Thunderstorms and larger storms
- Great cyclonic storms

d. The Gaussian storm type

- Smooth at the center
- Asymptotic approach to zero
- Valid for short-duration storms
- Convective storms (less than two hours)
- Probabilistic justification

Some remarks must be made.

. It seems not realistic to require 'smoothness' at the center. A vanishing first derivative can only be found on the basis of a model and so this property is included in the choice of the model. The phenomenon of rainfall often indicates the existence of isolated peak values. It is not clear how 'smoothness' should be defined on the basis of rainfall data alone.

. It seems not realistic to require an asymptotic approach to zero. Intervals in which this approach takes place are comparatively small when regarding the total storm diameter.

. The probabilistic justification of the Gaussian type is not a correct argument since the ordinate of the storm function is the rainfall amount and not the probability density. According to the theory developped here (equation (14)) the differential form for the density would read (see also section 8)

$$\frac{2}{B} \cdot \frac{1}{4bh} \frac{1}{\sqrt{\frac{1}{2b} \ln(H/h)}} \, dh$$

which is not Gaussian.

- Physical arguments are not given that plead for a specific storm function. The only thing that could be concluded is:

  Cold-front, wide spread rainfall storms occurring over large areas can be described by the rectangular storm function;

  Convective storms, thunderstorms, short-duration storms, great cyclonic storms can be described by storm functions that have a maximum at the center and a relatively steep slope to the edges.
- For an analytic approach integrals of the type given by (15) and (16) have to be solved with the chosen storm function. This argument probably is more selective than any physical or meteorological condition.

5. THE RECTANGULAR STORM TYPE

We can define the rectangular storm type by

$$h = {}^{1}f(x) = H \quad \text{and} \quad h = {}^{2}f(x) = H \qquad (18)$$

The total storm volume is

$$S = BH$$

and the mean value equals the mathematical expectation (STOL, 1977a)

$$\bar{h} = \mu = H$$

and because $h = H$ is constant we have

$$\sigma^2 = 0$$

All statistical parameters can be obtained and so this storm type is suitable for further elaborations.

6. THE TRIANGULAR STORM TYPE

Like (8) the rectangular type can be defined by

$$h = {}^{1}f(x) = \frac{2H}{B}x, \quad h = {}^{2}f(x) = 2H - \frac{2H}{B}x \tag{19}$$

and the total storm volume is

$$S = 2\int_{0}^{\frac{1}{2}B} \frac{2H}{B}\,x\,dx = \frac{4H}{B} \cdot \tfrac{1}{2}\,x^2 \Big|_{0}^{\frac{1}{2}B} = \tfrac{1}{2}BH$$

which gives for the mean value

$$\bar{h} = \frac{S}{B} = \tfrac{1}{2}H$$

From the definition of this storm type we have

$${}^{1}f^{-1}(h) = \frac{Bh}{2H}$$

and so, from (14):

$$i_o = \frac{B}{2H}$$

with density 1/H. Consequently

$$\mu = \int_{0}^{H} h.\frac{1}{H}\,dh = \frac{h^2}{2H}\Big|_{0}^{H} = \frac{H}{2}$$

The variance is obtained by

$$\sigma^2 = \int_{0}^{H} h^2.\,\frac{1}{H}\,dh - \mu^2$$

$$= \frac{h^3}{3H}\Big|_{0}^{H} - \frac{H^2}{4} = \frac{H^2}{12}$$

This storm type too can be used for further elaborations.

7. THE EXPONENTIAL STORM TYPE

According to (6) we can define the exponential storm type by

$$h = {}^{1}f(x) = He^{2b(x-\frac{1}{2}B)} \tag{20a}$$

$$h = {}^{2}f(x) = He^{2b(\frac{1}{2}B-x)} \tag{20b}$$

The total storm volume is

$$S = 2\int_{0}^{\frac{1}{2}B} He^{2b(x-\frac{1}{2}B)}\; d\,\frac{2b(x-\frac{1}{2}B)}{2b}$$

$$= \frac{H}{b}\, e^{2b(x-\frac{1}{2}B)}\Big|_{0}^{\frac{1}{2}B}$$

$$= \frac{H}{b}\,(1-e^{-bB})$$

which gives for the mean value

$$\bar{h} = \frac{S}{B} = \frac{H}{bB}\,(1-e^{-bB})$$

From the definition of this storm type we have

$${}^{1}f^{-1}(h) = i_o = \frac{1}{2}B + \frac{1}{2b}\ln\frac{h}{H}$$

and so, from (14):

$$i_o = \frac{1}{2bh}\,, \qquad {}^{1}f(0) \leq h \leq {}^{1}f(\tfrac{1}{2}B)$$

with density $\frac{1}{bBh}$. Consequently, with lower boundary ${}^{1}f(0)$ and upper boundary ${}^{1}f(\frac{1}{2}B)$, we have

$$\mu = \int_{He^{-bB}}^{H} h . \frac{1}{bBh}\, dh$$

$$= \frac{1}{bB}\, h \Big|_{He^{-bB}}^{H}$$

$$= \frac{H}{bB}\,(1-e^{-bB})$$

The variance is obtained by

$$\sigma^2 = \int_{He^{-bB}}^{H} h^2 . \frac{1}{bBh}\, dh - \mu^2$$

$$= \frac{1}{bB} \cdot \frac{1}{2}\, h^2 \Big|_{He^{-bB}}^{H} - \mu^2$$

$$= \frac{1}{2}\,\frac{H^2}{bB}\,(1-e^{-2bB}) - \frac{H^2}{b^2B^2}\,(1-e^{-bB})^2$$

or

$$\sigma^2 = \frac{H^2}{bB}\,\{\frac{1}{2}\,(1-e^{-2bB}) - \frac{1}{bB}\,(1-e^{-bB})^2\}$$

This storm type can also be used for further elaborations.

8. THE GAUSSIAN STORM TYPE

A few authors used a Gaussian type function to describe a storm. The equation reads

$$h = {}^{1}f(x) = He^{-2b(\frac{1}{2}B-x)^2} \qquad (21a)$$

$$h = {}^{2}f(x) = He^{-2b(x-\frac{1}{2}B)^2} \qquad (21b)$$

Since this function has no indefinite integral that can be expressed into elementary functions, the analytic approach breaks down here.

The inverse function reads

$$ {}^{1}f^{-1}(h) = \tfrac{1}{2}B - \sqrt{\frac{-1}{2b} \ln \frac{h}{H}} $$

and so i_o becomes

$$ \frac{+1}{4bh} \cdot \frac{1}{\sqrt{\frac{1}{2b} \ln \frac{H}{h}}}, \qquad {}^{1}f(0) \leq h < {}^{1}f(\tfrac{1}{2}B) $$

It is not possible to find expressions for μ and σ^2 on the basis of this integrand and so the Gaussian storm type is not used any further.

9. RELATIONSHIP BETWEEN PARAMETERS

The obtained results are collected in Table 1, where

$$ u = 1 - e^{-bB} \quad \text{and} \quad v = 1 + e^{-bB} \tag{22} $$

so

$$ uv = 1 - e^{-2bB} \quad \text{and} \quad u + v = 2 \tag{23} $$

Table 1. Characteristics of different storm types

Type	diameter	maximum	μ	σ^2	Volume S
Rectangular	B	H	H	0	BH
Triangular	B	H	$\frac{1}{2}H$	$\frac{1}{12} H^2$	$\frac{1}{2}BH$
Exponential	B	H	$\frac{H}{bB} u$	$\frac{H^2}{bB}(\frac{1}{2} v - \frac{1}{bB} u)u$	$\frac{H}{b} u$
Gaussian	B	H	-	-	

In order to be able to compare the different storm types mutually, some quantities have to be taken equal, to be sure that comparable storms have been used.

It can be proved (STOL, 1977c) that the correlation function in its simplest form does not depend on the maximum rainfall amount H in the center of the storm, while the storm diameter appears to be an important parameter. To be comparable with respect to their correlation function, storms should have equal diameters B, regardless the value of the maximum H. However, since the mathematical expectation occurs in the expression of the correlation function this parameter should be equallized too.

Apart from the arguments, suggested by the structure of the correlation function, how to compare storms, we can consider storms itselves in the same way.

Since the shape of the storms are different we can expect that they have different variances. We will consider storms matching if they have the same volume S of rain and we will not try to equallize their variances. Since we chose storms with equal diameter B, this means that we compare storms with the same value $\frac{S}{B} = \bar{h}$, so with the same mean value and in virtue of (17) also with the same expectation.

Let the subscript r refer to the rectangular type, t to the triangular type and e to the exponential type, then we have the following pairs for comparison.

9.1. Comparing the rectangular type with the triangular type

We take $\mu_r = \mu_t$ so

$$H_r = \tfrac{1}{2}H_t$$

or

$$H_t = 2H_r$$

and so take the maximum rainfall amount for the triangular storm equal to two times the maximum amount of the rectangular type to have storms with equal diameters and equal storm volumes.

9.2. Comparing the rectangular type with the exponential type

We take $\mu_r = \mu_e$ and so

$$H_r = \frac{H_e}{bB} (1-e^{-bB})$$

which yields

$$H_e = bBH_r \frac{1}{1-e^{-bB}}$$

Here we see that given H_r each value for b gives a solution for H_e which can be expressed by

$$H_e = H_e(b|B, H_r)$$

where the vertical bar means: 'given'.

No unique solution is obtained this way and further conditions have to be put forward. This will be done in the next section.

9.3. Comparing the triangular type with the exponential type

Since comparing the rectangular type with the triangular type does not give a unique solution we make use of the general shape of the triangular type and the exponential type which are approximately similar.

In conclusion we require for the exponential type to have

. the same volume and diameter as those for the rectangular type
. the same maximum amount as that for the triangular type with the same volume and diameter.

This means that we require that

$$H_e = H_t = 2H_r \qquad (24)$$

With this new condition we have to solve b from

$$2H_r = bBH_r \frac{1}{1 - e^{-bB}} \qquad (25)$$

or

$$bB = 2(1 - e^{-bB}) \qquad (26)$$

and, putting $bB = \beta$, this reduces to

$$\beta = 2(1 - e^{-\beta}) \qquad (26a)$$

This is a non-linear equation with one unknown, the storm constant β, which has to be solved iteratively.

10. THE SOLUTION OF THE STORM CONSTANT β

The solution of β can be obtained as follows
Consider the two functions given by (26) or (26a) and sketched in Fig. 3 viz.

$$l(\beta) = \beta \qquad (27)$$

$$c(\beta) = 2(1 - e^{-\beta}) \qquad (28)$$

This simultaneous system is assumed to be solved for $\beta = \beta_n$ if in sufficient approximation $l(\beta_n) = c(\beta_n)$.

First we have to prove that a solution, different from $\beta = 0$, exists. Solve both equations for β, then, introducing subscripts that are self-explanatory

$$\beta_l = l \quad , \quad \text{(from (27))}$$

$$\beta_c = \ln 2 - \ln(2-c) \; , \quad \text{(from (28))}$$

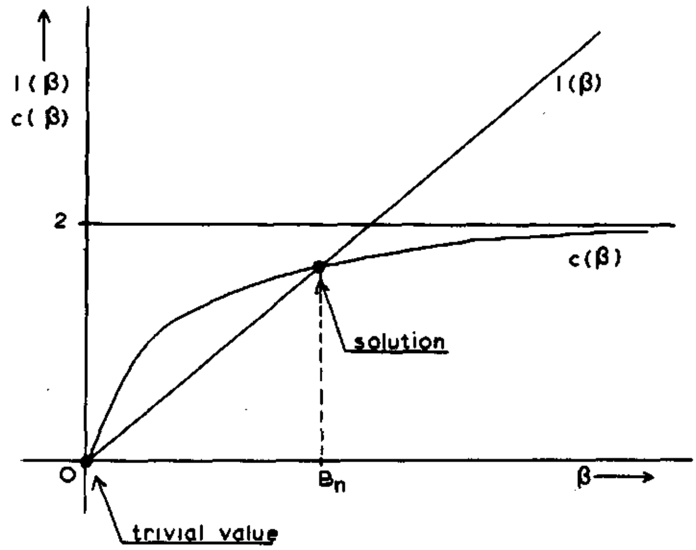

Fig. 3. Schematic illustration of the two functions of β, a linear and a curved function, whose points of intersection have to be determined

Take l = c and both equal to $(2 - e^{-N})$ which gives the particular solutions

$$\beta_1 = 2 - e^{-N} \tag{29}$$

$$\beta_c = N + \ln 2 \tag{30}$$

then, since $0 < \ln 2 < 1$, we note that for two particular values $N = N_i$ and $N = N_j$,

$$N_i = 0 \quad \text{gives} \quad \beta_1 > \beta_c \tag{31}$$

and

$$N_j = 2 \quad \text{gives} \quad \beta_1 < \beta_c \tag{32}$$

Since both functions (27) and (28) are continuous and monotonic increasing their must be a value N' with property $0 < N' < 2$ that gives

$$\beta_1 = \beta_c$$

We can solve the system by trial and error. For instance, from (31) and (32) we have

Let $N = N_k$ be $N_i < N_k < N_j$, then with this value inserted in (29) and (30) we decide about N_{k+1}

If $\beta_1 > \beta_c$ take $N_{k+1} > N_k$

If $\beta_1 < \beta_c$ take $N_{k+1} < N_k$

So on the basis of (31) and (32) we can decide to increase the last used value of N or to decrease it.

However, a method that gives in succession better approximations to the solution automatically can be constructed quite easily (see Fig. 4).

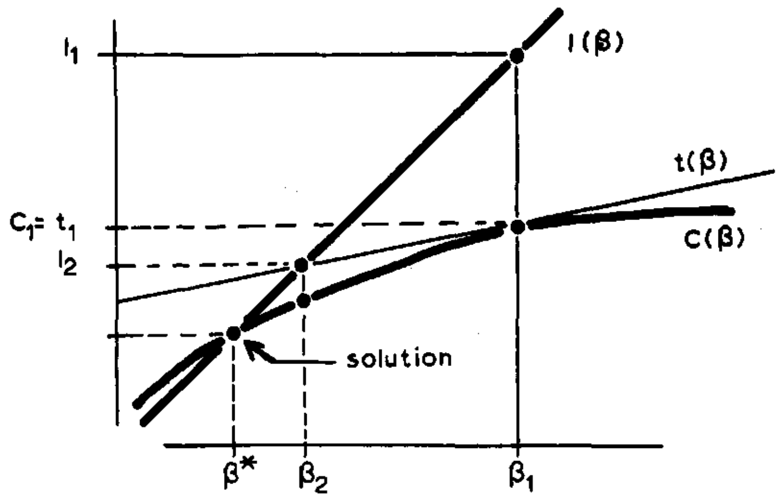

Fig. 4. Illustration of the iterative procedure to find the point of intersection β* which is the solution of the simultaneous system l(β) and c(β)

We start the solution with deriving the equation for the tangent t(β) to c(β). The slope to this curve at any point (β_1, c_1) reads

$$\left.\frac{dc}{d\beta}\right|_{\beta = \beta_1} = 2\, e^{-\beta_1}$$

so the tangent at (β_1, c_1) reads

$$t - c_1 = 2e^{-\beta_1} (\beta - \beta_1)$$

where according to (28) we have

$$c_1 = 2(1 - e^{-\beta_1})$$

and consequently

$$t(\beta) = 2e^{-\beta_1}(\beta-\beta_1) + 2(1 - e^{-\beta_1})$$

Now the procedure is

1. Choose a point (β_1, l_1) on $l(\beta)$
2. Determine the tangent to $c(\beta)$ at β_1 which gives

   $$t(\beta) = 2\, e^{-\beta_1}(\beta-\beta_1) + 2(1 - e^{-\beta_1})$$

3. Calculate the intersection of the tangent with the straight line $l(\beta)$, by putting $l_2 = t(\beta_2) = l(\beta_2)$ so by putting

   $$l_2 = \beta_2$$

   and

   $$l_2 = 2\, e^{-\beta_1}(\beta_2-\beta_1) + 2\,(1 - e^{-\beta_1})$$

   respectively.

   From both we have

   $$\beta_2 = 2\, e^{-\beta_1}\, \beta_2 - 2\, e^{-\beta_1}\beta_1 + 2(1 - e^{-\beta_1})$$

   which, solved for β_2, gives

$$\beta_2 = \frac{2 - 2e^{-\beta_1}(1+\beta_1)}{1 - 2e^{-\beta_1}} \tag{33}$$

and then

$$l_2 = \beta_2$$

4. Take the new point (β_2, l_2) to start with 1. again.

5. See 1, with $(\beta_1, l_1) = (\beta_2, l_2)$, etc.

The procedure is very simple and reads, in general: choose a starting value, insert this value in (33), insert the result in (33) again, etc. where (33) can be written with general subscripts

$$\beta_{i+1} = 2\,\frac{1 - e^{-\beta_i}(1+\beta_i)}{1 - 2e^{-\beta_i}} \tag{34}$$

Since $c(\beta) < 2$ we can start with $l_i = 2$ and from (27) $\beta_i = 2$, so for β_{i+1}, taking $i = 1$ we obtain a better approximation by

$$\beta_2 = 2\,\frac{1-3e^{-2}}{1-2e^{-2}} = 1.628878$$

Since the tangent t to c always is above c and has a positive slope, intersection with l in each following iteration cycle takes place at a lower lying point on l, but will not pass below the curve c since β is a point of the tangent.

This means that (34) produces a bounded row

$$\beta_1 < \beta_2 < \ldots < \beta_i < \beta_{i+1} < \ldots < \beta_n < \beta^*$$

which converges to β^*. The value β_n can be taken arbitrarily close to the solution. For practical reasons n will be taken relatively small.

The above described procedure converges very fast to the solution. In table 2 the results are given in 6 decimals.

Table 2. Consecutive solutions for the storm constant β

i	β_i	β_{i+1}
1	2	1.628878
2	1.628878	1.594030
3	1.594030	1.593624
n=4	1.593624	1.593624

$e^{-\beta_4} = 0.203188$

For practical purposes the results can be approximated by continued fractions according to

$$\frac{p}{q} = \frac{1}{\frac{q}{p}} = \frac{1}{n_1 + \frac{1}{n_2 + \text{etc}}}, \quad p < q$$

The smallest fractions that approximate the results best are determined by successively neglecting fractions that occur in the denominators. These approximations are given in Table 3.

Table 3. Approximation to main results obtained by continued fractions

$\beta = 1.593624$	$e^{-\beta} = 0.203188$
$1 = 1$	$\frac{1}{4} = 0.25$
$2 = 2$	$\frac{1}{5} = 0.20$
$1\frac{1}{2} = 1.5$	$\frac{12}{59} = 0.2034$
$1\frac{3}{5} = 1.6$	$\frac{13}{64} = 0.203125$
$1\frac{16}{27} = 1.5925$	$\frac{38}{187} = 0.203209$
$1\frac{19}{32} = 1.59375$	$\frac{5207}{1058} = 0.203188$
$1\frac{130}{219} = 1.593607$	
$1\frac{149}{251} = 1.593625$	
$1\frac{1471}{2478} = 1.593624$	

A practical optimal choice suggested by Table 3, seems to be

$$bB = \beta = 1\frac{3}{5} \quad \text{and} \quad e^{-\beta} = \frac{1}{5}$$

which gives, inserted in (22) and (23)

$$u = \frac{4}{5} \quad \text{and} \quad v = 1\frac{1}{5}$$

$$uv = \frac{24}{25} \text{ and } \quad u + v = 2$$

to be used in the applications. We observe that still

$$\frac{1 - e^{-bB}}{bB} = \frac{u}{bB} = \frac{4/5}{8/5} = \frac{1}{2}$$

the required solution to equallize the expectation or mean value.

11. APPLICATION OF THE STORM CONSTANT β

The condition expressed by (26) and (26a) produces, with the definition for u and v by (22) and (23) the equallity

$$\frac{u}{bB} = \frac{u}{\beta} = \frac{1}{2}$$

This means that the variance of the exponential storm type (Table 1) can be written

$$\sigma^2 = \frac{1}{2} H^2\{\frac{1}{2}(2-u) - \frac{1}{2}\}$$

$$= \frac{1}{4} H^2(1-u)$$

and so

$$\sigma^2 = \frac{H^2}{4} e^{-bB} = \frac{H^2}{4} e^{-\beta}$$

and in good approximation we obtained in the last section $e^{-\beta} = \frac{1}{5}$

giving

$$\sigma^2 = \frac{1}{20} H^2 = \frac{1}{5} H_o^2$$

For the triangular type we have (Section 6 and Table 1)

$$\sigma^2 = \frac{1}{12} H^2 = \frac{1}{3} H_o^2$$

Finally all results can be collected in a table which gives the parameters and characteristics of three storm types with equal diameter and the same storm volume. See Table 4.

Table 4. Characteristics of different storm types

Type	Diameter	Maximum H in center	Special parameter if any	$\bar{h}$ and μ	σ^2	Volume S
Rectangular	B	H_o	-	H_o	0	BH_o
Exponential	B	$2H_o$	$b = \frac{8}{5B}$	H_o	$\frac{1}{5} H_o^2$	BH_o
Triangular	B	$2H_o$	-	H_o	$\frac{1}{3} H_o^2$	BH_o

In this table the storms are ordered according increasing values of their variance.

12. SUMMARY OF STORM MODELS

When it is required to apply the rectangular, exponential and triangular storm model with equal storm diameter B and equal mean value H_o, storms have to be defined as follows.

a. the rectangular type

$$h = {}^1f(x) = H_o, \quad 0 \leq x < \frac{1}{2} B < B$$

$$h = {}^2f(x) = H_o, \quad 0 < \frac{1}{2}B \leqq x \leqq B$$

properties $\mu = H_o, \quad \sigma^2 = 0$

$${}^1f(0) = H_o, \quad {}^1f(\tfrac{1}{2}B) = {}^2f(\tfrac{1}{2}B) = H_o$$

$${}^2f(B) = H_o$$

b. the exponential type

$$h = {}^1f(x) = 2H_o \exp\{\frac{8}{5}(\frac{2x}{B} - 1)\}, \; 0 \leqq x < \tfrac{1}{2}B < B$$

$$h = {}^2f(x) = 2H_o \exp\{\frac{8}{5}(1 - \frac{2x}{B})\}, \; 0 < \tfrac{1}{2}B \leqq x \leqq B$$

properties $\mu = H_o, \; \sigma^2 = \frac{1}{5} H_o^2$

$${}^1f(0) = \frac{2}{5} H_o, \; {}^1f(\tfrac{1}{2}B) = {}^2f(\tfrac{1}{2}B) = 2H_o$$

$${}^2f(B) = \frac{2}{5} H_o$$

c. the triangular type

$$h = {}^1f(x) = \frac{4H_o}{B} x, \quad 0 \leqq x < \tfrac{1}{2}B < B$$

$$h = {}^2f(x) = 4 H_o(1 - \frac{x}{B}), \; 0 < \tfrac{1}{2}B \leqq x \leqq B$$

properties $\mu = H_o, \quad \sigma^2 = \frac{1}{3} H_o^2$

$${}^1f(0) = 0, \quad {}^1f(\tfrac{1}{2}B) = {}^2f(\tfrac{1}{2}B) = 2 H_o$$

$${}^2f(B) = 0$$

Finally a graph of the three storm functions with the above mentioned properties is given in Fig. 5.

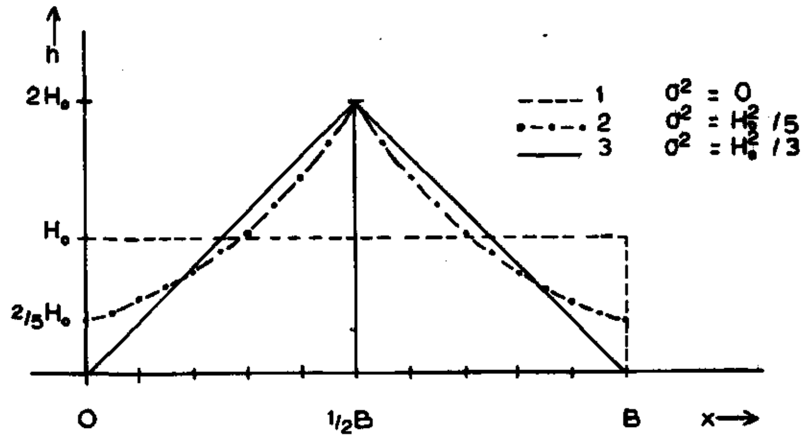

Fig. 5. Graphical representation of three storm models with equal diameter B and equal mean value H_o. They are 1: rectangular type; 2: exponential type; 3: triangular type. Storms are ordered according increasing values of their variance

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