The price of corn: short-term variations and long-term dynamics

Minor MSc Thesis (AEP-80424)

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Abstract

In this thesis a modeling approach introduced by Schwartz and Smith (2000) is followed in order to model the price of corn. The spot price of corn is modeled as a combination of two stochastic processes: a unit-root process for long-term dynamics, and a mean-reversion process for short-term variations. Futures prices are used to approximate spot prices, since futures prices reflect the expected value of spot prices under risk-neutral probabilities. To estimate the distribution of the vector of futures prices at time t conditional on the history of observed prices up to time $t-1$ the recursive Kalman filter is used. Given the conditional distribution for a vector of prices at time t the likelihood function of a time series of prices is computed as the product of conditional distributions, which allows applying maximum likelihood in estimation of the model parameters. Due to time constraints the model is estimated as a one-factor model only.

1. Introduction

Over the last couple of years agricultural commodity prices have fluctuated strongly due to weather circumstances, policies and the economic and financial crises. In the period 2007 to early 2008 these prices increased significantly and some of them even reached their highest levels in thirty years. However, in October 2008 the agricultural commodity prices decreased sharply while they increased again in 2009-2010 (FAO, 2010). Besides the strong fluctuations of agricultural commodity prices, their long-term equilibrium levels have increased over the last ten years. Food prices may have decreased in late 2008, nevertheless they stabilized at a higher level than the equilibrium level of 2005. This can also be seen in figures 1 to 3 below, which show weekly prices for the period January 2000 – November 2012 for soybeans, corn and wheat, respectively.

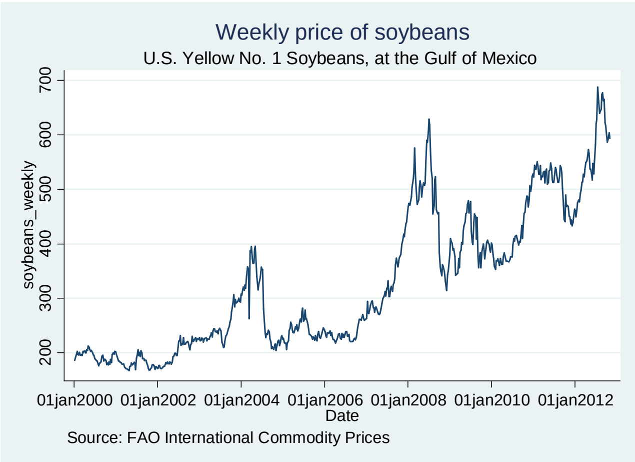

Figure 1: Weekly prices of soybeans

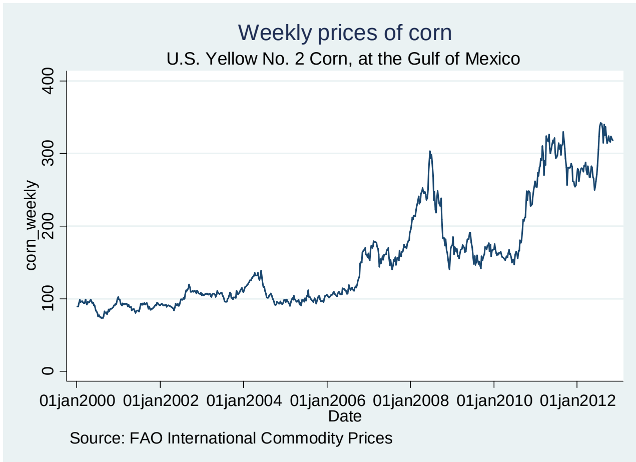

Figure 2: Weekly prices of corn

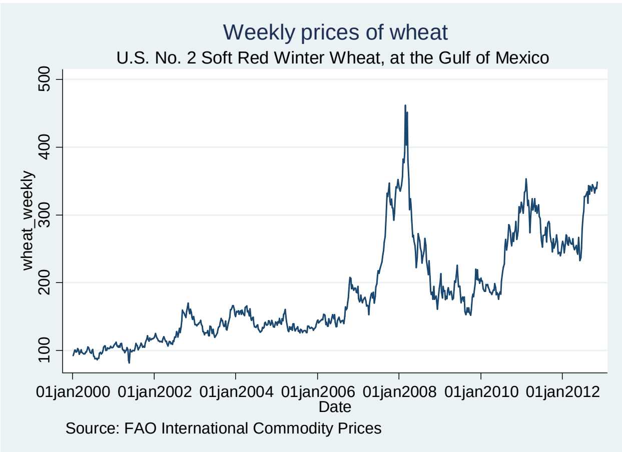

Figure 3: Weekly prices of wheat

Both spot and futures prices were subject to strong fluctuations in past years. The futures price is the price of a futures contract, which includes the specified quantity of a commodity to be delivered at a specified future date. The spot price is defined as the price of a commodity to be delivered immediately after purchase at a physical market. According to Hernandez and Torero (2010), the relationship between spot and futures prices could be described as futures prices leading spot prices due to the risk-transfer role, the price discovery role, and a higher degree of transparency of futures prices. Therefore, the difference between these prices is a risk premium and the futures price is a biased estimate of the spot price. An alternative popular view on the difference between higher futures price and lower spot prices is the theory of storage, in which the difference is explained by interest forgone in storing the commodity, warehousing costs and a convenience yield stemmed from inventory (Fama and French, 1987). Fluctuations in spot and futures prices of agricultural commodities are often thought to be driven by changes in demand or supply disturbances (e.g. due to weather circumstances or productivity increases), but could also be caused by changes in inventories, global food reserves, expectations, speculation, different trading mechanisms and macroeconomic policies (FAO, 2010).[1] Obviously, these elements can affect the dynamic relationship between spot and futures prices.

Strong fluctuations and an increasing long-term trend of agricultural commodity prices are both difficult to manage for market participants, in particular for developing countries that have become net food importers and are more vulnerable to price volatility. Food price volatility has adverse effects on developing countries' poverty, food security and welfare (FAO, 2010). Besides the impact on the poor, increasing and more volatile agricultural prices have led to more regulation in some countries and export bans in some major exporting countries, creating market failures (von Braun and Torero, 2008).

To analyze (agricultural) commodity prices, often standard time series models are used. Examples are autoregressive moving average (ARMA) models, vector autoregressive (VAR) models and vector error correction models (VECM), which take into account the (inter)dependencies of prices on previous prices, their variances and prices of other commodities, respectively. Another example is a general autoregressive conditional heteroscedasticity (GARCH) model (Bollerslev, 1986), which combines an ARMA model specification for the time series with an ARMA structure for the residual variance. In financial economics however, pricing models such as the Black-Scholes model and Markov models are used more often to price for example financial derivatives, options, exchange rates and interest rates. Another different class of models entails simulation models, for instance Computable General Equilibrium (CGE) models. CGE models are used to study an economy as a whole using actual data about policies, prices, technology, etc. Finally, since commodity prices can be considered as dynamic systems, the methodology of state space models is worth noting. State space models describe the behavior of dynamic systems in a general form and are able to capture unobserved components of time series (Lennox and Thornhill, 2009). They are estimated by the Kalman filter, a recursive algorithm, which constructs the likelihood function associated with a state space model (Pichler, 2007).

Financial economists concentrated mostly on the evolution of spot and futures prices of other commodities or stocks than agricultural ones, probably because these markets are much larger

[1] All other things being equal, e.g. transportation costs and negotiation costs.

than the food commodity markets. However, this enormous amount of literature about the stochastic behavior of commodity prices in general can also be used for agricultural commodity prices. Financial models differ in the number of components included (e.g. the equilibrium price level, short-term dynamics) and how these components are modeled. Initial models consider simple random walk with drift models (e.g. studied by Schwartz (1998), and Cortazar and Schwartz (1997)), while more advanced models include alternative dynamics like seasonality, time varying risk premiums and mean-reversion (e.g. Gibson and Schwartz (1990), Schwartz and Smith (2000), Cortazar et al. (2008)). Moreover, some models capture only long-term dynamics, while others capture short-term dynamics as well.

Schwartz (1997), for example, has developed three models which take into account the mean-reverting characteristic of commodity prices. Models which include also an overall upward trend, the so-called drift, of this mean-reverting process seem more desired. Cortazar and Schwartz (2003) have developed a three-factor model, which contains the commodity spot price, the convenience yield[2] and the expected long-term spot price return. In the two-factor model of Schwartz and Smith (2000), the spot price of a commodity is modeled as a combination of two processes: a unit root process of long-term dynamics and a mean-reversion process of short-term variation. The long-term dynamics or equilibrium price level is assumed to evolve according to a random walk model, while the short-term variation, defined as the difference between the spot and equilibrium price, is assumed to return towards zero according to a mean-reverting process. Simultaneously, uncertainty in the equilibrium price level is taken into account.

This thesis aims to examine the relationship between spot and futures prices of corn, one of the most important agricultural commodities, using the model of Schwartz and Smith. The main research objective is therefore to address theoretically and empirically the use of financial models in analyzing the evolution of corn prices. To accomplish this objective, the model of Schwartz and Smith and its estimation by state space models and the Kalman filter is studied in detail. Therefore, some research questions are defined:

1. What are key elements and assumptions of the model of Schwartz & Smith?
2. How to apply financial models in terms of data requirements and estimation methods?
3. How does the model of Schwartz & Smith perform?
4. What are advantages and drawbacks of the model of Schwartz & Smith?

It is chosen to start to study the rather basic one-factor model of Schwartz and Smith, followed by the two-factor model. Both models are estimated using simulated as well as real data, by means of the computing program Matlab. It is expected that the model of Schwartz and Smith can be applied to other agricultural commodities like wheat and soybeans later.

The remainder of this thesis is organized as follows. Chapter 2 describes the model of Schwartz and Smith in more detail. It starts with describing the two factors of the model separately and continues

[2] Convenience yields are defined as "the flow of services which accrues to the owner of a physical inventory but not to the owner of a contract for future delivery" (Brennan, 1991).

with explaining how risk is taken into account in this model. It also describes how futures prices are connected to spot prices in this model. Chapter 3 provides a thorough explanation of state space models in general, how these are estimated using the Kalman filter, and it discusses which other techniques are used to estimate the model parameters. Chapter 4 provides a description of the used data, whereas chapter 5 discusses the results of the estimation process. Finally, chapter 6 gives a critical discussion of the model and presents conclusions.

2. Model of Schwartz & Smith

In this chapter the model of Schwartz and Smith is presented in more detail. The goal is to model the spot price of corn, in which the long-term equilibrium price level is one factor, and short-term deviations from that long-term equilibrium level another factor. In order to understand both processes better, the two factors are first discussed separately and followed by a motivation for assuming a risk-neutral process. Finally, it is addressed how futures prices can be used to estimate spot prices.

2.1 One-factor model

In early models, for example Brennan and Schwartz (1985), commodity prices are assumed to be a simple random walk specification, i.e. a Brownian motion. A geometric Brownian motion is a random walk to which a drift term is added. Both the drift and the variance of the random walk specification are not constant but increase linearly with time (Stokey, 2008). Also, the empirical evidence shows the validation that trends in commodity prices are significant (FAO, 2010). Therefore, in this one-factor model the log of the spot price of corn is modeled as a long-term equilibrium price level which grows over time:

$$\ln S_t = \xi_t \tag{1}$$

where S_t denotes the spot price at time t and ξ_t the long-term equilibrium price level at time t. It is assumed that ξ_t evolves over time using the following geometric Brownian motion process:

$$d\xi_t = \mu_\xi dt + \sigma_\xi dz_\xi \tag{2}$$

where $d\xi_t$ denotes the increment in the long-term equilibrium price level, μ_ξ denotes the positive slope or drift of that long-term equilibrium price level, dt the increment in time, σ_ξ the standard deviation of the increment in the long-term equilibrium price level and dz_ξ is a standard normal random variation with mean zero and variance one. Therefore, the term $\sigma_\xi dz_\xi$ can be considered as an innovation to the development of the slope of the long-term equilibrium price level, because they represent the new part of the slope which cannot be predicted from past information (Durbin and Koopman, 2012). If innovations appear, they have a permanent effect on the long-term dynamics of the corn price. Since the innovations have a normal distribution, ξ_t will also be normally distributed according to the Central Limit Theorem. So, ξ_t captures the long-term dynamics of the spot price and permanent random innovations to these dynamics. An example of a permanent random innovation is a change in policies, which will influence the supply decisions of farmers on the medium to long run. This in turn affects the level of the long-term equilibrium corn price, because policies often have an effect for several years.

In Appendix A the derivation of the mean and variance of this one-factor model is presented, from which the expected spot price follows below. Given ξ_0, the log of the expected spot price is then also normally distributed:

$$E[\ln S_t] = E[\xi_t] = \xi_0 + \mu_\xi t \tag{3}$$

with variance $V[\ln S_t] = V[\xi_t] = \sigma_\xi^2 t$ (4)

As said before, due to the geometric Brownian motion specification, the variance of the spot price grows linearly over time. From equation (4), it follows that the spot price S_t is log-normally distributed with expected value:

$$E[S_t] = e^{E[\ln S_t]+0.5V[\ln S_t]} = e^{(\xi_0+\mu_\xi t)+0.5\sigma_\xi^2 t} \tag{5}$$

when t approaches infinity. This means that the expected spot price in this one-factor model depends on a specific starting point of the series (ξ_0) and increases by a drift term and variance, which both in itself increase linearly with time. This can be clearly seen in equations (3) and (4).

2.2 Two-factor model

Since about a decade, when it was learned that (futures) price volatility declines with maturity, a factor of mean reversion is added to the random walk specification to model commodity prices (Cortazar and Schwartz, 2003). One-factor models including mean-reversion can be found for example in Schwartz (1998) and Cortazar and Schwartz (1997). Unfortunately, these one-factor models with random-walk specifications and mean-reversion have strong correlations among their futures returns (Cortazar and Schwartz, 2003). Therefore, later studies used two-factor models, in which one factor is mean-reversion. Examples are Gibson and Schwartz (1990), Schwartz (1997) and obviously Schwartz and Smith (2000). Mean-reversion can be mathematically described as an Ornstein-Uhlenbeck process. Following the model of Schwartz and Smith (2000), in this two-factor model the log of the spot price of corn is modeled using two stochastic factors:

$$\ln S_t = \xi_t + \chi_t \tag{6}$$

where S_t again denotes the spot price at time t, ξ_t the long-term equilibrium price level at time t and χ_t denotes the short-term dynamics at time t. As in the one-factor model above, it is assumed that ξ_t evolves over time according to a geometric Brownian motion process. Moreover, it is assumed that χ_t evolves over time according to an Ornstein-Uhlenbeck process. Mathematically, the joint process is then as follows:

$$d\xi_t = \mu_\xi dt + \sigma_\xi dz_\xi \tag{7}$$

$$d\chi_t = -\kappa\chi_t dt + \sigma_\chi dz_\chi \tag{8}$$

In equation (8), $d\chi_t$ denotes the increment in short-term dynamics, $-\kappa$ denotes the mean-reverting coefficient, i.e. the rate at which short-term innovations seem to disappear and the system returns to its mean, which is the long-term equilibrium price level. These short-term innovations are defined by $\sigma_\chi dz_\chi$, in which σ_χ is the standard deviation of the increment in short-term dynamics and dz_χ a standard normal random variation with mean zero and variance one. As said before, if innovations appear, they have a temporary effect on the price dynamics. Again, using the Central Limit Theorem, χ_t has a normal distribution due to the normality of the innovations. Furthermore, dt denotes the increment in time and equation (7) is equal to equation (2) in the one-factor model. The standard normal variations dz_ξ and dz_χ are correlated using $dz_\xi dz_\chi = \rho_{\xi\chi} dt$.

Considering the process of χ_t isolated, when random innovations are absent in this process, $d\chi_t$ will be equal to zero because the only parameter left $(-\kappa)$ describes at which rate innovations will disappear. In other words, without innovations $d\chi_t$ will revert to its mean χ_t. From this, it follows that in the presence of innovations and if $t \to \infty$, $\chi_t \to 0$ because it is assumed that innovations cancel out over time. Also, when χ_t is not equal to zero at a particular point in time, it tends to zero over time. Hence, χ_t is a mean-reverting process and intends to capture short-term deviations from the equilibrium prices ξ_t. One of the causes of short-term deviations is a change in supply of futures contracts, which will occur due to entry and exit of suppliers at increasing and decreasing prices. In other words, the expectation of market participants about the corn futures price, either too high or too low, is corrected towards the long-term equilibrium price. Therefore, the impact of prices on the supply of futures contracts will induce mean-reversion in commodity prices (Schwartz, 1997).

Considering both processes together again and because of the interrelationships between the short-term and long-term dynamic factors, ξ_t and χ_t are jointly normally distributed, given ξ_0 and χ_0. With both means derived in Appendices A and B, the mean vector equals:

$$E[\xi_t, \chi_t] = [\, \xi_0 + \mu_\xi t,\ e^{-\kappa t}\chi_0] \qquad (9)$$

Here it is clearly shown that the short-term factor χ_t reverts to its mean, because when the time horizon increases $e^{-\kappa t}\chi_0$ will approach zero. In this two-factor model, the process will revert to $\xi_0 + \mu_\xi t$ over time.

As is the derivation of the variance of ξ_t presented in Appendix A, so are the derivations of the variance of χ_t and the covariance between χ_t and ξ_t presented in Appendices B and C, respectively. The covariance matrix equals:

$$Cov[\chi_t, \xi_t] = \begin{bmatrix} (1 - e^{-2\kappa t})\frac{\sigma_\chi^2}{2\kappa} & (1 - e^{-\kappa t})\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa} \\ (1 - e^{-\kappa t})\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa} & \sigma_\xi^2 t \end{bmatrix} \qquad (10)$$

The first element of this covariance matrix denotes the variance of χ_t, which depends on the mean-reversion coefficient κ. The second and third elements denote the covariance between both factors, in which interrelationships among the factors are visible. For instance, the process of χ_t reverts to the long-term equilibrium price level ξ_t from which it follows that the correlation between χ_t and ξ_t depends on the mean-reverting coefficient as well. The last element denotes the variance of ξ_t, also given in equation (4). Since all elements of this matrix depend on time, the interrelationships between the two factors can also change over time. This involves that the connection between the two factors is not as strong for all time periods, for example when short-term deviations (due to e.g. weather disturbances or inventory expectations) have a relatively large influence in the formation of the corn price.

From the above presented mean vector and covariance matrix, and given ξ_0 and χ_0, the log of the expected spot price is then normally distributed:

$$E[\ln S_t] = E[\xi_t] + E[\chi_t] = \xi_0 + \mu_\xi t + e^{-\kappa t}\chi_0 \quad (11)$$

with variance:

$$V[\ln S_t] = V[\xi_t] + V[\chi_t] + 2Cov(\chi_t, \xi_t) = \sigma_\xi^2 t + (1 - e^{-2\kappa t})\frac{\sigma_\chi^2}{2\kappa} + 2\left[(1 - e^{-\kappa t})\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}\right] \quad (12)$$

Again, it is clear from equations (11) and (12) that both the mean and the variance of the expected spot price vary with time. It follows that the spot price S_t is log-normally distributed by:

$$E[S_t] = e^{E[\ln S_t] + 0.5V[\ln S_t]} = e^{(\xi_0 + \mu_\xi t + e^{-\kappa t}\chi_0) + 0.5\left\{\sigma_\xi^2 t + (1 - e^{-2\kappa t})\frac{\sigma_\chi^2}{2\kappa} + 2\left[(1 - e^{-\kappa t})\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}\right]\right\}} \quad (13)$$

When the time horizon increases, $\ln(E[S_t])$ approaches to:

$$\ln(E[S_t]) = \left(\xi_0 + \frac{\sigma_\chi^2}{4\kappa} + \frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}\right) + \left(\mu_\xi + 0.5\sigma_\xi^2\right)t \quad (14)$$

As can be seen in equation (14), the correlation between the short-term and long-term factor also influences the expected log spot price, which again shows it is important to take into account both processes simultaneously in analyzing corn prices in the long run.

2.3 Risk-neutral process

As said before, the difference between higher futures prices and lower spot prices is a risk premium, since risk is transferred from spot markets to futures markets in hedging. This risk includes that commodity prices can either be high or low, which can have a large effect on profits of farmers. In the often used net present value concept, the risk of expected net cash flows (i.e. the uncertain spot price) is reflected in a discount rate. However, an alternative approach considers certainty-equivalent cash flows, which are the certain prices a farmer can get for his commodity, at an equal value as that uncertain spot price. This approach is easier to use in the case of commodities, because the futures price of commodities can be considered as certainty-equivalent cash flows (Schwartz, 1998). Hence, also according to Casassus and Collin-Dufresne (2001), futures prices reflect the expected value of spot prices under risk-neutral probabilities, or futures prices are a biased estimate of the spot price due to the risk premium (Hernandez and Torero, 2010).

Therefore, a risk-neutral process is taken into account by introducing risk premium coefficients into the two-factor model, which reflect the overall market prices of that risk. This market price of risk encloses several costs, e.g. storage costs and transport costs (Schwartz, 1997). The risk premiums will affect the long-term values of ξ_t and χ_t, i.e. the drifts of both stochastic processes are discounted at a positive risk-free rate λ_i. The risk-neutral model could then be considered as an arbitrage-free model with exogenous interest rates (Manoliu and Tompaidis, 2002). The joint process describing how ξ_t and χ_t evolve over time becomes then:

$$d\xi_t = (\mu_\xi - \lambda_\xi)dt + \sigma_\xi dz_\xi^* \quad (15)$$

$$d\chi_t = (-\kappa\chi_t - \lambda_\chi)dt + \sigma_\chi dz_\chi^* \quad (16)$$

in which dz_ξ^* and dz_χ^* are new standard normal variations, with $dz_\xi^* dz_\chi^* = \rho_{\xi\chi}dt$. From here on, the parameters of the risk-neutral process will be indicated with an asterisk. In Appendix D it is shown that χ_t reverts to $-\frac{\lambda_\chi}{\kappa}$ in the risk-neutral process rather than reverting to zero in the true process if the time horizon increases. This means that short-term deviations from equilibrium prices are driven by the risk-free rate and the mean-reversion coefficient. In other words, when costs are higher, the differences between actual and equilibrium prices are larger. This makes sense, because in most cases the difference between spot and futures markets is larger than a hedged risk due to e.g. arbitrage.

Again, given ξ_0 and χ_0, ξ_t and χ_t are jointly normally distributed with mean vector and covariance matrix (see for derivations Appendices D and E, respectively):

$$E^*[\xi_t, \chi_t] = \left[\xi_0 + (\mu_\xi - \lambda_\xi)t, e^{-\kappa t}\chi_0 - (1 - e^{-\kappa t})\frac{\lambda_\chi}{\kappa}\right] \quad (17)$$
$$Cov^*[\chi_t, \xi_t] = Cov[\chi_t, \xi_t] \quad (18)$$

The covariance matrix in the risk-neutral process is equal to the one in the true process, because the risk premium coefficients do not affect the variances and covariance of both factors. As said before, only the drifts of both factors are discounted.

The log of the expected spot price, $\ln S_t = \xi_t + \chi_t$, is then normally distributed with:

$$E^*[\ln S_t] = \xi_0 + \mu_\xi t + e^{-\kappa t}\chi_0 - \left((1 - e^{-\kappa t})\frac{\lambda_\chi}{\kappa} + \lambda_\xi t\right) \quad (19)$$
$$V^*[\ln S_t] = V[\ln S_t] \quad (20)$$

From equation (19) it can be shown that the risk premium coefficients reduce the expected spot price derived before by $\left((1 - e^{-\kappa t})\frac{\lambda_\chi}{\kappa} + \lambda_\xi t\right)$, regardless the value of the state variables. The risk premium coefficients vary with time, which is also argued for by Casassus and Collin-Dufresne (2001), and Fama and French (1987).

2.4 Linking futures prices to spot prices

In analyzing agricultural commodity prices, the goal is to model the evolution of commodity spot prices. However, this price is rather uncertain and formed in many different locations, which makes it difficult to observe. Futures prices, on the contrary, can be more easily observed since they are traded on a few exchanges and are more transparent. Therefore, the corresponding futures contract closest to maturity could serve as a proxy for the spot price. Also, according to Williams and Wright (2005), a futures price is the mathematical expectation of the corresponding spot price in a specific period conditional upon the information available in the current period. In addition, this is confirmed by Hernandez and Torero (2010), who state that price changes in futures markets lead price changes in spot markets more often than the reverse. Therefore, it is chosen to model the actual and previous futures prices as proxies for the unobservable spot price. The expression for futures prices is then a function of the state variables ξ_t and χ_t, and derived from the definition of the futures prices as the risk-neutral conditional expectation of the underlying spot price at the maturity of the corresponding futures contract (Manoliu and Tompaidis, 2002). Hence,

$$E^*[S_T] = F_{0,T} \text{ and } \ln E^*[S_T] = \ln F_{0,T} \quad (21)$$

in which $E^*[S_T]$ denotes the expected spot price in the risk-neutral process corresponding to a futures contract maturing at time T and $F_{0,T}$ the current futures price of a contract maturing at time T. With the derivation presented in Appendix F, it can be shown that:

$$\ln F_{0,T} = \xi_0 + e^{-\kappa T}\chi_0 + A(T) \quad (22)$$

with

$$A(T) = (\mu_\xi - \lambda_\xi)T - \left((1 - e^{-\kappa T})\frac{\lambda_\chi}{\kappa}\right) + \frac{1}{2}\left((1 - e^{-2\kappa T})\frac{\sigma_\chi^2}{2\kappa} + \sigma_\xi^2 T + 2(1 - e^{-\kappa T})\frac{\rho_{\xi\chi}\sigma_\chi\sigma_\xi}{\kappa}\right) \quad (23)$$

in which T denotes again the time of maturity of the futures contract. When $\ln F_{0,T}$ is considered as the curve for futures prices, $A(T)$ can be thought of as the change in this risk-neutral curve due to different maturities. From equation (23) it can be seen that the slope of this curve, the long-term equilibrium price μ_ξ, is weakened by the long-term risk premium coefficient λ_ξ. Besides that, as in equation (19), equation (23) shows that the difference between the futures price curve and the expected spot price curve is a short-term deviation, reflected by an amount discounted by the short-term risk premium coefficient. In other words, it is made explicitly that the futures price is a biased estimate of the spot price due to the risk premium. Moreover, the log futures price depends linearly on the stochastic factors, which is convenient in using the econometric technique to estimate the parameters (Schwartz, 1997).

3. Estimation strategy

To estimate the model of Schwartz and Smith, it is converted into a so-called state space model which is presented in detail below in paragraph 3.1. The Kalman filter combines filtering and maximum likelihood techniques and is presented as the estimation method in paragraph 3.2. Paragraph 3.3 discusses related maximum likelihood techniques separately.

3.1 State space models

In many cases the dynamics studied in an economic model are, or to some extent, immeasurable or unobservable; the structure of an underlying model or the underlying variables can be relatively unknown. This makes it difficult to apply standard autoregressive time series models to study the evolution of endogenous variables over time. However, these autoregressive models can be adapted to state space models.

State space models are able to connect dependent observable variables to explanatory, possibly unobservable variables, which can be considered as internal states. Observable variables are related to those internal state variables via an observation equation, which explicitly describes the assumed relationship between the observed time series and the possibly unobserved states. State equations describe the evolution of state variables as being driven by the stochastic process of innovations (Pichler, 2007). These innovations are assumed to have a zero mean, a Gaussian (normal) distribution, to be serially uncorrelated and to have no cross correlations (StataCorp., 2011). Overall, state space models are a method to describe the evolution of dynamic systems in a general form and are able to represent linear, nonlinear, time-invariant and time varying dynamics (Lennox and Thornhill, 2009). Using state space models, the dynamic relationships among multiple time series are emphasized in estimating them. In addition, explanatory variables can be included in both the state and observation equations. A practical advantage of state space models is that missing data can be easily filled in if series are driven by the same internal states.

Considering the risk-neutral model of Schwartz and Smith again, the joint process of ξ_t and χ_t in continuous time is [see equations (15) and (16)]:

$$d\xi_t = (\mu_\xi - \lambda_\xi)dt + \sigma_\xi dz_\xi^*$$

$$d\chi_t = (-\kappa\chi_t - \lambda_\chi)dt + \sigma_\chi dz_\chi^*$$

which leads to the log of the expected spot price proxied by futures prices as [see equations (22) and (23)]:

$$\ln F_{0,T} = \xi_0 + e^{-\kappa T}\chi_0 + A(T)$$

with

$$A(T) = (\mu_\xi - \lambda_\xi)T - \left((1 - e^{-\kappa T})\frac{\lambda_\chi}{\kappa}\right) + \frac{1}{2}\left((1 - e^{-2\kappa T})\frac{\sigma_\chi^2}{2\kappa} + \sigma_\xi^2 T + 2(1 - e^{-\kappa T})\frac{\rho_{\xi\chi}\sigma_\chi\sigma_\xi}{\kappa}\right)$$

from which the unobservable state variables ξ_t and χ_t can be converted into a state space model. As indicated by equation (22), those state variables are estimated from futures prices, using short-to-maturity and long-to-maturity traded contracts. Prices of long-to-maturity traded contracts give information about deviations in the long-term equilibrium price and prices of short-to-maturity traded contracts give information about short-term deviations in the price of corn. The powerful estimation method known as the Kalman filter seems especially suitable because in this model new market information arrives in a daily manner and parameters can be updated every new time period.

Mathematically, the model of Schwartz and Smith can be defined as a discrete time form state space model as follows. The state equation is (also presented in Appendix C):

$$x_t = c + G_t x_{t-1} + I_t \omega_t \tag{24}$$

for $t = 1, \dots, n$ and where $x_t \equiv \begin{bmatrix} \chi_t \\ \xi_t \end{bmatrix}$, a 2x1 vector of state variables; $c \equiv \begin{bmatrix} 0 \\ \mu_\xi^* \Delta t \end{bmatrix}$, a 2x1 vector of long-term equilibrium prices[3]; $G_t \equiv \begin{bmatrix} e^{-\kappa \Delta t} & 0 \\ 0 & 1 \end{bmatrix}$, a 2x2 matrix of dependencies on previous state values; $\omega_t = \begin{bmatrix} \omega_{\chi_t} \\ \omega_{\xi_t} \end{bmatrix}$, a 2x1 vector of independent and identically distributed (i.i.d.) innovations of the internal states; and W_t the covariance matrix of the internal innovations ω_t. Since the process of χ_t captures only short-term dynamics, it does not appear in vector c for long-term equilibrium prices. The corresponding value is therefore assumed to be zero. In matrix G_t the autoregressive effects of the state variables are presented, but since it is assumed there are no mutual effects among the state variables the off-diagonal elements of this matrix are assumed to be zero. The internal innovations are i.i.d., which results in mean vector $E[\omega_t] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and covariance matrix $W_t = V[\omega_t] = \begin{bmatrix} \sigma_\chi^2 & \rho_{\chi\xi}\sigma_\chi\sigma_\xi \\ \rho_{\chi\xi}\sigma_\chi\sigma_\xi & \sigma_\xi^2 \end{bmatrix} \Delta t$. Furthermore, t denotes time until maturity T of futures contracts and $\Delta t = \frac{t}{n}$ denotes the length of one time step, here one day. It follows that if e.g. $t = 2$, the system is at time $t_0 + \Delta t + \Delta t$. The state equation given in (24) is structured as a first order vector autoregressive model, in which the autoregressive coefficient matrix G_t and the innovation coefficient matrix I_t vary with time.

The observation equation relating the state variables ξ_t and χ_t to the observed futures prices is:

$$y_t = d_t + F_t x_t + v_t \tag{25}$$

[3] To obtain more accurate estimations, $\mu_\xi - \lambda_\xi$ is defined as a new parameter, μ_ξ^*.

for $t = 0, \ldots, n$ and where $y_t \equiv \begin{bmatrix} \ln F_{t,T_1} \\ \ln F_{t,T_2} \\ \vdots \\ \ln F_{t,T_n} \end{bmatrix}$, an nx1 vector of observed futures prices at time t with maturities $T_1, \ldots, T_n$; $d_t \equiv \begin{bmatrix} A_t[T_1] \\ A_t[T_2] \\ \vdots \\ A_t[T_n] \end{bmatrix}$, an nx1 vector of equation (23) for different maturities; $F_t \equiv \begin{bmatrix} e^{-\kappa T_1} & 1 \\ e^{-\kappa T_2} & 1 \\ \vdots & 1 \\ e^{-\kappa T_n} & 1 \end{bmatrix}$, an nx2 matrix of associations between the state and observation equations; and v_t, the observation innovation. The elements of vector y denote log futures prices with different maturities. Since the last day of a futures contract is fixed, for corn the 15th day of the corresponding month, maturities of the log futures prices change over time. Hence the amount of observed futures prices at time t varies. As indicated in paragraph 3.1, the innovations of the observable variables are i.i.d., which results in $E[v_t] = 0$ and $Cov\ [v_t] = V$. Although the innovations v_t are not directly resulting from the model of Schwartz and Smith but are resulting from the general state space model notation, they can be included as representatives of equation (25)'s disability to fit observed futures prices. The observation equation given in equation (25) is structured as a standard linear regression model, in which the association matrix F_t varies with time.

3.2 Kalman filter

State space models are often estimated using a powerful estimation technique, known as the Kalman filter. The Kalman filter was initially developed by R.E. Kalman, who first solved a discrete-data linear filtering problem recursively (Kalman, 1960).

Basically, the Kalman filter is a system of equations that applies a predictor-corrector type estimator that is optimal since it minimizes the estimated covariance matrix (Welch and Bishop, 2001). It connects the state equation (24) to the observation equation (25) by giving more or less weight, known as the Kalman gain, to either the state equation or the observation equation. This is done recursively, such that estimates are optimal and latest information available can be included. The steps of the Kalman filter are initialization, prediction, correction and likelihood construction (Pichler, 2007), also demonstrated in figure 4 on the next page. The Kalman filter establishes a 'best guess' or prior estimate of the state variable and a prior covariance matrix conditional upon all information available, predicts the new state and its covariance matrix, corrects the estimates by establishing a posterior estimate and posterior covariance matrix when new information is available and constructs the likelihood of that latest estimate in accordance with Bayes' rule. This last step will be considered in more detail in paragraph 3.3. The posterior state estimate then is a linear combination of the weighted prior estimate and the weighted actual measurement. The step of correcting estimates of state variables is also known as state smoothing (Durbin and Koopman, 2012). After constructing the posterior estimate, the procedure is repeated with the current posterior estimate as new prior estimate to develop the next state. In fact, the Kalman filter serves as a tool to optimize the filtering of new information in the market, which lies behind the notion of price discovery and price determination of a commodity price.

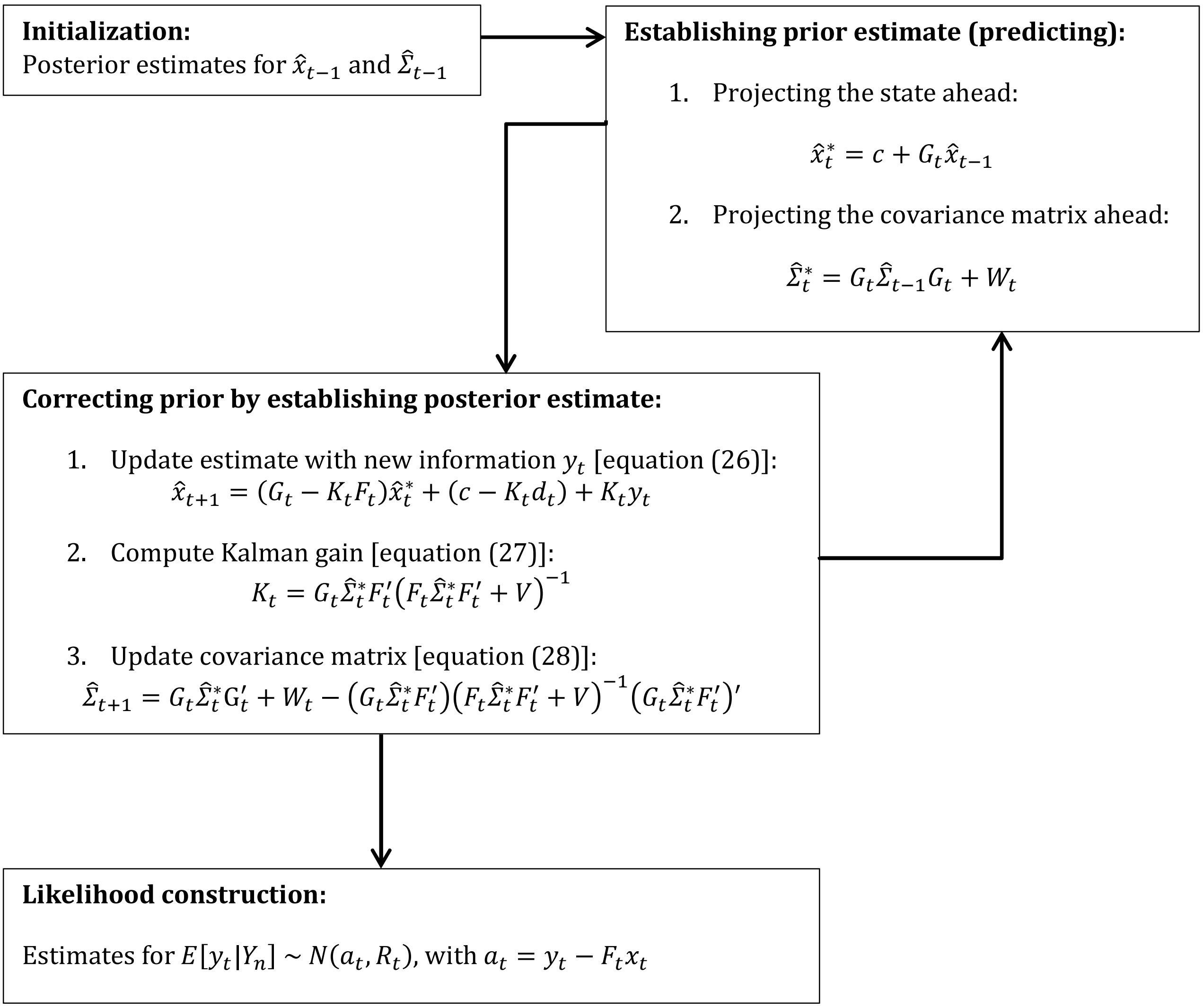

Figure 4: Kalman filter procedure

Following figure 4, the initial state x_0 is unobserved but it is assumed to have a Gaussian distribution with mean $\hat{x}_0$ and covariance matrix Σ_0. The Kalman filter is used to compute the mathematical expectation $E[x_t|y_t, \dots, y_0]$, in which x_t is the unobservable internal state and $y_t, \dots, y_0$ (i.e. Y_n) the information available. To run the discrete time model in equations (24) and (25) recursively using the Kalman filter and estimate the mean and covariance of the state vector $x_0 = [\chi_0, \xi_0]'$, the algorithm is then, using $\hat{x}_{t+1} = E[x_{t+1}|Y_n]$ and the notation of paragraph 3.1:

$$\hat{x}_{t+1} = (G_t - K_t F_t)\hat{x}_t^* + (c - K_t d_t) + K_t y_t \qquad (26)$$

with $K_t = G_t \Sigma_t F_t' (F_t \Sigma_t F_t' + V)^{-1}$ (27)

and $\Sigma_{t+1} = G_t \Sigma_t G_t' + W - (G_t \Sigma_t F_t')(F_t \Sigma_t F_t' + V)^{-1} (G_t \Sigma_t F_t')'$ (28)

Equation (26) describes how the posterior estimate or the next internal state is predicted using the prior estimate of the current state $\hat{x}_t$ (denoted by $\hat{x}_t^*$) weighted by dependencies on (previous) states G_t and F_t and the Kalman gain K_t [given by equation (27)], the estimate of the risk-adjusted long-term mean $c - d_t$ weighted by the Kalman gain, and the observed series, also weighted by the Kalman gain. The Kalman gain minimizes the posterior covariance matrix, defined recursively in equation (28). The matrices G_t and F_t, and the covariance matrix $V$are unknown and have to be estimated.

Using the state equation in the Kalman filter, its recursive mean and non-stationary variance are, with x_t derived in Appendix G and the derivation of the variance in Appendix H:

$$E[x_t] = \begin{bmatrix} e^{-(\kappa \Delta t)t} \chi_0 \\ \mu_\xi (\Delta t) t + \xi_0 \end{bmatrix} \tag{29}$$

$$V[x_t] = \begin{bmatrix} \left(1 - e^{-2(\kappa \Delta t)t}\right) \frac{\sigma_\chi^2}{2\kappa} & \left(1 - e^{-(\kappa \Delta t)t}\right) \frac{\rho_{\xi\chi} \sigma_\chi \sigma_\xi}{\kappa} \\ \left(1 - e^{-(\kappa \Delta t)t}\right) \frac{\rho_{\xi\chi} \sigma_\chi \sigma_\xi}{\kappa} & \sigma_\xi^2 (\Delta t) t \end{bmatrix} \tag{30}$$

As can be seen from equations (29) and (30), both the mean and variance of the recursive process depend on the overall t and the actual time step Δt, here days. This means that really all information available is included when estimating the parameters of the state equation.

As said above, the initial state vector x_0 is unobserved, but it is assumed that its distribution is known. However, when the distribution of the initial state vector is relatively unknown, one can use the diffuse Kalman filter. This version of the Kalman filter deals with so-called diffuse initial conditions and treats the distribution of the initial state vector as an unknown constant with infinite variance to be estimated by maximum likelihood (Durbin and Koopman, 2012).

When data points are missing, the Kalman filter can be used to linearly extrapolate these data points by setting the Kalman gain K_t equal to zero for those data points (Durbin and Koopman, 2012). The above described Kalman filter is also known as the discrete Kalman filter. Another version of the Kalman filter is the extended Kalman filter which deals with nonlinearity and non-normality of random variables (Welch and Bishop, 2001; Durbin and Koopman, 2012).

3.3 Maximum likelihood

When the innovations of the state space model as well as the initial states are normally distributed, the Kalman filter permits to continue to estimate the model with maximum likelihood. In fact, the parameters of the model are estimated by the Kalman filter but optimized using maximum likelihood. After running the Kalman filter algorithm, the likelihood of occurrence of the observations conditional upon the estimated parameters could be maximized, while the variance of the innovations is minimized. However, to use maximum likelihood the model should be stationary, i.e. the distributions of the random variables y_t and x_t should not depend on time (Durbin and Koopman, 2012).

Since there is a sample of observations for the range $t = 0, \dots, n$, i.e. y_0^n, the likelihood function can be defined as a joint probability distribution $f(y_n, y_{n-1}, \dots, y_0)$ in which $y_n, y_{n-1}, \dots, y_0$ are vectors containing several futures prices at a specific date t. Since futures prices at time t are conditioned upon futures prices at time $t-1$ until $t = 0$, futures prices at time $t-1$ are conditioned upon futures prices at time $t-2$ until $t = 0$, and so on. This joint density can be separated in several factors:

$$f(y_n, y_{n-1}, \dots, y_0) = f(y_n|y_{n-1}, \dots, y_0)f(y_{n-1}|y_{n-2}, \dots, y_0) \dots f(y_1|y_0)f(y_0) \quad (31)$$

As shown in figure 4, the Kalman filter algorithm gives estimates for the expected value of $[y_t|Y_n]$, which has a conditional normal distribution with mean $a_t = y_t - F_t x_t$ and variance R_t. In fact, the mean a_t is defined as the prediction error and R_t is defined as the variance of that prediction error. Furthermore, R_t can be expressed as the covariance matrix between the state equation [equation (24)] and the observation equation [equation (25)] and depends on the particular parameters that generated the data, the matrices $V[x_t]$, G_t and F_t.

Using a_t and R_t, the log likelihood of this joint density can be expressed as

$$\log L = \log f(Y_n) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{n}\left(\log R_t + \frac{a_t^2}{R_t}\right) \quad (32)$$

in which n the number of observations and by which the likelihood of occurrence of the estimates $V[x_t]$, G_t and F_t is maximized.

As said above, the parameters collected in the vector $E[x_t]$ and the matrices $V[x_t]$, G_t and F_t have to be estimated using both the Kalman filter and maximum likelihood, i.e. the six parameters κ, σ_χ, σ_ξ, $\rho_{\xi\chi}$, λ_χ, μ_ξ^*, plus the newly added residuals of the observation equation, collected in covariance matrix V, σ_{v2}.

4. Data

This chapter presents some key statistics of the data used to estimate the model of Schwartz and Smith. To check if the model was specified correctly and performing well, first simulated data was used for both the one- and two-factor model before continuing with real data.

4.1 Simulated data

In using simulated data, the values of the parameters are set beforehand to check if the model can reproduce those numbers. Also, the number of observations on which the model is allowed to reproduce the parameters is determined.

In this simulation, it is chosen to set the number of observations equal to 1,000, the number of trading days per year is set equal to 252 and the number of trading days per month equal to 21. This was done because there is no trade in futures contracts on weekend days and holidays. Moreover, the time periods were set to one day, the maximum number of days until maturity was fixed at three years and the maximum number of observable futures prices at one time period (day) was fixed at ten. Finally, the values of the parameters, the starting values of both states, their expected values and an error term are set according to the following table:

Table 1: Parameter values for generating simulated data

Parameter	Value
κ	0.3483
λ_χ	0.7063
σ_χ	0.7151
$\rho_{\xi\chi}$	0
μ_ξ	0.1721
μ_ξ^*	-0.0513
σ_ξ	0.3165
σ_{v2}	0.0396
$E[\xi_t]$	0
ξ_0	2.857
$E[\chi_t]$	0
χ_0	0.119

4.2 Real data

The data used to estimate the model of Schwartz and Smith contains daily observations of corn futures prices over the period January 1972 to November 2012, in which for every time period a maximum of ten contracts is used in estimation since at each date there are various contracts with different expiration dates. Corn futures contracts expire every year in the months March, May, July, September and December, which all are used in estimation. In addition, we use the contracts expiring in January and November over the period 2000-2002 and 2000-2001, respectively. The number of available futures contracts at a particular date ranges from four to eighteen contracts per date, with the highest number of contracts available in the years 2008-2010. The average time to maturity is approximately 351 days, in a range from 0 to 1,474 days. However, it should be noted that the contracts which expire in the months December and July are traded over a longer period than other contracts, which gives in the end a better estimation of the spot price in those periods since more contracts are available at a certain date. As said before, both short-to-maturity and long-to-maturity contracts are of interest, to have information about short-term variations and long-term dynamics of the corn price.

Since futures prices are used as a proxy for spot prices, which are expressed as a function of risk-neutral drifts in equation (19), the cross section and time series aspect of futures prices are exploited to estimate the risk premium parameters. This results in a total of 86,901 observations of close prices, in a range from 117.13 dollars per 5,000 bushels to 838.02 dollars per 5,000 bushels, which is the standard quantity of a futures contract. The mean close price is 313.62 dollars per 5,000 bushels, with a standard deviation of 120.03 dollars.

The above described data is obtained from different sources. The first part is obtained from the Chicago Board of Trade (CBOT), for the period January 1972 – December 2007. The second part, January 2008 till the start of August 2010, is obtained from the Chicago Mercantile Exchange (CME). The last part, August 2010 – November 2012, is obtained from Wikiposit, an online data warehousing service[4]. To validate the choice for these sources, the correlations between the datasets can be found in table 2 below. Here the CBOT data is compared to the CME data and the Wikiposit data, and the CME data is compared to the Wikiposit data, both for the CBOT period (January 1972 – December 2007). It turns out that all data sets are nearly identical.

Table 2: Correlations between data sets

	CBOT	CME	Wikiposit
CBOT	1.0000		
CME	0.9989	1.0000	
Wikiposit	0.9992	1.0000	1.0000

[4] See http://wikiposit.org

5. Results

This chapter presents the estimation results for the model of Schwartz and Smith. Paragraph 5.1 presents the results of the two-factor model using the simulated data, and paragraph 5.2 shows the results of the two-factor model using real futures prices.

5.1 Simulated two-factor model

In table 3 below, the results of the two-factor model using simulated data are presented. The first column mentions the concerned parameter, the second column gives the value of that parameter estimated with the Kalman filter and the final column states the true value used in simulating the data, which corresponds to the value in table 3.

Table 3: Results two-factor model with simulated data

Parameter	Kalman filter value	True value
μ_ξ	0.1101	0.1721
κ	0.3588	0.3483
σ_χ	0.7140	0.7151
σ_ξ	0.2848	0.3165
$\rho_{\chi\xi}$	0.1052	0.0000
σ_{v2}	0.0374	0.0396
μ_ξ^*	-0.0487	-0.0513
λ_χ	0.7167	0.7063

It turns out that using the Kalman filter to estimate the model of Schwartz and Smith gives reasonable results, although there is some space for improvement. In general, the parameters of the short-term mean-reversion process are estimated quite well, while the parameters of the long-term process are estimated somewhat worse, including the correlation between both processes. The long-term equilibrium price μ_ξ is lower estimated compared to the true value, i.e. 0.1101 versus 0.1721. So, some parts of this long-term equilibrium price are not explained by the current two-factor model of Schwartz and Smith. The variance of this long-term process σ_ξ is also lower estimated compared to the true value, which indicates that the model of Schwartz and Smith underestimates the uncertainty of the long-term equilibrium price.

Using above values of parameters, the state variables ξ_t and χ_t can be computed numerically. In turn, the log spot price can be computed using those state variables. In figure 5 on the next page, the values of ξ_t and χ_t are shown over time in the upper panels, respectively, while the combination of both processes, known as the log spot price, is presented in the lower panel of figure 5. It can be seen that the log price clearly follows the pattern of the short-term process χ_t, but it is captured on the level of the long-term equilibrium price ξ_t. This makes sense, since it proves that the log spot price is shaped by short-term deviations of the long-term equilibrium price, while it moves around that long-term equilibrium level. So, the mean-reversion process is clearly seen in the lower panel as well as in the upper panel where χ_t moves around zero. Moreover, it is clear from figure 5 that the dynamics of ξ_t are intensified by the pattern of χ_t, for example when years = 0.5 and when years = 3.5. Here, the joint process declines (increases) more due to the addition of χ_t, which declines (increases) also.

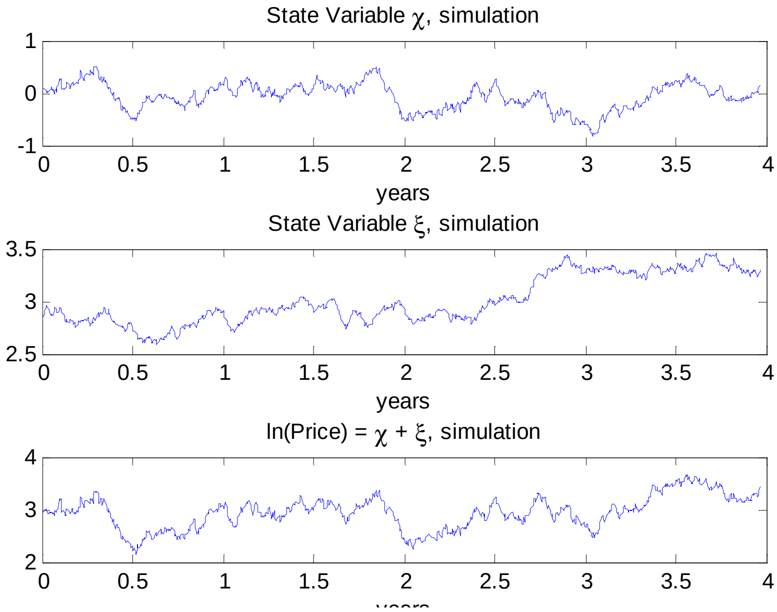

Figure 5: Short-term, long-term and joint processes over time using simulated data

In figure 6 below, the relation between spot and futures prices is presented, where the futures price closest to maturity is used as a proxy for the spot price. It can be clearly seen that both prices follow an equal, nearly at the same level, mean-reverting pattern, which shows the closest futures price is quite a good proxy for the spot price. However, futures prices tend to be slightly biased and explain the spot price a little lower at some points.

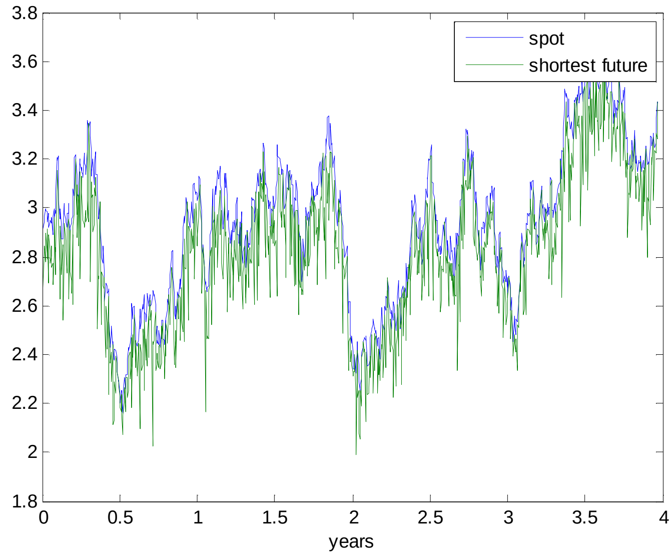

Figure 6: Relation between spot and futures prices using simulated data

5.2 Two-factor model with real data

When estimating the model with real data, first the one-factor model was used in estimation. The results of this one-factor model using all observations are shown in table 4 below.

Table 4: Results one-factor model with real data

Parameter	Kalman filter value
μ_ξ	0.0413
σ_ξ	0.2017
λ_ξ	0.0467
μ_ξ^*	-0.0054
σ_{v2}	0.6602

According to the results given above, the long-term equilibrium drift of the corn price turns out to be positive in the true process, i.e. 4.13%. However, in reviewing the long-term risk premium of 4.67%, the long-term equilibrium price level is slightly negative, i.e. -0.54%, in the risk-neutral process. This means that every time period the long-term equilibrium price level of corn decreases with 0.54%, while prices vary about 20% ($\sigma_\xi = 0.2017$) in the long run. Moreover, the futures prices are estimated with an error term of 0.6602, which indicates some space for improvement or adding a second factor.

To find out if parameters have changed over time, the one-factor model is estimated again for different subsections of observations. The results of that estimation process with the Kalman filter are shown below in table 5, in which the first column describes the concerned parameters and the other columns the value of that parameter for a particular time period.

Table 5: Results one-factor model with real data for different time periods

Parameter	1972-2012	1972-1980	1981-1990	1991-2000	2001-2012
μ_ξ	0.0413	0.1211	-0.0433	0.0020	0.0865
σ_ξ	0.2017	0.2301	0.1708	0.1594	0.2321
λ_ξ	0.0467	0.1332	-0.0475	-0.0061	0.1006
μ_ξ^*	-0.0054	-0.0021	0.0042	0.0081	-0.0141
σ_{v2}	0.6602	0.2861	0.3965	0.7796	0.8488

From table 5, it can be seen that the long-term equilibrium drift of the corn price was fairly positive in the period 1972-1980, while it was negative in the period 1981-1990 and positive again the last 20 years. This disputes the fact that equilibrium prices are always increasing, an assumption of the model of Schwartz and Smith. Even when considering the risk-neutral long-term equilibrium drift this is disputed, since this drift can be both positive and negative over time though not as strong as in the true process. In all periods, the price volatility was somewhat constant, i.e. around 20%. Furthermore, as for all observations, futures prices did not quite a good job in estimating spot prices, since the error terms σ_{v2} increased over time. They are very large for the last 20 years in particular, which argues for explaining spot prices using more than one factor.

When estimating the two-factor model with real data, it turned out there were some errors in the codes for programming the model. Unfortunately, due to time constraints, we were not able to detect all the errors and therefore show the results of the one-factor model only.

6. Conclusions and discussion

This chapter first presents the conclusions of this study in paragraph 6.1. A discussion of this study, focusing on some constraints and limitations of the model, the estimation method and the results is presented in paragraph 6.2.

6.1 Conclusions

This thesis aimed to examine the relationship between spot and futures prices of corn using the model of Schwartz and Smith, estimated as a state space model using the Kalman filter technique. The main conclusions with respect to the research questions can be found below.

1. *What are key elements and assumptions of the model of Schwartz & Smith?*

A fundamental assumption of the model of Schwartz and Smith is that spot prices are shaped by both short-term variations and long-term dynamics. The short-term variations are assumed to be mean-reverting, while the long-term dynamics are modeled as an increasing trend in spot prices to which the short-term variations revert. Another important element of this model is that the variance of both processes grows over time. Furthermore, futures prices are used to estimate the evolution of spot prices, in which the futures prices are a biased estimate of spot prices due to a risk premium.

2. *How to apply financial models in terms of data requirements and estimation methods?*

Financial models which study the evolution of commodity prices often require a large and extensive data set, i.e. many futures contracts with different maturities, since parameters are estimated recursively using state space models and the Kalman filter to obtain an accurate estimate of commodity spot prices. The use of state space models to estimate those financial models is valuable because it allows modeling unobserved dynamics.

3. *How does the model of Schwartz & Smith perform?*

It is unknown how the model of Schwartz and Smith performs for the two-factor model using actual data, but considering the two-factor model using simulated data it can be concluded that parameters of the short-term process are estimated quite well while the parameters of the long-term process are estimated somewhat worse. This proves there is some space for improvement in defining the model. This is also suggested by the one-factor model using actual data, since it shows large error terms. Nonetheless, the one-factor model with actual data seems to recommend a mean-reverting characteristic of the long-term equilibrium price over a longer time period, too.

4. *What are advantages and drawbacks of the model of Schwartz & Smith?*

A drawback of the model of Schwartz and Smith is its high degree of complexity, which makes the empirical applicability and interpretation of results more difficult compared to more traditional time series models. Advantages are the ability to study both factors separately and to include unobserved features of a time series.

6.2 Discussion

While much effort is put in making this model work, this thesis has its limitations which should be noted. First, as in every modeling application, the model used in this thesis is limited in its scope and empirical applicability. It is limited in its scope because it only deals with quite technical and abstract concepts about how spot prices evolve over time. For example, including explanatory variables like oil prices and stock indices could make the model more realistic but also more comprehensible, increasing its empirical applicability. The difficulty then is to find data that matches with futures prices data, i.e. data on a daily basis. More important is the fact that the model assumes that futures prices are a biased estimate of spot prices due to the risk-transfer role of futures prices, while the exact relationship between spot and futures prices is not taken into account thoroughly. For example, it is assumed that the basis[5] equals zero and is together with the convenience yield somehow absorbed in the risk premium. Also, the model assumes interrelationships between the two stochastic factors, which are carried out via the correlation between innovations of both processes but are not explored further. Moreover, the model neglects driving forces behind the determination of prices in detail, e.g. supply and storage decisions of farmers and the evaluation of those decisions in the risk-neutral approach. Possible extensions are to include explanatory variables and storage decisions, to take into account a stochastic long-term equilibrium price level and running the model for other commodities.

In considering estimation strategies, using state space models for this model is advantageous compared to more traditional time series models used in estimating agricultural commodity prices since the unobserved components of a time series could be easily modeled. However, a limitation of the state space model used in this model is that it does not deal with a structure for the error terms in examining price volatilities. To see if financial models are more appropriate than the traditional time series models, this should be taken into account. The method to estimate state space models typically proposed in the finance literature is the Kalman filter, which requires extensive data since its algorithm works recursively in predicting and minimizing the estimated covariance matrix. Nevertheless, according to Cortazar and Schwartz (2003), financial models like that of Schwartz and Smith could be more easily estimated using the idea of minimum variance and more simple regression methods.

Financial models, like the model of Schwartz and Smith, are related to traditional time series models, like an ARIMA model, since both types of models approximate the stochastic evolution of commodity prices given by the data. However, the use of financial models is limited in estimating agricultural commodity prices. Since agricultural commodities are traded more and more at futures markets, information about prices and production forecasts becomes available at a more frequent basis and expectations are established more frequently, estimating agricultural commodity prices using financial models could give valuable insights about the long-term behavior of commodity prices. In this case, those financial models certainly need some improvements to judge their results. In the end, financial models could be applied in forecasting price movements and providing better information, which can result in a better functioning market. Developing hybrid models which

[5] Defined as the price difference between a futures price and a spot price

combine stochastic behavior of commodity prices with non-stochastic exogenous variables could be a future extension.

Finally, due to time constraints we were not able to estimate the two-factor model completely since there are some errors in the programming codes for this model. As a consequence, the interpretations of the results are not quite elaborated, and a comparison with the minimum variance idea of Cortazar and Schwartz could not be made. The availability of more time would have enhanced the level of detail of used concepts, as well as the results of programming and estimating this model.

While the estimations of this model could not be completed, this thesis has met its objective in exploring financial models and their estimation methods theoretically to analyze the relationship between spot and futures prices for corn.

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Appendices

A. Derivation of first and sector moments of ξ_t

Over time, the mean of ξ_t equals:

$$\int_0^t d\,\xi_t = \xi_t - \xi_0 = \int_0^t \mu_\xi dt + \int_0^t \sigma_\xi\, dz_\xi \tag{A.1}$$

Rewriting gives:

$$\xi_t = \xi_0 + \mu_\xi t + \sigma_\xi \int_0^t dz_\xi \tag{A.2}$$

Taking expectations, it follows that:

$$E[\xi_t] = \xi_0 + \mu_\xi t + \sigma_\xi \int_0^t E\left[dz_\xi\right] \tag{A.3}$$

Assuming that $E\left[dz_\xi\right] = 0$:

$$E[\xi_t] = \xi_0 + \mu_\xi t \tag{A.4}$$

The variance of ξ_t equals, given that $E\left[\left(dz_\xi\right)^2\right] = V\left[dz_\xi\right] = 1$:

$$V[\xi_t] = \sigma_\xi^2 \int_0^t V\left[dz_\xi\right] = \sigma_\xi^2 t \tag{A.5}$$

B. Derivation of first and second moments of χ_t

Over time, the mean of χ_t equals:

$$\int_0^t d\,\chi_t = \chi_t - \chi_0 = \int_0^t -\kappa\chi_t dt + \int_0^t \sigma_\chi\, dz_\chi \tag{B.1}$$

Rewriting gives:

$$\chi_t = \chi_0 - \kappa \int_0^t \chi_t dt + \sigma_\chi \int_0^t E\left[dz_\chi\right] \tag{B.2}$$

Taking expectations, it follows that:

$$E[\chi_t] = \chi_0 - \kappa \int_0^t E[\chi_t] dt \tag{B.3}$$

Differentiating with respect to t and assuming $E\left[dz_\chi\right] = 0$ gives:

$$E'[\chi_t] = -\kappa E[\chi_t] \tag{B.4}$$

It follows that: $\chi_t = e^{-\kappa t}\chi_0$ (B.5)

The variance of χ_t can be derived using the following steps:

$$V[\chi_t] = E[\chi_t^2] - E[\chi_t]^2 \quad \text{(B.6)}$$

To derive $E[\chi_t^2]$ Ito's lemma is applied to $F(\chi_t) = \chi_t^2$ (Stokey, 2008):

$$\chi_t^2 = \chi_0^2 - 2\kappa \int_0^t \chi_t^2 dt + 2\sigma_\chi \int_0^t \chi_t \, dz_\chi + \sigma_\chi^2 t \quad \text{(B.7)}$$

Taking expectations:

$$E[\chi_t^2] = \chi_0^2 - 2\kappa \int_0^t E[\chi_t^2] dt + 2\sigma_\chi E\left[\int_0^t \chi_t \, dz_\chi\right] + \sigma_\chi^2 t \quad \text{(B.8)}$$

Assuming $E\left[\int_0^t \chi_t \, dz_\chi\right] = 0$:

$$E[\chi_t^2] = \chi_0^2 - 2\kappa \int_0^t E[\chi_t^2] dt + \sigma_\chi^2 t \quad \text{(B.9)}$$

Filling in equation (B.5) and rewriting gives:

$$E[\chi_t^2] = \left(\chi_0^2 - \frac{\sigma_\chi^2}{2\kappa}\right) e^{-2\kappa t} + \frac{\sigma_\chi^2}{2\kappa} \quad \text{(B.10)}$$

The variance of χ_t can be further derived:

$$V[\chi_t] = E[\chi_t^2] - E[\chi_t]^2 = \left(\chi_0^2 - \frac{\sigma_\chi^2}{2\kappa}\right) e^{-2\kappa t} + \frac{\sigma_\chi^2}{2\kappa} - e^{-2\kappa t}\chi_0^2 \quad \text{(B.11)}$$

Rewriting gives:

$$V[\chi_t] = (1 - e^{-2\kappa t}) \frac{\sigma_\chi^2}{2\kappa} \quad \text{(B.12)}$$

This is also confirmed by Stokey (2008), which states that the stationary distribution of an Ornstein-Uhlenbeck process is normal with mean zero and variance $\frac{\sigma^2}{2\alpha}$. In the model of Schwartz and Smith, the parameter κ can be considered equal to the parameter α in Stokey (2008).

C. Derivation of covariance matrix of ξ_t and χ_t

Following Schwartz and Smith (2000) in their derivation of the covariance matrix of χ_t and ξ_t, first a discrete-time covariance matrix is derived by using equations (7) and (8) from the main text.

$$x_t = c + Gx_{t-1} + \omega_t \quad \text{(C.1)}$$

in which the time steps are of length $\Delta t = t/n$, $x_t \equiv \begin{bmatrix}\chi_t \\ \xi_t\end{bmatrix}$, $c \equiv \begin{bmatrix}0 \\ \mu_\xi \Delta t\end{bmatrix}$, $G \equiv \begin{bmatrix}\varphi & 0 \\ 0 & 1\end{bmatrix}$, with $\varphi = 1 - \kappa\Delta t$, and $\omega_t = \begin{bmatrix}\omega_{\chi t} \\ \omega_{\xi t}\end{bmatrix}$, a 2x1 vector of error terms with mean $E[\omega_t] = \begin{bmatrix}0 \\ 0\end{bmatrix}$ and variance $W = V[\omega_t] = \begin{bmatrix}\sigma_\chi^2 & \rho_{\chi\xi}\sigma_\chi\sigma_\xi \\ \rho_{\chi\xi}\sigma_\chi\sigma_\xi & \sigma_\xi^2\end{bmatrix}\Delta t$.

Using equation (C.1), the next n-step mean vector (m_n) and covariance matrix (V_n) are given recursively by:

$$m_n = c + Gm_{n-1} \tag{C.2}$$

$$V_n = GV_{n-1}G' + W \tag{C.3}$$

with $m_0 = x_0 \equiv \begin{bmatrix}\chi_0 \\ \xi_0\end{bmatrix}$ and $V_0 = 0$. In this process, the next n-step mean vector is a function of the long-term equilibrium values of the states and its previous mean vector weighted by the mean-reversion coefficient for each state. The next n-step covariance matrix is a function of its previous covariance matrix weighted by the mean-reversion coefficient for each state and the error terms of the state equation. Applying this recursion gives:

$$m_n = \begin{bmatrix}\varphi^n \chi_0 \\ \xi_0 + \mu_\xi n\Delta t\end{bmatrix} \tag{C.4}$$

$$V_n = \begin{bmatrix}\sigma_\chi^2 \Delta t \sum_{i=0}^{n-1}\varphi^{2i} & \rho_{\chi\xi}\sigma_\chi\sigma_\xi\Delta t\sum_{i=0}^{n-1}\varphi^i \\ \rho_{\chi\xi}\sigma_\chi\sigma_\xi\Delta t\sum_{i=0}^{n-1}\varphi^i & n\Delta t\sigma_\xi^2\end{bmatrix} \tag{C.5}$$

The system of equations above can be rewritten, using $\sum_{i=0}^{n-1}\varphi^{2i} = \frac{1-\varphi^{2(n-1)}}{1-\varphi^2}$ and $\sum_{i=0}^{n-1}\varphi^i = \frac{1-\varphi^{n-1}}{1-\varphi}$ (Schwartz and Smith, 2000). The errors in this discrete time approximation are of an order smaller than Δt. To derive the continuous-time covariance matrix, the limit is taken as $n \to \infty$ and $\Delta t = \frac{t}{n} \to 0$, then $\varphi^n = \left(1 - \frac{\kappa t}{n}\right)^n$ approaches $e^{-\kappa t}$, φ^{2n} approaches $e^{-2\kappa t}$, and $\frac{1-\varphi^{n-1}}{1-\varphi}\Delta t \to \frac{1-e^{-\kappa t}}{\kappa}$ and $\frac{1-\varphi^{2(n-1)}}{1-\varphi^2}\Delta t \to \frac{1-e^{-2\kappa t}}{2\kappa}$. Substituting these limiting forms into the equations (C.4) and (C.5) and rewriting gives:

$$Cov[\chi_t, \xi_t] = \begin{bmatrix}(1-e^{-2\kappa t})\frac{\sigma_\chi^2}{2\kappa} & (1-e^{-\kappa t})\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa} \\ (1-e^{-\kappa t})\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa} & \sigma_\xi^2 t\end{bmatrix} \tag{C.4}$$

D. Derivation of first moment of χ_t in the risk-neutral process

When equation (16) in the main text is considered and innovations seem to appear in this process, over time the mean of χ_t equals:

$$\chi_t = \chi_0 - \kappa\int_0^t \chi_t dt - \lambda_\chi t + \sigma_\chi \int_0^t dz_\chi^* \tag{D.1}$$

Given that $E\left[dz_\chi^*\right] = 0$ and taking expectations, it follows that:

$$E[\chi_t] = \chi_0 - \kappa \int_0^t E[\chi_t]dt - \lambda_\chi t \qquad \text{(D.2)}$$

Differentiating with respect to t gives:

$$E'[\chi_t] = -\kappa\, E[\chi_t] - \lambda_\chi \qquad \text{(D.3)}$$

Rewriting gives:

$$E[\chi_t] = \left(\chi_0 + \frac{\lambda_\chi}{\kappa}\right) e^{-\kappa t} - \frac{\lambda_\chi}{\kappa} \qquad \text{(D.4)}$$

It follows that: $\chi_t = e^{-\kappa t}\chi_0 - (1 - e^{-\kappa t})\frac{\lambda_\chi}{\kappa}$ (D.5)

E. Derivation of first moment of ξ_t in the risk-neutral process

Over time, the mean of ξ_t equals:

$$\int_0^t d\,\xi_t = \xi_t - \xi_0 = \int_0^t (\mu_\xi - \lambda_\xi)dt + \int_0^t \sigma_\xi\, dz_\xi^* \qquad \text{(E.1)}$$

Rewriting gives:

$$\xi_t = \xi_0 + (\mu_\xi - \lambda_\xi)t + \sigma_\xi \int_0^t d\, z_\xi^* \qquad \text{(E.2)}$$

Taking expectations, it follows that:

$$E[\xi_t] = \xi_0 + (\mu_\xi - \lambda_\xi)t + \sigma_\xi \int_0^t E\left[dz_\xi^*\right] \qquad \text{(E.3)}$$

Assuming that $E\left[dz_\xi^*\right] = 0$:

$$E[\xi_t] = \xi_0 + (\mu_\xi - \lambda_\xi)t \qquad \text{(E.4)}$$

F. Derivation of relationship between spot and futures prices

Using equation (13) and (21) from the main text for maturity T gives:

$$\ln E^*[S_T] = \ln F_{0,T} = E[\ln S_t] - \left((1 - e^{-\kappa T})\frac{\lambda_\chi}{\kappa} + \lambda_\xi T\right) + \frac{1}{2}\left((1 - e^{-2\kappa T})\frac{\sigma_\chi^2}{2\kappa} + \sigma_\xi^2 T + 2(1 - e^{-\kappa T})\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}\right) \qquad \text{(F.1)}$$

Filling in equation (19) gives:

$$\ln F_{0,T} = e^{-\kappa T}\chi_0 + \xi_0 + \mu_\xi T - \left((1-e^{-\kappa T})\frac{\lambda_\chi}{\kappa} + \lambda_\xi T\right) + \frac{1}{2}\left((1-e^{-2\kappa T})\frac{\sigma_\chi^2}{2\kappa} + \sigma_\xi^2 T + 2(1-e^{-\kappa T})\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}\right) \quad \text{(F.2)}$$

Presenting this in a more simple way:

$$\ln F_{0,T} = e^{-\kappa T}\chi_0 + \xi_0 + A(T) \quad \text{(F.3)}$$

with $A(T) = (\mu_\xi - \lambda_\xi)T - \left((1-e^{-\kappa T})\frac{\lambda_\chi}{\kappa}\right) + \frac{1}{2}\left((1-e^{-2\kappa T})\frac{\sigma_\chi^2}{2\kappa} + \sigma_\xi^2 T + 2(1-e^{-\kappa T})\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}\right)$ (F.4)

G. State equation in recursive form

When considering the state equation in a recursive process, it turns out that vector c can be multiplied with a new matrix and vector ω_t becomes a recursive vector in itself:

$$x_t = \begin{bmatrix} 1 + e^{-\kappa\Delta t} + \cdots + e^{-(\kappa\Delta t)(t-1)} & 0 \\ 0 & 1+1+\cdots+1 \end{bmatrix} c + \begin{bmatrix} e^{-(\kappa\Delta t)t} & 0 \\ 0 & 1 \end{bmatrix} x_{t-1} + \begin{bmatrix} e^{-(\kappa\Delta t)(t-1)}\omega_{\chi 1} + e^{-(\kappa\Delta t)(t-2)}\omega_{\chi 2} + \cdots + \omega_{\chi t} \\ \omega_{\xi 1} + \omega_{\xi 2} + \cdots + \omega_{\xi t} \end{bmatrix} \quad \text{(G.1)}$$

H. Derivation of state variance in recursive form

The variance of x_t can be derived using the following steps:

$$V[x_t] = E[(x_t - E[x_t])(x_t - E[x_t])'] \quad \text{(H.1)}$$

Filling in equation (G.1), taking expectations and rewriting gives:

$$V[x_t] = \begin{bmatrix} \left(e^{-2(\kappa\Delta t)(t-1)} + e^{-2(\kappa\Delta t)(t-2)} + \cdots + 1\right)\sigma_\chi^2\Delta t & \left(e^{-2(\kappa\Delta t)(t-1)} + e^{-2(\kappa\Delta t)(t-2)} + \cdots + 1\right)\rho_{\chi\xi}\sigma_\chi\sigma_\xi\Delta t \\ \left(e^{-2(\kappa\Delta t)(t-1)} + e^{-2(\kappa\Delta t)(t-2)} + \cdots + 1\right)\rho_{\chi\xi}\sigma_\chi\sigma_\xi\Delta t & \sigma_\xi^2(\Delta t)t \end{bmatrix} \quad \text{(H.2)}$$

Rewriting and summing up as equation (C.5) gives

$$V[x_t] = \begin{bmatrix} \frac{1-e^{-2(\kappa\Delta t)t}}{1-e^{-2(\kappa\Delta t)}}\sigma_\chi^2\Delta t & \frac{1-e^{-(\kappa\Delta t)t}}{1-e^{-(\kappa\Delta t)}}\rho_{\chi\xi}\sigma_\chi\sigma_\xi\Delta t \\ \frac{1-e^{-(\kappa\Delta t)t}}{1-e^{-(\kappa\Delta t)}}\rho_{\chi\xi}\sigma_\chi\sigma_\xi\Delta t & \sigma_\xi^2(\Delta t)t \end{bmatrix} \quad \text{(H.3)}$$

Using $(1-e^{-x}) \approx -x$ gives:

$$V[x_t] = \begin{bmatrix} 1 - e^{-2(\kappa\Delta t)t}\frac{\sigma_\chi^2}{2\kappa} & 1 - e^{-(\kappa\Delta t)t}\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa} \\ 1 - e^{-(\kappa\Delta t)t}\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa} & \sigma_\xi^2(\Delta t)t \end{bmatrix} \quad \text{(H.4)}$$