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# On lower bounds using separable terms in interval B&B for one-dimensional problems *

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**Abstract** Interval Branch-and-Bound (B&B) algorithms are powerful methods which aim for guaranteed solutions of Global Optimization problems. Lower bounds for a function in a given interval can be obtained directly with Interval Arithmetic. The use of lower bounds based on Taylor forms show a faster convergence to the minimum with decreasing size of the search interval. Our research focuses on one dimensional functions that can be decomposed into several terms (sub-functions). The question is whether using this characteristic leads to sharper bounds when based on bounds of the sub-functions. This paper deals with separable functions in two sub-functions.

The use of the separability is investigated for the so-called Baumann form and Lower Bound Value Form (LBVF). It is proven that using the additively separability in the LBVF form may lead to a combination of linear minorants that are sharper than the original one. Numerical experiments confirm this improving behaviour and also show that not all separable methods do always provide sharper additively lower bounds. Additional research is needed to obtain better lower bounds for multiplicatively separable functions and to address higher dimensional problems.

**Keywords:** separable functions, Interval Arithmetic, Taylor forms, branch-and-bound lower bound

## 1. Introduction

Interval Branch-and-Bound methods aim for guaranteed solutions of Global Optimization problems. Consider the one dimensional generic interval constrained global optimization problem, which is to find

$$f^* = \min_{x \in S} f(x) \tag{1}$$

where $S \in \mathbb{I}$ is the search region and $\mathbb{I}$ stands for the set of all one-dimensional closed real intervals.

**Definition 1.** *Function $f : S \subset \mathbb{R} \to \mathbb{R}$ is additively separable, if it can be written as*

$$f(x) = \sum_{j=1}^{p} f_j(x),\ x \in S. \tag{2}$$

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We have

$$\min_{x \in S} f(x) \geq \sum_{j=1}^{p} \min_{S} f_j(x). \tag{3}$$

Let $\underline{F}_j$ be a lower bound of $f_j$ over $S$. Then we have

$$\min_{x \in S} f(x) \geq \sum_{j=1}^{p} \underline{F}_j. \tag{4}$$

To create a lower bound $\underline{F}$ of $f$ over interval $X$ in an interval B&B framework, can be done in several ways. Sharper bounds are better, i.e. higher values of $\underline{F}$ lead to more efficient performance of the B&B algorithm. Considering functions that have an additively separable structure (2), our research question is: for which cases

$$\underline{F} \leq \sum_{j=1}^{p} \underline{F}_j? \tag{5}$$

Alternatively, the question is to find ways to combine minorants on the separable terms, such that we get sharper bounds.

## 2. Taylor forms

Besides the standard IA bounding, called "natural interval extension" $F = [\underline{F}, \overline{F}]$ of $f$ [4, 5], one can obtain an inclusion function of $f$ using the inclusion function $F'$ of $f'$. Consider the first order Taylor expression

$$T(c, X) := f(c) + (X - c)F'(X), \tag{6}$$

where $c \in X$. Notice that this expression is mainly of interest if the function is not monotonous on $X$, so at least $0 \in F'(X)$. By taking for $c$ the middle $m = \frac{\underline{X}+\overline{X}}{2}$ of the interval, we have what is called a center form of the inclusion

In [1], Baumann proves that taking $c = b^-$ in the Taylor expression, leads to the best lower bound, where:

$$b^- = \begin{cases} \dfrac{\underline{X}_i \overline{F'}(X) - \overline{X}\underline{F'}(X)}{\overline{F'}(X) - \underline{F'}(X)} & , 0 \in F'(X) \\ \overline{X} & , \overline{F'}(X) \leq 0 \\ \underline{X} & , \underline{F'}(X) \geq 0 \end{cases}$$

So,

$$f(X) \geq \underline{T}(b^-, X). \tag{7}$$

An additively separable Baumann form bound $ASB(X)$ can be constructed in a straightforward way evaluating the Taylor expression (6) for the two sub-functions in their Baumann point and adding the resulting lower bounds,

$$f(X) \geq ASB(X) = \underline{T}_1(b_1^-, X) + \underline{T}_2(b_2^-, X). \tag{8}$$

**Example 2.** *Consider function $f(x) = f_1(x)+f_2(x) = (x+1)^2+(x-1)^2$ on the interval $X = [-2, 2]$. The minima of the sub-functions is 0, whereas the minimum of $f$ itself is $f(0) = 2$. Figure 1 illustrates this idea and also draws lower bounds of all functions based on Baumann point. $\underline{T}(b^-, X) = -14$ and $\underline{T}_i(b_i^-, X) = -6$ such that $\underline{T}_1(b_1^-, X) + \underline{T}_2(b_2^-, X) = -12$, illustrating question (5).*

## 3. Lower Bound Value Form

Another way to compose derivative based linear minorants is the so-called Lower Boundary Value Form (LBVF), ([6] p. 60 and [2, 3]) that uses the evaluation of the end-points of the interval. Consider the most left point of $X$. Function

$$\varphi l(x) = \underline{F}(\underline{X}) + \underline{F'}(X)(x - \underline{X}), \tag{9}$$

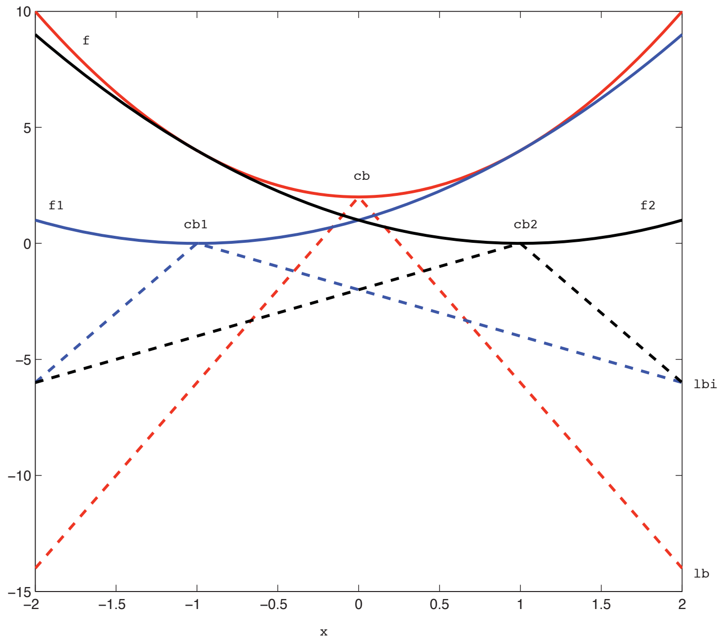


*Figure 1.* Quadratic illustration of (3) and (5)

provides an affine minorant. Similarly, the right most point of $X$ provides

$$\varphi r(x) = \underline{F}(\overline{X}) - \overline{F'}(X)(\overline{X} - x) = \underline{F}(\overline{X}) + \overline{F'}(X)(x - \overline{X}). \tag{10}$$

The values $\underline{\varphi l}(\overline{X})$ and $\underline{\varphi r}(\underline{X})$ are lower bounds of $f(X)$ over $X$. A sharper lower bound can be obtained when $0 \in \overline{F'}(X)$ by combining (9) and (10) in lower bounding function

$$\varphi m(x) = \max\{\varphi l(x), \varphi r(x)\}. \tag{11}$$

The Lower Boundary Value Form $\underline{\varphi m}(X)$ follows from finding $y$ for which (9) and (10) are equal

$$\underline{\varphi m}(X) = \varphi m(y) = \frac{\underline{F}(\underline{X})\overline{F'}(X) - \underline{F}(\overline{X})\underline{F'}(X)}{w(F'(X))} + \frac{w(X)\overline{F'}(X)\underline{F'}(X)}{w(F'(X))}. \tag{12}$$

So,

$$f(X) \geq \underline{\varphi m}(X),\ 0 \in F'(X). \tag{13}$$

An Additively Separable Lower Bound Value form can be constructed in the following way:

$$f(X) \geq ASLBV(X) = \underline{\varphi m}_1(X) + \underline{\varphi m}_2(X),\ 0 \in F_1'(X),\ 0 \in F_2'(X). \tag{14}$$

We focus further on the LBVF minorants of both sub-functions in order to obtain a sharper lower bound than $\underline{\varphi m}(X)$ without worrying about the monotonicity of the sub-functions for a given interval. Notice that only the case where the composite function $f$ is not monotonous, $0 \in F'(X)$ is interesting. Consider the addition of the separate minorant terms

$$\varphi(x) = \varphi m_1(x) + \varphi m_2(x),$$

where $\varphi m_i$ is defined by (11). First of all, notice that $\varphi$ is a piecewise linear minorant function and the maximum of four different affine terms:

$$\varphi(x) = \max \left\{ \begin{array}{l} \varphi l(x) := \varphi l_1(x) + \varphi l_2(x) \\ \varphi a(x) := \varphi l_1(x) + \varphi r_2(x) \\ \varphi b(x) := \varphi r_1(x) + \varphi l_2(x) \\ \varphi r(x) := \varphi r_1(x) + \varphi r_2(x) \end{array} \right\}. \tag{15}$$

Then, one can see that $\varphi(x)$ is a sharper minorant than $\varphi m(x)$.

**Theorem 3.** *Let* $\forall x \in X, f(x) = f_1(x) + f_2(x)$ *and* $\varphi l$, $\varphi r$ *and* $\varphi m$ *be defined by (9), (10) and (11).* $\forall x \in X, \varphi m_1(x) + \varphi m_2(x) \geq \varphi m(x)$.

*Proof.* Given equivalence (15), we have that

$$\varphi m(x) = \max\{\varphi l(x), \varphi r(x)\} \leq \max\{\varphi l(x), \varphi a(x), \varphi b(x), \varphi r(x)\} = \varphi(x).$$

$\square$

Theorem 3 provides us with a new Additively Separable Lower Bound $ASLB\varphi$ defined by

$$f(X) \geq ASLB\varphi(X) = \underline{\varphi}(X) \tag{16}$$

## 4. Summary

For $ASLBV\varphi$, it is proven that the corresponding minorant is sharper than the standard one for LBVF. How to evaluate $ASLB\varphi(X)$ and numerical experiments will be shown in GOW 2012. Numerical results confirm this improving behaviour, although monotonicity of the sub-function and the composite function over an interval reduces this effect. Numerical results also show that separable variant for the Baumann lower bound is usually worse than the original one.

Future investigation could focus on the question how to extend the $ASLBV\varphi$ lower bound for $n$-dimensional functions. Another question is the derivation of specific interval based bounds for multiplicative terms.

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