

NOTA 1082

Instituut voor Cultuurtechniek en Waterhuishouding  
Wageningen

TIME DEPENDENT, ONE-DIMENSIONAL LINEARIZED MOISTURE FLOW

INCLUDING WATER UPTAKE BY ROOTS IN THE PRESENCE OF

A SHALLOW WATER TABLE

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C O N T E N T S

	Page
1. INTRODUCTION	1
2. BASIC EQUATIONS AND ASSUMPTIONS	2
a. Differential equations	2
b. Boundary and initial conditions	4
c. Sink functions	5
d. The complete problem with dimensionless variables	5
3. SOLUTIONS	6
a. Deep water table	6
b. Shallow water table	10
4. SUMMARY AND CONCLUSIONS	19
5. LITERATURE	21

## 1. INTRODUCTION

The work contained in this note was performed while the author was on leave from the Mathematics Department at the University of Arizona for two months, the summer of 1978.

Solutions of the time dependent moisture flow equation with plant-water extraction are primarily obtained by numerical techniques. For example consider papers by NIMAH and HANKS (1973), NEUMAN, FEDDES and BRESLER (1975) and FEDDES and ZARADNY (1977). Analytical solutions are, by necessity, subject to more restrictive assumptions, but are generally easy and inexpensive to evaluate. They provide exact answers for which round-off and computational errors are negligible. Such solutions are valuable for checking complicated numerical simulations as well as for providing answers when the assumptions are satisfied or the input data are such that a more elaborate analysis would be unnecessary. In this report a solution of the problem of the title will be derived. Besides being of interest in its own right, it could be useful to check complicated subroutines of programs such as SWATR (FEDDES et al, 1978). The number of calculations and instructions needed in such a large scale program is such that it is nearly impossible to be sure all are given without error. Having an analytical solution to compare with can save a lot of time and effort on the part of the researcher as well as greatly increase his confidence in the output of his numerical approximation.

## 2. BASIC EQUATIONS AND ASSUMPTIONS

### a. Differential equations

The starting point for many investigations into moisture movement in soil is a relationship hypothesized to explain steady flow of a single viscous liquid through an isotropic saturated porous medium. This is called Darcy's law and says the vertical velocity of the liquid,  $v$ , is proportional to the gradient of the total hydraulic head,  $H$ . RICHARDS (1931), extended Darcy's law to an unsaturated medium where it is written as:

$$v = - K \nabla H \quad (1)$$

In eq. (1) the constant of proportionality is the unsaturated hydraulic conductivity,  $K = K(\Psi)$ , which is a function of the pressure head  $\Psi$ , ( $H = \Psi - z$ , with  $z$  positive downward into the soil).

The second equation which we use is the continuity equation:

$$\frac{\partial \theta}{\partial t} = - \frac{\partial v}{\partial z} - S \quad (2)$$

which is derived using conservation of mass. In eq. (2),  $\theta$  is the volumetric water content,  $t$  is time and  $S$  is the "water uptake" term representing the volume of water used by the roots per unit volume of soil per unit time.

If we combine eqs. (1) and (2) we obtain:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left[ K(\Psi) \frac{\partial \Psi}{\partial z} \right] - \frac{\partial K(\Psi)}{\partial z} - S \quad (3)$$

which is identical with (7) of FEDDES and ZARADNY (1977). There are two obstacles to obtaining solutions of (3), namely two dependent variables,  $\theta$  and  $\Psi$ , and the unsaturated hydraulic conductivity  $K$  which is a highly non-linear function of  $\Psi$ .

GARDNER (1958) eliminated the second obstacle for steady flows by defining a new dependent variable  $\phi$ , as:

$$\phi = \int_{-\infty}^{\Psi} K(\Psi) d\Psi \quad (4)$$

and assuming

$$K = K_0 \exp(\alpha\Psi) \quad (5)$$

where  $K_0$  and  $\alpha$  should be considered empirical constants (BEN-ASHER, LOMEN and WARRICK (1976)). The function  $\phi$  is called the "matrix flux potential" but was used by KIRCHHOFF as a "diffusivity potential" (see IRMAY (1966)). If we then assume  $\theta = \theta(\phi)$ , we can write eq. (3) as:

$$\frac{d\theta}{d\phi} \frac{\partial\theta}{\partial t} = \frac{\partial^2\phi}{\partial z^2} - \alpha \frac{\partial\phi}{\partial z} - S \quad (6)$$

(note that  $\frac{\partial\phi}{\partial z} = K(\Psi) \frac{\partial\Psi}{\partial z}$  and  $\phi = K/\alpha$ )

For steady state situations the left hand side of eq. (6) is zero and a relationship between  $\theta$  and  $\phi$  is not needed. Solutions of eq. (6) in that situation are given by WARRICK (1974) and LOMEN and WARRICK (1976) for sink functions (S) which are defined explicitly in terms of depth or implicitly through several functions of  $\phi$ . This was done for deep and shallow water tables as well as an impermeable barrier at a shallow depth.

Time dependent solutions of eq. (6) with no water uptake ( $S = 0$ ) have been given by PHILIP (1969), BRAESTER (1973) and WARRICK (1975) by assuming:

$$\frac{d\theta}{d\phi} = \frac{1}{D} \quad (7)$$

with D the soil moisture diffusivity ( $D = Kd\Psi/d\theta$ ). This is equivalent to having K linearly related to  $\theta$  as:

$$\frac{dK}{d\theta} = \frac{dK}{d\phi} \frac{d\phi}{d\theta} = \alpha D$$

and reduces eq. (6) to

$$\frac{1}{D} \frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial z^2} - \alpha \frac{\partial \phi}{\partial z} - S \quad (8)$$

In general  $K$  is not linear with  $\theta$ , but if the soil moisture varies over a limited range the assumption is more realistic. In evaluating the acceptability of all assumptions it is important to remember the natural uncertainty of all input parameters due to time and spatial variability and to experimental error. Major advantages to be gained by the linearizing assumptions are numerical accuracy and simplicity. Computational times are negligible compared with finite difference and finite element solutions of eq. (6). Of course as a tool for checking complicated numerical schemes, these reservations are unwarranted.

#### b. Boundary and initial conditions

If a time dependent surface flux is given by  $v(t)$ , we can use eq. (1) and the equations in parentheses following eq. (6) to obtain:

$$v(t) = -K \frac{\partial}{\partial z}(\Psi - z) = -K \frac{\partial \Psi}{\partial z} + K = -\frac{\partial \phi}{\partial z} + \alpha \phi \quad (9)$$

where the right hand side of eq. (9) is evaluated at  $Z = 0$ .

If a shallow water table is present at depth  $L$  we specify the potential there, namely:

$$\phi = \phi_0 \text{ (or } \Psi = \Psi_0 \text{) at } z = L \quad (10)$$

On the other hand for deep water tables it is more convenient to assume that:

$$\lim_{z \rightarrow \infty} \phi(z, t) = \phi_\infty \quad (11)$$

For either situation we need to specify an initial condition for time zero, namely:

$$\phi(z, 0) = g(z) \quad (12)$$

c. Sink functions

The technique of solution for a shallow water table will permit the sink function  $S = S(Z, T)$  to be any reasonable function of depth and time. For a specific choice of  $S$ , all that the user need do is to evaluate some integrals using, for example, DWIGHT (1961).

The solution for an infinitely deep water table uses a sink function which varies discretely in time as:

$$S(Z, T) = \begin{cases} S_1(Z) & 0 = T_0 & < T < T_1 \\ S_2(Z) & T_1 & < T < T_2 \\ \vdots & \vdots & \\ S_n(Z) & T_{n-1} & < T < T_n = \infty \end{cases} \quad (13)$$

d. The complete problem with dimensionless variables

It seems advantageous to introduce dimensionless variables  $Z$  and  $T$  by:

$$Z = \alpha z/2, \quad T = \alpha^2 Dt/4 \quad (14)$$

which reduce the boundary value problem (eqs. (8), (9), (10) and (11)) to:

$$\frac{\partial \phi}{\partial T} = \frac{\partial^2 \phi}{\partial Z^2} - 2 \frac{\partial \phi}{\partial Z} - \frac{4}{\alpha^2} S \quad T > 0, 0 < Z < \alpha L/2 \quad (15)$$

$$- \frac{\partial \phi}{\partial Z} + 2\phi = 2v(T)/\alpha \quad \text{at } Z = 0 \quad (16)$$

$$\phi = \phi_0 \quad \text{at } Z = \alpha L/2 \quad (17)$$

$$\phi = g(Z) \quad \text{at } T = 0 \quad (18)$$

### 3. SOLUTIONS

#### a. Deep water table

The solution of eqs. (15), (16), (17) and (18) for a deep water table ( $L = \infty$ ) has been given in a recent article (LOMEN and WARRICK (1978)). A summary of their conclusions is included in this report to give an indication of the types of results possible from a linear analysis. For a sink function given by eq. (13) we define:

$$\phi_{S,i}(Z) = U_i/\alpha - 4/\alpha^2 \int_Z^{\infty} \exp[2(Z - Z')] \int_0^{Z'} S_i(\xi) d\xi dZ',$$

$$T_{i-1} < T < T_i \quad (19)$$

where  $U_i$  is the "steady" velocity during the time period  $T_{i-1} < T < T_i$ ,

$$U_i = (2/\alpha) \int_0^{\infty} S_i(Z) dZ \quad (20)$$

If we also define two additional functions:

$$\begin{aligned} \phi_u(Z, T) = & - (Z + 2T + 1/2) \exp(2Z) \operatorname{erfc}(Z/2T^{1/2} + T^{1/2}) + \\ & + (4T/\pi)^{1/2} \exp[-(Z/2T^{1/2} - T^{1/2})^2] + 1/2 \operatorname{erfc}(Z/2T^{1/2} - T^{1/2}) \end{aligned} \quad (21)$$

and

$$\begin{aligned} R(Z, T, f) = & \exp(Z - T) \int_0^{\infty} [(4\pi T)^{-1/2} [\exp(-(Z - Z')^2/4T) + \exp(-(Z + Z')^2/4T)] - \\ & - \exp(T + Z + Z') \operatorname{erfc}((Z + Z')/2T^{1/2} + T^{1/2})] f(Z') \exp(-Z') dZ' \end{aligned} \quad (22)$$

and consider the surface flux to change with time as:



$$v(T) = \begin{cases} v_1 & 0 = T^0 & < T < T^1 \\ v_2 & T^1 & < T < T^2 \\ \vdots & \vdots & \vdots \\ v_m & T^{m-1} & < T < T^m = \infty \end{cases} \quad (23)$$

then the overall solution is:

$$\begin{aligned} \phi(Z, T) = & \phi_\infty [1 - \phi_u(Z, T)] + (1/\alpha) \sum_{k=1}^j (v_k - v_{k-1}) \phi_u(Z, T - T^{k-1}) + \\ & + \phi_{S,i} - (1/\alpha) \sum_{k=1}^i (U_k - U_{k-1}) \phi_u(Z, T - T_{k-1}) + \\ & + \sum_{k=1}^i R(Z, T - T_{k-1}, \phi_{S,k-1} - \phi_{S,k}), \quad T^{j-1} < T < T^j; \quad T_{i-1} < T < T_i \end{aligned} \quad (24)$$

where

$$\phi_{S,0} = g(Z) \quad \text{and} \quad v_0 = U_0 = 0 \quad (25)$$

Five numerical examples will now be presented to demonstrate the application of eq. 24.

**Example 1 - Drainage.** In all there are five terms in eq. 24. The first term arises from the boundary condition at infinity. If  $g(Z) = 0$  and no water was added or withdrawn after time zero, then simple drainage would occur and  $\phi$  would be given by the first term only. This situation is given in Fig. 1A showing drainage from an initially wet profile where  $\phi/\phi_\infty$  is plotted as a function of dimensionless depth  $Z$  for dimensionless times  $T = \alpha^2 Dt/4$  of 0.1, 0.5 and 1.5. As the time increases the upper part of the profile drains. For  $\alpha = 0.015 \text{ cm}^{-1}$  and  $D = 4000 \text{ cm}^2/\text{day}$ ,  $Z = 0.1$  corresponds to a 13 cm depth while  $T = 0.1$  is the same as 10.6 hours.

**Example 2 - Infiltration.** If a surface flux is introduced but no plant water uptake occurs, the first two terms of eq. 24 comprise

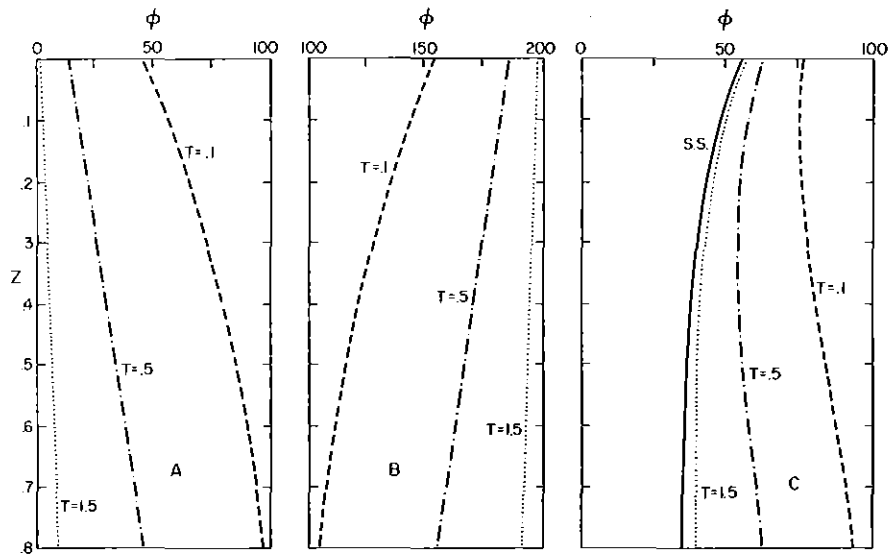


Fig. 1. Solution for drainage (A), infiltration (B) and infiltration with constant uptake (C). After LOMEN and WARRICK (1978)

the solution. If  $T^1 = \infty$ , then the solution is for a constant surface flux and equivalent to that of BRAESTER (1973). Such a result is shown in Fig. 1B for  $v = 3$  cm/day and  $\alpha\phi_\infty = 2$  cm/day.

Example 3 - Constant uptake. The third term  $\phi_{S,i}$  of eq. 24 is due to the sink function in effect at a given time. Fig. 1C depicts the matric flux potential with a water withdrawal function of the form  $a \exp(-bz)$  where  $a = 0.03$  day $^{-1}$  and  $b = 0.03$  cm $^{-1}$ ,  $v = 1.5$  cm/day and  $\phi_\infty = 100$  cm $^2$ /day. The curve for  $T = \infty$  is the steady-state limit and is identical with Warrick (1974, equation [1], table 1). These values of  $a$  and  $b$  result in 95% of the total uptake from the upper 100 cm of soil. The total uptake equals 1.0 cm/day.

Examples 4 and 5 - Cyclic uptake. Figs. 2 and 3 illustrate cyclic cases where the surface flux is assumed to be 3.0 cm/day for the first 24 h and 0.0 for the next 24 h. This pattern is continued with a 2-day cycle. The exponential sink function has  $b = 0.02$  cm $^{-1}$  for both cases with  $a = 0.02$  day $^{-1}$  in Fig. 2 while in Fig. 3 12-h active,  $a = 0.04$  day $^{-1}$ , 12-h inactive,  $a = 0.0$ , cycles are taken. The curves depicted are for depths  $Z = 0.1, 0.5$  and  $1.0$  ( $z = 13.3, 66.6$  and  $133$  cm) with  $\phi_\infty = 33.3$  cm $^2$ /day. As expected, the potential reaches

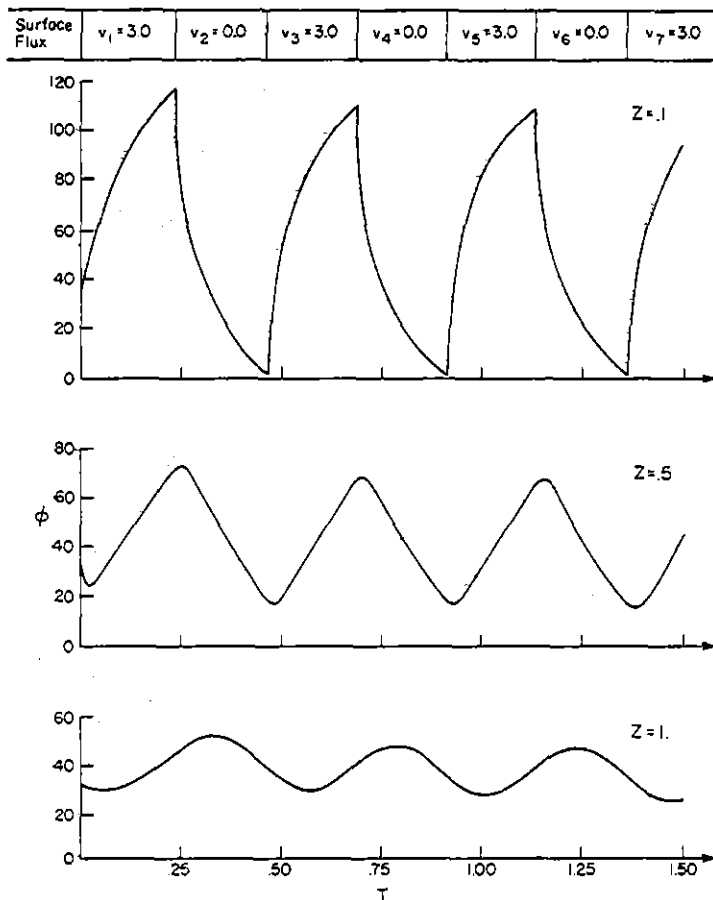


Fig. 2. Solution for cyclic input and constant uptake (exponential with depth). After LOMEN and WARRICK (1978)

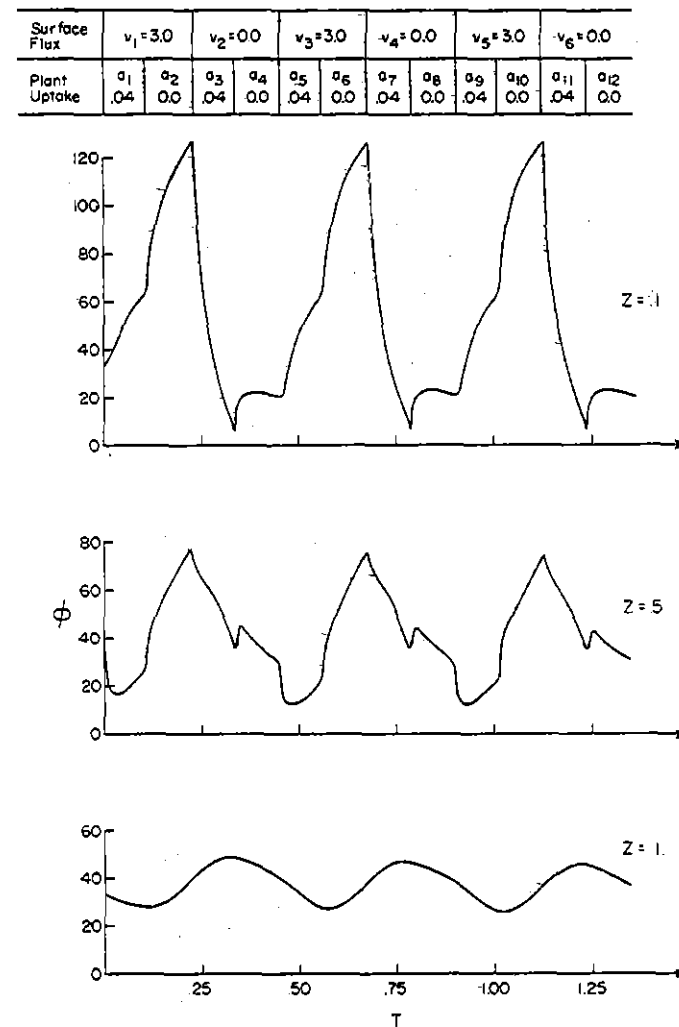


Fig. 3. Solution for cyclic input and cyclic uptake (exponential with depth). After LOMEN and WARRICK (1978)

its highest values when there is no water withdrawal, just before the surface flux stops. Notice also that for  $Z = 1.0$  there is very little difference between the two curves. The sink strength for Fig. 3 is twice that of Fig. 2 so the total uptake in the two cases is identical. The rapid fall in  $\phi$  after irrigation ceases has been demonstrated before in Fig. 3 of BEN-ASHER et al. (1978) but is even more pronounced here because of the water withdrawal (sink) term.

b. S h a l l o w w a t e r t a b l e

The Laplace transform will be used to obtain the solution of an problem defined by eq. (15) through (18). Some useful results about Laplace transforms are listed in Table 1 (from Chapter 13 of KUIPERS and TIMMAN (1963)).

Table 1. Properties of Laplace transforms

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$\bar{f}(p) = \mathbf{L}(f(t))$	$= \int_0^{\infty} f(t) e^{-pt} dt$
$\mathbf{L}^{-1}\{\mathbf{L}(f(t))\}$	$= \begin{cases} 0 & t < 0 \\ f(t) & t > 0 \end{cases}$
$\mathbf{L}(f'(t))$	$= p\mathbf{L}(f(t)) - f(0)$
$\mathbf{L}^{-1}\{\bar{f}(p) \bar{g}(p)\}$	$= \int_0^t f(t - \tau) g(\tau) d\tau$
$\mathbf{L}^{-1}\{\bar{f}(p - a)\}$	$= e^{at} f(t)$
$\mathbf{L}^{-1}\{e^{-bp} \bar{f}(p)\}$	$= f(t - b) H(t - b)$
$\mathbf{L}^{-1}\{(p - a)^{-1}\}$	$= e^{at}$

---

If we define the Laplace transform of  $\phi$  by eq. (26) and

$$\bar{\phi}(Z, p) = \mathbf{L}(\phi(Z, t)) = \int_0^{\infty} \phi(Z, t) e^{-pt} dt \quad (26)$$

use it on the differential eq., (15), we obtain;

$$p\bar{\phi} - g(Z) = \bar{\phi}_{ZZ} - 2\bar{\phi}_Z - (4/\alpha^2) \bar{S}(Z, p) \quad (27)$$

where

$$\bar{S}(Z, p) = \int_0^{\infty} S(Z, T) e^{-pT} dT \quad (28)$$

We can also use it to transform the boundary conditions (16) and (17) to:

$$-\bar{\phi}_Z + 2\bar{\phi} = 2\bar{v}/\alpha \quad \text{at } Z = 0 \quad (29)$$

$$\bar{\phi} = \bar{\phi}_0 \quad \text{at } Z = \alpha L/2 = Z_0 \quad (30)$$

Eq. (27) coupled with eqs. (29) and (30) defines a Sturm-Liouville boundary value problem. To obtain eigenvalues and eigenfunctions we need a transformation

$$\bar{\phi} = \bar{\chi} + Z\bar{\phi}_0/Z_0 + A(Z - Z_0) \quad (31)$$

to make the boundary conditions homogeneous. If we choose

$$A = (-\bar{\phi}_0/Z_0 - 2\bar{v}/\alpha)/(1 + 2Z_0) \quad (32)$$

the new dependent variable  $\bar{\chi}$  will satisfy

$$-\bar{\chi}_Z + 2\bar{\chi} = 0 \quad \text{for } Z = 0 \quad (33)$$

$$\bar{\chi} = 0 \quad \text{for } Z = Z_0 \quad (34)$$

as well as the differential equation

$$\begin{aligned} \bar{\chi}_{ZZ} - 2\bar{\chi}_Z - p\bar{\chi} = & (4/\alpha^2) \bar{S}(Z, p) - g(Z) + 2(\bar{\phi}_0/Z_0 + A) + \\ & + p[Z\bar{\phi}_0/Z_0 + A(Z - Z_0)] \end{aligned} \quad (35)$$

Since the right hand side does not contain the unknown function  $\bar{\chi}$ , we should seek solutions of the homogeneous differential equation

$$\bar{\chi}_{ZZ} - 2\bar{\chi}_Z + \lambda^2 \bar{\chi} = 0 \quad (36)$$

satisfying eqs. (33) and (34). Such solutions have been obtained by WARRICK and LOMEN (1977) as

$$\rho_n(Z) = 2 e^Z \sin \mu_n (Z_0 - Z) \left[ \frac{\mu_n}{2\mu_n Z_0 - \sin 2\mu_n Z_0} \right]^{\frac{1}{2}} \quad n = 1, 2, 3, \dots \quad (37)$$

with the  $\mu_n$  determined from

$$\mu_n = -\tan \mu_n Z_0, \quad n = 1, 2, 3, \dots \quad (38)$$

The eigenvalues are given by  $\lambda_n^2 = 1 + \mu_n^2$ ,  $n = 1, 2, 3, \dots$ .

Values of  $\mu_n$  appear in Table 4.19 of ABRAMOWITZ and STEGUN (1964).

The last factor in eq. (37) is chosen so the  $\rho_n(Z)$  form an orthonormal set, i.e.

$$\int_0^{Z_0} [\rho_n(Z)]^2 dZ = 1 \quad (39)$$

If we assume a solution of the form

$$\bar{\chi} = \sum_{n=1}^{\infty} c_n(p) \rho_n(Z) \quad (40)$$

and substitute it into eq. (35) we obtain

$$\sum_{n=1}^{\infty} c_n(p) [-1 - \mu_n^2 - p] \rho_n(Z) = (4/\alpha^2) \bar{S}(Z, p) - g(Z) + 2(\bar{\phi}_0/Z_0 + A) + p[Z\bar{\phi}_0/Z_0 + A(Z - Z_0)] \quad (41)$$

If we make use of the fact that the  $\rho_n(Z)$  are orthonormal on the interval  $(0, Z_0)$ , we obtain

$$c_n(p) = \frac{-1}{1 + \mu_n^2 + p} \int_0^{Z_0} \left\{ (4/\alpha^2) \bar{S}(Z', p) + g(Z') + 2(\bar{\phi}_0/Z_0 + A) + p[Z'\bar{\phi}_0/Z_0 + \Lambda(Z' - Z_0)] \right\} \rho_n(Z') dZ' \quad (42)$$

Combining eqs. (31), (32) and (40) yields the solution to our original problem as

$$\phi(Z, T) = \sum_{n=1}^{\infty} L^{-1}\{c_n(p)\} \rho_n(Z) + \frac{(2Z + 1)}{(2Z_0 + 1)} \phi_0 + \frac{(Z - Z_0)}{(2Z_0 + 1)} (2v(T)/\alpha) \quad (43)$$

Now we decompose  $c_n(p)$  into the following parts:

$$L^{-1}\{c_n(p)\} = \frac{-4}{\alpha^2} L^{-1}\left\{ \frac{1}{1 + \mu_n^2 + p} \int_0^{Z_0} \bar{S}(Z', p) \rho_n(Z') dZ' \right\} \quad (1)$$

$$+ e^{-(1+\mu_n^2)T} \int_0^{Z_0} g(Z') \rho_n(Z') dZ' \quad (2)$$

$$- L^{-1}\left\{ \frac{1}{1 + \mu_n^2 + p} [\bar{\phi}_0] \left[ \frac{2}{Z_0} - \frac{2}{Z_0(1 + 2Z_0)} \right] \int_0^{Z_0} \rho_n(Z') dZ' \right\} \quad (3)$$

$$- L^{-1}\left\{ \frac{1}{1 + \mu_n^2 + p} p\bar{\phi}_0 \left\{ \left[ \frac{1}{Z_0} - \frac{1}{Z_0(1 + 2Z_0)} \right] \int_0^{Z_0} Z' \rho_n(Z') dZ' + \frac{1}{1 + 2Z_0} \int_0^{Z_0} \rho_n(Z') dZ' \right\} \right\} \quad (4)$$

$$+ L^{-1}\left\{ \frac{1}{1 + \mu_n^2 + p} \bar{v} \left[ \frac{\int_0^{Z_0} \rho_n(Z') dZ'}{\alpha(1 + 2Z_0)} \right] \right\} \quad (5)$$

$$+ \mathcal{L}^{-1} \left\{ \frac{1}{1 + \mu_n^2 + p} p \bar{v} \right\} \left[ \frac{2 \int_0^{Z_o} Z' \rho_n(Z') dZ'}{\alpha(1 + 2Z_o)} - \frac{2Z_o \int_0^{Z_o} \rho_n(Z') dZ'}{\alpha(1 + 2Z_o)} \right] \quad (6)$$

(44)

In eq. (44)

- ① represents the effect of the sink
- ② represents the effect of the initial profile
- ③ + ④ together with  $(2Z + 1)/(2Z_o + 1) \phi_o$  from (43) represent the effect of the water table
- ⑤ + ⑥ together with the  $v(t)$  term in eq. (43) represent the effect of the surface flux (infiltration or evaporation)

If we now use the results of Table 1 we can rewrite some terms of eq. (44) as follows:

① becomes

$$- \frac{4}{\alpha} \left[ \int_0^T e^{-(1+\mu_n^2)\tau} \int_0^{Z_o} S(Z', \tau) \rho_n(Z') dZ' d\tau \right] e^{-(1+\mu_n^2)T}$$

③ becomes

$$\frac{-4}{1 + 2Z_o} \left[ \int_0^T e^{-(1+\mu_n^2)\tau} \phi_o(\tau) d\tau \int_0^{Z_o} \rho_n(Z') dZ' \right] e^{-(1+\mu_n^2)T}$$

④ becomes

$$\frac{-1}{1 + 2Z_o} \left[ \int_0^T e^{-(1+\mu_n^2)(\tau-T)} \mathcal{L}^{-1} \{ p \bar{\phi}_o \} d\tau \right] \left[ \int_0^{Z_o} 2 \int_0^{Z'} \rho_n(Z') dZ' + \int_0^{Z_o} \rho_n(Z') dZ' \right]$$



⑤ becomes

$$\frac{4}{\alpha(1+2Z_o)} \left[ \int_0^T e^{(1+\mu_n^2)\tau} v(\tau) d\tau \int_0^{Z_o} \rho_n(Z') dZ' \right] e^{-(1+\mu_n^2)T}$$

⑥ becomes

$$\frac{2 e^{-(1+\mu_n^2)T}}{\alpha(1+2Z_o)} \left[ \int_0^T e^{(1+\mu_n^2)\tau} \mathcal{L}^{-1}\{p\bar{v}\} d\tau \left[ \int_0^{Z_o} Z' \rho_n(Z') dZ' - Z_o \int_0^{Z_o} \rho_n(Z') dZ' \right] \right] \quad (45)$$

To make further progress let  $\phi_o(T) = \phi_o$ , a constant, giving ③ as

$$\frac{-4\phi_o(1 - e^{-(1+\mu_n^2)T})}{(1+2Z_o)(1+\mu_n^2)} \int_0^{Z_o} \rho_n(Z') dZ' \quad (46)$$

and ④ as

$$\frac{-\phi_o e^{-(1+\mu_n^2)T}}{(1+2Z_o)} \left[ 2 \int_0^{Z_o} Z' \rho_n(Z') dZ' + \int_0^{Z_o} \rho_n(Z') dZ' \right]$$

A second possibility is to let

$$\phi_o(T) = \begin{cases} \phi_1 & 0 < T < 1_T & \phi_T = 0 \\ \phi_2 & 1_T < T < 2_T \\ \vdots & \\ \phi_i & i-1_T < T < i_T \end{cases} \quad \text{with } \phi_1, \phi_2, \dots, \phi_i \text{ constants} \\ i = 1, 2, 3, \dots, I$$

This gives (3) as

$$\frac{-4}{1+2Z_0} \int_0^{Z_0} \rho_n(Z') dZ' \left[ \frac{e^{-(1+\mu_n^2)T}}{1+\mu_n^2} \left[ \sum_{k=1}^{i-1} \phi_k \left[ e^{(1+\mu_n^2)^k T} - e^{(1+\mu_n^2)^{k-1} T} \right] \right. \right. \\ \left. \left. + \phi_i \left[ e^{(1+\mu_n^2) T} - e^{(1+\mu_n^2)^{k-1} T} \right] \right] \right] \quad (47)$$

$i-1 T < T < T^i$

while (4) should be rewritten using

$$\mathbf{L}^{-1} \left\{ \frac{p \bar{\phi}_0}{1 + \mu_n^2 + p} \right\} = \mathbf{L}^{-1} \left\{ \frac{\sum_{k=1}^I \phi_n (e^{-p^{k-1} T} - e^{-p^k T})}{1 + \mu_n^2 + p} \right\}$$

$$= \sum_{k=1}^I \phi_k \left[ e^{-(1+\mu_n^2) (T^{k-1} T)} H(T - k-1 T) - e^{-(1+\mu_n^2) (T^k T)} H(T - k T) \right]$$

The usual function for  $v$  will model infiltration and evaporation, so

$$v(T) = \begin{cases} v_1 & 0 = T^0 < T < T^1 \\ v_2 & T^1 < T < T^2 \\ \vdots & \\ v_j & T^{j-1} < T < T^j \end{cases} \quad j = 1, 2, 3, \dots, J \quad (48)$$

$$p \mathbf{L}(v(t)) = \sum_{j=1}^J v_j \left[ e^{-p T^{j-1}} - e^{-p T^j} \right] \quad (49)$$

from (5) eq. (45) we need

$$\int_0^T e^{(1+\mu_n^2)\tau} v(\tau) d\tau = \sum_{k=1}^{j-1} v_k \left[ \frac{e^{(1+\mu_n^2)T^k} - e^{(1+\mu_n^2)T^{k-1}}}{1 + \mu_n^2} \right] + v_j \left[ \frac{e^{(1+\mu_n^2)T} - e^{(1+\mu_n^2)T^{j-1}}}{1 + \mu_n^2} \right] \quad T^{j-1} < T < T^j \quad (50)$$

from (6) eq. (44) we need

$$\mathbf{L}^{-1} \left\{ \frac{\bar{p}v}{1 + \mu_n^2 + p} \right\} = \mathbf{L}^{-1} \left\{ \frac{1}{1 + \mu_n^2 + p} \sum_{j=1}^J v_j \left[ e^{-pT^{j-1}} - e^{-pT^j} \right] \right\} = \sum_{j=1}^J v_j \left\{ \frac{e^{-(1+\mu_n^2)(T-T^{j-1})}}{H(T - T^{j-1})} - e^{-(1+\mu_n^2)(T - T^j)} \right\} H(T - T^j) \quad (51)$$

The only task left before one can use eq. (43) is to specify the sink function i.e. evaluate:

$$\int_0^T e^{(1+\mu_n^2)\tau} \int_0^{Z_0} S(Z', \tau) \rho_n(Z') dZ' d\tau$$

specify the surface fluxes,  $v_j$  (see eq. (48), and the value (or values) of  $\phi$  at depth  $z = L$ . The remaining integrals can be evaluated using DWIGHT (1961) as

$$\int_0^{Z_0} \rho_n(Z') dZ' = 2\mu_n N_n e^{Z_0} / (1 + \mu_n^2) \quad (52)$$

where

$$N_n^2 = \mu_n / (2\mu_n Z_0 - \sin(2\mu_n Z_0))$$

$$\int_0^{z_o} z' \rho_n(z') dz' = \frac{2\mu_n N_n}{1 + \mu_n^2} \left[ e^{z_o} \left( z_o - \frac{2}{1 + \mu_n^2} \right) + \cos(\mu_n z_o) \right] \quad (53)$$

while simple algebra gives

$$2 \int_0^{z_o} z' \rho_n(z') dz' + \int_0^{z_o} \rho_n(z') dz' = \frac{4\mu_n N_n}{1 + \mu_n^2} \left[ e^{z_o} \left( .5 + z_o - \frac{2}{1 + \mu_n^2} \right) + \cos(\mu_n z_o) \right] \quad (54)$$

$$\int_0^{z_o} z' \rho_n(z') dz' - z_o \int_0^{z_o} \rho_n(z') dz' = \frac{2\mu_n N_n}{1 + \mu_n^2} \left[ \frac{2e^{z_o}}{1 + \mu_n^2} - \cos(\mu_n z_o) \right] \quad (55)$$

#### 4. SUMMARY AND CONCLUSIONS

A solution of the linearized one-dimensional moisture flow equation has been derived including water uptake by plant roots in the presence of a shallow water table. The major assumptions are listed as follows:

- 1) Hydraulic conductivity vs. pressure head is given by

$K = K_0 \exp(\alpha \Psi)$ , (Note that considering  $K_0$  and  $\alpha$  to be empirical constants is equivalent (away from near saturation) to having  $K = K_{sat} \exp(\alpha(\Psi - \Psi_a))$  as proposed by RIJTEMA (1965), see also WESSELING and WIT (1966)).

- 2) The hydraulic conductivity is proportional to the volumetric water content. (This is a severe requirement and care must be taken in applying the results).

- 3) A shallow water table is present.

Eq. (46) represents the parts of the solution valid for a water table which stays at  $Z=L$  over the time period in question. Since I specified the value of  $\phi$  at  $Z=L$ , a fluctuating water table can be approximated by changing this value as a function time. The solution in this case will use data from eq. (47). If the fluctuating water table is to be accounted for exactly the method of solution would be much more complicated.

- 4) A surface flux is specified.

$v(t) > 0$  implies infiltration while  $v(t) < 0$  represents evaporation.

- 5) An arbitrary sink function can be specified as a function of depth and time. Having specified the sink function, the first integral in eq. (45) must be evaluated. For types of sink functions usually used this might result in lengthy computations but they are surely possible.

The solution as given by eq. (43) and (44), is quite lengthy but is completely determined. Time is not currently available to illustrate the behavior of this solution by means of examples. However graphical presentations of the solution for the same type of situations illustrated in Section IIIa are possible. The purpose of those illustrations (IIIa) was to show how linearized solutions can be used in modelling infiltration, drainage and periodic irrigation with diurnal water uptake.

In the future a computer program will be written to easily calculate the value of the solution for many values of  $Z$ . This is not needed as an approximation device, but as a tool to add, subtract, multiply and divide the many expressions in eqs. (43) and (44). Then examples, as mentioned previously, can be given which also include the effects of a shallow water table.

*Homogeen profiel ?*

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