Application of chance constraint programming to water system control

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ABSTRACT: In the Netherlands water system control limits the flooding of polders. Applying adaptive control can reduce the frequency of system failure. The estimation and predictions of the state of the polder system in on line application are uncertain. A Gaussian distribution represents the information about the uncertainties. The state of the system should be kept within predetermined boundaries. Exceedance probabilities of these boundaries are obtained using a linear model of the water system. Exceedance is accepted for small probabilities. The optimisation problem is called chance constraint. The objective is to minimise operational costs. A genetic algorithm is used to solve the optimisation problem. This method has been successfully applied to simulate control of a water reservoir with unknown inflow while taking account of the uncertainties involved.

1 INTRODUCTION

This study is concerned with the decrease in the frequency of water system failure by applying real-time control. The subject of the paper is the adaptive optimal control of a Dutch polder system. The Netherlands can be divided into a hilly region and a flat and low-lying region with each being of approximately the same size (figure 1). The low-lying region mainly consists of polders, the majority of which are below mean sea level. Pumping stations are used to drain these polders. The capability to control the water level in a polder depends mainly on the *pumping* ca-pacity and *storage* capacity (Schultz 1992). A couple of times per year an undesirable rise

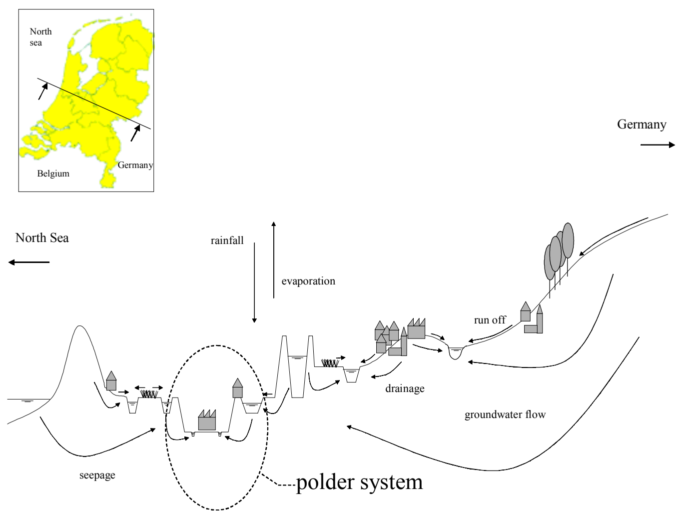

Figure 1: Schematic crossection of the hydrological system of the Netherlands from the North Sea to Germany.

in the.water level occurs, due to limited capacity. Flooding of polder sections occurs once every 10-50 years. These events will here be called *system failure*. The frequency of system failure or even better the design of pump and storage capacity depends on the social and economic acceptance of the effects. Research (Schilling 1991, Lobbrecht 1997) shows that usually not all the capacity available is used at the moment of failure and that unused capacity remains in the system. System failures can be reduced considerably by installing adaptive control systems, which reallocate available storage and discharge capacity before the moment of failure.

2 ADAPTIVE OPTIMAL CONTROL

In current practice, an operator in a pumping station controls the polder. Generally, he uses measurements and his experience to decide upon control actions. Lobbrecht (1997) suggested semi-automatic adaptive optimal control. An on-line computer application determines a control strategy for the near future. It runs on a centrally placed computer that receives monitoring data and sends the control actions supervised by the operator. The application:

- predicts the drainage discharge in the polder;
- determines the associated system behaviour and
- optimises the control strategy.

There are similarities in several optimal control methods (Martin-Garcia 1996). The strategy is determined by building and solving an optimisation problem, which incorporates a model of both the system and the objectives. In this article we follow the same approach. Within this approach, the evaluation of the state of the system is complicated by the uncertainty in the information. Estimates or predictions of the system behaviour incorporate uncertainties. Even measurements or not without uncertainties. Neglecting this aspect, Lobbrecht (1997) found a considerable decrease in the frequency of system failure by applying adaptive control.

3 UNCERTAINTY

3.1 *Uncertainty of state estimation*

One of the reasons why optimal control has not always been successful in practice is that the model of the system does not always represent the actual state of the polder system. When a discrepancy occurs between the estimated and actual state, it cannot be guaranteed that the solutions of the optimisation problem indeed reflect the optimal control. Filtering methods can help. Filtering methods compare the simulated state with real-time monitoring data. For example, based on this comparison the well-known Kalman filter (Kalman 1960, Anderson & Moore 1979) estimates the state of the system and yields a measure for the uncertainty in the information.

3.2 *Uncertainty of state prediction*

Predictions of the drainage discharge in a polder system can play an important role in the determination of the optimal control strategy. For polders that react slowly (e.g. a day or longer), it has been proven that only an indicative prediction of the drainage is needed to enhance system performance considerably (Lobbrecht 1997). For polders with rapid runoff characteristics (e.g. a couple of hours), the suitability of the control strategy depends much more on the accuracy of the prediction available (Botterhuis 1998). Weather forecasts are input for such drainage predictions. These forecasts are generally inaccurate and very dependent on the type of weather, the time of year, the intensity predicted and the period the prediction is made for (Jilderda 1995, van Meijgaard 1994, de Rooy 1992). The forecast accuracy is far from ideal to effectively use the anticipative control in fast-reacting polder systems (Lobbrecht 1997).

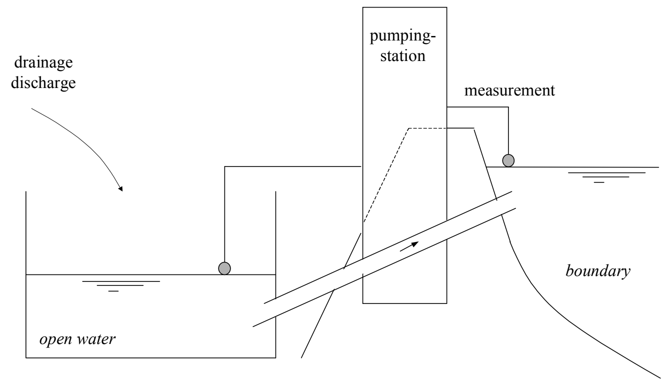

Figure 2: Simple model of a rural water system.

4 ADAPTIVE OPTIMAL CONTROL OF STOCHASTIC SYSTEMS

4.1 *Decision under uncertainty*

In semi-automatic adaptive optimal control the operator has to choose between different strategies for the near future. As previously stated, the accuracy of the state estimation should be known. The accuracy of the weather forecast should be supplied with the forecast itself. This permits the operator to use this assessment of the uncertainty in his decision. An optimisation method, which includes uncertainty, is described in this paper to advise the operator. In this approach, the amount of water stored in the polder and the drainage discharge are stochastic in nature. The solution of the optimisation gives the probability of the water level being at a certain level, as a result of control strategies and uncertain system behaviour.

4.2 *State estimation*

A polder system can be modelled as a system of two reservoirs: the open water reservoir and the boundary reservoir. These are connected as shown in figure 2. The open water reservoir represents a network of canals. The excess water is pumped to a boundary, which could represent the sea or a river. In this paper, the real system has been simplified in order to concentrate on the optimisation problem. The state estimation gives the initial values for the state prediction. This estimation is calculated with a Kalman filter. As a result the estimation of the current drainage discharge $X_Q\,(k|k)$ and the water level in the open water $X_h\,(k|k)$ have a Gaussian distribution.

$$\hat{X}_Q\,(k|k)\;\sim\;N\left(\mu_{X_Q}(k),\sigma_{X_Q}(k)^2\right),\;k=1,\ldots,n \qquad (1)$$

$$\hat{X}_h\,(k|k)\;\sim\;N\left(\mu_{X_h}(k),\sigma_{X_h}(k)^2\right),\;k=1,\ldots,n \qquad (2)$$

In these equations k is the time index.

4.3 *State prediction*

The drainage discharge $X_Q(k|k)$ is the outcome of the transformation of the rainfall into a discharge that drains from the agricultural fields to ditches, streams and canals. This discharge can be predicted with equation (3). In this equation l is the prediction index.

$$\hat{X}_Q\left(k+l|k\right)\;=\;\hat{X}_Q\left(k+l-1|k\right)+\Upsilon_{\Delta Q}(l),\;l=1,\ldots,m \qquad (3)$$

The predicted drainage $X_Q(k+l|k)$ is considered to be dependent on the previous discharge. The operator assumes the discharge for the coming period is equal to the discharge of the previous period. From one period to another the discharge is changed by a small increment $\Upsilon_{\Delta Q}(l)$. This adjustment is related to the measured or predicted amount of rainfall and evaporation. $\Upsilon_{\Delta Q}(l)$ is considered to be a random variable, Gaussian distributed and independent of the water level or discharge.

$$\Upsilon_{\Delta Q}(l)\;\sim\;N\left(\mu_{Y_{\Delta Q}},\sigma_{Y_{\Delta Q}}{}^2\right),\;l=1,\ldots,m \qquad (4)$$

The water level in the open water $X_h(k|k)$ can be predicted with equation (5). $X_h(k+l|k)$ is dependent on the previous value, the drainage discharge and the discharge of the pumping station $u(k)$.

$$\hat{X}_h\left(k+l|k\right)= \hat{X}_h\left(k+l-1|k\right)+\frac{\Delta t}{a}\hat{X}_Q\left(k+l-1|k\right)-\frac{\Delta t}{a}u\left(k+l\right) \qquad (5)$$

In equation (5) a stands for the surface area of the open water and Δt stands for the time step of the prediction.

4.4 *Control strategy optimisation*

4.4.1 *Chance constraint programming*

Chance constraint programming (CCP) is a type of mathematical programming, with stochastic elements incorporated into the constraint functions (Birge 1997). Charnes and Cooper (1959) introduced CCP as a means of handling randomness or uncer-tainty in data. CCP models the decision by specify-ing a confidence level α for each constraint. The constraint can only be violated $(1-\alpha)$ percent of the time. α is provided as an appropriate safety margin by the operator. The objective function in CCP may contain expected-value functions. However in our application the objective function is deterministic. The method has been used in many stochastic pro-gramming problems, for example inventory control, portfolio selection and reservoir management. The application for offline reservoir management (Meter et al. 1972, Arunkumar & Yeh 1973) is most closely related to our application for on-line polder system control.

4.4.2 *Objective function*

The aim of polder system control is to minimise the difference between the predicted water level $X(k+l|k)$ and the preferred water level in the open water reservoir against the lowest pumping costs. Therefore the objective function can be defined as:

$$\min\sum_{l=1}^{m}u(k+l) \qquad (6)$$

The restrictions concerning minimising the difference between the preferred and predicted water level will be included in the stochastic constraints.

4.4.3 *Stochastic state constraints*

For any given time-step, the water level should not be allowed to rise Δh above or below the preferred water level h_{pref}. In fact, the water level $X_h(k+l|k)$ should be equal to h_{pref}. These constraints are not deterministic due to the stochastic behaviour of $X_h(k+l|k)$; there is still a possibility that they will be broken. Therefore, incorporating the probabilities for which these constraints must hold, the constraints can be defined as stochastic constraints, as follows:

$$
\begin{aligned}
&\Pr\left\{\hat{X}_h(k+l|k) > h_{pref} + \Delta h = h_{upper}\right\} \le 1-\alpha_1 \\
&\Pr\left\{\hat{X}_h(k+l|k) \le h_{pref}\right\} \le \alpha_2 \\
&\Pr\left\{\hat{X}_h(k+l|k) > h_{pref}\right\} \le 1-\alpha_3 \\
&\Pr\left\{\hat{X}_h(k+l|k) \le h_{pref} - \Delta h = h_{lower}\right\} \le \alpha_4, l = 1,\ldots,m
\end{aligned} \tag{7}
$$

In other words, the first stochastic constraint states that the probability of the water level remaining below the upper boundary should be greater than α_1. The second states that the level should not fall below h_{pref} any more than α_2 percent of the time. The probability of the predicted level being less than the preferred level has to be greater than or equal to α_3. The last constraint states that the probability of falling below the lower bound has to be less than α_4. The probability distribution of $X_h(k+l|k)$ can be drawn, as shown in figure 3 below.

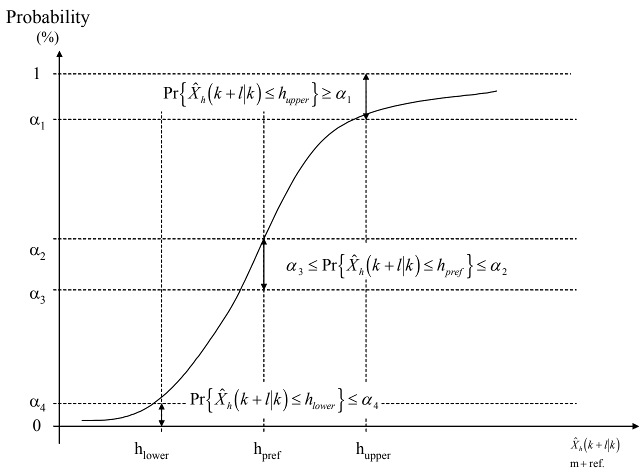

Figure 3: Probability distribution of the water level compared with the constraints.

It helps illustrate a clearer picture of how the values for probabilities are calculated. In order for the problem to have a feasible solution, the distribution has to cross the continuous lines between the arrows. In figure 3 it can be seen that the drawing of the probability distribution of $X_h(k+l|k)$ satisfies the first three water level constraints. It does not satisfy the fourth constraint; therefore this is not a feasible solution.

4.4.4 *Deterministic equivalents of stochastic constraints*

The stochastic constraints of equation 9 can be rewritten using equations 3, 4, 5 and state estimations 1 and 2.

$$
\hat{X}_h(k+l|k) = \hat{X}_h(k+l-1|k) + \frac{\Delta t}{a}\hat{X}_Q(k+l-1|k) - \frac{\Delta t}{a}u(k+l) \le h_{pref} \tag{10}
$$

With mathematical induction it can be proven (appendix 1) that this is equal to:

$$
\hat{X}_h(k|k) + l\frac{\Delta t}{a}\hat{X}_Q(k|k) + \frac{\Delta t}{a}\sum_{i=1}^{l}(l\text{-}i)Y_{\Delta Q}(i) - \frac{\Delta t}{a}\sum_{i=1}^{l}u(k+i) \le h_{pref} \tag{11}
$$

$$
\Pr\left\{\frac{\Delta t}{a}\sum_{i=1}^{l}u(k+i) \ge h_l(\cdot, h_{pref})\right\} \le \alpha_2 \tag{12}
$$

with

$$
\begin{aligned}
h_l(\cdot, h_{pref}) &= h_l\left(k, \hat{X}_Q(k|k), \hat{X}_h(k|k), Y_{\Delta Q}(k), h_{pref}\right) = \\
&\hat{X}_h(k|k) + l\frac{\Delta t}{a}\hat{X}_Q(k|k) + \frac{\Delta t}{a}\sum_{i=1}^{l}(l\text{-}i)Y_{\Delta Q}(i) - h_{pref}
\end{aligned}
$$

The mean and the variance of $X_Q(k|k)$, $X_h(k|k)$ and $Y_{\Delta Q}(k)$ can be used to define the deterministic equivalents of the stochastic constraints. It is assumed that these three stochastic variables are independent. Since the system is assumed to be linear, it is proven that the stochastic behaviour of $h_l(\cdot, h_{pref})$, $h_l(\cdot, h_{upper})$ and $h_l(\cdot, h_{lower})$ is Gaussian. The mean $\mu_{h_l(\cdot,h_{pref})}$ and standard deviation $\sigma_{h_l(\cdot)}$ of $h_k(\cdot, h_{pref})$ is derived in appendix 2 and written in equation (13). The mean of $h_k(\cdot, h_{upper})$ or $h_k(\cdot, h_{lower})$ can be derived when h_{pref} is substituted with h_{upper} or h_{lower}. The standard deviations of the functions $h_k(\cdot, h_{upper})$ and $h_k(\cdot, h_{lower})$ are equal to $\sigma_{h_l(\cdot)}$.

$$
h_k(\cdot, h_{pref}) \sim N\left(\mu_{h_l(\cdot,h_{pref})}(k), \sigma^2_{h_l(\cdot)}(k)\right) \tag{13}
$$

with

$$\mu_{h_l(\cdot,h_{pref})} = \mu_{X_h}(k) + l\frac{\Delta t}{a}\mu_{X_Q}(k) + \tfrac{1}{2}l(l-1)\frac{\Delta t}{a}\mu_{Y_{\Delta Q}} - h_{pref}$$

$$\sigma^2_{h_l(\cdot)} = \sigma_{X_h}(k)^2 + l^2\left(\frac{\Delta t}{a}\right)^2 \sigma_{X_Q}(k)^2 + \tfrac{1}{6}l(l-1)(2l-1)\left(\frac{\Delta t}{a}\right)^2 \sigma_{Y_{\Delta Q}}{}^2$$

As the probability levels can be chosen using figure 2, the deterministic equivalents can then be computed and the constraint (12) can be rewritten as:

$$\Pr\left\{h_l(\cdot,h_{pref}) \le \frac{\Delta t}{a}\sum_{i=1}^{l} u(k+i)\right\} = F_{h_l(\cdot,h_{pref})}\left(\frac{\Delta t}{a}\sum_{i=1}^{l} u(k+i)\right) \le \alpha_2$$

where $F_{h_l(\cdot,h_{pref})}(\cdot)$ is the distribution of $h_l(\cdot,h_{pref})$.

$$\frac{\Delta t}{a}\sum_{i=1}^{l} u(k+i) \le F_{h_l(\cdot,h_{pref})}{}^{-1}(\alpha_2)$$

$$\frac{\Delta t}{a}\sum_{i=1}^{l} u(k+i) \le \mu_{h_l(\cdot,h_{pref})} + \Phi^{-1}(\alpha_2)\sigma_{h_l(\cdot)}$$

with Φ is the cumulative distribution function of the standard normal distribution.

Deterministic equivalents of the chance constraint problem of equation 9 are written in equation 14.

$$\begin{aligned}
&\frac{\Delta t}{a}\sum_{i=1}^{l} u(k+i) - \mu_{h_l(\cdot,h_{upper})} - \Phi^{-1}(\alpha_1)\sigma_{h_k(\cdot)} \ge 0 \\
&-\frac{\Delta t}{a}\sum_{i=1}^{l} u(k+i) + \mu_{h_l(\cdot,h_{pref})} + \Phi^{-1}(\alpha_2)\sigma_{h_l(\cdot)} \ge 0 \\
&\frac{\Delta t}{a}\sum_{i=1}^{l} u(k+i) - \mu_{h_l(\cdot,h_{pref})} - \Phi^{-1}(\alpha_3)\sigma_{h_l(\cdot)} \ge 0 \\
&-\frac{\Delta t}{a}\sum_{i=1}^{l} u(k+i) + \mu_{h_l(\cdot,h_{lower})} + \Phi^{-1}(\alpha_4)\sigma_{h_l(\cdot)} \ge 0, \\
&l = 1,\ldots,m
\end{aligned} \tag{14}$$

4.4.5 *Deterministic pumping station constraints*

The pumping station restrictions involved in this problem relate to the capacity of the pumps. The feasible pumping rate is not continuous between zero and the maximum rate of the station $Max_{station}$. The possible options are discrete between no pumps in operation and all pumps in operation. In polder system control the possible strategies are limited. The decision problem can be simplified to a selection of the optimal strategy from a set of known strategies. We use a *genetic algorithm* (GA) to search within the population of feasible strategies. For this article we have used the work of Goldberg (1989) to design the algorithm. For an introduction and description of GAs the interested reader is referred to the book of Goldberg.

Each possible strategy is coded in a string where each position can have a value of 0 or 1. For example, a string with two positions has four possible codes (00, 10, 01 and 11). The code 00 could represent no pumps in operation, and the code 11 could represent three pumps in operation. The number of string positions per time step is related to the number of possible control actions per time step. Due to the string having a fi-nite number of positions, the possible strategies are limited. The string can only represent strategies that do not violate the pumping station constraints. All possible codes are linked to a strategy.

4.4.6 *Transformation into unconstrained maximisation problem*

The constraint problem in section 4.4.4 is transformed into an unconstrained problem by associating a penalty with all constraint violations. Because of the lack of capacity (section 1), it is not possible to keep the state of the system between the upper and lower boundary. The formulation of an unconstrained problem makes constraint violation possible within the search for the optimal strategy. Even when the drainage discharge causes the water level to rise above the upper boundary, the algorithm still finds an optimal strategy. The penalty for violating a constraint is calculated as follows:

$$g_{h_l(\cdot,h_{upper}),\alpha_1}\big(u(k+1),\ldots,u(k+l)\big) = \frac{\Delta t}{a}\sum_{i=1}^{l} u(k+i) - \mu_{h_l(\cdot,h_{upper})} - \Phi^{-1}(\alpha_1)\sigma_{h_l(\cdot)} \ge 0 \tag{15}$$

$$\Psi\left(g_{h_l(\cdot,h_{upper}),\alpha_1}(\cdot)\right) = \begin{cases} g_{h_l(\cdot,h_{upper}),\alpha_1}(\cdot)^2 & \textit{if } g_{h_l(\cdot,h_{upper}),\alpha_1}(\cdot) < 0 \\ 0 & \textit{otherwise} \end{cases} \tag{16}$$

Likewise, the penalty function $\Psi(g_{h_l(\cdot,h_{upper}),\alpha_1}(\cdot))$ is used to transform the other constraints in (14). This results in the following minimisation problem:

$$
\begin{aligned}
\min \sum_{l=1}^{m} u(k+l) &+ \lambda_1 \sum_{l=1}^{m} \Psi\Big(g_{h_l(\cdot,h_{upper}),\alpha_1}(\cdot) \Big) \\
&+ \lambda_2 \sum_{l=1}^{m} \Psi\Big(g_{h_l(\cdot,h_{pref}),\alpha_2}(\cdot) \Big) \\
&+ \lambda_3 \sum_{l=1}^{m} \Psi\Big(g_{h_l(\cdot,h_{pref}),\alpha_3}(\cdot) \Big) \\
&+ \lambda_4 \sum_{l=1}^{m} \Psi\Big(g_{h_l(\cdot,h_{lower}),\alpha_4}(\cdot) \Big)
\end{aligned} \tag{17}
$$

The unconstrained problem is considerably more suited to being handled by a GA (Goldberg 1989). A GA maximises an objective function that is guaranteed to be nonnegative. To transform a minimisation problem to a maximisation problem (suitable for GAs) the following objective transformation is commonly used (Goldberg 1989):

$$
\begin{aligned}
\min\ C^{*}_{\max} - \sum_{l=1}^{m} u(k+l) & \\
&+ \lambda_1 \sum_{l=1}^{m} \Psi\Big(C^{1}_{\max} - g_{h_l(\cdot,h_{upper}),\alpha_1}(\cdot) \Big) \\
&+ \lambda_2 \sum_{l=1}^{m} \Psi\Big(C^{2}_{\max} - g_{h_l(\cdot,h_{pref}),\alpha_2}(\cdot) \Big) \\
&+ \lambda_3 \sum_{l=1}^{m} \Psi\Big(C^{3}_{\max} - g_{h_l(\cdot,h_{pref}),\alpha_3}(\cdot) \Big) \\
&+ \lambda_4 \sum_{l=1}^{m} \Psi\Big(C^{4}_{\max} - g_{h_l(\cdot,h_{lower}),\alpha_4}(\cdot) \Big)
\end{aligned} \tag{18}
$$

$C^{(\cdot)}_{\max}$ is the largest possible value of the summation following it in equation (18). When the minimum of the constraint function (15) is smaller then zero, the square of this minimum is equal to maximum of the penalty function. These maximums have to be summed to get the largest possible value:

$$
C^{1}_{\max} = \sum_{l=1}^{m} \Big(-\mu_{h_l(\cdot,h_{upper})} - \Phi^{-1}(\alpha_1)\sigma_{h_l(\cdot)} \Big)^2
$$

$$
C^{2}_{\max} = \sum_{l=1}^{m} \Big(-l\frac{\Delta t}{a} Max_{station} + \mu_{h_l(\cdot,h_{pref})} + \Phi^{-1}(\alpha_2)\sigma_{h_l(\cdot)} \Big)^2
$$

$$
C^{3}_{\max} = \sum_{l=1}^{m} \Big(-\mu_{h_l(\cdot,h_{pref})} - \Phi^{-1}(\alpha_3)\sigma_{h_l(\cdot)} \Big)^2
$$

$$
C^{4}_{\max} = \sum_{l=1}^{m} \Big(-l\frac{\Delta t}{a} Max_{station} + \mu_{h_l(\cdot,h_{lower})} + \Phi^{-1}(\alpha_4)\sigma_{h_l(\cdot)} \Big)^2
$$

$$
l = 1, \dots, m
$$

The largest possible pumping costs $C^{*}_{\max}$ is equal to maximum pumping rate of the station multiplied by the amount of prediction steps m.

$$
C^{*}_{\max} = m\frac{\Delta t}{a} Max_{station}
$$

4.5 *Tests of algorithm*

To solve the unconstrained maximisation problem the GA code of Lindfield and Penny (1995) was used. For the simple problem in figure 2 it provides good results. In the test the mean and variances of $X_Q(k|k)$, $X_h(k|k)$ and $\Upsilon_{\Delta Q}(l)$ were chosen arbitrarily. Further testing of the algorithm is required. To compare the results with current polder system control, system identification methods have to be used to identify the expectations and variances. An Extended Kalman Filter can be used to increase the complexity and subsequently increase the accuracy of the state estimation and prediction. Such a filter will make it possible to translate uncertain rainfall forecasts into a stochastic prediction of system behaviour in the near future.

5 CONCLUSIONS

It is possible to model the control problem for a polder system as a chance constraint optimisation problem. The state prediction based on persistence of the drainage discharge can be translated to a stochastic dynamical system. An increase of the complexity of the model of a rural water system will make the prediction of the system state more accurate. Physically based modelling of the rainfall-runoff process can be translated with the Extended Kalman Filter to a stochastic dynamical system. Chance Constraint Optimisation of such estimations and/or predictions will make the method applicable to adaptive control for a broad variety of rural water systems.

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APPENDIX 1

Let $\hat{X}_Q(k+l-1|k) = \hat{X}_Q(k|k)$,
$\hat{X}_h(k+l-1|k) = \hat{X}_h(k|k)$, $l=1$, $k=1,\ldots,n$

as well as

$$\hat{X}_Q(k+l|k) = \hat{X}_Q(k+l-1|k) + \Upsilon_{\Delta Q}(l),$$

$$\hat{X}_h(k+l|k) =$$

then $\hat{X}_h(k+l-1|k) + \frac{\Delta t}{a}\hat{X}_Q(k+l-1|k) - \frac{\Delta t}{a}u(k+l)$,

$l = 1,\ldots,n$

$$\hat{X}_Q(k+l|k) = \hat{X}_Q(k|k) + \sum_{i=1}^{l}\Upsilon_{\Delta Q}(i)$$

$$\hat{X}_h(k+l|k) = \hat{X}_h(k|k) + l\frac{\Delta t}{a}\hat{X}_Q(k|k) + \frac{\Delta t}{a}\sum_{i=1}^{l}(l-i)Y_{\Delta Q}(i) - \frac{\Delta t}{a}\sum_{i=1}^{l}u(k+i),$$

$l = 1,\ldots,n$

The proof follows by mathematical induction. As the basis of the induction,

$$\hat{X}_Q(k+1|k) = \hat{X}_Q(k|k) + \Upsilon_{\Delta Q}(1)$$

$$\hat{X}_h(k+1|k) = \hat{X}_h(k|k) + \frac{\Delta t}{a}\hat{X}_Q(k|k) - \frac{\Delta t}{a}u(k+1)$$

Using the induction hypothesis it follows that,

$$\hat{X}_Q(k+l|k) = \hat{X}_Q(k|k) + \sum_{i=1}^{l-1}\Upsilon_{\Delta Q}(i) + \Upsilon_{\Delta Q}(l) = \hat{X}_Q(k|k) + \sum_{i=1}^{l}\Upsilon_{\Delta Q}(i)$$

$$\begin{aligned}
&\hat{X}_h(k+l|k)\\
&= \hat{X}_h(k+l-1|k) + \frac{\Delta t}{a}\hat{X}_Q(k+l-1|k) - \frac{\Delta t}{a}u(k+l)\\
&= \hat{X}_h(k|k) + (l-1)\frac{\Delta t}{a}\hat{X}_Q(k|k) + \frac{\Delta t}{a}\sum_{i=1}^{l-1}(l-1-i)\Upsilon_{\Delta Q}(i) - \frac{\Delta t}{a}\sum_{i=1}^{l-1}u(k+i)\\
&\quad + \frac{\Delta t}{a}\left(\hat{X}_Q(k|k) + \sum_{i=1}^{l-1}\Upsilon_{\Delta Q}(i)\right) - \frac{\Delta t}{a}u(k+l)\\
&= \hat{X}_h(k|k) + l\frac{\Delta t}{a}\hat{X}_Q(k|k) - \frac{\Delta t}{a}\sum_{i=1}^{l}u(k+i)\\
&\quad + \frac{\Delta t}{a}\left(\sum_{i=1}^{l}(l-1-i)\Upsilon_{\Delta Q}(i) - (l-1-l)\Upsilon_{\Delta Q}(l)\right)\\
&\quad + \frac{\Delta t}{a}\left(\sum_{i=1}^{l}\Upsilon_{\Delta Q}(i) - \Upsilon_{\Delta Q}(l)\right)\\
&= \hat{X}_h(k|k) + l\frac{\Delta t}{a}\hat{X}_Q(k|k) + \frac{\Delta t}{a}\sum_{i=1}^{l}(l-i)\Upsilon_{\Delta Q}(i)\\
&\quad - \frac{\Delta t}{a}\sum_{i=1}^{l}u(k+i)
\end{aligned}$$

APPENDIX 2

The mean of $n_l(\cdot, n_{pref})$ is

$$\mu_{h_l(\cdot, h_{pref})}$$

$$= E\left(\hat{X}_h(k|k) + l\frac{\Delta t}{a}\hat{X}_Q(k|k) + \frac{\Delta t}{a}\sum_{i=1}^{l}(l-i)\Upsilon_{\Delta Q}(i) - h_{pref} \right)$$

$$= \mu_{X_h}(k) + l\frac{\Delta t}{a}\mu_{X_Q}(k) + \frac{\Delta t}{a}E\left(\sum_{i=1}^{l}(l-i)\Upsilon_{\Delta Q}(i)\right) - h_{pref}$$

The expectation of the summation of the increment of the drainage discharge $\Upsilon_{\Delta Q}(l)$ can be calculated as follows:

$$E\left(\sum_{i=1}^{l}(l-i)\Upsilon_{\Delta Q}(i)\right) = \sum_{i=1}^{l}(l-i)E\left(\Upsilon_{\Delta Q}(i)\right) = \sum_{i=1}^{l}(l-i)\mu_{\Upsilon_{\Delta Q}}$$

$$= \tfrac{1}{2}l(l-1)\mu_{\Upsilon_{\Delta Q}}$$

The last step in this denotation can be proven by mathematical induction. As the basis of the induction,

$$\sum_{i=1}^{1}(1-i)\mu_{\Upsilon_{\Delta Q}} = \tfrac{1}{2}1(1-1)\mu_{\Upsilon_{\Delta Q}} = 0$$

Using the induction hypotheses it follows that,

$$\sum_{i=1}^{l+1}(l+1-i) = \sum_{i=1}^{l+1}(l-i) + \sum_{i=1}^{l+1}1$$

$$= \sum_{i=1}^{l}(l-i) + (l-l-1) + (l+1) = \sum_{i=1}^{l}(l-i) + l$$

$$= \tfrac{1}{2}l(l-1) + l = l\left(\tfrac{1}{2}(l-1)+1\right) = l\left(\tfrac{1}{2}l + \tfrac{1}{2}\right)$$

$$= \tfrac{1}{2}l(l+1) = \tfrac{1}{2}(l+1)(l+1-1)$$

The variance of $n_l(\cdot, n_{pref})$ is

$$\sigma_{h_l(\cdot, h_{pref})}$$

$$= VAR\left(\hat{X}_h(k|k)\right) + VAR\left(l\frac{\Delta t}{a}\hat{X}_Q(k|k)\right)$$

$$+ VAR\left(\frac{\Delta t}{a}\sum_{i=1}^{l}(l-i)\Upsilon_{\Delta Q}(i)\right) - VAR\left(h_{pref}\right)$$

$$= \sigma_{X_h}(k)^2 + l^2\left(\frac{\Delta t}{a}\right)^2\sigma_{X_Q}(k)^2$$

$$+ \left(\frac{\Delta t}{a}\right)^2 VAR\left(\sum_{i=1}^{l}(l-i)\Upsilon_{\Delta Q}(i)\right)$$

The variation of the summation of the increment $\Upsilon_{\Delta Q}(l)$ can be calculated as follows:

$$VAR\left(\sum_{i=1}^{l}(l-i)\Upsilon_{\Delta Q}(i)\right) = \left(\sum_{i=1}^{l}(l-i)\right)^2 \sigma_{\Upsilon_{\Delta Q}}{}^2$$

$$= \tfrac{1}{6}l(l-1)(2l-1)\sigma_{\Upsilon_{\Delta Q}}{}^2$$

The last step in this denotation can also be proven by mathematical induction. As the basis of the induction,

$$\left(\sum_{i=1}^{1}(l-1)\right)^2 = \tfrac{1}{6}1(1-1)(2\cdot 1-1) = 0$$

Using the induction hypothesis it follows that,

$$\left(\sum_{i=1}^{l}(l-i)\right)^2 = \tfrac{1}{6}(l+1)(l+1-1)(2(l+1)-1)$$

$$= \tfrac{1}{6}l(l-1)(2(l+1)-1) + \tfrac{1}{6}l(2(l+1)-1)$$

$$+ \tfrac{1}{6}(l+1-1)(2(l+1)-1)$$

$$= \tfrac{1}{6}l(l-1)(2l-1) + \tfrac{2}{6}l(l-1) + \tfrac{2}{6}(2(l+1)-1)$$

$$= \tfrac{1}{6}l(l-1)(2l-1) + \tfrac{2}{6}l^2 + \tfrac{4}{6}l^2 + \tfrac{2}{6}l$$

$$= \left(\sum_{i=1}^{l}(l-i)\right)^2 + l^2 = \left(\sum_{i=1}^{l+1}(l+1-i)\right)^2$$