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# Methods for computing Nash equilibria of a location-quantity game*

*M. Elena Sáiz, Eligius M.T. Hendrix*

**Abstract**

A two stage model is described where firms take decisions on where to locate their facility and on how much to supply to which market. In such models in literature, typically the market price reacts linearly on supply. Often two competing suppliers are assumed or several that are homogeneous, i.e. their cost structure is assumed to be identical. The focus of this paper is on developing methods to compute equilibria of the model where more than two suppliers are competing that each have their own cost structure, i.e. they are heterogeneous. Analytical results are presented with respect to optimality conditions for the Nash equilibria in the two stages. Based on these analytical results, algorithms are developed to find equilibria. Numerical cases are used to illustrate the results and the viability of the algorithms. The method finds an improvement of a result reported in literature.

## 1 Introduction

Many studies in literature describe a so-called non-cooperative game where competing firms decide on production locations and supply quantities to markets. To make a game theoretic analysis tractable, often a limited number of suppliers are considered, or alternatively homogeneous firms and markets are assumed. We focus on situations where companies can be as well similar as not similar. In supply chains, farm cooperatives, etc., many decisions appear in which preferences cannot be assumed to be homogeneous. Also symmetric behaviour, finite strategy set or a two or few actors setting are strong assumptions in literature. Decisions are influenced by differences on prices or cost ("player" depending) between actors and between the location of the facilities. Our focus is on constructing solution methods for games in which players are: asymmetric, heterogeneous and facing multiple decisions in several stages.

Cournot (1838) introduced the idea of a Cournot oligopoly equilibrium, where two firms compete on the same market. Due to price reaction of the market on the total quantity offered, a price equilibrium

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appears. Hotelling (1929) added the idea of having a freedom in choice of location, where the possible location area is a simple line in between the markets. A generally applicable concept is that of a Nash equilibrium (Nash (1951)) which is defined by the situation where none of the firms (players) is better off by changing its current (equilibrium) strategy. Because choice of location is usually prior to decision on quantities, in the model under consideration, this concept is applied to a supply chain study where two nested levels of decisions are at stake: that of supply quantity and location choice. The corresponding two-stage solution is called a *subgame perfect Nash equilibrium*.

The basis of the model has been introduced by Bulow et al. (1985) who consider a game with 2 markets and 2 firms. Later, Farrell and Shapiro (1990) studied a game on quantity decisions with one market and $n$ firms where decisions are simultaneous and products are homogeneous. Labbé and Hakimi (1991) consider a two-stage location-quantity simultaneous game with $m$ markets and 2 firms with linear demand. Sarkar et al. (1997) extend these results considering a 2-stage static and simultaneous game with $m$ markets and $n$ firms in a network. They only consider a case with a fixed number of firms entering in the market, i.e., the quantities offered by each firm in all markets are strictly positive. Rhim et al. (2003) extend the work of Sarkar et al. (1997) by considering free entry (simultaneous and sequential) with symmetric cost (site specific) and capacity limitations. Their setting is a 3-stage game with $m$ markets and $n$ firms with production capacity and quantity decisions, and final stage is the location choice in a network. Recently, Dorta-González et al. (2004) apply the Stakelberg equilibrium in a two-stage non-cooperative Cournot game with location and quantity choice with $n$ markets located at the vertices of a network and $r$ firms. They use the Nash equilibrium concept in the location stage. In all of these studies, cases applied are small, most are symmetric, and no computational experience is reported.

This paper extends the studies of Sarkar et al. (1997), Rhim et al. (2003). Free entry is possible as in Rhim et al. (2003), i.e., the number of firms entering the markets is not known in advance, but in our case costs are asymmetric (firm-specific). We provide conditions for the supplying decisions (second-stage of the game). Moreover, as firms will be affected by the timing and level of entry on the market, properties on how to determine the size of the market are derived. Another difference with the study of Rhim et al. (2003) is the procedure on how to find the equilibrium of the game. We consider not only the possibility of leaving a market but also the possibility of that the supplier moves its facility to another location. Doing so, a firm has to re-think the quantity decision on how much to supply to which markets. By applying the method in the cases of Sarkar et al. (1997), a mistake is found in the outcome given in their study. Their reported possible equilibrium appears to be wrong as is shown in Section 4. Moreover, a sequential analysis is followed in this paper. It appears that starting with the cheapest firm, one can successively arrive at the size of each of the markets. When market sizes are determined, the optimum quantities each supplier delivers to each markets they enter can be computed.

In Section 2, a model is outlined consisting of a non-cooperative game where quantity decisions and location decisions take place. Furthermore, theoretical results concerning the optimum decisions in these models are derived. In Section 3, methods for computing the Nash equilibria on quantity-location decisions and for computing the size of a market are described. Numerical illustrations of model and methods can be found in Section 4. Finally, Section 5 discusses the conclusions.

## 2 Location-Quantity game: problem formulation

The model describes a two-stage non-cooperative game. In the first stage of the game, firms take a simultaneous decision about where to locate a supplying facility in a network, i.e., each firm chooses a location-strategy without knowledge of the strategy chosen by the other firms. In the second stage of the game, firms decide about the quantity to be produced at these facilities and how much to supply to each market.

The model on quantity decisions and location choice is described by the following notation. Firms are denoted by an index $i \in N = \{1, \dots, n\}$ and markets are denoted by an index $h \in M = \{1, \dots, m\}$ each demanding a quantity of a good, depending on its price. In game theory, usually a linear price reaction model is assumed. We will follow this tradition. The demand is fulfilled by the supply of a quantity $Q_{ih}$ from the facility of firm $i$ to market $h$. The location $x_i$ of the facility of firm $i$ determines its marginal production cost $c_i(x_i)$. The regional dispersion effect comes in when every market appears to be situated at one location and, an important assumption, each supply firm can open a facility at only one of the locations. The relations are formalised as follows.

Let $G = (V, E)$ be an undirected graph with $V$ and $E$ as its sets of nodes and edges respectively, $|V| = m$. Given two nodes $v_i, v_j \in V$, $d(v_i, v_j)$ is the length of a shortest (with respect to the sum of edge lengths) path on $G$ connecting $v_i$ and $v_j$. There are $m$ markets located each at one node on the network ; there are $n$ firms that open a facility each at one node with $n \leq m$. Let $x_i \in V = \{v_1, \dots, v_m\}$ be the location decision by firm $i$ on the network. The cost of establishing a facility by firm $i$ at $x_i$ is $w(x_i) \geq 0$. The quantity decision matrix $\underline{Q}$ for all firms and all markets is given by:

$$\underline{Q} = \begin{pmatrix} Q_{11} & \dots & Q_{1h} & \dots & Q_{1m} \\ \dots & \dots & \dots & \dots & \\ Q_{i1} & \dots & Q_{ih} & \dots & Q_{im} \\ \dots & \dots & \dots & \dots & \\ Q_{n1} & \dots & Q_{nh} & \dots & Q_{nm} \end{pmatrix}$$

where the sum of a row indicates the quantity supply by firm $i$ over all markets $h \in \{1, \dots, n\}$, $s_i =$

$\sum_{h=1}^{m} Q_{ih}$ and the sum of a column indicates the quantity supplied by all firms $i \in \{1, \dots, n\}$ to market $h$, $q_h = \sum_{i=1}^{n} Q_{ih}$. The price $p_h(q_h)$ at market $h$ is assumed to depend on the quantity according to the relation:

$$p_h(q_h) = \max\{0, \alpha_h - \beta_h q_h\}, \quad q_h \geq 0 \tag{1}$$

with price parameters $\alpha_h, \beta_h \geq 0$. The price at market $h$ depends on the quantity decision of all firms that supply to market $h$.

The $n$ firms interact over two stages. In the first stage, firms simultaneously choose the locations of their facilities, $x_i$, $i = 1, \dots, n$. In the second stage, depending on the location decisions $x_i$, firms choose quantities $Q_{ih}$ to be supplied to markets, which results in the quantity decision matrix $\underline{Q}$. The profit firm $i$ wants to maximise is denoted by $\pi_i(x_i, \underline{Q})$. A strategy for firm $i$ at market $h$, $[x_i, Q_{ih}]$, comprises a choice of $x_i$ for stage 1 and a choice of $Q_{ih}$ for stage 2; $[x_i, Q_{i\cdot}]$ for all markets, where $Q_{i\cdot}$ denotes the row vector $(Q_{i1}, \dots, Q_{im})$.

The game is solved by backward induction. First the second stage is solved. Firm $i$ chooses optimally the vector of quantities $Q_{i\cdot} = (Q_{i1}, \dots, Q_{im})$, based on what the others deliver and depending on the chosen location $x_i$:

$$Q_{i\cdot}^* = \arg\max_{Q_{i\cdot}} \pi_i(x_i, \underline{Q^*}(x)) \tag{2}$$

By induction it becomes a one stage problem where matrix $\underline{Q^*}$ is defined for each location vector $x_1, \dots, x_n$. Now firm $i$ chooses a location strategy $x_i^*$ such that:

$$x_i^* = \arg\max_{x_i} \pi_i(x_i, \underline{Q}^*(x))$$

quantity. We model price and quantity decisions as a Cournot quantity game. The unit transportation cost between the location $x_i$ of the facility of firm $i$ and location $v_h$ of market $h$, is represented by $t_{ih} = T(d(x_i, v_h))$, where $T$ is concave and increasing in the distance [1]. The total cost of the location and supply decision of firm $i$ is given by:

$$\begin{aligned} TC_i(x_i, Q_{i\cdot}) &= \sum_{h=1}^{m} t_{ih} Q_{ih} + c_i(x_i) s_i + w(x_i) \\ &= \sum_{h=1}^{m} t_{ih} Q_{ih} + c_i(x_i) \sum_{h=1}^{m} Q_{ih} + w(x_i) \\ &= \sum_{h=1}^{m} (t_{ih} + c_i(x_i)) Q_{ih} + w(x_i) \end{aligned}$$

[1] This assumption also appears on the studies of Lederer and Thisse (1990), Labbé and Hakimi (1991), Sarkar et al. (1997) among others

For the convenience of notation we represent the total unit cost of firm $i$ at market $h$ by

$$TCu_{ih} = t_{ih} + c_i(x_i).$$

Profit is denoted by $\pi_i$ and defined as:

$$\pi_i(x_i, \underline{Q}) = \sum_{h=1}^{m} p_h(q_h)Q_{ih} - TC_i(x_i, Q_{i\cdot})$$

Price at market $h$ is given by equation (1). Firms determine quantities for the markets to maximize profit. Substituting the price reaction of the markets into (3) gives

$$\pi_i(x_i, \underline{Q}) = \sum_{h=1}^{m} (\alpha_h - \beta_h \sum_{j=1}^{n} Q_{jh})Q_{ih} - TC_i(x_i, Q_{i\cdot}) \tag{3}$$

Table 1 summarises the notation used.

Tab. 1: Notation

| | |
|---|---|
| $N, M$ | Set of firms and markets, respectively |
| $x_i$ | Location of firm $i$ |
| $v_h$ | Location of market $h$ |
| $Q_{ih}$ | Quantity supply by firm $i$ at market $h$ |
| $Q_{i\cdot} = (Q_{i1}, \ldots, Q_{im})$ | Quantity decision vector for firm $i$ |
| $s_i = \sum_{h \in M} Q_{ih}$ | Total quantity supplied by firm $i$ |
| $q_h = \sum_{i \in N} Q_{ih}$ | Total quantity supply at market $h$ |
| $\alpha_h, \beta_h$ | Price parameters |
| $p_h(q_h) = \alpha_h - \beta_h q_h$ | Price at market $h$ |
| $t_{ih} = T(d(x_i, v_h))$ | Unit transportation cost |
| $w(x_i)$ | Cost of establishing a centre at $x_i$ |
| $c_i(x_i)$ | Marginal production cost |
| $TCu_{ih} = t_{ih} + c_i(x_i)$ | Total unit cost |
| $TC_i(x_i, Q_{i\cdot})$ | Total cost of location and supply |
| $\pi_i(x_i, \underline{Q})$ | Profit for firm $i$ depending on location and quantities |

In Section 2.1, properties are given of the equilibrium prices and quantities depending on the location decision of the firms. Section 2.2 describes the criterion for selecting optimal location decisions, $x^*$, based on the optimal quantity decisions, $\underline{Q}^*(x)$.

## 2.1 Quantity decision

The Nash equilibrium is the solution concept used in the quantity-stage of the game. From (2), the Nash column of the $\underline{Q}$ matrix can be determined by an iterative process. Nash equilibrium quantities shipped by firm $i$ to market $h$ follow from the first order condition optimising (3) over $Q_{ih}$:

$$Q^*_{ih} = \max\left\{0, \frac{\alpha_h - \beta_h \sum_{j=1,j\neq i}^{n} Q^*_{jh} - t_{ih} - c_i(x_i)}{2\beta_h}\right\} \tag{4}$$

At equilibrium, one can distinguish for each market $h$ a group of active firms, $A_h$, that deliver to $h$; $Q^*_{ih} > 0$ for $i \in A_h$ and $Q^*_{ih} = 0$ for $i \in \overline{A}_h = N \backslash A_h$.

Proposition 1 provides the equilibrium quantity for each firm $i \in A_h$.

Proposition 1: Let $A_h$ be the set of firms which supply market $h$, $|A_h| = k_h$. The positive equilibrium quantities are given by

$$\begin{aligned} Q^*_{ih} &= \frac{\alpha_h - k_h(c_i(x_i) + t_{ih}) + \sum_{j \in A_h \backslash \{i\}} (c_j(x_j) + t_j)}{(k_h+1)\beta_h} \\ Q^*_{ih} &> 0 \quad \forall i \in A_h \end{aligned} \tag{5}$$

$Q^*_{ih}$ depends on production and transportation cost of the active suppliers.

**Proof.**

From equation (4) follows for $i \in A_h$

$$Q_{ih} = \frac{\alpha_h - t_{ih} - c_i(x_i)}{2\beta_h} - \frac{1}{2} \sum_{j \in A_h \backslash \{i\}} Q_{jh} \tag{6}$$

Let $a_{ih} = \frac{\alpha_h - t_{ih} - c_i(x_i)}{2\beta_h}$, then (6) can be written as

$$Q_{ih} = a_{ih} - \frac{1}{2} \sum_{j \in A_h \backslash \{i\}} Q_{jh}$$

In vector notation

$$\begin{pmatrix} Q_{1h} \\ \ldots \\ Q_{ih} \\ \ldots \\ Q_{k_h h} \end{pmatrix} = \begin{pmatrix} a_{1h} \\ \ldots \\ a_{ih} \\ \ldots \\ a_{k_h h} \end{pmatrix} - \frac{1}{2}\left[\underline{1_{k_h}}\,\underline{1'_{k_h}} - \underline{I}\right] \begin{pmatrix} Q_{1h} \\ \ldots \\ Q_{ih} \\ \ldots \\ Q_{k_h h} \end{pmatrix}$$

$$\underline{Q}_h = \underline{a}_h - \frac{1}{2}\left[\underline{1_{k_h}}\,\underline{1'_{k_h}} - \underline{I}\right] \underline{Q}_h$$

where $\underline{1_{k_h}}$ is the all ones vector and $\underline{I}$ is the $k_h \times k_h$ unit matrix. By linear algebra,

$$
\begin{aligned}
\underline{I}\underline{Q}_h &= \underline{a}_h - \frac{1}{2}\left(\underline{1_{k_h} 1_{k_h}}' - \underline{I}\right)\underline{Q}_h \\
\underline{a}_h &= \frac{1}{2}\left[\underline{1_{k_h} 1_{k_h}^{'}} + \underline{I}\right]\underline{Q}_h
\end{aligned}
$$

$$
\underline{Q}_h = \underline{B}^{-1}\underline{a}_h \tag{7}
$$

where $\underline{B}$ is the $k_h \times k_h$ matrix

$$
\underline{B} = \frac{1}{2}\left[\underline{1_{k_h} 1_{k_h}^{'}} + \underline{I}\right]
$$

having the following form,

$$
B = \begin{pmatrix}
1 & \dots & 1/2 & \dots & 1/2 \\
\dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & 1/2 & \dots \\
1/2 & \dots & 1 & \dots & 1/2 \\
\dots & 1/2 & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots \\
1/2 & \dots & 1/2 & \dots & 1
\end{pmatrix}
$$

The inverse matrix can be derived to be,

$$
\underline{B}^{-1} = 2\left(\underline{I} - \frac{1}{k_h+1}\underline{1_{k_h} 1_{k_h}^{'}}\right)
$$

$$
B^{-1} = 2\begin{pmatrix}
k_h/k_h+1 & \dots & -1/k_h+1 & \dots & -1/k_h+1 \\
\dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & -1/k_h+1 & \dots \\
-1/k_h+1 & \dots & k_h/k_h+1 & \dots & -1/k_h+1 \\
\dots & -1/k_h+1 & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots \\
-1/k_h+1 & \dots & -1/k_h+1 & \dots & k_h/k_h+1
\end{pmatrix}
$$

The equivalence of equations (4) and (5) for each market $h$ now follows from (7):

$$
\begin{aligned}
\underline{Q}_h &= \underline{B}^{-1}\underline{a}_h = 2\left[\underline{I} - \frac{1}{k_h+1}\underline{1_{k_h}1_{k_h}^{'}}\right]\underline{a_h} \\
&= 2\begin{pmatrix}
k_h/k_h+1 & \dots & -1/k_h+1 & \dots & -1/k_h+1 \\
\dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & -1/k_h+1 & \dots \\
-1/k_h+1 & \dots & k_h/k_h+1 & \dots & -1/k_h+1 \\
\dots & -1/k_h+1 & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots \\
-1/k_h+1 & \dots & -1/k_h+1 & \dots & k_h/k_h+1
\end{pmatrix}
\begin{pmatrix}
a_{1h} \\ \dots \\ a_{ih} \\ \dots \\ a_{k_h h}
\end{pmatrix}
\end{aligned}
$$

and for each firm $i$ we obtain:

$$
\begin{aligned}
Q_{ih} &= \frac{2k_h}{k_h+1}a_{ih} - \frac{2}{k_h+1}\sum_{j\in A_h\setminus\{i\}} a_{jh} \\
&= \frac{2k_h}{k_h+1}\frac{\alpha_h - t_{ih} - c_i(x_i)}{2\beta_h} - \frac{2}{k_h+1}\sum_{j\in A_h\setminus\{i\}}\frac{\alpha_h - (t_{jh}+c_j(x_j))}{2\beta_h} \\
&= \frac{2k_h\alpha_h - 2k_h(t_{ih}+c_i(x_i))}{2(k_h+1)\beta_h} - \frac{(k_h-1)\alpha_h}{(k_h+1)\beta_h} + \sum_{j\in A_h\setminus\{i\}}\frac{t_{jh}+c_j(x_j)}{(k_h+1)\beta_h} \\
&= \frac{\alpha_h - k_h(t_{ih}+c_i(x_i)) + \sum_{j\in A_h\setminus\{i\}}(t_{jh}+c_j(x_j))}{(k_h+1)\beta_h}
\end{aligned}
$$

which corresponds to equation (5). ∎

Consequently, the total quantity supplied to market $h$ is:

$$
q_h^* = \sum_{j\in A_h} Q_{jh}^* = \frac{1}{(k_h+1)\beta_h}\left(k_h\alpha_h - \sum_{j\in A_h}(c_j(x_j)+t_{jh})\right) \tag{8}
$$

which means that higher average marginal cost and transportation costs decrease the total quantity supplied. The optimal price at each market can now be derived by substituting (8) into (1):

$$
p_h^* = \frac{1}{k_h+1}\left(\alpha_h + \sum_{j\in A_h}(c_j(x_j)+t_{jh})\right) \tag{9}
$$

Optimal prices at each market proportionally rise with average marginal cost and transportation cost over the firms supplying the market. Higher costs leads to a higher equilibrium price and lower costs leads to higher quantity supplied.

In order to have any delivery at market $h$ in (8) a necessary condition is that $\exists j \in N$ such that $TCu_{jh} < \alpha_h$.

From Proposition 2 also follows the result for the symmetric case.

Theorem 1: Let unitary costs be symmetric (the same) for all the suppliers at market $h$. If unitary costs are lower than $\alpha_h$, all the suppliers will enter the market.

**Proof.**

Let the $n$ firms entering the market have the same cost, $TCu_{1h} = TCu_{2h} = \ldots = TCu_{nh} = Cu_h$ and $Cu_h < \alpha_h$, the optimal quantity and price can be derived from equation (5) and (9),

$$Q^*_{ih} = \frac{\alpha_h - nTCu_{ih} + \sum_{j=1,j\neq i}^{n} TCu_{jh}}{(n+1)\beta_h} = \frac{\alpha_h - Cu_h}{(n+1)\beta_h}$$

and

$$p^*_h = \alpha_h - \beta_h q_h = \alpha_h - \beta_h n \frac{\alpha_h - Cu_h}{(n+1)\beta_h} = \frac{\alpha_h + nCu_h}{n+1}$$

The necessary condition for a firm to serve to market $h$ is:

$$\begin{aligned} & Cu_h < p_h = \frac{\alpha_h + nCu_h}{n+1} \\ & \Leftrightarrow (n+1)Cu_h < \alpha_h + nCu_h \\ & \Leftrightarrow Cu_h < \alpha_h \qquad (10) \\ & \Rightarrow Q_{ih} > 0 \quad \forall i \in N \end{aligned}$$

which means that the $n$ firms will enter and supply to market $h$. ■

Corolary 2: Let unitary costs be symmetric (the same) for all the suppliers at market $h$. If unitary costs are higher than $\alpha_h$, no suppliers will enter the market.

From Proposition 1 can also be derived when a firm would be interested to enter market $h$, given that a set of firms $A_h$ is already delivering.

Proposition 2: Let $A_h$ be a set of firms supplying market $h$. A firm $i$ is interested in supplying market $h$ if $TCu_{ih} < p_h$.

**Proof.**

Follows from the partial derivative of $\pi_i$ with respect to $Q_{ih}$ for $Q_{ih} = 0$. ■

Proposition 3: In the optimum $\underline{Q}^*$, $\forall i \in A_h$, $TCu_{ih} < p_h$.

**Proof.**

From equation (9) the equilibrium price is

$$\begin{aligned} p_h^* &= \frac{1}{k_h+1}\left(\alpha_h + \sum_{j\in A_h}(c_j(x_j)+t_{jh})\right) \\ &= \frac{1}{k_h+1}\left(\alpha_h + \sum_{j\in A_h} TCu_{jh}\right) \end{aligned}$$

From equation (5) equilibrium quantities are given by

$$\begin{aligned} Q_{ih}^* &= \frac{\alpha_h - k_h(c_i(x_i)+t_{ih}) + \sum_{j\in A_h\setminus\{i\}}(c_j(x_j)+t_j)}{(k_h+1)\beta_h} \\ &= \frac{\alpha_h - k_h TCu_{ih} + \sum_{j\in A_h\setminus\{i\}} TCu_{jh} + TCu_{ih} - TCu_{ih}}{(k_h+1)\beta_h} \\ &= \frac{\alpha_h - (k_h+1)TCu_{ih} + \sum_{j\in A_h\setminus\{i\}} TCu_{jh} + TCu_{ih}}{(k_h+1)\beta_h} \\ &= \frac{\alpha_h - (k_h+1)TCu_{ih} + \sum_{j\in A_h} TCu_{jh}}{(k_h+1)\beta_h} \\ &= \frac{\alpha_h + \sum_{j\in A_h} TCu_{jh}}{(k_h+1)\beta_h} - \frac{(k_h+1)TCu_{ih}}{(k_h+1)\beta_h} \\ &= \frac{p_h}{\beta_h} - \frac{TCu_{ih}}{\beta_h} \\ &= \frac{p_h - TCu_{ih}}{\beta_h} \end{aligned}$$

From equation (5), at equilibrium $Q_{ih}^* > 0 \;\; \forall i \in A_h$, $\frac{p_h - TCu_{ih}}{\beta_h} > 0$ such that $p_h > TCu_{ih}$. ■

Consequently, for $i \in A_h$

$$c_i(x_i) + t_{ih} < \frac{1}{|A_h|+1}\left(\alpha_h + \sum_{j\in A_h}[c_j(x_j)+t_{jh}]\right) \tag{11}$$

For all $j \notin A_h$, $Q_{jh}^* = 0$ and

$$c_j(x_j) + t_{jh} \geq \frac{1}{|A_h|+1}\left(\alpha_h + \sum_{i\in A_h}[c_i(x_i)+t_{ih}]\right) \tag{12}$$

Proposition 4: The relation between the firm with the highest total unit costs in the active set, $i \in A_h$, with any firm $j \in \overline{A}_h$ which is not entering the market is

$$TCu_{ih} < \frac{\alpha_h + \sum_{r\in A_h} TCu_{rh}}{|A_h|+1} \leq TCu_{jh}$$

**Proof.**

The first inequality follows from $TCu_{ih} < p_h$ and is satisfied by any firm in the active set $A_h$. The last inequality is satisfied by any firm $j \in \overline{A}_h$ following from $TCu_{jh} \geq p_h$. ■

Proposition 4 shows that

$$\max_{i \in A_h} TCu_{ih} < p_h \leq \min_{j \in \overline{A_h}} TCu_{jh}$$

This is used in the algorithm in Section 3 to determine the number of active firms $|A_h|$. Firms are ordered on the basis of total unit costs, such that $TCu_{(1)h} \leq TCu_{(2)h} \leq \ldots \leq TCu_{(n)h}$. The rule that is used is the following

1. Initialise $p = \alpha$, $|A| = 0$.

2. while $TCu_{(k)} \leq p$, $(k)$ enters the market and the price is updated.

More details of the algorithm are given in Section 3.

Let $M_i$ be the set of markets in which firm $i$ is active, $M_i = \{h \in M | i \in A_h\}$. The total quantity supplied by each firm is

$$s_i = \sum_{h \in M_i} Q^*_{ih} = \sum_{h \in M_i} \frac{\alpha_h - k_h(c_i(x_i) + t_{ih}) + \sum_{j \in A_h \setminus \{i\}}(c_j(x_j) + t_{jh})}{(k_h + 1)\beta_h} \tag{13}$$

Total cost for each firm is

$$TC_i = \sum_{h \in M_i} (c_i(x_i) + t_{ih}) \frac{\alpha_h + \sum_{j \in A_h \setminus \{i\}}(c_j(x_j) + t_{jh}) - k_h(c_i(x_i) + t_{ih})}{(k_h + 1)\beta_h} + w(x_i) \tag{14}$$

Using (5) and (8), the final payoff for each firm given location vector $x$ is:

$$\begin{aligned}
\pi_i(x) &= \sum_{h \in M_i} (p_h - (c_i(x_i) + t_{ih})Q_{ih} - w(x_i) \\
&= \sum_{h \in M_i} \frac{\left[\alpha_h + \sum_{j \in A_h}(c_j(x_j) + t_{jh}) - (k_h + 1)(c_i(x_i) + t_{ih})\right]^2}{(k_h + 1)^2 \beta_h} - w(x_i) \\
&= \sum_{h \in M_i} \frac{\left[\alpha_h + \sum_{j \in A_h \setminus \{i\}}(c_j(x_j) + t_{jh}) - n(c_i(x_i) + t_{ih})\right]^2}{(k_h + 1)^2 \beta_h} - w(x_i) \qquad (15) \\
&= \sum_{h \in M_i} \beta_h (Q^*_{ih})^2 - w(x_i)
\end{aligned}$$

Proposition 5: The optimum $Q^*_{ih}$, $q^*_h$ and $p^*_h$ in equation (5), (8) and (9), respectively, is a Nash Equilibrium for the competitive second stage of the game given location vector $x$.

## 2.2 Location decision

Given the optima of the second stage, focus is on the first stage of the game. Considering the equilibrium supply quantity choice in the second stage, $Q^*(x)$, each firm $i$ maximizes the profit function $\pi_i$ by selecting a location on the network. We assume that several firms can be located at the same site. At equilibrium, no other location decision is better off for each firm.

The strategy $x^* = (x^*_1, \ldots, x^*_n)$ is a **Nash Equilibrium** if for each firm $i$, $x^*_i$ is the best response to the strategies specified by the $n-1$ other firms:

$$\pi_i(x^*_i, \underline{Q}^*(x^*)) \geq \pi_i(x_i, \underline{Q}^*(\hat{x}))$$
$$\text{with } \hat{x} = (x^*_1, \ldots, x_i, \ldots, x^*_n) \quad \forall x_i$$

for every feasible strategy $x_i$. That is, $x^*_i$ solves

$$\max_{x_i} \pi_i(x_i, \underline{Q}^*(\hat{x}))$$

The method and algorithms used to select optimal locations and quantities for the firms are described in Section 3.

# 3 Methods for selecting Nash quantities and Nash locations

This section describes an algorithm derived from the theoretical results in Section 2. It systematically enumerates all location possibilities for which equilibrium quantities are computed. After that it tries to detect which location vectors correspond to a Nash equilibrium by checking whether it is better for a firm to relocate its facility.

## 3.1 Main Algorithm

The main algorithm is sketched in "Algorithm for searching equilibria" (Algorithm 1) which calls iteratively to a subroutine called `Quantity` described in Section 3.2. This procedure computes the Nash equilibrium quantities for each location vector based on the size of the market and equilibrium price. Once the optimal quantities have been determined, the subroutine called `Profit` computes the profit for the firms at all the possible location vectors based on Nash quantities. The output is the profit (payoff) matrix $\Pi$. Finally, a subroutine determining the Nash equilibria on location decisions, called `Equilibria`, is described in Section 3.3.

**Algorithm 1**: Algorithm for searching Nash equilibria

```
input  : Number of firms n; number of markets m; parameters α and β ; distance matrix
         d(x_i, v_h); marginal costs c_i(x_i), opening costs w(x_i), transportation costs function T
output: Nash equilibria of the non-cooperative game
L ⟵ m^n ;                                             /* all possible locations */
/* Generate location matrix X_{l,i} iteratively                                  */
x_{1,i} = 1 ∀i to x_{L,1} = 1, x_{L,2} = 2, ..., x_{L,n} = m;
for each location l do
    Q* ⟵ Quantity(x_l) ;   /* computation of the Nash equilibrium quantities for
    each location vector based on the size of the market and equilibrium price
    */
    Π*(l) ⟵ Profit(x_l, Q*) ; /* computation of the profit matrix for each location
    vector based on Nash quantities */
end
E* ⟵ Equilibria(Π*) ;      /* computation of all (if many) Nash equilibria on
location decisions */
```

## 3.2 Procedure `Quantity` for computing Nash equilibrium on Quantities

**Procedure** `Quantity(x)` : Procedure to compute Nash equilibrium quantities, $Q_l^*$

```
input  : location vector x_l and global variables
output: Nash equilibrium Quantity decisions: Q_l^*
for h ∈ M do
    TCu_{·h} ⟵ t_{·h} + c(x_l);                              /* Total unit cost */
    STCu_{·h} ⟵ Sort(TCu_{·h}) ;     /* Procedure to order the firms on total unit
    costs, TCu_{(1)h}, ..., TCu_{(n)h} */
    [k_h, p_h] ⟵ SizeMarket(STCu_{·h}, α_h, β_h, k_h, p_h, A_h) ;    /* Procedure to determine
    the size of market h */
    Q*_{·h} ⟵ OptQ(STCu_{·h}, p_h, α_h, β_h, k_h, A_h) ;       /* Procedure to compute optimal
    quantities for firms entering the market h */
end
```

Procedure `Quantity` is called by the main algorithm for each of the possible location vectors for the suppliers. Every time the procedure is called, total unit costs are computed for each firm at each market. Results derived in Section 2.1 (Proposition 1, Proposition 4 and equation (5)) are applied to derive optimal quantities depending on the size of the market, that is, the number of firms entering. The computation generates this results by the following two procedures:

1. Procedure `SizeMarket`: this procedure determines the size of the active set $A_h$ and the equilibrium price $p_h$;

2. Procedure `OptQ`: this procedure computes optimal quantities for firms entering the market $h$. From

equation (5) and depending on the asymmetry of the firms, the method finds the optimal quantities for the active firms.

**Procedure** `SizeMarket`($TC, \alpha, \beta, k, p, A$): Procedure to determine the size of market $k$ and equilibrium price $p$

**input** : Total unitary costs $TC$, parameters $\alpha, \beta$
**output**: Size and price of market $h$: $k$, $p$; and active set, $A$

$p \longleftarrow \alpha_h$ ; `/* Initial price at the market */`
$k \longleftarrow 0$ ; `/* Initial size of the market */`
**while** $k \leq n$ *and* $TC_{k+1} < p$ **do**
  $k \longleftarrow k + 1$;
  $p \longleftarrow \frac{\alpha+\sum_{j=1}^{k} TC_j}{(k+1)}$;
**end**

**Procedure** `OptQ`($TC, p, \alpha, \beta, k, A$): Procedure to compute the optimal quantities for firms at market $h$

**input** : Total unitary costs $TC$,price at market, $p$, parameters $\alpha, \beta$ and size of the market, $k$, active set, $A$
**output**: Optimal quantities: $Q^*_{\cdot h}$

**if** $k == 0$ **then**
  $Q \longleftarrow 0 \ \ \forall i$;
**else**
  **for** $i \in A$ **do**
    $Q_i \longleftarrow \frac{\alpha-(k+1)TC_i+\sum_{j=1}^{k} TC_j}{(k+1)\beta}$;
  **end**
  **for** $i \in \overline{A} = N \backslash A$ **do**
    $Q_i \longleftarrow 0$;
  **end**

## 3.3 Algorithm Location Stage

In the location-stage, the problem is to maximize profit by selecting a node where to locate the facility in the network. Given $n$ supplier firms and $m$ markets, the feasible set $\underline{X}$ has $L = m^n$ elements. A Nash equilibrium can be identified for the first stage game by testing each element of $\underline{X}$ in the following way. Consider location vector $X_l = (x_{l1}, \ldots, x_{li}, \ldots, x_{ln})$. Note that it is possible that $x_{li} = x_{lj}$ for $i \neq j \in N$. For each firm, one should test whether firm $i$ located at $x_{li}$ is better off leaving its current location choosing another, $x_{ki} \in V$ with $k \neq l$ and $k \leq L$. This step generates a set of possible configurations for firm $i$ where only firm $i$ has a different location with respect to $X_l$. The firm knows that its decision could generate changes in the quantity choice by the other supplier firms. If none of the other locations is profitable for firm $i$, this firm is so-called "at equilibrium" with respect to its current location decision, $x_{li}$. As values of $n$ and $m$ are given in the input of the algorithm, the set of feasible locations is finite and the search process stops with an optimum, equilibrium, if it exists.

If firm $i$ finds a location $x_{ki} \in V, x_{ki} \neq x_{li}$, in which it is better off than at $x_{li}$, then location vector $X_l = (x_{l1}, \ldots, x_{li-1}, x_{li}, \ldots, x_{ln})$ is not at equilibrium. Firm $i$ will prefer to move and to locate its facility at $x_{ki}$. The procedure follows with another possible location vector with all the possible deviations. If no firm $j$ improves by moving to another site from $x_{lj}$ ($\forall j \in N$), then $X_l$ is at equilibrium. The procedure proceeds checking all possible location vectors one by one whether or not it is Nash equilibrium.

For solving this stage of the game, an exact enumeration algorithm is used in which all feasible locations are enumerated knowing that an optimal solution will be at one of the nodes of the network (Labbé and Hakimi (1991) and Sarkar et al. (1997)).

**Procedure** `Equilibria(Π*)` : Procedure to compute all the equilibria in the game

**input** : Profit Matrix, $\Pi(1 \ldots L, 1 \ldots n)$, for all location configurations in $X(1 \ldots L, 1 \ldots n)$ and all firms
**output**: Set of Equilibria

```
for each location l do
    NE ⟵ TRUE ;    /* boolean variable, it indicates whether location
    vector x_l is at equilibrium */
    while i ≤ n and NE == TRUE do
        ε ⟵ set of configurations where only firm i has a different location with
        respect to X(l, i);
        s ⟵ 1;
        while s <= |ε| and NE == TRUE do
            xa ⟵ ε(s) ; /* xa is the alternative location vector for firm i
            */
            if π(X(l, i), Q*) ≥ π(xa(i), Q*) then
                s ⟵ s + 1;
            else
                NE ⟵ FALSE;
        end
    end
end
```

## 4 Numerical illustration

Two cases are elaborated to illustrate the procedure and the analytical results. The first case is taken from Sarkar et al. (1997) with $n = 3$ firms and $m = 6$ markets. A mistake in the output of Sarkar et al. (1997) is found when the algorithm outlined in the last section is applied. In their study an extra location vector is obtained as equilibrium. In Section 4.1 we show why this location vector can not be an equilibrium. The second numerical example consists of 4 different cases. It is used to show the viability of the algorithm when bigger and more sophisticated cases are applied. This is illustrated in Section 4.2.

## 4.1 Network with 6 markets and 3 firms

The location decisions are represented by the node of the market in which firms are located (for example, $(1, 2, 1)$ means firms one and three are located at the same node that of market 1 and firm two at market 2, see Figure 1). Market locations are denoted by $v_h, h = \{1, \ldots, 6\}$.

In this example, $T(d(x_i, v_h)) = d(x_i, v_h)$ for all $x_i, v_h \in V$. At each of the vertices $h$, marginal cost are $c_i(x_i) = 10$ and $\beta_h = 1$. Sarkar et al. (1997) describe four different configurations for parameter $\alpha = (\alpha_1, \ldots, \alpha_6)$. The algorithm described in Section 3 has been implemented in Matlab and applied to each of the different cases.

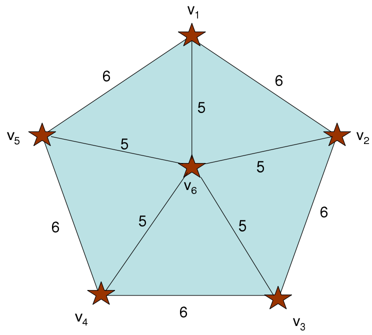


Fig. 1: Example 1. Network

Tab. 2: Location equilibria for example 1

| Configuration $\alpha$ | Corresponding Nash location $x^*$ |
|---|---|
| $(50, 50, 400, 400, 50, 250)$ | $(3, 4, 6)$,$(3, 6, 4)$,$(4, 3, 6)$,$(4, 6, 3)$,$(6, 3, 4)$,$(6, 4, 3)$ |
| $(50, 50, 500, 500, 40, 50)$ | $(3, 3, 4)$,$(3, 4, 3)$,$(4, 3, 3)$ |
| $(50, 500, 50, 50, 500, 50)$ | $(6, 6, 6)$ |
| $(1000, 1000, 1000, 1000, 1000, 0)$ | $(6, 6, 6)$ |

At equilibrium, for each configuration $\alpha$ a location vector and all its permutations are Nash equilibria because all $c_i$ have the same value.

Tab. 3: Size of markets and profits of firms for the equilibria

| Configuration | Location equilibria | No. of entrants $\|A_h\|$ | | | | | | Profit | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | $x_l^*$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $f_1$ | $f_2$ | $f_3$ |
| 1 | $(3, 4, 6)$ | 3 | 3 | 3 | 3 | 3 | 3 | 22392 | 22392 | 22388 |
| 2 | $(3, 3, 4)$ | 3 | 3 | 3 | 3 | 3 | 3 | 29895 | 29895 | 29901 |
| 3 | $(6, 6, 6)$ | 3 | 3 | 3 | 3 | 3 | 3 | 29733 | 29733 | 29733 |
| 4 | $(6, 6, 6)$ | 3 | 3 | 3 | 3 | 3 | 0 | 303200 | 303200 | 303200 |

Table 2 shows the location vectors at equilibrium for each configuration of $\alpha$ and Table 3 shows the number of entrants at each of the markets (size of the markets) and the corresponding payoffs. Only in the case of the last configuration for $\alpha$ there is no firm supplying market number 6. In all other cases, all firms supply all markets. This shows that Sarkar et al. (1997) choose relatively easy configurations; they did not have to determine the number of active firms. For the first configuration, $\alpha = (50, 50, 400, 400, 50, 250)$, and all the equilibria, the maximum total unit costs for a firm is $\max_{i\in N} TCu_{ih} = \max_{i\in N}[c_i(x_i)+t_{ih}] = 20$ ($c_i(x_i) = 10\ \forall i \in N$, $\max_{i\in N} t_{ih} = 10$) and $\min_{h\in M}[(\alpha_h + \sum_{i\in N} TCu_{ih})/(n+1)] = 22.5$, then from Proposition 4 all firms will supply all markets:

$$\max_{h\in M}\max_{i\in N} TCu_{ih} = 20 < 22.5 = \min_{h\in M} \frac{\alpha_h + \sum_{i\in N} TCu_{ih}}{n+1}$$

The same applies for the second and third configurations,

$$\max_{h\in M}\max_{i\in N} TCu_{ih} = 20 < 23.75 = \min_{h\in M} \frac{\alpha_h + \sum_{i\in N} TCu_{ih}}{n+1}$$
$$\max_{h\in M}\max_{i\in N} TCu_{ih} = 15 < 20 = \min_{h\in M} \frac{\alpha_h + \sum_{i\in N} TCu_{ih}}{n+1}$$

For the last configuration of $\alpha$ and markets from 1 to 5,

$$\max_{h\in M}\max_{i\in N} TCu_{ih} = 15 < 261.25 = \min_{h\in\{1\dots,5\}} \frac{\alpha_h + \sum_{i\in N} TCu_{ih}}{n+1}$$

In case of market 6, no firm will supply since from Proposition 2, firm $i$ ($\forall i$) is not interested in supplying market 6 since $TCu_{i6} = 10\ \forall i$, and the initial price at market 6 is $p_6 = \alpha_6 = 0$, such that $TCu_{i6} > p_6$.

For configuration 2, Sarkar et al. (1997) describe an additional stable location for the firms, namely $(3, 4, 4)$ and its corresponding permutations. Our algorithm does not find that location structure as equilibrium. To show that consider the profits for the firms locating at $(3, 4, 4)$:

Firm 1 $\longmapsto$ 29926

Firm 2 $\longmapsto$ 29880

Firm 3 $\longmapsto$ 29880

A firm is in Nash equilibrium if it does not have an incentive to move to another location. Consider firm 2 and suppose the others do not change of location. Five possible strategies should be evaluated to determine a possible improvement of the profit. Evaluation $\pi_2$ where $x_2 \in \{v_1, \dots, v_6\}$ results in Table 4.

Tab. 4: Firm 2 profits for each location while firm 1 and 3 are fixed

| | Equilibrium in Sarkar et al. (1997) | Alternatives for firm 2 | | | | |
|---|---|---|---|---|---|---|
| Location | $(3, 4, 4)$ | $(3, 1, 4)$ | $(3, 2, 4)$ | $(3, 3, 4)$ | $(3, 5, 4)$ | $(3, 6, 4)$ |
| Profit firm 2 | 29880 | 27585 | 28251 | 29895 | 28214 | 29368 |

One can observe that firm 2 is better off changing strategy by moving to market 3. This means that $(3, 4, 4)$ is not an equilibrium as wrongly included by Sarkar et al. (1997).

## 4.2 Network with 15 markets and 5 firms

For the illustration of the viability of the algorithms, four cases have been generated with 15 markets and 5, 4, 3 and 2 firms, respectively, where data were randomly generated. Figure 2 shows the network and table 5 shows the location points of the markets.

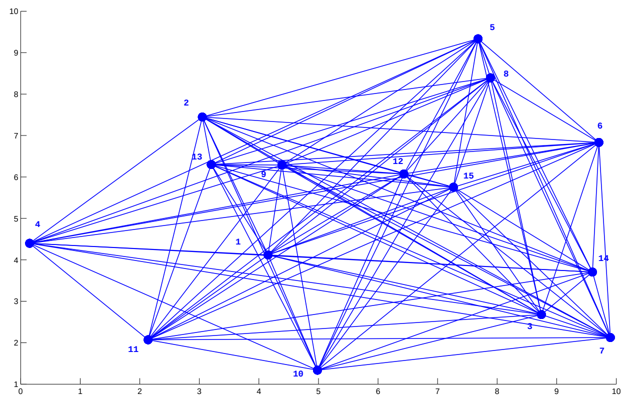


Fig. 2: Network Example 2

Tab. 5: Location of 15 markets randomly generated

| $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ |
|---|---|---|---|---|---|---|---|
| $\begin{pmatrix}4.1537\\4.1195\end{pmatrix}$ | $\begin{pmatrix}3.0500\\7.4457\end{pmatrix}$ | $\begin{pmatrix}8.7437\\2.6795\end{pmatrix}$ | $\begin{pmatrix}0.1501\\4.3992\end{pmatrix}$ | $\begin{pmatrix}7.6795\\9.3338\end{pmatrix}$ | $\begin{pmatrix}9.7084\\6.8333\end{pmatrix}$ | $\begin{pmatrix}9.9008\\2.1256\end{pmatrix}$ | $\begin{pmatrix}7.8886\\8.3924\end{pmatrix}$ |
| $v_9$ | $v_{10}$ | $v_{11}$ | $v_{12}$ | $v_{13}$ | $v_{14}$ | $v_{15}$ | |
| $\begin{pmatrix}4.3866\\6.2878\end{pmatrix}$ | $\begin{pmatrix}4.9831\\1.3377\end{pmatrix}$ | $\begin{pmatrix}2.1396\\2.0713\end{pmatrix}$ | $\begin{pmatrix}6.4349\\6.0720\end{pmatrix}$ | $\begin{pmatrix}3.2004\\6.2989\end{pmatrix}$ | $\begin{pmatrix}9.6010\\3.7048\end{pmatrix}$ | $\begin{pmatrix}7.2663\\5.7515\end{pmatrix}$ | |

Tab. 6: Parameters $\alpha$, $\beta$, $w$

| Parameter | Values for each of the 15 markets | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ | $v_9$ | $v_{10}$ | $v_{11}$ | $v_{12}$ | $v_{13}$ | $v_{14}$ | $v_{15}$ |
| $\alpha$ | 976 | 615 | 804 | 743 | 946 | 881 | 728 | 509 | 911 | 722 | 808 | 896 | 961 | 869 | 588 |
| $\beta$ | 3 | 5 | 5 | 3 | 5 | 1 | 2 | 5 | 1 | 1 | 2 | 1 | 4 | 2 | 1 |
| $w$ | 145 | 216 | 252 | 206 | 228 | 142 | 176 | 257 | 236 | 192 | 214 | 259 | 111 | 221 | 110 |

The input parameters $\alpha$, $\beta$, $w$ at each market are given in Table 6. Table 10 (appendix) shows the distance matrix which defines $T(d(x_i, v_h))$. Marginal costs, $c_i(v_j)$, are detailed in Table 11 (appendix).

When the case with 5 firms is considered, the algorithm found one equilibrium in location: $(1, 10, 9, 10, 2)$. Table 4.2 shows the total unit costs when firms are located at equilibrium, the Nash quantity matrix, the number of entrants at each market and corresponding payoffs.

Tab. 7: Case 15 markets, 5 firms - Nash location $(1, 10, 9, 10, 2)$

Total Unit Costs for the firms at each market ($TCu_{ih}$)

| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ | $v_9$ | $v_{10}$ | $v_{11}$ | $v_{12}$ | $v_{13}$ | $v_{14}$ | $v_{15}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_1^*$ | 113 | 116.50 | 117.81 | 117.01 | 119.29 | 119.18 | 119.08 | 118.67 | 115.18 | 115.90 | 115.87 | 116.01 | 115.38 | 118.46 | 116.51 |
| $x_2^*$ | 237.90 | 241.41 | 238.99 | 240.72 | 243.44 | 242.25 | 239.98 | 242.63 | 239.99 | 235 | 237.94 | 239.95 | 240.27 | 240.19 | 239.97 |
| $x_3^*$ | 309.18 | 308.77 | 312.66 | 311.64 | 311.49 | 312.35 | 313.91 | 311.09 | 307 | 311.99 | 311.78 | 309.06 | 308.19 | 312.82 | 309.93 |
| $x_4^*$ | 257.90 | 261.41 | 258.99 | 260.72 | 263.44 | 262.25 | 259.98 | 262.63 | 259.99 | 255 | 257.94 | 259.95 | 260.27 | 260.19 | 259.97 |
| $x_5^*$ | 113.50 | 110 | 117.43 | 114.21 | 115 | 116.69 | 118.67 | 114.93 | 111.77 | 116.41 | 115.45 | 113.65 | 111.16 | 117.54 | 114.54 |

Quantity Matrix - Supply from each firm to each market

| Firms | Markets | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ | $v_9$ | $v_{10}$ | $v_{11}$ | $v_{12}$ | $v_{13}$ | $v_{14}$ | $v_{15}$ |
| $f_1$ | 73.86 | 30.47 | 37.93 | 59.37 | 42.76 | 203.10 | 87.03 | 25.53 | 208.97 | 172.96 | 95.58 | 206.43 | 54.33 | 100.62 | 147.28 |
| $f_2$ | 32.26 | 5.49 | 13.69 | 18.14 | 17.93 | 80.04 | 26.58 | 0.74 | 84.17 | 53.86 | 34.55 | 82.48 | 23.11 | 39.76 | 23.83 |
| $f_3$ | 8.47 | 0 | 0 | 0 | 4.32 | 9.94 | 0 | 0 | 17.15 | 0 | 0 | 13.38 | 6.13 | 3.44 | 0 |
| $f_4$ | 25.56 | 1.49 | 9.69 | 11.47 | 13.93 | 60.04 | 16.58 | 0 | 64.17 | 33.86 | 25.55 | 62.48 | 18.11 | 29.76 | 3.83 |
| $f_5$ | 73.69 | 31.77 | 38.01 | 60.31 | 43.62 | 205.60 | 87.23 | 26.28 | 212.39 | 172.46 | 95.79 | 208.78 | 55.39 | 101.08 | 149.26 |

No. of entrants for each market

| $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ | $v_9$ | $v_{10}$ | $v_{11}$ | $v_{12}$ | $v_{13}$ | $v_{14}$ | $v_{15}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 4 | 4 | 4 | 5 | 5 | 4 | 3 | 5 | 4 | 4 | 5 | 5 | 5 | 4 |

Profit for each firm

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ |
|---|---|---|---|---|
| 295653.69 | 39470.23 | 818.54 | 21239.80 | 301487.76 |

Following Proposition 4, in the equilibrium, markets $v_1, v_5, v_6, v_9, v_{12}, v_{13}$ and $v_{14}$ satisfy

$$\max_{i\in N} TCu_{ih} < \frac{\alpha_h + \sum_{i\in N} TCu_{ih}}{n+1} \quad (h = 1, 3, 5, 6, 9, 12, 13, 14)$$

For markets $v_2$, $v_3$, $v_4$, $v_7$, $v_{10}$, $v_{11}$ and $v_{15}$:

$$\max_{i\in N} TCu_{ih} < \frac{\alpha_h + \sum_{i\in N} TCu_{ih} - \max(TCu_{ih})}{n} \quad (h = 2, 3, 4, 7, 10, 11, 15)$$

And for market $v_8$:

$$\max_{i\in N} TCu_{ih} < \frac{\alpha_h + \sum_{i\in N} TCu_{ih} - Max_1 - Max_2}{n-1} \quad (h = 8)$$

where $Max_1$ and $Max_2$ are given by

$$Max_1 = \max_{i\in N}(TCu_{ih})$$
$$Max_2 = \max_{i\in N\setminus\{Max_1\}}(TCu_{ih})$$

Table 8 shows the Nash location for each of the cases with 4, 3 and 2 firms. Finally, table 9 shows the computational CPU times for the four cases and the complexity when the number of firms goes from 2 to 5. The algorithm has been implemented in Fortran and run on a core-duo Pentium IV processor.

Tab. 8: Equilibrium. Case 15 markets and 4, 3, 2 (respectively) supplier firms

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Firm | $f_1$ | $f_2$ | $f_4$ | $f_5$ | No. of entrants for each market | | | | | | | | | | | | | | |
| Chosen Location | 1 | 3 | 7 | 8 | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ | $v_9$ | $v_{10}$ | $v_{11}$ | $v_{12}$ | $v_{13}$ | $v_{14}$ | $v_{15}$ |
| Resulting Profit | 311846.92 | 319044 | 12580.02 | 45953.31 | 4 | 3 | 4 | 4 | 4 | 4 | 4 | 3 | 4 | 4 | 4 | 4 | 4 | 4 | 3 |
| Firm | | $f_1$ | $f_2$ | $f_4$ | No. of entrants for each market | | | | | | | | | | | | | | |
| Chosen Location | | 4 | 14 | 9 | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ | $v_9$ | $v_{10}$ | $v_{11}$ | $v_{12}$ | $v_{13}$ | $v_{14}$ | $v_{15}$ |
| Resulting Profit | | 379668.31 | 362520.9 | 30005.05 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 2 |
| Firm | | | $f_3$ | $f_5$ | No. of entrants for each market | | | | | | | | | | | | | | |
| Chosen Location | | | 8 | 4 | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ | $v_9$ | $v_{10}$ | $v_{11}$ | $v_{12}$ | $v_{13}$ | $v_{14}$ | $v_{15}$ |
| Resulting Profit | | | 663895.26 | 62272.21 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |

Tab. 9: Complexity

| Number of markets | Number of firms | No. of location vectors | CPU Time (seconds) |
|---|---|---|---|
| 15 | 5 | 759375 | 58.10938 |
| 15 | 4 | 50625 | 2.703125 |
| 15 | 3 | 3375 | 0.1718750 |
| 15 | 2 | 225 | 1.5625000E-02 |

## 5 Conclusions

A competitive location and quantity "a la Cournot" game has been described in this paper to study the oligopolistic competition between $n > 2$ heterogeneous firms. Firms have to decide where to locate a facility and then decide on how much to supply to all or some of $m > 2$ spatially separated markets from these facilities. The following results were derived with respect to the optimal supply decisions where we are dealing with possibly heterogeneous firms:

- A necessary condition to have any delivery to a market
- Analytic expression of the equilibrium quantities of the firms that supply to a market
- Necessary condition for a firm to supply to a market
- Based on the former, a new procedure has been developed to identify those firms that are supplying to a market, the active set, which determines the size of the market

Based on these results an algorithm is designed to find Nash equilibria of the game. The results and algorithm are illustrated numerically. By using the algorithm as a systematic computation instrument to cases reported in literature, a mistake was detected in (Sarkar et al. 1997). In that paper a solution is given that appears not to be an equilibrium of the model. Furthermore, tests on larger generated instances show the viability of the approach.

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# A Appendix

## A.1 Input data for example two

Distance matrix $d(x_i, v_h)$ and marginal costs $c_i(v_j)$ are given in Table 10 and Table 11, respectively.

Tab. 10: Marginal Production Costs for each firm (depending on location) $c_i(v_j)$

| | Possible locations for firms | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ | $v_9$ | $v_{10}$ | $v_1 1$ | $v_1 2$ | $v_1 3$ | $v_1 4$ | $v_1 5$ |
| $f_1$ | 113 | 772 | 501 | 939 | 519 | 477 | 862 | 573 | 282 | 705 | 855 | 117 | 713 | 441 | 849 |
| $f_2$ | 553 | 739 | 486 | 374 | 270 | 274 | 714 | 372 | 488 | 235 | 728 | 440 | 874 | 869 | 634 |
| $f_3$ | 547 | 910 | 840 | 681 | 836 | 694 | 408 | 361 | 307 | 581 | 755 | 378 | 855 | 611 | 433 |
| $f_4$ | 733 | 592 | 500 | 725 | 659 | 816 | 462 | 570 | 693 | 255 | 982 | 344 | 327 | 889 | 764 |
| $f_5$ | 223 | 110 | 905 | 279 | 369 | 695 | 356 | 522 | 158 | 990 | 625 | 481 | 564 | 400 | 490 |

Tab. 11: Distance Matrix

| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ | $v_9$ | $v_{10}$ | $v_{11}$ | $v_{12}$ | $v_{13}$ | $v_{14}$ | $v_{15}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $v_1$ | 0 | | | | | | | | | | | | | | |
| $v_2$ | 3.50 | 0 | | | | | | | | | | | | | |
| $v_3$ | 4.81 | 7.43 | 0 | | | | | | | | | | | | |
| $v_4$ | 4.01 | 4.21 | 8.76 | 0 | | | | | | | | | | | |
| $v_5$ | 6.29 | 5.00 | 6.74 | 9.00 | 0 | | | | | | | | | | |
| $v_6$ | 6.18 | 6.69 | 4.26 | 9.86 | 3.22 | 0 | | | | | | | | | |
| $v_7$ | 6.08 | 8.67 | 1.28 | 10.01 | 7.54 | 4.71 | 0 | | | | | | | | |
| $v_8$ | 5.68 | 4.93 | 5.78 | 8.71 | 0.96 | 2.40 | 6.58 | 0 | | | | | | | |
| $v_9$ | 2.18 | 1.77 | 5.66 | 4.64 | 4.49 | 5.35 | 6.91 | 4.09 | 0 | | | | | | |
| $v_{10}$ | 2.90 | 6.41 | 3.99 | 5.72 | 8.44 | 7.25 | 4.98 | 7.63 | 4.99 | 0 | | | | | |
| $v_{11}$ | 2.87 | 5.45 | 6.63 | 3.06 | 9.13 | 8.94 | 7.76 | 8.54 | 4.78 | 2.94 | 0 | | | | |
| $v_{12}$ | 3.00 | 3.65 | 4.10 | 6.50 | 3.49 | 3.36 | 5.25 | 2.74 | 2.06 | 4.95 | 5.87 | 0 | | | |
| $v_{13}$ | 2.38 | 1.16 | 6.62 | 3.59 | 5.41 | 6.53 | 7.89 | 5.13 | 1.19 | 5.27 | 4.36 | 3.24 | 0 | | |
| $v_{14}$ | 5.46 | 7.54 | 1.34 | 9.48 | 5.95 | 3.13 | 1.61 | 4.99 | 5.82 | 5.19 | 7.64 | 3.95 | 6.91 | 0 | |
| $v_{15}$ | 3.51 | 4.54 | 3.41 | 7.24 | 3.61 | 2.67 | 4.48 | 2.71 | 2.93 | 4.97 | 6.31 | 0.89 | 4.10 | 3.11 | 0 |