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On determining the cover of a simplex by spheres centered at its vertices

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Abstract The aim of this work is to study the Simplex Cover (SC) problem, which is to determine whether a given simplex is covered by spheres centered at its vertices. We show that the SC problem is equivalent to a global optimization problem. We investigate its characteristics.

Keywords Covering · Simplex · Spheres

1 Introduction

This study is motivated by earlier work on testing whether simplices representing the design area of blending problems may contain feasible solutions. The spheres represent areas where certainly no feasible solution can be located. This means, that if a simplex is covered, then

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the simplex can be excluded from the search, because it certainly does not contain a feasible solution [2,5]. This paper studies the characteristics of the SC problem. The reasoning is illustrated with graphical examples and validated by several lemmas and a theorem.

We formulate the problem and its ingredients. A simplex is a set of convex combinations of n affine independent vertices $v_1, \ldots, v_n$,

$$S = \left\{ x = \sum_{j=1}^{n} \lambda_j v_j, \ \sum_{j=1}^{n} \lambda_j = 1; \ \lambda_j \in [0, 1] \right\} \tag{1}$$

Usually the vertices are defined in Euclidean space where the dimension d should at least be $n - 1$. Regular simplices are special cases where all the edges have the same length. The next ingredient of the SC problem are the spheres or balls with radius r_i around the vertices,

$$B_i = \left\{ x \in R^d \ : \|x - v_i\| \le r_i \right\} \tag{2}$$

In our investigation we are specifically interested in the Euclidean distance norm $\|.\|_2$. Now the Simplex Cover problem can be formulated.

Simplex cover (SC) problem: Given an instance of a simplex S and a set of spheres $B_1, \ldots, B_n$ around its vertices. Certify if

$$\forall x \in S, \quad x \in \cup_i B_i \tag{3}$$

or alternatively, check whether

$$\exists x \in S, \quad x \notin \cup_i B_i. \tag{4}$$

An equivalent representation of the SC problem can be used that focuses far more on distance from the vertices. This comes closer to the concept of Laguerre Voronoi diagrams (or Power diagrams) that is elaborated in Sect. 2. We focus on squared Euclidean distance. Define the function

$$\varphi(x) = \min_i \{\|x - v_i\|^2 - r_i^2\} \tag{5}$$

where the squared distance is additively weighted [1]. Notice that function φ being a minimum of strict convex functions is not convex. Moreover, in the specific case where none of the spheres B_i is covering one of the other vertices $v_j, i \neq j$, φ has local minima of $-r_i^2$ in each vertex; $\varphi(v_i) = -r_i^2$. The SC problem is equivalent to

$$\forall x \in S \ \varphi(x) \le 0. \tag{6}$$

Examples of the φ function can be seen in Figs. 1 and 2.

Notice that for a regular instance of S, if one of the B_i covers another vertex, it covers them all and SC is solved.

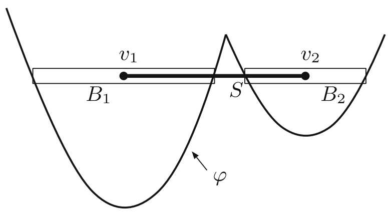

Fig. 1 Instance of SC problem with two vertices, where S is not covered

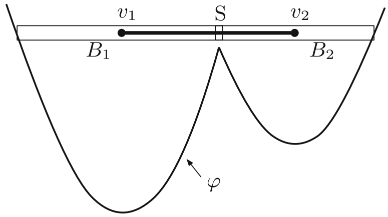

Fig. 2 Instance of SC problem with two vertices, where S is covered

Equivalence (6) shows that it is interesting to study problem SCO (Simplex Cover Optimization):

$$\Phi := \max_{x \in S} \varphi(x). \tag{7}$$

The SC problem has a certificate if $\Phi \leq 0$.

A lot of research is available on so-called Power sets [3,6] that relates to φ and mainly focuses on lower dimensional analysis. Our research deals with the general dimensional case. We elaborate this in Sect. 2. Problem SCO is not necessarily easier to solve than SC (3) or (4). One of the most obvious observations on the SC problem is that the simplex S is covered if one of the spheres B_j covers it. If this is not the case, we require properties on the SC problem of covering and the SCO problem of maximising φ over S. In Sects. 3 and 4, we derive mathematical properties of the SC and SCO problems that can be used to solve them, which we illustrate by numerical examples. Specifically, Sect. 4.1 analyses the case of regular simplices, which is easier to solve and useful for the blending problem. Finally Sect. 5 summarises the found results.

2 Distances, Voronoi diagrams and simplices

Voronoi diagrams for a set of points p_i, divide the space into regions $D(p_i)$ where all points in $D(p_i)$ are closer to p_i than to $p_j,\ j \neq i$. A survey of Voronoi diagrams can be found in [1]. There are several distance functions that can be used with respect to Voronoi diagrams. The usual is to consider the euclidean distance:

$$d_e(p_1, p_2) = \|p_1 - p_2\|_2. \tag{8}$$

A Voronoi cell of p_i for Euclidean distance are those points closest to p_i;

$$D(p_i) = \left\{x \in R^d | d_e(x, p_i) \leq d_e(x, p_j), \quad j \neq i,\ j = 1, \dots, n\right\}. \tag{9}$$

Each sphere B_i in the SC problem has a radius r_i. This radius can be considered as a weight. Two ways to weight the distance are commonly considered and could be used for what are called weighted points, p_i:

– Weight distance additively:

$$d_a(x, p_i) = d_e(x, p_i) - r_i \tag{10}$$

– Weight the so-called power distance:

$$d_p(x, p_i) = d_e(x, p_i)^2 - r_i^2 \tag{11}$$

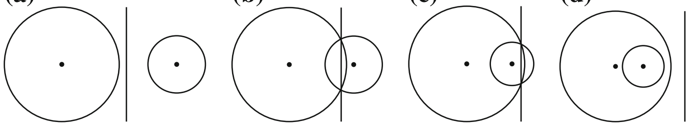

Fig. 3 Power planes of two weighted points on 4 different instances

Using d_a in the Voronoi concept leads to diagrams with edges that are hyperbolic curves. Also so-called multiplicative Voronoi diagrams exist that have hyperbolic curves [1].

Life is a bit easier using d_p; the separating shapes are planes. One obtains a power diagram with power planes

$$\Pi_{ij} = \left\{x \in R^d \mid d_p(x, p_i) = d_p(x, p_j)\right\} \tag{12}$$

and power cells

$$D_p(p_i) = \left\{x \in R^d \mid d_p(x, p_i) \le d_p(x, p_j), \quad j \ne i \; j = 1, \dots, n\right\}. \tag{13}$$

For a point x outside sphere B_i, $d_p(x, p_i)$ is positive and $\sqrt{d_p(x, p_i)}$ represents the distance from the sphere with radius r_i around v_i to the point x outside the sphere along the tangent line through the point x [4]. The *power Voronoi diagram* (or *power diagram*) cells are convex polyhedra. Figure 3 shows the power diagram edges produced by two weighted points. The difference between a Euclidean Voronoi diagram and a Power diagram is that in the latter a cell can be empty, i.e., there is not a point inside (see Fig. 3d).

An important property of weighted and unweighted Voronoi diagrams is:

Property 1 Given $n + 1$ affinely independent points with $n > 1$. The Voronoi diagram contains one vertex of degree $n + 1$ (number of edges incident to the vertex).

This has been shown for low dimensional cases in [1]. In Sect. 3 we will focus on the power diagram for the general dimensional case and derive a procedure to determine the corresponding so-called θ point. We will show that it plays a major role in the solution of SC and SCO problems.

3 Covering a simplex, analysis based on the θ-point

The function

$$\varphi(x) = \min_i\{d_p(x, v_i)\}, \tag{14}$$

is piecewise convex in each power cell. This implies that a maximum of φ over a power cell is attained in one of its extreme points. It means that looking for the maximum value, Φ (see Eq.(7)), we can focus on the power planes Π_{ij} that constitute the edges of power diagrams. Let us denote the interior of a set by int. Lemma 1 tells us that the solution of SCO should be looked for on the boundary of the power cells.

Lemma 1 *A point* $x^* \in \text{argmax}_{x \in S}\, \varphi(x) \notin int S \cap \cup_i int D_p(v_i)$.

Proof For an interior point x of a power cell around v_i,

$$\varphi(x) = d_p(x, v_i) < d_p(x, v_j), \quad \forall j \ne i.$$

This means that one can increase the value of φ by going in the direction $\nabla\varphi(x) = 2(x - v_i)$. Therefore x cannot be a maximum point. □

It is easy to check whether one sphere covers the complete simplex. In this case the maximum of $\varphi(x)$ may be attained at one or several vertices v_i of S (see Figs. 3c, d). For the case the maximum is not in a vertex, the lemma shows us that we can concentrate on the power planes Π_{ij}. Actually, it gives us the information that if the intersection of the power planes with the simplex are covered, i.e. $\varphi \leq 0$ on $S \cap \cup_{ij}\Pi_{ij}$, then all the simplex is covered. To illustrate these observations, we consider the line between the centers of the spheres of Fig. 3 as a simplex. One can see that in Fig. 3a the spheres do not intersect and the simplex is not covered in contrast to Fig. 3b. Notice that the intersection point of power plane and simplex corresponds to the global maximum point x^*. Figure 3c illustrates the case where one of the spheres covers the complete simplex and in Fig. 3d also the other sphere. Notice that in the last two cases, the power plane does not intersect with the simplex and the maximum point x^* is found in a vertex of S.

We now focus on the situation where the maximum is attained at an interior point of S; i.e. $x^* = V\lambda$, where matrix $V = [v_1, v_2, \ldots, v_n]$ and $\lambda_i > 0,\ \sum_i \lambda_i = 1$.

Lemma 2 *Consider problem SCO on a simplex with vertices $v_1, v_2, \ldots, v_n$ with corresponding B_i. Let $x^* = V\lambda$ be an interior maximum point of SCO.*

$$d_p(x^*, v_1) = d_p(x^*, v_i), \quad i = 2, \ldots, n \tag{15}$$

i.e. the power function of each vertex has the same value in x^*.

Proof Let the index set I of active vertices be defined by $d_p(x^*, v_i) = \varphi(x^*)$ for $i \in I$. Assume that the lemma is not true: x^* is interior maximum point and $\exists j,\ d_p(x^*, v_j) > \varphi(x^*)$ or $j \notin I$. In that case, one can construct a direction u which is the projection of $v_j - x^*$ on the linear space where the active set remains active; $d_p(x^* + u, v_i) = \varphi(x^* + u)$. Direction u is an ascent direction of φ in the point x^* and feasible, because x^* is interior. This is a contradiction with x^* being a maximum point. □

The so-called θ point satisfying equation (15) fulfills the necessary condition for an interior optimum. Notice that θ can also be located outside S. The recipe to compute the θ point equating planes is derived from setting the power function values equal:

$$(\theta - v_1)^T(\theta - v_1) - r_1^2 = (\theta - v_i)^T(\theta - v_i) - r_i^2, \quad i = 2, \ldots, n. \tag{16}$$

Elaboration and bringing the terms with θ to the left hand side gives

$$2(v_i - v_1)^T\theta = r_1^2 - r_i^2 + v_i^T v_i - v_1^T v_1, \quad i = 2, \ldots, n. \tag{17}$$

Equation (17) describes the plane separating v_1 and v_i on equal power function value. The edge $(v_i - v_1)$ is perpendicular to the plane and the right hand side of the equation describes a constant. Focusing on a solution θ, one can read (17) as linear equation $(v_i - v_1)^T\theta = \mathit{constant}$. As point θ has to be in all the planes (17) for $i = 2, \ldots, n$, we have $n - 1$ linear equalities in either $\theta = V\lambda$ or in λ which is n-dimensional.

$$2(v_i - v_1)^T V\lambda = r_1^2 - r_i^2 + v_i^T v_i - v_1^T v_1, \quad i = 2, \ldots, n. \tag{18}$$

We are interested in the intersection with the plane through the points $v_1, v_2, \ldots, v_n$. It means that θ should move in that plane, such that the last linear equality is given by

$$(1, 1, \ldots, 1)^T\lambda = 1. \tag{19}$$

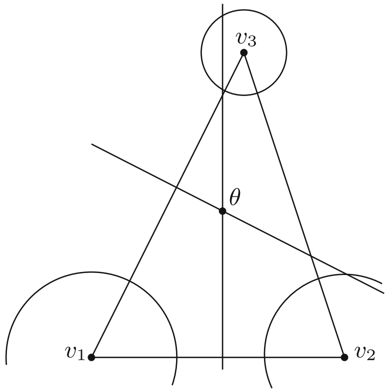

Fig. 4 Graphical illustration of Example 1

This means that finding θ can be done by solving a set of n linear equalities. Given an interior θ, we have the unique maximum point of SCO and one can simply check $d_e(\theta, v_1)^2 \leq r_1^2$ to verify the solution of SC.

Example 1 Consider the following instance of three spheres in 2-dimensional space (see Fig. 4):

$$v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \; v_3 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}, \; r_1^2 = 4, r_2^2 = 3, r_3^2 = 1.$$

Point $\theta = \begin{pmatrix} 2.6 \\ 2.7 \end{pmatrix}$ can be determined equating the two power planes: Π_{12} between v_1 and v_2 and Π_{13} between v_1 and v_3. The corresponding solution $\lambda = (0.3, 0.25, 0.45)^T$ is in the interior of the corresponding simplex. The point is on equal power function value $\varphi(\theta) = 10.05$ for the three vertices.

Notice that if θ is covered by one of the spheres, it is also covered by all of them; $\theta \in B_j$ gives $\theta \in \cap B_j$. This is related to an earlier found result that says that if $\exists x \in S$ covered by all the spheres then the simplex is covered. It means that if θ is covered by one of the spheres then the simplex is fully covered. The result as proven in [2], is the following. Let $conv$ denote the convex hull of a set.

Lemma 3 *Given polytope $S = conv\{v_1, \ldots, v_h\}$, with $v_1, \ldots, v_h$ the extreme points of S (vertices) with corresponding spheres B_j, $j = 1, \ldots, h$. If $\exists y \in \cap_j B_j \cap S$ then $S \subset \cup_j B_j$.*

Figure 5 shows illustrative examples of 2-simplices. In Fig. 5a S is not covered because the θ point is not covered. Figure 5b illustrates a simplex which is covered.

The next question is what to do if the computed θ is not an interior point of S; i.e. $\exists j\ \lambda_j \leq 0$, at least one of the λ_j is not positive. It means that the maximum point x^* of SCO is found at the boundary of S. We will investigate this further in Sect. 4.

Focusing on θ, after finding θ, it may be outside S, $\theta \notin S$, but covered by all spheres, $\theta \in \cap_j B_j$. The following lemma combined with Lemma 3, shows that this is also a sufficient condition to certify the covering of S and no further analysis is required. It means that we do not need to solve problem SCO.

Lemma 4 *Given a point $x = V\lambda$, $x \notin S$ and $x \in \cap_j B_j$. Then $\exists y \in \cap_j B_j \cap S$.*

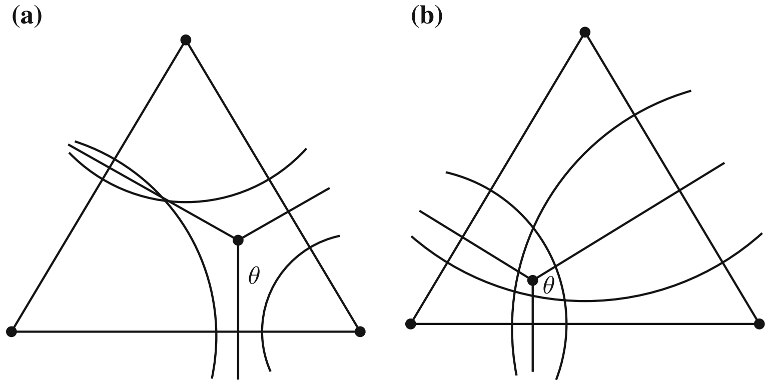

Fig. 5 Two instances of SC with corresponding θ point

Proof Without loss of generality, let $\lambda_1, \ldots, \lambda_m \geq 0$ and $\lambda_{m+1}, \ldots, \lambda_n < 0$, so that $v_{m+1}, \ldots, v_n$ are at the other side of the facet where x is closest to. We now make an orthogonal projection of $(y - v_1) = P(x - v_1)$ on the closest facet. One can do so by constructing an $n \times (m-1)$ matrix X with elements $v_j - v_1$ for $j = 2, \ldots, m$. An important observation is that in the construction of X for the starting vector v_1, one can take any $v_i, i = 1, \ldots, m$. The projection $y = v_1 + X(X^T X)^{-1} X^T (x - v_1)$ results into $y \in S$. Notice that $\|y - v_i\|^2 = \|P(x - v_i)\|^2 \leq \|x - v_i\|^2, i = 1, \ldots, m$ and y is closer to $v_j, j = m+1, \ldots, n$. So $(y - v_j)^T (y - v_j) \leq (x - v_j)^T (x - v_j)\ \forall j$ such that $y \in \cap B_j$.

A consequence of combining Lemmas 1–4, is the following theorem:

Theorem 1 *Let θ be a solution of* (18) *and* (19). *If $\varphi(\theta) \leq 0$ then S is covered.*

Notice that this theorem is valid even for $\theta \notin S$. However, if $\theta \notin S$ and $\varphi(\theta) > 0$, we have to investigate the boundary solutions of SCO in a more detailed way. As shown in the next section, also there the θ-point plays a role.

4 SCO has a boundary optimum

If the maximum of φ is not interior in S, it can be found on its boundary. In an extreme case, the maximum can be in a vertex of S if one big sphere covers all the simplex S. Alternatively, maximum point x^* can be attained at an edge, a triangular set or even higher dimensional face C of S. Again the θ-point plays a big role in the determination of face C. Consider the set $T = conv\{v_1, \ldots, v_n, \theta\}$. Now one can see that

$$\theta = \underset{x \in T}{\operatorname{argmax}}\, \varphi(x) \tag{20}$$

because for all $x \in T$, φ increases in the direction of θ such that θ is the global maximum point of (20). Moreover, θ is the common vertex of the power diagram. As $S \subset T$ also $\varphi(\theta) \geq \Phi$. That reconfirms the argumentation that a negative $\varphi(\theta)$ tells us that S is covered.

If $\varphi(\theta)$ is positive, we are still interested in solving SCO. Figure 6 illustrates a simplex for which the θ point is located outside. The corresponding values of λ_1 and λ_2 are positive and λ_3 has a negative value. S is covered because the intersection of S with the power planes are covered.

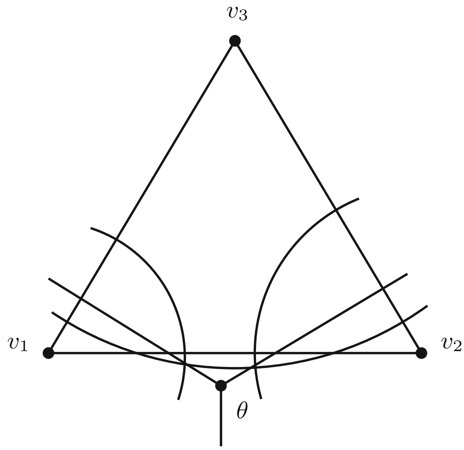

Fig. 6 Instance of SC with corresponding θ point not covered and outside of S

The previous analysis also helps us to determine the binding facet C of S where maximum points x^* with $\Phi = \varphi(x^*)$ can be found. Let $\lambda_1, \ldots, \lambda_m \geq 0$, then the set C where the maximum Φ is attained is given by $C = conv\{v_1, \ldots, v_m\}$. It means that for a (local or global) maximum point $x^* = V\mu$ of SCO $\mu_{m+1}, \ldots, \mu_n = 0$. As $n - m$ elements of μ have a value of zero, SCO is now equivalent to

$$\max_C \varphi(x) = \max \left\{ \varphi(\hat{V}\hat{\mu}) | \ \sum_1^m \hat{\mu}_j = 1, \hat{\mu} \geq 0 \right\} \tag{21}$$

where $\hat{\mu}$ and $\hat{V}$ now consist of the first m elements of μ and V respectively.

One can consider the problem as a lower dimensional covering verification. Unfortunately, we are not dealing with an equivalent SC problem. Besides the spheres $B_1, \ldots, B_m$ we have to deal with the cut $\hat{B}_j$ of spheres B_j, $j = m+1, \ldots, n$ with the plane of C. We will describe the plane and the intersection of the spheres that correspond to points that are not vertices of face C.

The plane can be described by

$$\Gamma = \{x \in R^d | x = v_1 + Xy, y \in R^{m-1}\} \tag{22}$$

where X is again the matrix $X = [v_2 - v_1, \ldots, v_m - v_1]$. The orthogonal projection of $v_{m+1}, \ldots, v_n$ on Γ via projection matrix $P = X(X^TX)^{-1}X^T$ gives centre points $\hat{v}_j = v_1 + P(v_j - v_1)$. Notice that due to the projection, the power function value of a point $z \in \Gamma$ to one of the vertices v_j is $\|z - v_j\|^2 = \|z - \hat{v}_j\|^2 + \|\hat{v}_j - v_j\|^2$. This means that the radii of the cuts of the spheres with plane Γ are given by $\hat{r}_j^2 = r_j^2 - \|\hat{v}_j - v_j\|^2$. A negative value means that B_j does not cut through plane Γ, so it does not have to be taken into account.

Problem (21) is lower dimensional than the original SC problem. Unfortunately, it is not easier as now we are dealing with a problem with m vertices and $n - m$ other centres in the plane, not necessarily in C. We know that the maximum of φ over C can be found on the vertices of the power diagram in C or at the intersection of the power diagram and the boundary of facet C. The power planes have the shape

$$2(v_i - v_j)^T \hat{V}\hat{\mu} = \hat{r}_i^2 - \hat{r}_j^2 + \hat{v}_i^T v_i - v_j^T v_j \rightarrow a_k^T \hat{\mu} = b_k, \ \ k = 1, \ldots, K \tag{23}$$

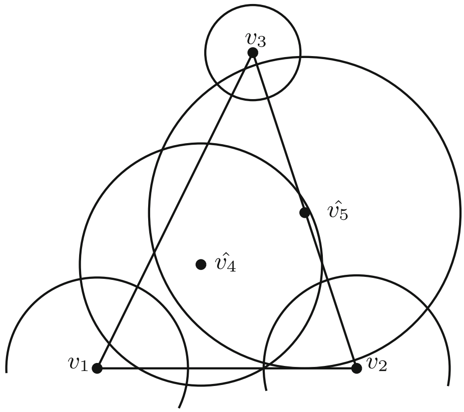

Fig. 7 Cross-cut of Example 2 with the plane through v_1, v_2, v_3

where not all $K = \frac{1}{2}n(n-1)$ power planes are of interest. One can argue that the power planes in between the vertices of C are of less interest, as x^* is not situated on those power planes and at the boundary of C, as one can improve the value of φ by going to the interior.

Example 2 Consider the following instance of 5 spheres in 6-dimensional space:

$$v_1 = \begin{pmatrix} 0\\0\\0\\0\\0\\0 \end{pmatrix},\ v_2 = \begin{pmatrix} 5\\0\\0\\0\\0\\0 \end{pmatrix},\ v_3 = \begin{pmatrix} 3\\6\\0\\0\\0\\0 \end{pmatrix},\ v_4 = \begin{pmatrix} 2\\2\\0\\1\\1\\1 \end{pmatrix},\ v_5 = \begin{pmatrix} 4\\3\\1\\0\\2\\1 \end{pmatrix}$$

and $r_1^2 = 4, r_2^2 = 3, r_3^2 = 1, r_4^2 = 9, r_5^2 = 15$. Now solving system (18) and (19) gives $\lambda = (1.3422, 0.8467, 1.3444, -2.2333, -0.3000)^T$. The negative values in the vector mean $\theta = V\lambda = (2.6, 2.7, -0.3, -2.233, -2.833, -2.533)^T$ is situated outside simplex S. It is at the same weighted power function value $\varphi(\theta) = 29.57$ of $v_1, \ldots, v_5$, so θ is not covered. The positive values in the λ vector indicate that we can look for the maximum of φ on the convex hull C of the first 3 vertices, as given by problem (21). Correspondingly, the cut of B_4 and B_5 can be found by projections $\hat{v}_4 = (2, 2, 0, 0, 0, 0)^T$, $\hat{v}_5 = (4, 3, 0, 0, 0, 0)^T$ and reduced radii $\hat{r}_4 = 6$ and $\hat{r}_5 = 9$. Figure 7 gives a 2D impression. The figure does not show us exactly whether C is covered. However, using a solver one can verify that the global maximum $\Phi = -0.22$ can be found in the point $x^* = (3.333, 0, 0, 0, 0, 0)^T$, and this means that S is covered.

Global optimum x^* is attained at one of the feasible basic solutions of a polyhedral set which describe the vertices of the power planes intersected with set C. To describe the polyhedral set we introduce free slack variables z_k of the power planes and define set H as

$$H = \left\{ \hat{\mu} \in R^m, z \in R^K \mid \sum_1^m \hat{\mu}_j = 1, \hat{\mu} \geq 0, a_k^T \hat{\mu} - b_k = z_k, \quad k = 1, \ldots, K \right\}. \quad (24)$$

As φ is piecewise convex it will have local optima at the feasible basic solutions of H, but one does not know which one corresponds to the global maximum Φ. This shows that SC is a problem which is hard to solve. Mainly this will be the case for instances with a large number of vertices representing an irregular simplex where none of the spheres covers other vertices.

4.1 The regular case

The question is whether the SC problem is easier to solve for regular simplices. The first check in the verification is of course to see whether one of the radii is big enough to cover the complete simplex. The next step is to generate the θ point and to check whether it is covered. If the latter is not the case, one should consider problem (21) on the lower dimensional plane C. The regular case reveals a specific property here. Due to the equal distance of a vertex v_j, $j = m+1, \ldots, n$ to all vertices v_i, $i = 1, \ldots, m$, the projection $\hat{v}_j = Pv_j$ on C also gives an equal distance point. This means that all projected vertices $\hat{v}_j$ coincide at the centroid of C.

Example 3 Consider the following instance of 5 spheres in 5-dimensional space:

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} -3 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; v_3 = \begin{pmatrix} -3 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \; v_4 = \begin{pmatrix} -3 \\ 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \; v_5 = \begin{pmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}$$

and $r_1^2 = 4, r_2^2 = 4, r_3^2 = 4, r_4^2 = 15, r_5^2 = 12$. The power function value between each pair of points is 18.

Now solving system (18) and (19) gives $\lambda = (.41, .41, .41, -.2, -.0333)^T$. Point $\theta = V\lambda = (-1.7667, 1.2333, 1.2333, -.6, -.1)^T$ is outside simplex S at the same weighted power function value $\varphi(\theta) = 2.533$ from $v_1, \ldots, v_5$, so θ is not covered. Projecting now via the P matrix determined by $X = [v_2 - v_1, v_3 - v_1]$ gives the centroid of $C = conv\{v_1, v_2, v_3\}$: $\hat{v}_4 = \hat{v}_5 = (-2, 1, 1, 0, 0)^T$. In the reduced problem, only $\hat{r}_4^2 = 15 - \|\hat{v}_4 - v_4\|^2 = 15 - 12 = 3$ is relevant, as $\hat{r}_5 = 0$. Due to the symmetry in this instance, the global maximum $\Phi = -1.2778$ is attained at several points on the boundary; vector $\hat{\mu}$ shows a permutation of the values 0, 0.611 and 0.389.

5 Conclusions

The Simplex Cover (SC) problem to determine whether a given simplex in d dimensional space is covered by n spheres centered at its vertices has been investigated. It is shown that this problem is equivalent to solving a Global Optimization problem SCO. The following has been found.

- Depending on the instance, SCO may have a unique interior optimum x^* which equals the so-called vertex point θ of a power diagram. A procedure is described to find this θ point.
- If the θ point is covered by the spheres, the simplex is covered, independently of θ being located in or outside the simplex.
- If SCO has a boundary optimum it may have local non-global optima.
- In the latter case, the θ point determines the face C of the simplex where the global optimum points of SCO can be found. A global optimum point is a feasible basic solution of a polyhedral set which is determined by C and so-called power planes.
- In the latter case, the SC problem appears to be equivalent to a problem of covering C with more spheres than the ones centered at the vertices.

– For a regular instance with all distances between vertices equal, the optimum point of SCO is either unique, or SCO is equivalent to the question of covering C with the spheres at its vertices plus a sphere at its centroid.

As future research we are interested in developing efficient procedures (algorithms) which can be used as infeasibility tests in the context of the solution of blending problems by Branch-and-Bound approaches, where simplices appear as a logical partition set.

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