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Phosphorus Extraction under Changing Population

By:

Shyam Kumar Basnet
(Regd. No. 801009037140)

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Supervisor:

Dr. Hans-Peter Weikard

De Leeuwenborch
Hollandseweg 1
Wageningen University
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Abstract

Phosphorus is a fundamental element to living organisms but it is exhaustible and there is no substitute for it in agriculture. Burgeoning population and their rising affluence levels have led to raised demand of phosphorus. Recycling of municipal and industrial wastes is a new approach to increase the supply of phosphorus and to decrease the extraction of mine phosphate. This study was targeted to examine the phosphate extraction path and investigate on its perturbation with an intervention of close substitutes through recycling. The Critical-Level Utilitarianism theory was followed to deal with social welfare maximization problem. Furthermore, an optimal control technique: the Maximum Principle was employed to analyze the problem.

A log-linear utility function with a Stone-Geary preference was considered to analyze the mine extraction path in a perfect competitive market. Under a constant marginal extraction cost, the extraction path decreases over time asymptotically to a subsistence level. As cost decreases over time due to technological advancement, mine extraction increases in the beginning until it reaches a peak and then, decreases continuously, and stock effect furthermore lowers it. A quasi-arithmetic progression gives a more consistent population growth over an exponential growth under a constraint of exhaustible natural resources. In a finite time horizon, the former case provides more per capita resource to consume than the latter case. However, it does not hold in the case of infinite time. A higher marginal utility of children raises population growth rate and ultimately exaggerates the problem of mine extraction. But the discount rate deters investment in population and tends to lower population growth rate and lower per capita resource extraction. Though phosphorus is non-destroyable in nature, a fraction of extracted mine would not be available for re-consumption through recycling from the ecosystem. Recycling supply substitutes from the waste of ecosystem and thus reduce phosphate extraction from mine. Phosphate extraction path rises in the beginning and after a peak, it starts to fall over time period. In case of phosphate mines only as stock of phosphorus, mine extraction path does not reveal a direct relation with recycling rate, and thus concluded that the entire ecosystem can be considered as the stock of phosphorus while considering recycling as a supply source.

Key words: Phosphorus, Population Growth, Recycling, Non-destroyable resource, Ecosystem, Stone-Geary preference.

Chapter 1. Introduction

1.1 Background

Phosphorus is a fundamental element to living organisms, and sufficient intake is necessary for growth, maintenance and repair of the body tissues. It supplies nutrient to the human beings through either plants (crops, vegetables and fruits), or animals (meat). Phosphorus is usually applied in the form of phosphatic fertilizer and bone meal into the soil to make it available to the plants. It is an indispensable requisite to meet the increasing demand of crop production. But it is an exhaustible natural resource because of its limited intertemporal flow of services. An optimal use of phosphorus is thus getting a prime importance due to an increasing population pressure on food production.

It would be easy to solve a problem of an optimal extraction of such exhaustible resources over a finite horizon. But the question of what next resource will fill the void space arises. Obviously, there is no substitute for this mine phosphate, and a complete extraction of it ensues the end of industrial revolution and economic growth. Thus, the time horizon for phosphate extraction would be large enough to support economic growth and well being of generations. Koopmans (1965) suggested that optimal programs won't be affected very much by the choice of time horizon. Arrow (1968) also suggested to proceed with the limit of mathematical infinity horizon in the model to capture a mathematical approximation of the real world. Withagen (1984) mentioned that Gray (1914) was the first to formulate a theory of mine, and Hotelling (1931) for theory of exhaustible resources on which almost all the present studies are based. In the field of natural resource economics and exhaustibility of resources, a large number of questions has been raised for discussion and some of the prominent ones include: what is the optimal depletion rate of a given fixed reserve of a phosphate mine with a given technology and population size, how does the extraction pattern get affected from a changing population pattern, how does research and development of close substitutes of this mine phosphate influence extraction pattern, and more questions similar to these. These questions take into account resource problems on a world scale or within closed economies. This study deals with exploitation of mine phosphate in a single economy with a partial equilibrium model and also extended to the changing population scenarios.

This study analyzes an optimal extraction pattern, and addresses the issue of how the consumption of phosphate yields maximal welfare of a given population size. It is extended to further study how growing population affects the phosphate extraction pattern, and furthermore, focuses on studying the sensitivity of extraction pattern with a variation in expected population growths and generation of close substitutes. Exhaustible resource problems are, by nature, dynamic and have a time dimension, and the optimizing behaviour over time is analyzed with the use of optimal control techniques.

1.2 Problem of Statement

Phosphorus fertilizer is an indispensable requisite to meet the increasing demand of crop production but it is an exhaustible resource. It is estimated that there were about 18 billion metric tonnes higher-grade phosphate rock reserves out of 24.3 billion tones (USGS, 2008a). Morrison (2009) reported that about 15 billion metric tons phosphorus resource are a profitable higher-grade mine, and its life-time was estimated at about 100 years, but Sameus (2009) estimated it at only 60-90 years at current market prices.

According to FAO (2008), there is no substantial room to increase food production in developed countries. Agricultural intensification has become essential to meet an increasing per capita food consumption of fast growing population in developing nations. This population demands more food and fiber crop to feed as well as clothe the additional people which ultimately leads to increased fertilizer demand. Meanwhile, hiking oil prices induced a high fertilizer demand for more agricultural production which is used to produce alternative energy sources (bio-fuels). Fertilizer demand may be even larger if production of bio-fuels continues to expand, and economic growth of developing countries further soars up.

But there is no substitute for phosphorus in agriculture; plus, it cannot be manufactured or synthesized (Cordell, 2008), (USGS, 2009). Thus, this increased demand trimmed down the phosphate rock reserves over time which raised the world price of mine phosphate due to high processing cost of lower quality rock (Jasinski, 2008). It is only a matter of time before the phosphate mines go out of stock in the future if the current trend of consumption continues. Adoption of improved technology such as recycling of organic phosphorus sources and improving nutrient use efficiency are likely to lead to a reduction in fertilizer demand.

This research is thus targeted to examine the optimum phosphate mine extraction path and investigate on how an increasing food requirement invites phosphorus crisis in future.

1.3 Theoretical Framework

Phosphorus is a primary nutrient for agricultural production whose consumption is dependent on human population. The fast growing population in developing nations has led to the rise in demand for more phosphorus in order to grow food products. Meanwhile, soil erosion, removal of crop residues and wastage of manures from agricultural field are the main causal factors of loss of phosphorus in agriculture. Recycling of municipal and industrial wastes and struvite crystallization are some of the new technologies implemented to recover phosphorus loss. In addition, management of cropping practices such as composting is a way to lessen the pressure of increasing demand from phosphorus poor countries.

1.3.1 Phosphorus Production

Today, optimization of phosphorus extraction is a matter of vital significance given its indispensable importance to human life and finite resource constraint. About 90 per cent of global phosphate rock is extracted for food production and the remaining for industrial application. The Hubbert Linearization theory postulated that the production of non-renewable resources follows a continuous inverted-U curve. White *et al.* (2008) used phosphorus production data between 1900-2007 from US Geological Survey and fitted to a Hubbert-type bell-shaped curve to analyze a peak phosphorus production. It is predicted that the peak production will be at around 2033 and start to drop in consecutive years even in case of maintaining 2007's consumption rate. But the US Geological Survey estimation model postulated the most likely peak production at around 1990 USGS (2008b). It was debated on endowment of phosphate rock reserve, its peak production and the lasting period. Ward (2008) showed a better fit of Hubbert-type bell curve on historical data set of both annual and cumulative production. It was postulated that the peak phosphate production of "easy" reserves would be at around 1990, and at around 2033 in the case of "hard" reserves.

1.3.2 Population Growth and Phosphorus requirement

The global population has been increasing sharply and it has been forecasted to reach about 9.2 billion in 2050 at a current average fertility rate. Population growth rate of developed countries by 2050 is expected to be more or less stationary, but in case of least developed and less developed countries, it is expected to be at around 1.15 and 0.5 percent, respectively (UNPD, 2009). Since more crop production exaggerates the removal of primary nutrients from soil, increasing food requirement will accelerate phosphorus depletion from agricultural soil.

Vlek *et al.* (1997) estimated that nutrient requirement for developing world would double by 2020. Zhang *et al.* (2007) reported that the growing world population causes to intensify the use of fertilizer nutrients and increase world's total fertilizers consumption by 32.1 per cent and of phosphorus by 25.8 per cent by 2030 against the current level. According to Matthews *et al.* (1999), fertilizer use in developing countries was increasing rapidly due to rapid population growth and growing food demand, but it was marginal in developed nations due to very low population growth during the period of 1970-96. In Asia, phosphorus consumption has doubled over the past 10 years and was forecasted to grow at an annual rate of 5 per cent over the coming 5 years (Belmehdi *et al.*, 1989). But Asia and the developing nations are poor in phosphorus and have to depend on imports to sustain their increasing food requirements. Thus, population control has become an indispensable requisite to reduce the pressure of food demand and hence preserve a potential phosphorus crisis in the future.

1.3.3 Recycling and phosphorus extraction

Recycling of industrial wastes and urban sludge provides substitute for scarce mine phosphates. Recycling and reusing of crop and animal wastes, though it is less sizeable, recovers phosphorus and reduce mine phosphorus requirement (Cordell *et al.*, 2009). The fertilizer-cost to crop-yield ratio can be minimized through the improvements in farming systems and recycling methods of municipal and industrial wastes (Liang *et al.*, 1989). Struvite crystallization from animal waste is a successful and promising technology for the recovery of phosphorus (Burns *et al.*, 2002). The problem of phosphorus scarcity has recently appeared with growing industrial agriculture. Phosphorus recovery from organic urban waste and manure has the potential to reduce demand for phosphorus in food production. Slade (1987) argued that the discoveries of new reserves and the supply of substitutes through recycling help to keep away the depletion of phosphate mine. Thus, recycling process may introduce the substitutes of mine phosphate and extend the life of reserves.

1.3.4 Ethical Framework

Understanding the principle of economics to bring a balance between supply and demand of natural resources do not always give an accurate description of real market principles. This reveals market failure to allocate scarce resources efficiently to generate the maximum social welfare. Many economists wish to give an equal space to normative economics and address a question of what *should* be done in a particular set of circumstances. It is suggested to seek intergenerational equity for allocating scarce resources optimally. It is inevitable to link resource economics with ethic or moral philosophy. Welfare economics assesses the consequences of individual welfare that is related to the ethics of natural resource depletion from the 'consequentialist' and 'welfarist' approach developed by Bentham, Mill, and others. This approach evaluates the actions in terms of their consequences for consumption of goods and services by individuals. John Rawls in *A Theory of Justice* stressed on equal rights of each individuals to the most extensive basic liberty and redistribution of utility from individuals with higher utility to those with lower utility for *Pareto* improvement of social welfare (Perman, 2003). Defining social welfare was dominated by the economists' analyses and claimed that individuals only care about their own consumption of produced goods and services.

Depletion of exhaustible natural resources and its impact on future generation is necessary to consider. It raises the questions of rights as a heart of the analysis of intergenerational equity and justice. Unwise use of such essential resources transfers its impact on future generations, and the future generations have an equal right to enjoy natural resources and have a standard of living no lower than the current one. Current generations have a responsibility to protect the resources from its inequitable consumption and have no right to worsen their successor's life. This generation ought to preserve endowment of resources for the next generation in a similar manner in which they received from the previous generation.

1.3.5 Theory of Population Ethics

It is important to discuss about population size from an ethical ground as well. The basic intuition that explain the ethical logic of increase in population size is based on valuing the population. It can be argued that adding additional people will be worthwhile if the marginal value of the additional people is positive. In literatures, welfare economists point out that there would be nothing wrong to have additional people if they add to the well-being leading to good lives, and make no harm to the existing population.

Social choice theory was developed as a theoretical framework to measure individual interests, values or welfares as an aggregate in welfare economics. In utilitarian ethical theory, it is assumed that each individual always puts efforts to maximize their individual utilities and have the same lives. Welfare economics concerns with the allocation of a good, that is, well-being and makes the people socially indifferent in a distribution of well-being as like dividing a cake fairly among a given group of people. Different social choice theories have been discussed below:

1.3.5.1 Classical Utilitarianism / Total Utilitarianism

Total utilitarianism (TU) identifies an optimal population that maximizes the sum of everybody's lifetime welfare. Thus, the total utilitarian social welfare function can be written as

$$W(C/L) = \int_0^\infty L_t . U(C/L).dt$$

where L_t is the size of the population at a time t, and $U(C/L)$ is the individual utility derived from a consumption.

Classical utilitarianism – propounded by Jeremy Bentham – suggested to have an analysis of good and bad sides of an action in terms of utility in the same way as economists do the cost-benefit analysis, and compare the interpersonal utility. Critics who do not believe in cardinal utility disagree to the above intuition of assessing good and bad aspects of people. Broome (1996) highlighted the inconsistency in Bentham's theory and discussed the issue of the repugnant conclusion as defined by Derek Parfit.

1.3.5.2 Average Utilitarianism

The Average Utilitarianism (AU) aims to maximize the average well-being of the existing people but is not concerned about having a large number around as in the case of Classical Utilitarianism. This can be done by two ways: firstly, by making existing people better-off and secondly, by increasing the population size of those people whose well-being is above the average well-being. Dasgupta (1988) mentioned that the AU in case of large population sizes avoids the repugnant conclusion, but it does not distribute to the well-being of existing people even in case of adding people with a higher well-being than the average, and identified it as a main drawback of average utilitarianism. In an intertemporal framework, the AU is expressed as

$$W(C/L) = \frac{1}{L_t}\int_0^{\infty} U(C, L, n).dt$$

The AU does not provide compensation to a large number population whose average well-being is less than the new persons. But the TU accepts the addition of another person until the total welfare is increased. Thus, the AU will have a smaller population as compared to the TU, and does not lead to the Parfit's repugnant conclusion (Ponthiere, 2003).

1.3.5.3 Critical-level utilitarianism

Critical-Level Utilitarianism (CLU) is a recently developed social choice theory and it was postulated by Charles Blackorby, David Donaldson and Walter Bossert (Broome, 1996). The CLU defines the optimal population as one that maximizes the sum of differences between individual utilities and a constant non-negative critical level of utility (Ponthiere, 2003). It is expressed as,

$$W(C/L) = \int_0^{\infty} [U(C, L, n) - \alpha].dt$$

where, α is the critical level of well-being and assumed to be constant over time.

According to this theory, a person having well-being above the critical level adds value in total, and otherwise in the opposite case. Similar to the classical utilitarianism, the critical-level utilitarianism also aims to maximize total utility. This theory supports having an additional person if s/he has a well-being above the critical level. (Broome, 1996) puts on a defensive argument in favor of the CLU theory since it is free from the problem of repugnant conclusion. However, it is pointed out that the well-being level of existing people though it is above the critical level will be reduced with the entry of additional people. This suggests to fix the critical-level at high. But in literatures, it is also commented that the fixing critical-level at high makes sacrifices of the people having modestly good well-being level, and it also possesses the drawbacks of average utilitarianism as discussed above. In case of low critical-level, it is equally probable to have the Parfit's repugnant conclusion if a situation is characterized by a mass of population having a utility level closer to the critical level. Thus, there is a serious debate in literatures on fixing the critical-level and this argument creates doubts about adopting the critical level theory.

1.3.5.4 Critical-band utilitarianism

Critical-Band Utilitarianism (CBU) is an extended theory of the CLU theory and it works with an assumption of a critical band of well-being, say for an example [a,b], rather than a single critical level. It supposes that adding people with a well-being above the band is good and is bad in case of people with a well-being below the band are added. A person with a well-being within the band is regarded as a neutral. According to this theory, adding a neutral person is not a bad, and thus it preserves the basic intuition of adding a person with neutral characteristic that enhances the greatest happiness of the greatest number. It considers an option equally as good as the other, but does not rank them. Thus this theory connotes that adding a person is equally as good as not adding, and it leads to contradiction and choosing a band randomly would not be appropriate. As long as the net gain in total utility is positive, the increase in population is desirable under the TU. Temkin (1992) was also in favor of the TU and argued that it supports the intuition of preferring more of the Good to less of the Good. Michel *et al.* (1998) identified a situation of a drop in the optimal level of consumption with an increase in economy in the TU, and tagged it as a relative repugnant conclusion. Thus, Michel *et al.* (1998) raised doubts on the adequacy of total utilitarianism.

In favor of the TU, Ng (1989) argued that a person's choice does not influence anyone else's happiness, and it strongly supports the utilitarian goal: the promotion of greatest happiness of the greatest number. But the AU excludes a considerable mass of people having utility just below the average but still positive. The CLU and CBU theories avoid the undesirable character of the AU since it considers to add people with utility higher than the critical level as a socially profitable. These theories with a positive critical level provide a safety cushion at least to some extent from the Parfit's repugnant conclusion over the classical utilitarianism. But both of them are highly suffering from the problem of the choice of the critical level of utility. In our study, we follow the critical-level utilitarianism theory to deal with our social welfare maximization problem under a resource constraint.

1.3.6 Partial Equilibrium

In this section, all theoretical issues to be considered in this study are brought together in a common framework model. We develop a simple, partial equilibrium model explicitly designed to analyze the effects of changing population. In a general economic equilibrium, extraction and transformation of phosphate mine into final goods, supply of those goods to the consumers for final use, and recycling or removal of wastes produced in that production and consumption process are reconciled, and the maximum utility is achieved. An economy is characterized by given knowledge about the following,

Consumers' preferences: A rational consumer always prefers to maximize utility derived from consumption of resources. A society maximizes a social welfare in the utilitarian sense, and welfare maximization does take place in a dynamic environment.

Resource endowments: Resource endowment of the economy is a stock of natural resources.

Technology: Technological advancement in resource mining reduces cost per unit resources and vice-versa. Meanwhile, depletion of existing stock also increases cost as extraction goes further deeper and deeper. In our study, we consider a constant technological effect on resource extraction industry.

Market forms: The study is confined in a single closed economy to make the model simple. This model only considers the intergenerational flow of resources, but not in- and out- flow of resources from an economy.

We concentrate on economic models of phosphate extraction in equilibrium. In this study, the ends is to maximize the consumer utility, welfare or profit under a set of constraints that are given by the environment and recognized by economic decision makers. However, these constraints do not capture all theoretical issues of an economic decision process. Our model is, therefore, by nature partial equilibrium model, and it assumes all economic conditions outside phosphate extracting sector as exogenously given.

1.4 Basic Modeling Framework

Resource economics in general analyzes alternative consumption paths of extracted natural resources, and finds an optimal path that maximizes the present value of utility, profits or welfare of resource consumption. Since phosphorus is an exhaustible resource, an optimal price of a unit of phosphate encompasses both marginal extraction cost and its opportunity cost of its endowment.

1.4.1 Benchmark model

In our benchmark setting, it is assumed that all phosphate-extracting firms are operating in perfect competition for both input and output markets with perfect information. By assumption, an individual owner cannot influence a phosphate's market price, no replenishment of stock is possible, and no exploration of new deposits occurs. Demand for phosphate is given and known to all participants in a resource market. Each phosphate-extracting firm faces an identical cost of extraction, and is aware of the opportunity cost of operating mine-extraction.

The assumed goal of firms is to maximize the discounted sum of all profits that are made from the date at which extraction begins to the terminal date at which phosphate stocks are depleted. The key question to be addressed here is: how much mine phosphate should be extracted in each period? An optimal extraction path is characterized by a path that reveals indifference between extracting an additional unit today and leaving it in the ground for extraction in the future. It does not satisfy the boundary conditions - either extreme of extracting all today or all in the future. Ruth (2002) thus pointed that opportunity costs, marginalism, substitution and time preference play a crucial role for taking economic decisions of resource extraction. In this dynamic model, an opportunity cost reflects the extent to which a marginal unit of resource extracted today rather than tomorrow influences overall utility or profits. Marginalism explains the impacts of marginal changes of resource extraction or consumption on utility or profits. The concepts of opportunity cost and marginalism are useful to compare a wide range of alternative consumption paths, and find an optimal one. A concept of substitution reflects the ways of using resources to achieve desired ends.

Consumers choose actions that maximize utility not only within a period of time but over a set of periods. It is highlighted to consider temporal distribution of consequences of their actions, thereby reflecting time preferences. Thus, an appropriate discount rate gives a reasonable trade-off between these decisions. According to Hotelling rule, an optimal resource extraction path must satisfy the condition that the price of an exhaustible resource minus marginal extraction costs rises over time at exactly the rate of discount, that is,

$$\frac{\dot{P}}{P_t} = r \ ,$$

where, r is the consumption discount rate and P is the in-situ value of phosphate.

1.4.2 Intertemporal Appraisals and Discounting

Choice of different consumption patterns over time periods depends on their consequences for individuals now and in the future. For an intertemporal analysis of consequences of consumption patterns, three factors have to be considered. First, a modeling of the consequences of resource consumption, second an aggregation of the consequences of individuals into an overall welfare, and third, an aggregation of overall welfare of individuals across generations. To keep the analysis simple and to focus on intertemporal ethics, we compare consequences of different consumption patterns with a judgment of overall welfare under the assumption of existence of an individual (or a group of identical individuals) at each point in time. It is focused on analyzing the behavioural pattern of current generation with respect to future generation.

Discounting over a substantial time horizon (several decades or more) is like a gamble with the welfare of future generations in the context of uncertain future costs and benefits. Stern (2006) defined discounting as marginal perturbations around a given consumption path, and not for comparing across the paths. Discounting puts less value on utility of future generation (as opposed to consumption) because of uncertainty on existence of the world. Discounting factor and discount rate evaluate the changes in marginal utility of consumption over time. Growing consumption is a reason for discounting utility. Weikard *et al.* (2005) argued that decreasing marginal utility of consumption over time, impatient nature of human beings and growing productivity of time are the reasons for discounting. In sum, diminishing marginal utility of consumption, and myopic behavior of human beings bring on discounting. However, choice of discount rate is controversial in environmental economics. Some literatures argued that constant discount rate is not justifiable for intergenerational equity, and it depends on underlying pattern of resource consumption. According to the classical Ramsey model, the consumption discount rate is the sum of pure rate of time preference, and the consumption growth rate multiplied by the elasticity of marginal utility of consumption. It is mathematically written as,

$$r = \eta \frac{\dot{C}}{C} + \rho$$

where,

ρ – Utility discount rate

η - elasticity of marginal utility of consumption

$\dot{C}/C$ - consumption growth rate along a path, and

The Ramsey formula as depicted above indicates that a pure rate of time preference is a utility discount rate, and a consumption discount rate is adjusted utility discount rate with a diminishing marginal utility of increasing consumption. Hotelling (1931), Farzin (1984), Daily *et al.* (1992) and Speck (1997) mentioned that discounting essential resources at higher rate induces faster depletion of resources, and encourages the public to underestimate the importance of future costs. Perman (2003) on the ethical basis argued that utility of each generation should be treated equally, and a positive discount rate gives less weight to the well-being of future generations. But we are uncertain about the existence of future generation, so less weight should be given to the future generations than the present. Thus, choice of discount rate is vital to the evaluation of economic activities.

Once a discount rate is chosen for the evaluation of alternative consumption paths, the second issue of whether this rate can be assumed to remain constant over time. Asheim (1994) stated that discounting utility at a constant rate does not give a constant utility path. Later, Pezzey *et al.* (1998)also supported Asheim (1994) and used a constant discount rate but a higher value. It is reported that a higher rate of discounting gives higher initial consumption, and an optimal consumption initially rises and then start to fall with a single peak. Weikard *et al.* (2005) stressed on giving consideration to the environmental amenity value of a stock or resources and discussed about dual-rate discounting. It is argued that exponential discounting is a standard approach for intertemporal evaluation because of uncertainty in the long-run interest rate and growth rate of economy. Rackwitz (2006) suggested to use about 1 % as an intergenerationally acceptable rate of time preference of consumption. Thus, researchers have their own stands with argument in the cause-and-effect relationship between different discount rate and phosphorus extraction.

1.5 Research aim/objective

The main goal of the research is to examine phosphorus extraction path over time period under growing population dynamics.

Specific objectives:

1. To analyze the effect of changing population on phosphorus extraction.
2. To analyze the effect of recycling on phosphorus extraction.

1.6 Research Questions

The research mainly focuses on optimization of phosphorus extraction under changing population dynamics over time period. Thus general research question is: how phosphorus extraction occurs under changing circumstances of population. This research looks for changing pattern of population under the present issues of food insecurity and its effect on phosphorus extraction. The main questions that will be answered by the research are as follows:

1. How does an increasing population affect phosphorus extraction over time?
2. How does recycling affect on optimizing phosphorus extraction over time?

1.7 Conceptual Framework

Phosphorus is a finite and exhaustible resource, and it is only a matter of time before it gets completely exhausted in the future. But the matter of debate is only on time duration. Different researchers are putting forward different conclusions from an analysis based on an exogenously given Hubbert-type production curve (Ward, 2008). This research study also applies the Hubbert-type production curve to get insights on the pattern of phosphorus production. In addition, this research looks for the production curve that is endogeniszed by population, and makes prediction of optimal extraction path. In the meantime, this study also considers recycling and reusing of organic urban waste and manure as a substitute for mine phosphorus in food production. It is thus hypothesized that phosphorus extraction increases over time period to feed the increasing population. The specific hypotheses are as below:

1. Population growth positively affect mine phosphate extraction.
2. Recycling of organic urban wastes extends mine phosphate use.

Chapter 2: Analysis and Discussion

2.1 The Model

In economic literatures, studies and discussions about exhaustible resources are becoming common. The intertemporal sum of services provided by a stock of such resource is finite. It leads to the scarcity of this resource because of diminishing availability of stock of services due to previous consumption. The problem of exhaustible resource is like a cake-eating problem, and consumption at present increases the scarcity of resource in future. Thus, an optimal extraction of such resource is important for intertemporal allocation. The Hotelling's approach is an influential approach to model such an intertemporal depletion and it derives an optimal consumption path that maximizes the present value of consumption.

Let a rational consumer aim to maximize the benefits obtained from the consumption of extracted resources. It is supposed that the utility function, $U(X(t),t)$, derived from resource consumption gives a measure of consumer's benefits in terms of utility. It can be mathematically expressed as,

$$B(X) = U(X(t),t)$$

Since a gross consumer surplus provides an assessment of utility gain, the utility driven inverse demand curve gives an indication of consumers' willingness to pay for phosphorus resource. Thus, a demand curve is derived and the utility is an area under demand curve,

$$W(X(t),t) = \int_0^{X(t)} D^{-1}(X,t)dX \qquad \text{………….. (2.1)}$$

where, $D^{-1}(.,.)$ is the inverse demand curve with respect to price. This area under inverse demand curve gives the measurement of gross consumer surplus in terms of utility gained, and is a measure of consumers' willingness to pay for phosphorus resource. At market clearance condition, the price $P(t)$ equates demand and supply of phosphorus.

$$D(P(t),t) = X(t) \qquad \text{………… (2.2)}$$

where, *X(t)* is the amount of resource extracted to supply.

In a perfectly competitive market, a constant marginal utility of income is equal to the resource price

$$P(t) = \frac{\partial U(X(t))}{\partial X}. \qquad \text{…………….. (2.3)}$$

According to the Hotelling rule, the optimal intertemporal allocation of resource satisfies the condition below

$$\frac{\dot{U}'(t)}{U'(t)} = \rho \qquad \text{…………… (2.3a)}$$

where, ρ is the discount rate.

The marginal utility of consumption rises at the agent's discount rate when the marginal utility of last unit consumption is the same in each time period.

By Eq (2.3) and (2.3a), we get

$$\frac{\dot{P}(t)}{P(t)} = \rho \qquad \text{………. (2.3b)}$$

This is called the "Hotelling rule". It says that the investment in resource extraction goes till the current production will earn the same if it is invested in other assets. If $\frac{\dot{P}(t)}{P(t)} > \rho$, it encourages reduction of the current production because the returns to scale in the future would be higher than the present value, and vice-versa. Schmidt (1988) supported the Hotelling theory on prices of exhaustible resources. It was argued that the changes in demand and technological advancement makes the resource constraint less binding and suppresses the pressure for rising real resource prices.

Consider a natural resource stock, *S,* can be depleted only when $\dot{S}$ is negative (*i.e.*, consumption is positive). At present price of extracted resource (P_t), price of resource stock (μ_t) and positive utility discount rate (ρ), a path $(X_t^*, S_t^*)_{t=0}^{\infty}$ will be feasible only when X_t^* maximizes $\int_0^{\infty} [U(X_t) - C(X,S,t)].e^{-\rho.t}.dt$ for each t subject to the constraint $\dot{S} = -X_t$. The present price of resource stock is an imputed rent equal to the marginal product of stock. In case of non-renewable resources, it is equal to the profits or rents that arise when such resources are depleted. Because of scarcity, prices always exceed per unit production cost throughout the period of its consumption. It is assumed that a feasible competitive path $(X_t^*, S_t^*)_{t=0}^{\infty}$ satisfies the following two conditions,

a) $\int_0^{\infty} U(X_t^*)e^{-\rho.t}dt$ exists and is finite

b) $\mu_t S_t^* \to 0$ to $t \to \infty$.

In our model, we maximize the welfare of representative individuals or households, which includes the present value of utility of consumption over time. For an allocation of natural resources to be optimal, the model maximizes a social welfare function, which is obtained from individual's well-being. Consider, an utilitarian social welfare function as

$$W = \int_0^{\infty} [U(X_t) - C(X, S, t)]e^{-\rho.t}dt \qquad \text{………… (2.4)}$$

It is assumed that the utility in each period is a concave function of the level of consumption in that period with $U'(X) > 0$ and $U''(X) < 0$. The diminishing marginal utility of consumption reflects the value of an additional unit of consumption declining as the society gets richer. Since the resource is non-renewable in nature and finite in stock, the total use of the resource over time is constrained to be equal to its initial stock. Denoting the initial stock (at $t = 0$) as S_0, and a rate of extraction and consumption of resource at a time t as X_t, the two constraints to be satisfied by an objective function can be written down as

i) $\dot{S}_t = - X_t$, where $X_t \geq 0$ and

ii) $S(0) = S_0$

where, the dot over S indicates a time derivative.

2.2 Benchmark Case with Constant Population Size

In our benchmark case, we assume that the economy has an exogenously given fixed size of population (labour force) that lives for an infinitely long period of time. The economy is inhabited by a single dynasty with an initial population of unity. All the individuals born at any date live forever. It is assumed that there is no growth of population in size. So the size of dynasty at initial time t_0 is normalized to one, that is, $L_0 = 1$. Not a single member of dynasty gets children in a period. The utility of each individual of a dynasty solely depends on consumption of phosphorus resource and the objective function of the dynasty is the welfare function for society. Since the economy has a constant population size, it is not necessary to distinguish between total and per capita well-being. Phosphorus resource is an essential one and there is no backstop technology or renewable substitutes. Thus, the feasible consumption must necessarily decline to zero in the long run. Let *U* be an instantaneous utility function satisfying *U*: strictly increasing, strictly concave, twice continuously differentiable, and $\partial U(X)/\partial X \to \infty$ as $X \to 0$. The society maximizes the total utility of the individuals derived from the consumption of exhaustible resource over a given time period [0,∞). For analysis, we considered an intertemporal welfare function as,

$$W = \int_0^\infty U(X_t).e^{-\rho.t}.dt$$

This formulation assumes that the utility at time t_0 is a weighted sum of all future flows of utility, $U(X_t)$. We assume a strictly positive rate of pure time preference or impatience ($\rho > 0$) because it values the utility less the latter and the people prefer current consumption than the future. The parents prefer a unit of consumption over a unit of their children's consumption. But Ramsey (1928) suggested to consider $\rho = 0$ for the reason that a social planner, rather than competitive household, should choose a smooth consumption for today's generation as well as for the future generation. Thus, the positive rate of time preference, also called as utility discount rate, reveals the selfishness of parents towards their children and some form of impatience. While supporting the above arguments, Weikard *et al.* (2005) stated that the value of a unit of consumption may change over time with a consumption discount rate, *r*, even if an utility discount factor is nil to discount future consumption. In our case, the households are competitive and take a positive rate of time preference. In definition, consumption discount rate *r* is the rate at which the marginal utility of a small increment of consumption falls as time changes. According to Asheim *et al.* (2007) and Weikard *et al.* (2009), a consumption trajectory obtained from the utility function to be an optimal does satisfy the necessary conditions,

$$\frac{\frac{d}{dt}(U'(X)}{U'(X)} = r \text{ (Hotelling's rule)}$$

Barro *et al.* (2004) and Weikard *et al.* (2005) talked about the basic condition for choosing consumption over time as

$$r = \rho + \eta.\frac{\dot{X}}{X} \text{ (Ramsey's rule)}$$

where, η is the elasticity of marginal utility of consumption.

According to the above relation, it can be said that the household chooses a consumption path that satisfies the equality relation between the consumption discount rate and the utility discount rate plus the rate of change of marginal utility due to changing consumption. If $\frac{\dot{X}}{X} > 0$, X is low relative to tomorrow. But the second term of right hand side of Ramsey's rule explains that the consumers like to smoothen their consumption over time because $U''(X) < 0$. The optimization problem makes these two rates of return equal and indifferent to consume resource or preserve in the ground. For $\dot{X}/X$ be a constant, the elasticity of intertemporal substitution, that is, the reciprocal of elasticity of marginal utility, must be constant (Barro *et al.*, 2004). Thus, an instantaneous utility function with a constant elasticity of marginal utility is considered in our study,

$$U(X_t) = \frac{X_t^{1-\eta} - 1}{1-\eta} \qquad \text{…………… (2.5)}$$

where, the elasticity of marginal utility equals a constant, $-\eta$, and the intertemporal elasticity of substitution, $\sigma, = \frac{1}{\eta}$. As η is getting higher, the proportionate decline in $U'(X)$ in response to increase in X becomes more rapid and the consumers are less willing to deviate from a uniform pattern of X over time. A higher elasticity of marginal utility implies a higher degree of consumption smoothing over time.

The optimality condition from Ramsey's formula for utility function is,

$$\frac{\dot{X}}{X} = \frac{1}{\eta}.(r - \rho).$$

The above equation explains the relation between r and ρ that determines the flow of consumption pattern over time. When the case of $r = \rho$, the $\dot{X}/X = 0$, that is, the consumers choose a flat consumption path. When $r > \rho$, the consumers like to move out from the flat consumption path. For sustainability of resource consumption, the condition $\dot{X}/X > 0$ induces the consumers to sacrifice some consumption today for more consumption tomorrow, and this holds only when $r > \rho$. The second term of the right hand side of Ramsey formula gives a compensation to stand an interest rate, r, sufficiently above the discount rate, ρ.

Phosphorus is an essential resource for living organisms and it is non-renewable and non-substitutable resource. Hence, the elasticity of marginal utility becomes equal to one, that is, $\eta = 1$, and utility function becomes $U(X_t) = \log X_t$. Dasgupta (1969) and Stiglitz (1974) claimed that the individual welfare level will be negative at a very low consumption level of resource, and thus there exists a minimum consumption level (X_{min}) at which life is just about enjoyable. The consumption at X_{min} level of resource gives a zero level of utility, and it is worthless to live with a negative utility. All living organisms need at least a minimum positive level of consumption for positive utility, which is characterized as a "welfare subsistence level". A Stone-Geary function gives the measure of utility from consumption with the subsistence levels. This function characterizes the necessity of consumption and existence of subsistence consumption level irrespective of its price or the consumer's income, and hence the consumer gets utility from that part of consumption which exceeds the subsistence level.

$$U(X) = \frac{(X_t - X_{\min})^{1-\eta} - 1}{1-\eta} \qquad \text{……………. (2.5a)}$$

where, the parameter X_{min} describes the necessities of commodities. As $\eta \to 1$, the $U(X)$ approaches $\log (X - X_{min})$, and this logarithmic preference gives a stable, indeterminate parameter values (Koskela *et al.*, 2007). In this study we use the natural logarithmic function to model Stone-Geary utility function.

$$U(X) = \ln(X_t - X_{\min}) \qquad \text{……………. (2.6)}$$

Here, we have assumed the subsistence consumption, X_{min} as exogenously given. The elasticity of marginal utility, $-X.U''/U' = X/(X - X_{\min})$, is one with logarithmic preferences, and in case of greater than one, it is a decreasing in consumption and increasing in subsistence level with Stone-Geary preferences. Thus, the modeling a subsistence level of consumption implies the features of a time-varying elasticity of marginal utility. Seyhan *et al.* (2009) proposed a functional form of utility with a minimum level of consumption as

$$U(X_t) = \ln[(X_t - X_{\min}) / X_{\min}] \qquad \text{…………… (2.6a)}$$

Comparing Eq (2.6) and (2.6a), the latter induces a higher extraction than the former because Eq (2.6a) gives a higher minimum level of resource consumption. Since phosphorus is an essential nutrient for life and a key non-renewable and non-substitutable resource, it will be worthwhile to minimize its extraction and consumption for intergenerational equity. In this study, we considered the utility function as suggested in Eq (2.6) for our further analysis.

2.3 Extraction Path under a Constant Marginal Extraction Cost

In a perfect competition, it is assumed that the competitive pricing of factors and resource output in the absence of uncertainty produce zero profit, and a price of output equal to a unit cost of production as measured by the capital rent. For simplicity of our analysis, the absence of resource scarcity rent in a cost function was assumed. Thus

$$P_t X_t = C X_t \qquad \text{…………. (2.7)}$$

where, P is the price of current flow of output and C is the capital rent per unit of product. According to the Hotelling’s rule, the shadow price of an exhaustible resource, price minus marginal extraction cost, grows at a market interest rate in a competitive resource market. It would be the case when marginal extraction cost is equal to zero, but grows as a rate less than the interest rate in case of non-zero constant marginal extraction cost. In general, the resource reserves occur in varying grades, and the higher quality are exploited first. Following the Hotelling’s rule, we assumed the case of an absence of stock effect on extraction cost. Furthermore, we also assumed the possibility of an exploration of new reserves and stock due to research and development, and thus there won’t be a resource scarcity due to the consumption. In this thesis research, we did not go for the study of an exploration activity and its effects on resource extraction. We assumed here the existence of constant marginal extraction cost *(C).*

Now we defined Net benefit *(NB (X,C,t))* of extracting phosphorus as total benefit minus total cost for an economic analysis and mathematically written as,

$$NB(X,S,t) = \ln(X_t - X_{\min}) - C.X_t$$

In a partial economy of phosphorus, a rational owner always maximizes the present value of net benefits stream by choosing an optimal quantity to be extracted in each time, $\{X(t)\}_{t=0}^{\infty}$, given the demand and cost schedules, and the stock size.

It was assumed that all the extracted resources were consumed in the economy within the same time period. The benefits and cost streams earned and made by resource owners were measured in utility terms, and the net benefit gives the welfare function of a society. Thus, in our optimization problem we maximize the social welfare to optimize the phosphorus extraction with given economic constraints.
The dynamic optimization problem is,

$$\underset{X}{Max.}\ \ W = \int_0^{\infty} [\ln(X_t - X_{\min}) - C.X_t].e^{-\rho.t}dt \qquad \text{…………..} (2.8)$$

subject to,

i) $\dot{S}_t = -X_t$,
ii) $X_t \geq 0$
iii) $S(0) = S_0$ S(T) free (S_0 given)

In recent literatures, the economists were pointing out that the constraint of exhaustibility of natural resources was a limit to the growth of an economy. They were raising concerns on how the limited supply of resource is best to be allocated over time. In this study, we use the Optimal Control Theory to tackle above question. The calculus of variations, a classical method of solving dynamic optimization problems, can handle only interior solutions with a single control variable. In contrast, optimal control theory works with one or more control variables and also deals with non-classical features such as corner solutions. Besides, the optimal control theory works to find an optimal path for a control variable, while the calculus of variations does for a state variable. The optimal control theory believes that we can find the optimal time path for a state variable once we determine it for a control variable since it is a steering variable for the state variable.

2.3.1 The Maximum Principle

The optimal control theory makes the use of first order necessary conditions which is known as the maximum principle. The maximum principle comprises the concepts of the Hamiltonian function and costate variable.

In the model above, the rate of resource extraction at a time *(X_t)* was identified as a control variable since this variable was subject to our discretion and the choice impinged upon the state variable *(S_t)*. The control variable worked as a steering variable that drives the state variable in a path via the equation of motion *dS/dt = -X(t)*. In our problem, we therefore optimized first the control variable to find the optimal extraction path of a finite exhaustible resource with the current-value Hamiltonian function.

$$H_t^c = \ln\left(X_t - X_{\min}\right) - C.X_t - \mu_t \cdot X_t \qquad \text{………. (2.9)}$$

The Hamiltonian function is nonlinear in X_t. On plotting H_t^c against X_t, the Hamiltonian function rises in the beginning with its peak at a *X* value between *X =0* and *X = X_T* as shown in the graph below.

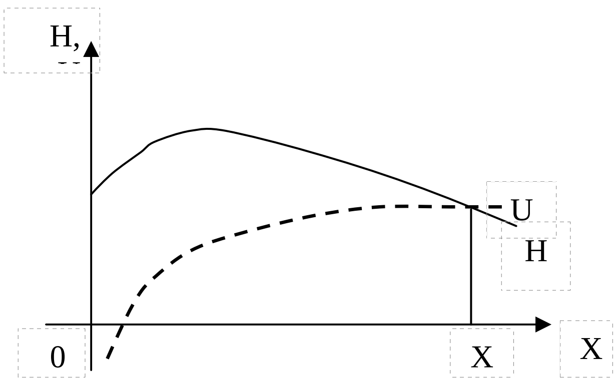

Fig 1: The Hamiltonian and Utility function against consumption.

The maximum principle conditions for optimization problem are,

$$\underset{X}{Max} \quad H^c(t, X, \mu) \quad \text{for all } t \in [0,\infty). \qquad \text{…………. (2.9a)}$$

$$-\frac{\partial H^c}{\partial S} = \dot{\mu} - \rho.\mu_t \qquad \text{………….. (2.9b)}$$

$$\frac{\partial H^c}{\partial \mu} = \dot{S} \qquad \text{………….. (2.9c)}$$

$\mu(T) = 0$ (Transversality condition) ………….. (2.9d)

The maximization of the Hamiltonian with respect to *X* under the maximum principle gives an optimal path of resource extraction (X^*). This holds

$H(t, X^*, \mu) \geq H(t, X, \mu)$

because the maximization problem follows the first-order condition $\partial H_t^c / \partial X = 0$, and yields an interior solution. It is

$$\frac{\partial H^c}{\partial X} = \frac{1}{(X_t - X_{\min})} - C - \mu_t = 0 \qquad \text{…………….. (2.10)}$$

The first term of Eq (2.10) measures the effect of a change in *X* on *U*, which is the marginal utility of phosphorus use through consumption. Similarly, the second term represents the marginal extraction cost, and the last one is the shadow price of phosphorus stock.

Rearranging Eq (2.10), we get,

$$\frac{1}{(X_t - X_{\min})} - C = \mu_t \qquad \text{…………….. (2.10a)}$$

The optimal X_t^* balances any marginal increase in the current benefit, profit or utility made by a policy (left hand side expression of above equation) against the marginal decrease (μ_t is a negative value) in the future benefit that the policy will induce via the change in the resource stock.

Solution of above Eq (2.10) gives X_t in terms of μ_t,

$$X_t = X_{\min} + \frac{1}{(C + \mu_t)} \qquad \text{……………. (2.10b)}$$

Eq (2.10b) implies the control path of phosphorus extraction from a mine.

To make sure that Eq (2.10) maximizes rather than minimizes the Hamiltonian, we checked up the sign of second order derivative of the Hamiltonian function.

$$\frac{\partial^2 H^c}{\partial X^2} = -\frac{1}{(X_t - X_{\min})^2} < 0$$

Since $\partial^2 H / \partial X^2 < 0$, the control path Eq (2.10b) maximizes the Hamiltonian at every point of time as the maximum principles requires. It is necessary to search for an optimal μ_t path to get the optimal control path in Eq (2.10b).

2.3.2 The Co-state Variable

The co-state variable is the shadow price that measures the change in the objective value of the optimization problem by relaxing a resource stock constraint by one unit. It is the value of the Lagrange multiplier at optimal solution that gives a measure of a change in objective function with a change in resource availability constraint infinitesimally. The co-state variable is used to bring the equation of motion, $\left[-X_t - \dot{S}_t = 0 \right]$ for all t in the interval [0,∞), into the objective function of optimization problem. With the Lagrange multiplier (μ_t), it can be expressed as

$\mu_t(-X_t - \dot{S}_t) = 0$ for all *t*. the sum over *t* becomes

$$\int_0^T \mu_t(-X_t - \dot{S}_t)dt = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.11)$$

Now, the new objective function, lets say *V*, is

$$V = \int_0^T \left\{ (\ln(X_t - X_{\min}) - C.X_t + \mu_t(-X_t - \dot{S}_t) \right\} dt$$

By substituting the H_t function, new functional form become

$$V = \int_0^T (\ln(X_t - X_{\min}) - C.X_t - \mu_t X_t)\, dt - \int_0^T (\mu_t \dot{S})\, dt$$

This becomes

$$V = \int_0^T (\ln(X_t - X_{\min}) - C.X_t - \mu_t X_t)\, dt - \mu(T)S_T + \mu(0)S_0 \qquad \ldots\ldots\ldots\ldots (2.12)$$

Taking the derivatives with respect to S_T and S_0, we get

$$\frac{\partial V}{\partial S_0} = \mu_0 \quad \text{And} \quad \frac{\partial V}{\partial S_T} = -\mu(T)$$

The initial co-state value (μ_0) measures the sensitivity of the optimal total benefit or utility to the given initial resource stock. It is the imputed value or shadow price of a unit of initial resource stock. While the negative terminal co-state value $\mu(T)$ explains the amount of sacrifice of total benefit or utility to be made for preserving one more unit of resource stock. It is the shadow price of a unit of the terminal resource stock.

2.3.3 The Optimal Co-state Path

The maximum principle gives an equation of motion for μ as

$$-\frac{\partial H^c}{\partial S} = \dot{\mu} - \rho.\mu_t = 0 \qquad \text{………… (2.13)}$$

Since Eq (2.13) is a first-order differential equation with a constant coefficient but a variable term, we can employ the standard methods of solution to find a complementary function μ_c and the integral $\bar{\mu}$ [1](Chiang, 2000).

$\mu_c = \mu_0 e^{\rho t}$ and $\bar{\mu} = 0$.

The general solution for μ_t is

$$\mu_t^* = \mu_c + \bar{\mu} = \mu_0 e^{\rho.t} \qquad \text{…………… (2.14)}$$

2.3.4 The Optimal Control Path

We have μ_t^*, and now the optimal control path from Eq (2.10b) and (2.14) would be,

$$X^* = X_{\min} + \frac{1}{\left(\mu_0 e^{\rho.t} + C\right)} \qquad \text{………….. (2.15)}$$

In this case, the value of X_{min}, μ_0, ρ and C are constant.

Taking a time derivative of Eq (2.15),

$$\frac{\partial X^*}{\partial t} = -\frac{\rho.\mu_0.e^{\rho.t}}{\left(\mu_0.e^{\rho.t} + C\right)^2} < 0 \qquad \text{…………….. (2.16)}$$

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[1]See also Chiang, A. C. (2000). Elements of Dynamic Optimization. Ch. 7, pp: 196-97.

$\dot{\lambda} - b(1-a).\lambda_t = hae^{r.t}$

Complementary functions,

$\lambda_c = Ae^{b(1-a).t}$

$\bar{\lambda} = \frac{ha}{B} e^{r.t}$ (B= r – b + ab)

General solution for λ is $\lambda_t = \lambda_c + \bar{\lambda} = Ae^{b(1-a).t} + \frac{ha}{B} e^{r.t}$

The optimal extraction path *(X^*)* is a decreasing function of *t* as suggested by Eq (2.16). Since the resource is exhaustible in nature, the scarcity rent increases over time at a rate of time preference, ρ, with an increase in phosphorus extraction. The marginal extraction cost is constant throughout the extraction period. Thus, the optimal extraction path declines continuously approaching to X_{min} and the stock of resources will be exhausted after a finite time. In this case, X_{min} indicates the minimum subsistence level of consumption. The optimal resource extraction pattern is shown in Figure 2 for a discount rate of 0.7 percent and a constant initial extraction cost of 0.1. The figures were considered in reference to (Seyhan *et al.*, 2009).

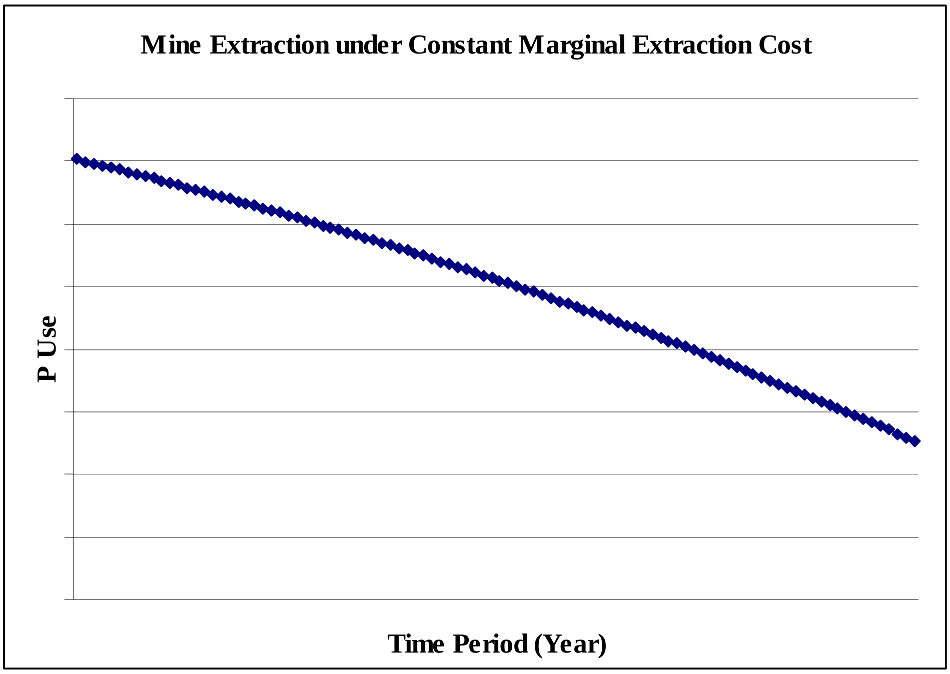


Fig 2: Extraction path under a constant marginal extraction cost (X_{min} = 0.7).

Now, the optimal level of resource extraction at time t_0 and t_T was determined as,

$$X^*(0) = X_{\min} + \frac{1}{(\mu_0 + C)} \qquad \text{…………… (2.17)}$$

$$X^*(T) = X_{\min} + \frac{1}{(\mu_0 e^{\rho.T} + C)} \qquad \text{……………. (2.17a)}$$

It is important to mention that the lowest point on the entire $X^*(t)$ path is $X^*(T)$, which was asymptotically close to the minimum consumption of phosphorus *(X_{min})*. Since phosphorus is an essential resource for living beings, there will be no life without its consumption. The consumption of phosphorus at a minimum subsistence level (X_{min})

gives a zero utility and people consume phosphorus worth at least $X^*(T)$ to have a positive utility.

This optimum phosphorus extraction path $X^*(t)$ is decreasing over time. In Eq (2.10) when shadow price is equal to zero, optimal extraction keeps a balance between marginal utility of resource use and marginal extraction cost as the familiar marginal cost *(MC)* equal to marginal revenue *(MR)* rule of a production firm that balances the cost and revenue effects of production.

Taking the derivative of Eq (2.10a),

$$\frac{\partial \mu}{\partial X} = -\frac{1}{(X_t - X_{\min})^2} < 0 \quad \text{and} \quad \frac{\partial^2 \mu}{\partial X^2} = \frac{2}{(X_t - X_{\min})^3} > 0 \qquad \text{…………… (2.18)}$$

By Eq (2.16) and (2.18), it can be deduced that the value of co-state variable is a monotonically decreasing function of X_t and increasing function of time. This reflects that the finite stock size of phosphorus decreases over time at a decreasing rate due to the increasing shadow price of resource. This is revealed from below equation,

$$\dot{S} = -X_{\min} - \frac{1}{C + \mu_t}. \qquad \text{……………. (2.19)}$$

Now taking derivative of the current-value Hamiltonian with respect to time, we get,

$$\frac{\partial H_c^*}{\partial t} = \frac{\partial H_c}{\partial S}\dot{S} + \frac{\partial H_c}{\partial \mu}\dot{\mu}$$

$$\frac{\partial H_c^*}{\partial t} = \rho \mu_t \dot{S}\text{, that is,}$$

$$\frac{\partial H_c^*}{\partial t} = -\rho \mu_t X_t \qquad \text{……………. (2.20)}$$

Furthermore, from Eq (2.10a) and (2.20), we derived the below Eq. as

$$\frac{\partial H_c^*}{\partial t} = -\rho \left(\frac{X_t}{X_t - X_{\min}} - C.X_t \right) \qquad \text{……………. (2.20a)}$$

From the definition in Eq. (2.3),

$$\frac{\partial H_c^*}{\partial t} = -\rho (P_t X_t - C.X_t) \qquad \text{…………… (2.20b)}$$

The above formulation explains that the current-value Hamiltonian ($\dot{H}_c^*$) moves along the same pattern as the optimal resource extraction path moves over time. It is in general non-zero and H_c^* is not constant over time unless $\rho = 0$.

2.3.5 The Optimal State Path

From Eq (2.9c) and (2.15), the equation of motion for *S* becomes,

$$\dot{S}^* = -X_{\min} - \frac{1}{\left(\mu_0 e^{\rho.t} + C\right)} \quad \text{…………….. (2.21)}$$

In integral form, Eq (2.21) becomes

$$S_t^* = \int \left(-X_{\min} - \frac{1}{\left(\mu_0 e^{\rho.t} + C\right)} \right).dt \quad \text{………… (2.22)}$$

An integration of Eq (2.22) gives an optimal state path as

$$S_t^* = \frac{1}{\rho.C}.\ln\left(\mu_0 + C.e^{-\rho.t}\right) - X_{\min}.t + K \quad \text{………. (2.23)}$$

where, *K* is a arbitrary constant

The optimal state path continuously decreases over time. As indicated by Eq (2.23), higher the discount rate, lower will be the stock size. The Hotelling rule of Eq (2.3b) fits well to explain the relationship between the discount rate and resource stock size. When the discount rate, which we simply call market interest rate, is higher, resource owner goes to extract more resource and enjoys with a higher present value. In the same way, the initial cost has a negative relationship with a resource stock size. When the initial cost becomes high, investors hesitate to invest for resource extraction and less resource will be harvested. Higher shadow price keeps the resource stock away from depletion and this motivates to keep the resource underground for future benefits. However, the subsistence level of resource consumption as it gets improves, the stock size decreases.

2.3.6 The Optimal Time Path

The extraction of an exhaustible resource depends on the initial extraction level. A higher start-up of extraction gives more extraction and shortens the time duration of exhaustion of finite quantity. It is mathematically explained as

$$S_0 = \int_0^T X_0.e^{\rho.t}.dt \quad \text{………. (2.24)}$$

Solving the integration for *T*, we get

$$T = \frac{1}{\rho}\ln\left(\rho\frac{S_0}{X_0} + 1\right) \quad \text{………. (2.25)}$$

Substituting X_0 from Eq (2.17) into Eq (2.25), we get,

$$T = \frac{1}{\rho}\ln\left(\rho\frac{S_0}{X_{\min} + \frac{1}{(\mu_0 + C)}} + 1\right)$$ ……… (2.25a)

Eq (2.25a) reveals that the terminal time of resource depletion has an inverse relation with the utility discount rate. As utility discount rate becomes higher, the resource owners are encouraged to extract more resource at current time since the returns to scale in the future would have a lower present value, and this invites the condition of $\frac{\dot{P}(t)}{P(t)} < \rho$. On the other hand, the investment in resource extraction at current time decreases when $\frac{\dot{P}(t)}{P(t)} > \rho$ and it allows resource owners to keep it in the ground. Thus, a lower utility discount rate prolongs the terminal time of resource exhaustion from the mine. From the view of Ramsey formula, as ρ gets higher, the resource consumption growth rate $\dot{X}/X$ decreases and consumers prefer to consume more today.

2.4 Extraction path under a Decreasing Marginal Extraction Cost

2.4.1 The Optimal Control Path

In this section, we relaxed the assumption of constant marginal extraction cost and assumed that it does vary over the time period. We also assumed that there is no exploration activity performed, and that a finite extractable quantity of reserves is known. Seyhan *et al.* (2009) and Cynthia Lin *et al.* (2007) suggested that the marginal cost of extraction (C_t) decreases exponentially over time with a rate of technological progress (γ), that is,

$C_t = C_0\, e^{-\gamma t}$ ………….. (2.26)

where, C_0 is a initial resource extraction cost.

Now the total cost of a resource extraction becomes

$$C_t = C_0 e^{-\gamma t} X_t,$$ ………… (2.27)

The dynamic optimization problem of Eq (2.8) becomes

$$\underset{X}{Max.}\ \ W = \int_0^\infty [\ln(X_t - X_{\min}) - C_0 e^{-\gamma t} X_t].e^{-\rho t}\,dt$$ ………….. (28)

subject to the same constraints as explained in Eq (2.8).

The optimal control theory was deployed to find out an optimal path for different variables. We came to the solution of an optimization problem by the maximum principle as suggested by Hamiltonian under optimal control theory. The maximum principle gives the current-value Hamiltonian function as

$$H_t^c = \ln\left(X_t - X_{\min}\right) - C_0 e^{-\gamma \cdot t} X_t - \mu_t \cdot X_t \qquad \text{……… (2.29)}$$

For maximizing the Hamiltonian function, we satisfied the first order conditions as mentioned in Eq (2.9a) – (2.9d).

The first order condition of Eq (2.9a) yields

$$\frac{\partial H^c}{\partial X} = \frac{1}{(X_t - X_{\min})} - C_0 e^{-\gamma \cdot t} - \mu_t = 0 \qquad \text{…………….. (2.30)}$$

Eq (2.30) gives the same interpretation as of Eq (2.10) but with decreasing marginal extraction cost.

It can be rewritten into the form,

$$\frac{1}{(X_t - X_{\min})} - C_0 e^{-\gamma \cdot t} = \mu_t \qquad \text{…………….. (2.30a)}$$

In this case, the shadow price of phosphorus resource would be higher and it increased faster than the case of constant marginal extraction cost. It might be due to the higher extraction cost resulted from a decreasing marginal cost.

Solving Eq (2.30) for X_t gives

$$X_t = X_{\min} + \frac{1}{\left(C_0 e^{-\gamma \cdot t} + \mu_t\right)} \qquad \text{……………. (2.30b)}$$

Equation (2.30b) implies the control path for phosphorus extraction under the condition of decreasing MEC.

The first order condition for state variable as suggested by Eq (2.9b) would be

$$-\frac{\partial H^c}{\partial S} = \dot{\mu} - \rho.\mu_t = 0 \qquad \text{…………(2.31)}$$

As suggested by Eq (2.14), the general solution for μ_t would be

$$\mu_t^* = \mu_0 e^{\rho.t} \qquad \text{…………… (2.31a)}$$

Eq (2.30b) and (2.31b) gives the optimal control path as,

$$X^* = X_{\min} + \frac{1}{\left(\mu_0 e^{\rho.t} + C_0 e^{-\gamma \cdot t}\right)} \quad \text{………….. (2.32)}$$

In this case, the value of X_{min}, μ_0, ρ, γ and C_0 are constant.

Time derivative of Eq (2.32) yields

$$\frac{\partial X^*}{\partial t} = -\frac{\rho.\mu_0.e^{\rho.t} - \gamma.C_0 e^{-\gamma.t}}{\left(\mu_0.e^{\rho.t} + C_0 e^{-\gamma \cdot t}\right)^2} < 0 \quad \text{…………….. (2.32a)}$$

The optimal extraction path (X^*) is a decreasing function of t as suggested by Eq (2.32a). But in the beginning, the second term of the numerator of Eq (2.32a), that is, the marginal extraction cost, would be higher than the first term – the marginal user cost. So the extraction of resource increases in the beginning and the optimal extraction path rises, and reaches to a peak when they are equal. Since phosphate is exhaustible in nature, its marginal user cost increases over time as extraction increases. On the other hand, the marginal extraction cost decreases over time due to the technological advancement. Thus beyond the peak, the optimal extraction path declines to X_{min} asymptotically and the mine stock will be exhausted after a finite time. The optimal phosphate extraction pattern is shown in Figure 3. In this case, we considered a rate of technological progress of 10 percent.

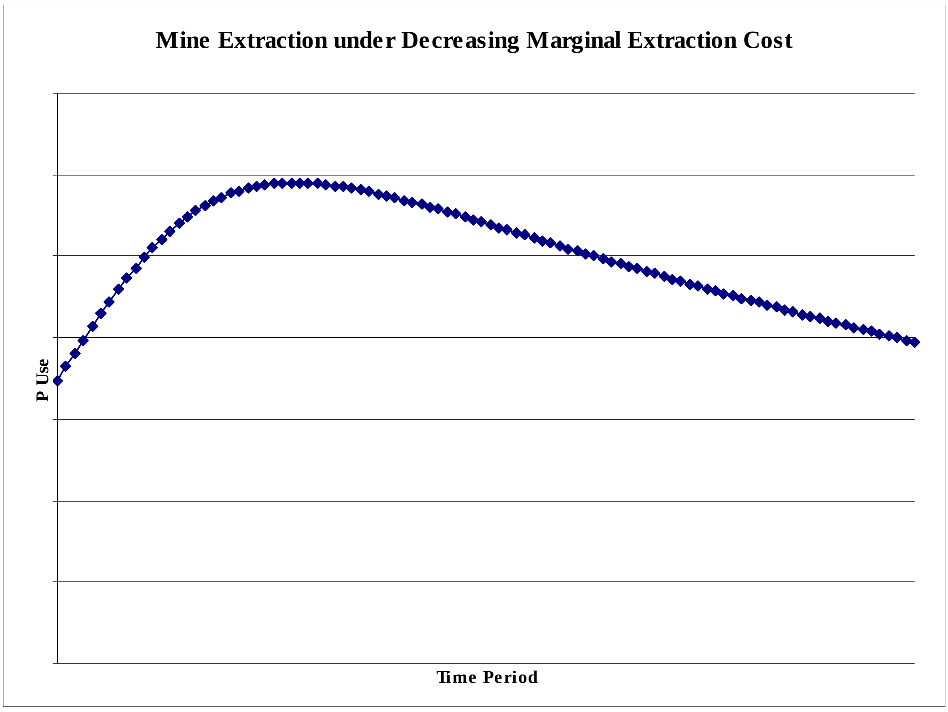


Fig 3: Extraction path under a decreasing marginal extraction cost.

For determining the initial level of extraction, Eq (2.32) at a time *0* and *T* in case of finite time horizon would be

$$X^*(0) = X_{\min} + \frac{1}{(\mu_0 + C_0)} \qquad \text{…………… (2.33)}$$

$$X^*(T) = X_{\min} + \frac{1}{(\mu_0 e^{\rho.T} + C_0 e^{-\gamma.T})} \qquad \text{……………. (2.33a)}$$

In this case, the initial level of mine extraction was the same in the previous case - constant marginal extraction cost, but the terminal level of extraction would be higher than the previous case. It might be due to the continuous decrease of marginal extraction due to technological advancement. In case of absence of phosphate scarcity, the declining marginal extraction cost induced more extraction of phosphorus at each time. In case of infinite time horizon, $T = \infty$, the extraction path goes asymptotically to X_{min}.

The change in stock size decreases over time as the stock shadow price increases. The effect of shadow prices on stock depletion would be offset if and only if a rate of technological advancement was large enough than a rate of utility discount rate of resource consumption. This is revealed from the equation below,

$$\dot{S} = -X_{\min} - \frac{1}{C_0 e^{-\gamma.t} + \mu_0 e^{\rho.t}}. \qquad \text{………. (2.34)}$$

Furthermore, Eq (2.20a) in this case becomes

$$\frac{\partial H_c^*}{\partial t} = -\rho\left(\frac{X_t}{X_t - X_{\min}} - C_0 e^{-\gamma.t} X_t\right) \qquad \text{………… (2.35)}$$

and

$$\frac{\partial H_c^*}{\partial t} = -\rho\left(P_t X_t - C_0 e^{-\gamma.t} X_t\right) \qquad \text{………. (2.35a)}$$

From Eq (20b) and (2.35a), it can be concluded that the current-value Hamiltonian ($\dot{H}_c^*$) decreases faster in this case than in the case of constant marginal extraction cost. The reason might be a faster growing scarcity due to a higher extraction under decreasing marginal extraction cost condition.

2.4.2 The Optimal State Path

In integral form, Eq (2.34) becomes,

$\dot{S} = -X_{\min} - \frac{1}{C_0 e^{-\gamma.t} + \mu_0 e^{\rho.t}}$ and it can written as

$$S_t^* = \int \left(-X_{\min} - \frac{1}{\left(\mu_0 e^{\rho.t} + C_0 e^{-\gamma.t}\right)} \right).dt \ .$$

An integral solution gives an optimal sate path as

$$S_t^* = \frac{1}{\rho.C_0.e^{-\gamma.t}}.\ln\left(\mu_0 + C_0 e^{-(\rho+\gamma).t}\right) - X_{\min}.t + K \qquad \ldots\ldots\ldots \ (2.36)$$

Eq (2.36) holds the same properties as Eq (2.23) does. Besides, it explains that the stock depletion would increase as marginal extraction cost decreases over time period. Lesser marginal extraction due to technological advancement attracts investment in the resource industry and induces to go for higher extraction.

2.4.3 The Optimal Time Path

For an optimal time path for phosphorus extraction, Eq (2.25a) becomes,

$$T = \frac{1}{\rho}\ln\left(\rho \frac{S_0}{X_{\min} + \frac{1}{(\mu_0 + C_0)}} + 1 \right) \qquad \ldots\ldots\ldots\ldots\ldots \ (2.37)$$

Both cases – constant and decreasing marginal extraction cost, have the same initial level of mine extraction. As ρ gets higher, there would be higher consumption at each time and the terminal time of resource exhaustion shortens. In previous case of constant marginal extraction cost, there would be continuous decline of mine phosphate consumption at a constant proportional rate. The marginal resource user cost mainly determines the speed of stock depletion. In this case, an initial peak of extraction was offset by the slow consumption in the period when the economy tends to reach the terminal time. This holds only in the case of a finite time period, but it does not apply in case of an infinite time period.

2.5 Extraction Path with stock effect

2.5.1 The Optimal Control Path

In this section, we included phosphorus scarcity rent in cost function, and now the price of extracted phosphorus becomes equal to a unit cost of extraction as given by a capital rent and scarcity rent in a competitive market. It is thus written as

$$P_t X_t = C_t X_t + \varsigma S_t \qquad \ldots\ldots\ldots\ldots \ (2.38)$$

where, ς is a scarcity rent per unit time for a use of stock. Eq (2.38) suggests the zero-profit condition in the competitive resource market.

We are limited to deal with a constant quality of resource. It was assumed that the cost of extraction is negatively related to the stock of resource, that is, it increases as the stock size decreases ($\varsigma \leq 0$).

The total cost Eq (2.27) now becomes

$$C_t = C_0 e^{-\gamma \cdot t} X_t - \varsigma S_t \qquad \text{………… (2.39)}$$

We can re-write the current-value Hamiltonian, Eq (2.29) as

$$H_t^c = \ln\left(X_t - X_{\min}\right) - C_0 e^{-\gamma \cdot t} X_t + \varsigma S_t - \mu_t \cdot X_t \qquad \text{………. (2.40)}$$

For maximizing the Hamiltonian function, Eq (2.9a) gives

$$\frac{\partial H^c}{\partial X} = \frac{1}{(X_t - X_{\min})} - C_0 e^{-\gamma \cdot t} - \mu_t = 0 \qquad \text{…………….. (2.41)}$$

Eq (2.41) is the same as Eq (2.30) because in the former case, it was assumed that the resource stock directly functions as a variable in the Hamiltonian function. Thus, the marginal value of the Hamiltonian function with a unit change in control variable would be the same in both cases.

Rearranging Eq (2.41), we get,

$$X_t = X_{\min} + \frac{1}{\left(C_0 e^{-\gamma \cdot t} + \mu_t\right)} \qquad \text{……………. (2.41a)}$$

Now, Eq (2.9b) gives the equation of motion of state variable as

$$-\frac{\partial H^c}{\partial S} = \dot{\mu} - \rho . \mu_t = -\varsigma \qquad \text{………… (2.42)}$$

From Eq (2.42), a general solution for μ_t would be

$$\mu_t^* = \mu_0 e^{\rho . t} + \frac{\varsigma}{\rho} \qquad \text{…………… (2.42a)}$$

Eq (2.41a) and (2.42a) yield an optimal control path for phosphorus extraction that maximizes the Hamiltonian function as,

$$X^* = X_{\min} + \frac{1}{\left(\dfrac{\varsigma}{\rho} + \mu_0 e^{\rho . t} + C_0 e^{-\gamma \cdot t}\right)} \qquad \text{………….. (2.43)}$$

In this case, the value of X_{min}, ς, μ_0, ρ, γ and C_0 are constant, and exogenously given. Taking a time derivative of Eq (2.43), we have

$$\frac{\partial X^*}{\partial t} = -\frac{\rho.\mu_0.e^{\rho.t} - \gamma.C_0 e^{-\gamma.t}}{\left(\mu_0.e^{\rho.t} + \frac{\varsigma}{\rho} + C_0 e^{-\gamma.t}\right)^2} < 0 \quad \text{................ (2.43a)}$$

In this case, the optimal control path of phosphorus extraction follows the same trend as in the case of decreasing marginal cost without stock effect. But the level of path would be lower than the previous case because of presence of resource scarcity rent.

$$X^*(0) = X_{\min} + \frac{1}{\left(\frac{\varsigma}{\rho} + \mu_0 + C_0\right)} \quad \text{.............. (2.44)}$$

$$X^*(T) = X_{\min} + \frac{1}{\left(\frac{\varsigma}{\rho} + \mu_0 e^{\rho.T} + C_0 e^{-\gamma.T}\right)} \quad \text{............... (2.44a)}$$

As suggested by Eq (2.44) and (2.44a), the initial level of phosphate extraction, *X(0)*, would be lower than the case of Eq (2.33) because of a lower level of phosphorus extraction due to the consideration of resource scarcity rent. The optimal control path *(X^*)* follows the same pattern as it was in case of Eq (2.32), but it would be lower than the former case throughout the time period. The control path in this optimization problem would be single-peaked and asymptotically declining to X_{min}. Dasgupta (1979), Pezzey *et al.* (1998) and Seyhan *et al.* (2009) also defined the same pattern of consumption path with a positive discount rate that eventually approaches to zero and is single peaked. Pezzey *et al.* (1998) suggested that the single peak moves outward as a discount rate decreases. It is thus concluded that in an efficient path, consumption must be increasing in an initial phase, and from the peak, start to decrease approaching to the minimum subsistence level of consumption. The optimal resource extraction pattern is shown in Figure 4. Here, we considered a stock effect of 1 percent from (Seyhan *et al.*, 2009).

Now, a time derivative of the current-value Hamiltonian with respect to time yields,

$$\frac{\partial H_c^*}{\partial t} = -\rho\mu_t X_t \quad \text{............... (2.45)}$$

Substituting the value of μ_t from Eq (2.41) into Eq (2.45), we have

$$\frac{\partial H_c^*}{\partial t} = -\rho\left(\frac{X_t}{X_t - X_{\min}} - C_0 e^{-\gamma . t} X_t\right) \quad \text{……………. (2.45a)}$$

This is exactly the same as Eq (2.35a). The stock effect was compensated by the co-state variable and thus it gave a null effect on the Hamiltonian function. It made Eq (2.45a) exactly same as of Eq (2.35a).

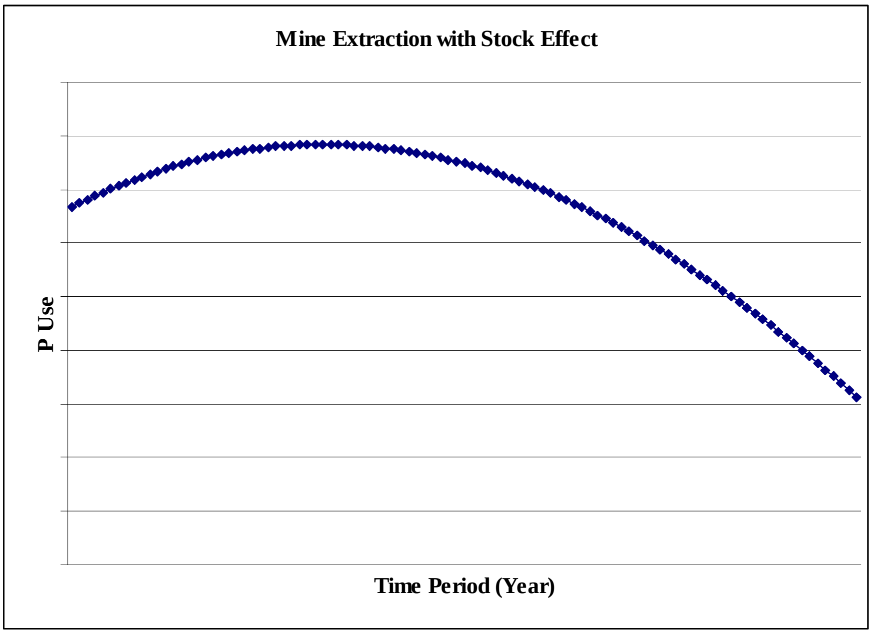


Fig 4: Phosphorus Extraction over time period.

2.5.2 The Optimal State Path

From Eq (2.9c) and (2.43), the equation of motion for S becomes,

$$\dot{S}^* = -X_{\min} - \frac{1}{\left(\frac{\varsigma}{\rho} + \mu_0 e^{\rho . t} + C_0 e^{-\gamma . t}\right)} \quad \text{…………….. (2.46)}$$

While comparing to Eq (2.34), the rate of stock depletion over time was less due to the presence of phosphorus scarcity rent.

In integral form, Eq (2.46) becomes

$$S_t^* = \int\left(-X_{\min} - \frac{1}{\left(\frac{\varsigma}{\rho} + \mu_0 e^{\rho . t} + C_0 e^{-\gamma . t}\right)}\right).dt$$

By integration, the equation of motion $\dot{S} = -X_t^*$ yields the optimal state path as

$$S_t^* = \frac{1}{\varsigma + \rho.C_0.e^{-\gamma.t}}.\ln\left(\frac{\varsigma}{\rho}e^{-\rho.t} + \mu_0 + C_0 e^{-(\rho+\gamma).t}\right) - X_{\min}.t + K \qquad \text{.... (2.46a)}$$

On the top of Eq (2.23) and Eq (2.36), Eq (2.46a) reveals a relationship of resource scarcity rent and the stock depletion. As stock is getting deeper and deeper, the cost of extraction becomes higher and keeps investors away from the resource industry. Consequently, the resource harvest would be less and industrialists look for the production of substitutes.

2.5.3 The Optimal Time Path

In this case, an optimal time path of phosphorus extraction and stock exhaustion obviously gets prolonged than the previous cases because of lower level of resource extraction due to the increasing resource scarcity. It is mathematically shown in equation derived from Eq (2.37) as

$$T = \frac{1}{\rho}\ln\left(\rho\frac{S_0}{X_{\min} + \frac{1}{\left(\frac{\varsigma}{\rho} + \mu_0 + C_0\right)}} + 1\right) \qquad \text{................ (2.47)}$$

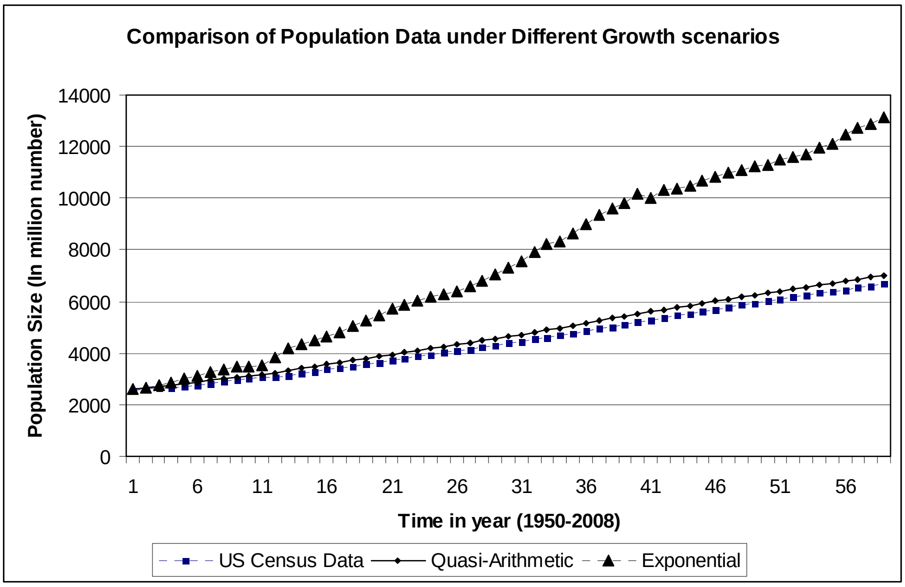
Eq (2.47) revealed the terminal time of phosphorus stock depletion. The utility discount rate would have the same impact as explained in Eq (2.37) while adhering to the Hotelling rule mentioned in Eq (2.3b).

2.6 Conclusion

For an analysis of mine phosphate extraction, a log-linear utility function with a Stone-Geary preference was considered to give a measure of utility from phosphate consumption with a subsistence level. We analyzed the optimal extraction path under different cost scenarios in a perfect competitive market. Under a constant marginal extraction cost, the extraction path decreases over time approaching to the subsistence level of consumption. But it would be more realistic to assume a decreasing marginal extraction cost over time due to the technological advancement in phosphate extraction technology. Under the situation, the optimal extraction increases in the beginning and reaches to the peak till the marginal extraction cost getting equal to marginal user cost. The path then starts to decreases continuously over time approaching to minimum consumption at subsistence level because of increasing marginal user cost. Meanwhile, the depletion of an exhaustible resource invites a stock effect and makes the extraction cost higher as stock getting deeper. Under the prevalence of stock effect on cost function, the mine extraction path follows the same directional path as in case of decreasing marginal cost, but it would be lower at each time period.

Chapter 3: Population Growth and Phosphate Extraction

Population policy is a purely conceptual problem, and the present human activity leads to the depletion of natural resources and reduces future consumption possibilities as well. Thus, a concern about the optimum size of population and an extraction of exhaustible resource is getting a lot of attention in the present days. Dasgupta (1969) defined the optimum population as the population size that would secure the largest per capita output with given natural resources, state of arts, and standard of working time. An economic welfare of a community depends on the policies that determine a rate of capital accumulation and a rate of population growth. In this study, we only discussed on population growth and its effect on community's welfare by identifying an optimal extraction path subject to the constraints of finiteness of resource stock. Arrow *et al.* (2003) and Asheim (2004) presented the exponential and logistic growth function of population $\{L(t)\}_{t=0}^{\infty}$ as $\dot{L} = n.L_t$ and $\dot{L} = n.L_t\left(1 - \frac{L_t}{L^c}\right)$ respectively, where n denotes population growth rate, and L^c denotes the maximum population carrying capacity of the finite world. Dorfman (1969), Stiglitz (1974) and many literatures made use of an exponential growth rate of population. In literatures, it was argued that it makes the analysis and mathematical calculation simple. But Mitra (1983) had discussed on consistent patterns of population growth with the attainment of maximum social welfare amidst exhaustible resource constraints. It was argued that an exponential growth was an inappropriate idealization because of finiteness of resources on which this population growth depends upon. It was reported that a quasi-arithmetic progression rather than a geometric one gives a consistent population growth path. The quasi-arithmetic growth function of population as suggested by Mitra (1983) was $\dot{L} = \frac{n.L_t}{(1+t)}$, and later Asheim *et al.* (2007) also agreed on it. Asheim *et al.* (2007) argued that a quasi-arithmetic population growth path was consistent with population growth trend and appropriate for an estimation of future development of the world's population with decreasing global population growth rate. This decreasing population growth rate indicates that the population is a concave function of time. Figure 5 showed that the estimated population data under quasi-arithmetic progression was close to the figures given by the US Census Bureau. The exponential growth function gives a higher level of population than the US Census data and may not be possible due to the exhaustibility of natural resources. However, we started to work with an exponential growth pattern for a general understanding and save our analysis from clotty. Besides, we also carried out the analysis on the basis of quasi-arithmetic progression and compared with the outcome of an exponential growth path.



Source: U.S. Census Bureau, International Data Base.

Fig 5: Population Growth Scenarios under different functional form.

In this section, we assume a constant growth rate of population in an economy, but our assumption seems implausible in the light of the presence of constraint of finite mine phosphate in production. We make many assumptions in the model to abstract the reality as simple as possible. To make an analysis simple and save from clutter, initial population is normalized to one, that is, $L(0) = 1$ (measured in thousands of millions of people).

3.1 Extraction path under an Exponential Population Growth

Suppose that $L_t = L_0 e^{nt}$ with no possibility of social control of natural increase at the rate of n.

From the above, we have a population growth rate as

$$\frac{\dot{L}}{L_t} = n$$

As we know, $\dot{S}_t = -X_t$ and the per capita stock size, $s_t = \frac{S_t}{L_t}$,

From the time derivative of per capita stock size, we get,

$$\dot{s} = \frac{\dot{S}}{L_t} - ns_t$$

and thus, finally,

$$\dot{s} = -x_t - ns_t \qquad \text{…………….. (3.1)}$$

The time path for consumption per head, *(x_t)*, delineates a time path for s_t through the above differential equation. It is also supported by Solow (1974). Once consumption is recognized, the rest of resource stock is added to the inherited stock of resource to give the next instant's stock of capital.

For exogenously given population and initial stock of resource, the path is feasible if $\int_0^T L_t x_t dt \le S_0$ is satisfied.

3.1.1 The Utility Function

Ethics of basic welfare economics focuses on the consequences of policy (as consequentialism approach) for the consumption of goods and services by individuals in a community. An individual obtains an utility or welfare from the consumption of goods and services, and the objective is to set a policy on behalf of the community that maximizes overall social welfare. This social welfare depends on the welfare of each individual in the community and the function gives a relationship between the measure of social well-being and the consumption of goods and services by each household. Let an individual's instantaneous welfare depend on phosphate consumption per capita and be represented by an iso-elastic utility function

$$u(X_t / L_t) = \ln(x_t - x_{\min}) \qquad \text{……………….. (3.2)}$$

where, x_t is the phosphate consumption per capita *($x_t = X_t/L_t$)*, and x_{min} is a minimum per capita consumption level.

It is assumed that there is no left over of extracted mine phosphate in each consumption period. It says that the economy consumes all the phosphates that are extracted in that particular time period. An instantaneous social welfare depends on the consumption per capita, and the size of total population. Mirrlees (1967) and Aarrestad (1979) presented a resource extraction model with population, and argued that the total utility at a time t will be $L_t.u(X_t/L_t)$. Thus, the objective function of an optimization problem is to maximize the sum (or integral) across all individuals and all time of the utility of consumption and it is,

$$W = \int_0^\infty L_t u(X_t / L_t) e^{-\rho.t} dt \qquad \text{……………… (3.3)}$$

In above formulation, $L_t\ u(X_t/L_t)$ is concave in L and X.

3.1.2 The Cost Function

In this section, we transformed the marginal extraction cost incurred by the global population for phosphorus extraction into per capita terms (C_t/L_t) by dividing cost function with a population size. This was done to incorporate the population factor in our analysis.

From Eq (2.26), the per capita marginal extraction cost was formulated as below. It is also a decreasing path over time with a rate of technological progress (γ),

$$\frac{C_t}{L_t} = \frac{C_0.e^{-\gamma.t}}{L_t} \qquad \text{…………….. (3.4)}$$

Eq (3.4) now becomes,

$$c_t = c_0 e^{-\gamma.t} \qquad \text{…………….. (3.4a)}$$

where, c_t is per capita marginal extraction cost, and c_0 is per capita initial extraction cost.

Now the total per capita cost of a resource extraction becomes

$$c_t = c_0 e^{-\gamma.t} x_t - \varsigma.s_t \qquad \text{……………… (3.5)}$$

Eq (3.5) gives the same definition of cost function as in case of Eq (2.39), but only the difference is that the former was expressed in per capita terms.

3.1.3 The Optimization Problem

For a resource extraction optimization problem, we maximized the total Net Benefit *(NB)* as we explained in section 2.3. We defined the net benefit as function of per capita phosphate consumption, stock size, time period and population size, $NB\ (x,s,t,L_t)$ and mathematically expressed as total per capita benefit minus total per capita cost,

$$NB(x,s,t) = L_t \ln(x_t - x_{\min}) - \left(c_0 e^{-\gamma \cdot t} x_t L_t - \varsigma . s_t . L_t\right) \qquad \text{………… (3.6)}$$

Now, the optimization problem is written as,

$$\underset{X}{Max.} \;\; W = \int_0^\infty [L_t \ln(x_t - x_{\min}) - c_0 e^{-\gamma \cdot t} x_t L_t + \varsigma s_t L_t].e^{-\rho . t} dt \qquad \text{……….. (3.7)}$$

subject to,

i) $\dot{s}_t = - x_t - n.s_t,$

ii) $L_t = L_o.e^{n.t}$

iii) $x_t \geq 0$

iv) $s\ (0) = s_0$ s(T) free (s_0 given)

The current-value Hamiltonian function is

$$H_t^c = L_t \ln\left(x_t - x_{\min}\right) - L_t c_0 e^{-\gamma \cdot t} x_t + \varsigma s_t . L_t - \mu_t (x_t + n.s_t) L_t \qquad \text{……………. (3.8)}$$

Maximizing the Hamiltonian function satisfies the first order conditions as explained in Eq (2.9a) – (2.9d). Eq (2.9a) yields,

$$\frac{\partial H^c}{\partial x} = \frac{L_t}{(x_t - x_{\min})} - c_0 e^{-\gamma \cdot t}.L_t - \mu_t.L_t = 0 \qquad \text{…………….. (3.9)}$$

Rearranging Eq (3.9)

$$\frac{1}{(x_t - x_{\min})} - c_0 e^{-\gamma \cdot t} = \mu_t \qquad \text{…………….. (3.9a)}$$

Solution of above Eq (3.9a) gives x_t in terms of μ_t,

$$x_t = x_{\min} + \frac{1}{\left(c_0 e^{-\gamma \cdot t} + \mu_t\right)} \qquad \text{…………….(3.9b)}$$

Equation (3.9b) gives the control path for per capita phosphate mine extraction.

Satisfying Eq (2.9b) in the Hamiltonian function – Eq (3.8), we get

$$-\frac{\partial H^c}{\partial S} = \dot{\mu} - \rho.\mu_t = -\varsigma.L_t + \mu.n.L_t \qquad \text{…………(3.10)}$$

Rearranging Eq (3.10),

$$\dot{\mu} - \mu_t (\rho + n.L_t) = -\varsigma.L_t \qquad \text{…………. (3.10a)}$$

The general solution of a differential equation for μ becomes

$$\mu_t^* = \mu_0 e^{(\rho + nL).t} + \frac{\varsigma}{\rho + nL} \qquad \text{…………… (3.10b)}$$

3.1.4 The Optimal Control Path

We now have μ_t^*, and we can get the optimal control path when we substitute μ_t^* for μ_t in (3.9b). This would be,

$$x_t^* = x_{\min} + \frac{1}{\left(\mu_0 e^{(\rho+nL).t} + \frac{\varsigma.L_t}{\rho + n.L_t} + c_0 e^{-\gamma \cdot t}\right)} \qquad \text{………….. (3.11)}$$

In this case, the value of x_{min}, ς, μ_0, ρ, γ, n and c_0 are exogenously given. The population variable follows an exponential growth path.

Time derivative of Eq (3.11) was obtained as

$$\frac{\partial x^*}{\partial t} = -\frac{(\rho + nL_t).\mu_0.e^{(\rho+nL).t} + \frac{\varsigma.L_t}{\rho + n.L_t} - \gamma.c_0 e^{-\gamma.t}}{\left(\mu_0.e^{(\rho+nL).t} + \frac{\varsigma.L_t}{\rho + nL_t} + c_0 e^{-\gamma \cdot t}\right)^2} < 0 \qquad \text{….. (3.11a)}$$

The optimal extraction path (x^*) is a continuous decreasing function of t as suggested by (3.11a).

Rearranging Eq (3.11a),

$$\frac{\partial x^*}{\partial t} = -\frac{(\rho + \dot{L}).\mu_0.e^{(\rho+\dot{L}).t} + \frac{\varsigma.\dot{L}}{\rho + n.\dot{L}} - \gamma.c_0 e^{-\gamma.t}}{\left(\mu_0.e^{(\rho+\dot{L}).t} + \frac{\varsigma.L_t}{\rho + \dot{L}} + c_0 e^{-\gamma \cdot t}\right)^2} < 0 \qquad \text{……. (3.11b)}$$

Eq (3.11a) reveals that the per capita consumption of phosphorus decreases continuously over time approaching x_{min}. Because of the population factor, the marginal user cost inflated with population growth at exponential rate. This increased scarcity rent of finite stock prolonging the terminal life of mine resource exhaustion. On the other hand, the marginal extraction cost is getting low due to the technological improvement, but there is no direct effect of population on reducing the cost. However, the increased population generates more ideas and knowledge through research and development that provides cost-effective technologies. The higher population stimulates the stock effect on phosphorus extraction, and makes the extraction cost higher. Thus the population-induced stock effect invites to extract much more phosphorus from the ground over time. Figure 6 gives a picture of mine phosphate extraction path under an exponential growth for a population growth rate of

1.5 percent. Maximum population is residing in the developing and least developed countries and the population growth rate is at around 2 percent. In this case, we considered 1.5 percent while considering the sluggish population growth in the developed countries.

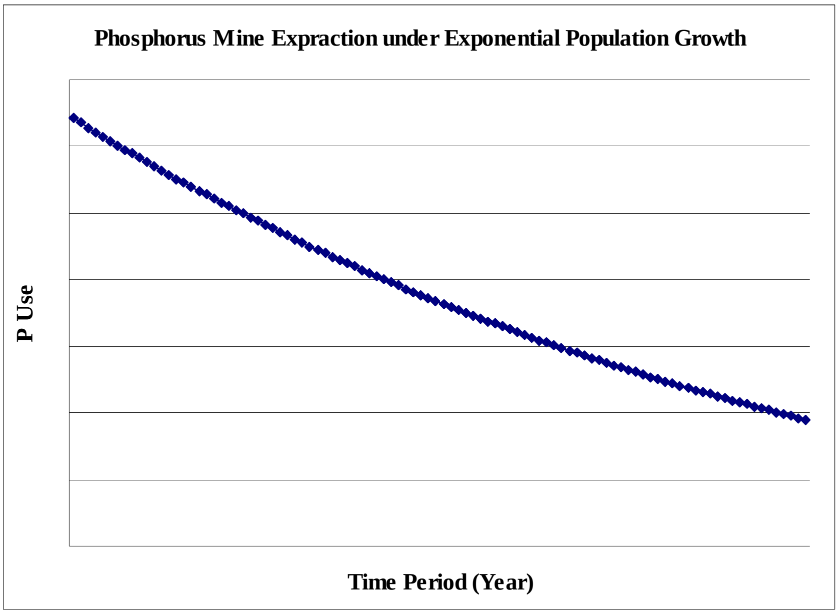


Fig 6: Extraction path under an Exponential population growth

The optimal per capita resource consumption at t_0 is higher and it fall steadily throughout the extraction period [0,∞). Now, the optimal level of mine phosphate extraction at time t_0 and time t_T can be exactly determined as,

$$x^*(0) = x_{\min} + \frac{1}{\left(\frac{\varsigma}{\rho + n} + \mu_0 + c_0\right)} \qquad \text{.............. (3.11c)}$$

$$x^*(T) = x_{\min} + \frac{1}{\left(\frac{\varsigma . L_T}{\rho + n.L_T} + \mu_0 e^{(\rho + n.L)T} + c_0 e^{-\gamma . T}\right)} \qquad \text{............... (3.11d)}$$

The harvesting of a phosphate at time t_0 is started from a higher level than in the case of without population. The population growth rate does affect on shifting the starting level from the benchmark model. The extraction path from the higher starting level called upon more extraction and a greater scarcity of phosphate.

From Eq (2.25) and (3.11c) , the solution for terminal time can be derived as,

$$T = \frac{1}{\rho} \ln\left(\rho \frac{s_0}{x_{\min} + \frac{1}{\left(\frac{\varsigma}{\rho + n} + \mu_0 + c_0 \right)}} + 1 \right) \qquad \text{………….. (3.12)}$$

From Eq (3.12), it can be figured out that the terminal time of mine exhaustion is lengthened with the higher values of initial shadow price and cost of extraction, and the stock effect. But the population growth rate, the utility discount rate and the per capita minimum consumption trimmed down the terminal time of finite resources. Moreover, the terminal time is reduced in this case as compared to the benchmark model because of the plummeting effect of population growth.

3.2 Extraction Path under a Quasi-Arithmetic Population Growth

In this section, we considered a quasi-arithmetic population growth path since it gave a more closer figure to the real world population data. We assumed that the population size increases at an exogenously given constant rate. The quasi-arithmetic population growth path is,

$$L_t = L_0 (1 + t)^n \qquad \text{………….. (3.13)}$$

From above, we get

$$\frac{\dot{L}}{L_t} = \frac{n}{1 + t} \qquad \text{…………. (3.13a)}$$

As we know, $\dot{S}_t = -X_t$ and the per capita stock size, $s_t = \frac{S_t}{L_t}$,

From the definition,

$$\dot{s} = \frac{\dot{S}}{L_t} - \frac{n s_t}{1 + t}$$

and thus finally,

$$\dot{s} = -x_t - \frac{n s_t}{1 + t} \qquad \text{..……….. (3.14)}$$

Now, the optimization problem becomes

$$\underset{x}{Max} \;\; U = \int_0^{\infty} [L_t \ln(x_t - x_{\min}) - c_0 e^{-\gamma . t} x_t L_t + \varsigma . s_t L_t] e^{-\rho . t} dt \qquad \text{…….. (3.15)}$$

Subject to the constraints,

i) $\dot{s} = -x_t - \frac{ns_t}{1+t}$

ii) $x_t \geq 0$

iii) $s\ (0) = s_0$ s(T) free (s_0 given)

In this problem, the current-value Hamiltonian function becomes

$$H_t^c = L_t \ln(x_t - x_{\min}) - L_t c_0 e^{-\gamma \cdot t} x_t + \varsigma . s_t . L_t - \mu_t (x_t + \frac{n.s_t}{1+t}) L_t \quad \ldots\ldots\ldots\ldots\ldots (3.16)$$

The first-order conditions of Maximum principle gives an optimal resource consumption path as

$$x_t^* = x_{\min} + \frac{1}{\left(\mu_0 e^{(\rho + \frac{n}{1+t}).t} + \frac{\varsigma . L_t}{\rho + \frac{n}{1+t}} + c_0 e^{-\gamma \cdot t} \right)} \quad \ldots\ldots\ldots (3.17)$$

The time derivative of Eq (3.17) gives

$$\frac{\partial x^*}{\partial t} = -\frac{\left(\rho + \frac{1}{(1+t)^2} \right) \mu_0 . e^{(\rho + \frac{n}{1+t}).t} + \varsigma . \dot{L} \left(\rho + \frac{1+n}{1+t} \right) - \gamma . c_0 e^{-\gamma . t}}{\left(\mu_0 e^{(\rho + \frac{n}{1+t}).t} + \frac{\varsigma . L_t}{\rho + \frac{n}{1+t}} + c_0 e^{-\gamma \cdot t} \right)^2} < 0 \quad \ldots\ldots (3.17a)$$

Replacing $\dot{L} = n(1+t)^{n-1}$ and $L = L_0 (1+t)^n$ in Eq (3.17a), we get

$$\frac{\partial x^*}{\partial t} = -\frac{\left(\rho + \frac{1}{(1+t)^2} \right) \mu_0 . e^{(\rho + \frac{n}{1+t}).t} + \varsigma . n(1+t)^{n-1} \left(\rho + \frac{1+n}{1+t} \right) - \gamma . c_0 e^{-\gamma . t}}{\left(\mu_0 e^{(\rho + \frac{n}{1+t}).t} + \frac{\varsigma . (1+t)^n}{\rho + \frac{n}{1+t}} + c_0 e^{-\gamma \cdot t} \right)^2} < 0 \quad \ldots (3.17b)$$

In case of quasi-arithmetic progression of population growth, the per capita phosphorus extraction and consumption also moves in the same pattern as it was in exponential growth. However, the marginal user cost curve is more flatter in this case as compared to an exponential growth and the increment in scarcity rent is less steeper. The extraction of mine phosphate would decrease continuously over time but the slope would be more flatter than the exponential case. The optimal extraction path under quasi-arithmetic progression is depicted in figure 7 below.

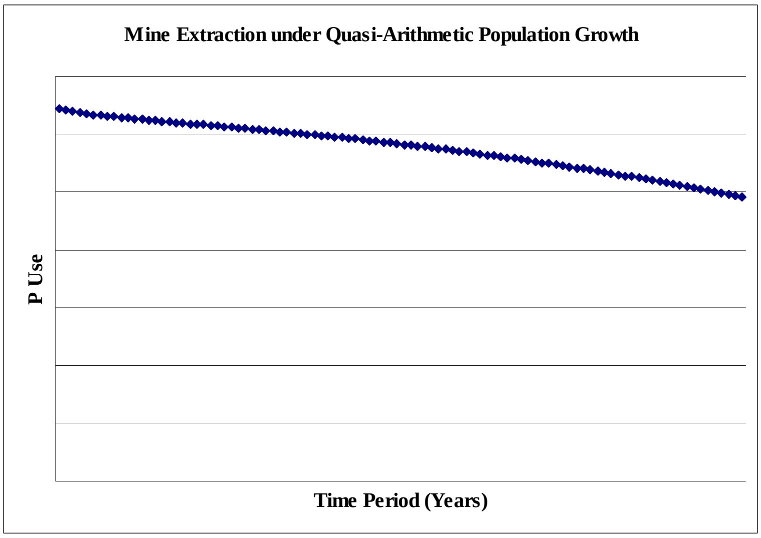


Fig 7: Extraction path under a quasi-arithmetic population growth

The starting level of per capita phosphate extraction is the same as in the previous case. However, the consumption of phosphorus would be high at each time period due to less scarcity of mine phosphate. In finite time horizon, the quantity consumed would be higher in each time period in quasi-arithmetic case for the completion of mine stock in given time. But in case of infinite time horizon, the terminal time would be extended and may not hold the above situation.

Under both population growth paths, the initial level of mine phosphate extraction would be the same point as mention in Eq (3.11c), but they move in different paths. The extraction path under a quasi-arithmetic progression would move from a higher level. It is depicted in figure 8 below.

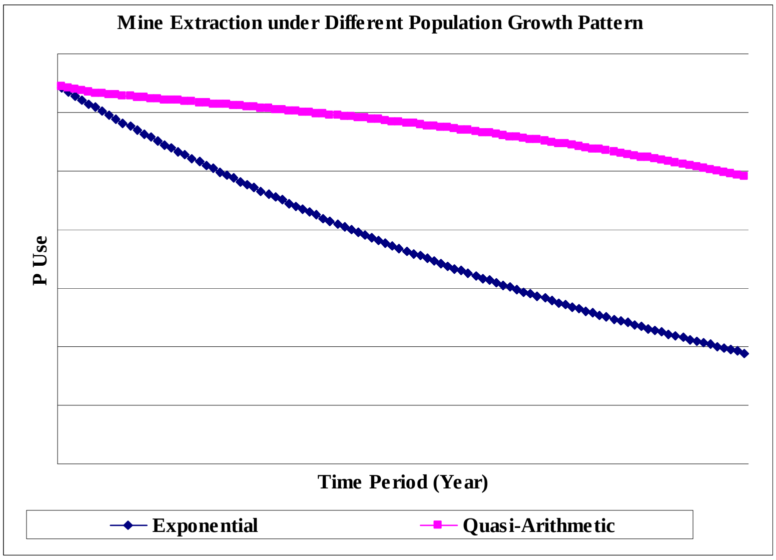


Fig 8: Comparison of phosphate extraction paths under different population growth scenarios.

Since the case was of finite time horizon, per capita consumption of mine phosphate would be higher and there would be more phosphate extraction from the stock in quasi-arithmetic population growth.

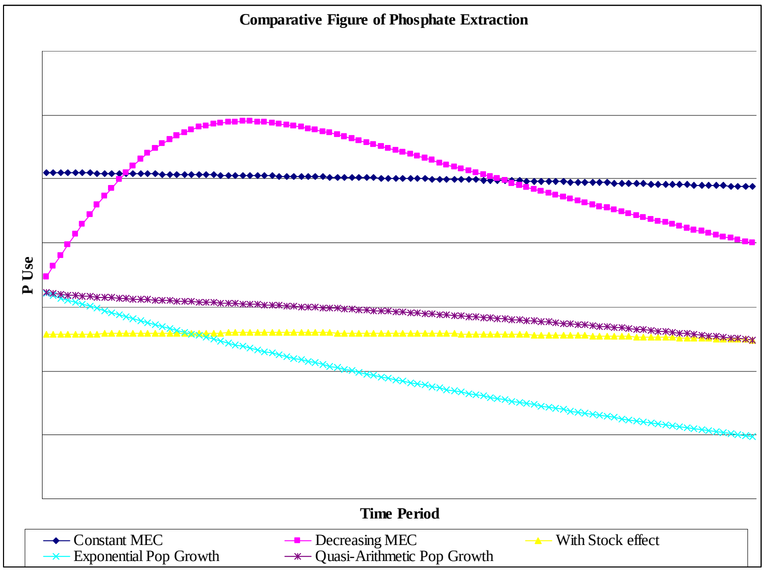


Fig 9: Comparative figure of Phosphate Extraction path under different conditions.

3.3 Conclusion

Optimum population size and consumption of exhaustible resource is getting a lot of attention among the resource economists. In literatures, we found that a quasi-arithmetic progression of population growth gives a more consistent population growth than the exponential under a constraint of exhaustible natural resources. We discussed population growth and optimal resource extraction path under both population growth scenarios.

Per capita consumption of mine phosphate decreases continuously over time and approaches to the subsistence level. With increasing population, resource scarcity rent increases at a faster pace than the decreasing rate of marginal extraction cost due to the technological advancement. With less population growth in quasi-arithmetic progression, more resource is available to consume at per capita level in finite time horizon and people go for a higher level of consumption than the exponential population growth.

Chapter 4: Endogenous Population and Phosphate Extraction

4.1 The Model of Utility Function

In this section, we focused on an endogenization of population growth rate and its influences on phosphorus extraction. We assumed that parents take decision on number of children based on the choices on their own consumption and intergenerational transfers. Becker *et al.* (1988) considered a fertility choice to maximize an utility function without considering a child-bearing cost. Abio (2003) highlighted the issues of interiority in the analysis of the optimal population growth rate with endogenous fertility, and addressed this issue in his paper by introducing both the cost and taste for children. It was argued that the cost of children play a crucial role to avoid a corner solution. It was assumed the diminishing marginal utility of parents with respect to the number of children and increasing child-rearing cost for an additional number. Barro *et al.* (2004) extended it to explain the population change and fertility choice with an Overlapping-Generations (OLG) model. The OLG model assumes that people live in two periods; childhood and adulthood. An adult or parent maximizes the utility not only from their own consumption, but also from their children's consumption and the number of children. The utility function is

$$U_0 = u(x_0, n_0) + a(n_t).U_1 n_0 \qquad \text{………………} (4.1)$$

where, U_0 is the adult's utility in current time, that is, present generation, x_0 is own consumption of each parent during adulthood, n_0 is the number of children per parent. The term $u(x_0,n_0)$ represents the utility of a parent obtained from consumption in his adulthood, and the presence of children. This utility function satisfies the properties of concavity, that is, $u_x > 0$ and $u_{xx} < 0$. But this utility function does not explain separately the consumption of children during their childhood, and it is included in the consumption of parent. The term U_1 is the utility obtained from the perspective happiness of children when they become adult, and the term $U_1 n_0$ is the aggregate utility of children in next generation. In this study, we also assumed that the parents are altruistic to all children equally and attain the same utility, U_1. The term $a(n_t)$ corresponds to the degree of parent's altruism toward each child, and it converts the children's utility into the parents' utility. Thus, the above formulation explains that the utility of each parent depends on its own consumption (x_0), the number of child *(n_0)*, and the utility *(U_1)* of own children. In the same way, the utility of children when they become parents U_1 would be generated from the same factors as mentioned above - own consumption *(x_1)*, their number of children *(n_1)* and their children's utility *(U_2)*. Likewise, the utility of next generation *(U_2)* will also be produced in the same way. Thus, it can be deduced that a dynastic utility function depends on its consumption and number of descendants in all generations. Mathematically, it is

$$U_t = \sum_{t=0}^{\infty} \psi_t L_t u(x_t, n_t) \qquad \text{…………….. (4.2)}$$

where, L_t is the number of descendants in the t^{th} time. Each time period in discrete series represents every generation. It was assumed that each individual was endowed with one unit of labour per period, and thus the population size in current generation *($t = 0$)* is normalized to unity, that is, $L_0 = 1$. The term ψ_t is the altruism function of the parents towards each child in the t^{th} time generation. Becker *et al.* (1988) assumed that the null effect of fertility on current utility function, and the altruism function had constant elasticity with respect to the number of children. But Barro *et al.* (2004) rule out the assumption made by Becker *et al.* (1988) and argued that the presence of children affects parents' consumption. The logic behind Barro's argument was that the OLG model does not distinguish between children and parents' consumption separately. In this study we only confined on the latter assumption made by Becker *et al.* (1988).

Similar to the previous settings, we assumed the functional form of *$u(x_t,n_t)$* with constant elasticity of marginal utility with respect to x_t and n_t as

$$u(x_t, n_t) = \frac{[x_t.n_t^{\phi}]^{1-\theta} - 1}{(1-\theta)} \qquad \text{……………… (4.3)}$$

where, $0 < \phi < 1$ and $\theta > 0$. The parameter ϕ represents the diminishing marginal utility of parents towards the presence of children, and θ shows the essentiality of control variables, *x* and *n*.

From (4.2) and (4.3), we get

$$U_t = \sum_{t=0}^{\infty} \psi.L_t.\frac{[x_t.n_t^{\phi}]^{1-\theta} - 1}{(1-\theta)} \qquad \text{………………. (4.4)}$$

Becker *et al.* (1988) considered as a complete model with constraints of cost for having and raising children, and an intergenerational budget that transfers to each child. In the above equation, the adults in each generation choose consumption and fertility to maximize U_t in Eq (4.4).

4.2 Model in continuous time

Contrary to the ILG (Infinitely-Lived Generation) model, Barro *et al.* (2004) accepted the OLG model to endogenize the fertility choice because it considered the length of a generation which was not in the ILG model. The assumption of infinite lives is natural in OLG model because this makes an altruistic linkage between each generation. In this, the degree of altruism represents the rate of time preference, and it depends on the number of children and the resources used to rear child. Now the utility function in continuous time is

$$U_t = \int_0^{\infty} \frac{e^{-\rho.t}}{(1-\theta)} L_t \left\{ [(x_t - x_{\min}).n_t^{\phi}]^{1-\theta} - 1 \right\} dt \qquad \text{.................. (4.5)}$$

where, the term $e^{-\rho.t}$ represents the altruism factor ψ of Eq (4.4).

In Eq (4.5), n_t represents the net growth rate of population. In above utility function, the stock of population L was considered as a given. We considered $x_t = x_t - x_{min}$, where the parameter $x_{min} > 0$ denotes the subsistence level of consumption in the economy. Jones (1999) also worked with an utility function to endogenize the population growth that was defined by a subsistence consumption and fertility choice, but it was independent with population size.

4.3 Population Change

In this section, we endogenize the population growth rate to explore its role in the resource economy. Jones (2001) discussed the Malthusian theory on population growth with this fertility model, and argued that population size becomes asymptotically constant with zero population growth because of a fixed supply of natural resources and its availability for consumption at subsistence level. To model the dynamics of demography, it is considered that a change in population is equal to the number of births minus number of deaths. Thus the law of motion for population is intimately tied to a linear differential equation. This linearity of the law of motion for population is a biological fact of nature that people have children in proportion to their number. By choosing the number of children to have, individuals choose the proportional rate of increase in population. To explain this, we imagined a world consisting of L_t individuals at a time *t*. It is supposed that each individual has a certain number of children, denoted by $\tilde{n}_t$, and exogenously given a constant population death rate, *d*, at each point in time.

4.4 The Model with an Exponential Population Growth

Thus, the law of motion for the aggregate population in a continuous time is given by

$$\dot{L}_t = (\tilde{n}_t - d)L_t \qquad \text{……………..} (4.6)$$

We considered the population growth rate, $n_t = \tilde{n}_t - d$ so that it makes it simple for mathematical purposes. In this study, we are only concerned with the population growth rate, and thus we considered it as.

$$\dot{L}_t = n_t L_t \qquad \text{…………….} (4.7)$$

In this case, the population birth rate, n_t, was a choice variable over time. But a death rate, d was considered as exogenously given to make the analysis simple though it depends on family or public expenditures on medical care, sanitation, and so on. The stock of population size, L_t, is now an additional state variable for households' optimization problem.

The optimization problem is to maximize the utility, U_t, in equation (4.5) to choose the path of the control variables x and n, and this maximization is subject to the constraints of Eq (4.7) and the per capita resource constraint of Eq (3.1). The problem is written as

$$\underset{x,n}{Max}. \quad U = \int_0^\infty \left[L_t \left\{ \frac{[(x_t - x_{\min}).n_t^{\phi}]^{1-\theta} - 1}{(1-\theta)} \right\} - c_0 e^{-\gamma.t} x_t L_t + \varsigma.s_t L_t \right].e^{-\rho.t} dt \qquad \text{……}(4.8)$$

subject to,

i) $\dot{s}_t = -x_t - n_t.s_t$

ii) $\dot{L}_t = n_t L_t$

iii) $x_t \geq 0$, $n_t \geq 0$

iv) $s(0) = s_0$ $\quad$ *s(T) free* $\quad$ *(s_0 given)*

v) $\lim_{t \to \infty}$, $\mu L = 0$, Transversality condition for population.

The Hamiltonian function of above optimization problem becomes

$$H_t = L_t \frac{\{[(x_t - x_{\min}).n_t^{\phi}]^{1-\theta} - 1\}}{(1-\theta)} - c_0 e^{-\gamma.t} x_t L_t + \varsigma.s_t L_t - \mu_1 (x_t + n_t s_t) L_t + \mu_2 n_t L_t \quad \text{…….} (4.9)$$

where, μ_1 and μ_2 are the shadow prices associated with the two state variables, s and L. The household maximization problem satisfies the usual first order-conditions and we concentrated the results and discussion under log utility when $\theta = 1$.

By the condition $\partial H / \partial x = 0$, we get

$$\mu_1 = (x_t - x_{\min})^{-\theta} n_t^{\phi(1-\theta)} - c_0 e^{-\gamma.t} \quad \text{……………. (4.10)}$$

When $\theta = 1$, Eq (4.10) becomes

$$\mu_1 = (x_t - x_{\min})^{-1} - c_0 e^{-\gamma.t} \quad \text{…………... (4.10a)}$$

Rearranging Eq (4.10a) for x_t,

$$x_t = x_{\min} + \frac{1}{\mu_1 + c_0 e^{-\gamma.t}} \quad \text{…………. (4.10b)}$$

Again the first order condition for a state variable, s_t, is

$$-\partial H / \partial s = \dot{\mu}_1 - \mu_1.\rho$$

$$-\varsigma.L = \dot{\mu}_1 - \mu_1(\rho + n_t.L_t) \quad \text{………….. (4.11)}$$

The general solution of a differential equation (4.11) for μ_1 is

$$\mu_1 = \mu_{1(0)} e^{(\rho + n_t L_t)t} + \frac{\varsigma.L_t}{\rho + n_t L_t} \quad \text{…………… (4.11a)}$$

From Eq (4.10b) and (4.11a),

$$x_t^* = x_{\min} + \frac{1}{\mu_{1(0)} e^{(\rho + n_t L_t)t} + \frac{\varsigma.L_t}{\rho + n_t L_t} + c_0 e^{-\gamma.t}} \quad \text{……… (4.11b)}$$

Eq (4.11b) gives an optimal mine phosphate extraction path under endogenous population growth.

Again for second control variable, n_t, and the state variable, L_t,

The first order condition $\partial H / \partial n = 0$ gives,

$$\mu_2 = (x_t - x_{\min})^{-\theta} n_t^{\phi(1-\theta)} \left[s_t - (x_t - x_{\min}).\phi.n_t^{-1} \right] - c_0 e^{-\gamma.t} s_t \quad \text{…….. (4.12)}$$

When $\theta = 1$, Eq (4.12) becomes

$$\mu_2 = (x_t - x_{\min})^{-1} s_t - \phi.n_t^{-1} - c_0 e^{-\gamma.t} s_t \quad \text{………… (4.12a)}$$

Eq (4.12a) can be expressed in the present-value Hamiltonian as

$$\mu_2 = \left[(x_t - x_{\min})^{-1} s_t - \phi . n_t^{-1} - c_0 e^{-\gamma . t} s_t\right] e^{-\rho . t} \quad \text{………… (4.12b)}$$

Rearranging Eq (4.12b) for n_t,

$$n_t = \frac{\phi}{(x_t - x_{\min})^{-1} s_t - c_0 e^{-\gamma . t} s_t - \mu_2} \quad \text{………… (4.13)}$$

From (4.10a), the first order condition $-\partial H / \partial L = \dot{\mu}_2 - \mu_2 . \rho$ becomes

$$-\frac{[(x_t - x_{\min}) . n_t^{\phi}]^{1-\theta} - 1}{(1-\theta)} + c_0 e^{-\gamma . t}(x_t - x_{\min}) - \varsigma . s_t + \mu_1 (x_t + n_t s_t) = \dot{\mu}_2 - \mu_2 (\rho - n_t) \quad \text{…..(4.14)}$$

When $\theta = 1$, Eq (4.14) becomes

$$c_0 e^{-\gamma . t} - \varsigma . s_t + [(x_t - x_{\min})^{-1} + c_0 e^{-\gamma . t}](x_t + n_t s_t) = \dot{\mu}_2 - \mu_2 (\rho - n_t) \quad \text{….. (4.14a)}$$

The general solution of a differential equation (4.14a) for μ_2 is

$$\mu_2 = \mu_{2(0)} e^{(\rho - n_t) t} - \left[\frac{c_0 e^{-\gamma . t} - \varsigma . s_t + [(x_t - x_{\min})^{-1} + c_0 e^{-\gamma . t}](x_t + n_t s_t)}{(\rho - n_t)}\right] \quad \text{.. (4.15)}$$

When the optimization problem satisfies the transversality condition of population - $\lim_{t \to \infty}$, $\mu L = 0$, the $\dot{\mu}_2 = 0$ for all t, and μ_2 always equals its steady-state value,

$$\mu_2 = \frac{c_0 e^{-\gamma . t} n_t s_t - \varsigma . s_t + (x_t - x_{\min})^{-1} x_t - (x_t - x_{\min})^{-1} n_t s_t}{(\rho - n_t)} \quad \text{…. (4.16)}$$

Solving Eq (4.13) with value of μ_2 from Eq (4.16), we get

$$n_t = \frac{\phi \rho}{\phi + s_t \left\{\rho\left[(x_t - x_{\min})^{-1} - c_0 e^{-\gamma . t}\right] - \varsigma\right\} + (x_t - x_{\min})^{-1} x_t} \quad \text{………….. (4.17)}$$

With the increase in the shadow price of mine, its scarcity rent increases and induces lower population growth rate. Higher the value of ϕ, the higher will be the marginal utility from the number of children, $\tilde{n}$ and thereby increase the population growth rate, n in case of constant death rate, d. Barro *et al.* (2004) explained that a higher discount rate, ρ, deters investment in population and tends to lower population growth rate. We also found the same relation as explained by Barro *et al.* (2004). A higher interest rate tends to lower population growth. When the consumption availability of phosphate decreases, the marginal utility, $(x_t - x_{\min})^{-1}$, increases and gives a higher which induces the willingness-to-pay for a unit resource. Thus, the cost of rearing child increases and the population growth rate, n_t, decreases.

When the parents get a higher level of marginal utility from the number of children, they get larger number of children and the extraction of mine phosphate for their consumption increases. As stock size getting a large that might be due to exploration of new reserves, the total cost of mine extraction decreases. It induces parents to have higher number of children and thus, more consumption of resource.

4.5 The Model with a quasi-arithmetic population growth

As we explained earlier, a quasi-arithmetic progression of population growth gives an estimation of world population figures close to the actual census data collected by the US Census Bureau, International Data Base (USCB, 2010). According to the literatures discussed in previous chapter 3.0, the quasi-arithmetic progression gives a more close to real picture though it makes our mathematical calculations complex.

Even with a quasi-arithmetic progression, the objective function of our optimization problem would be the same as in exponential case, but it was subjected to different constraints of the law of motion of stock variable and population. The dynamic optimization problem would be

$$\underset{x}{Max.}\ \ U = \int_0^\infty \left[L_t \left\{ \frac{[(x_t - x_{\min}).n_t^{\phi}]^{1-\theta} - 1}{(1-\theta)} \right\} - c_0 e^{-\gamma.t} x_t L_t + \varsigma.s_t L_t \right] .e^{-\rho.t} dt \quad \ldots\ldots\ldots (4.18)$$

subject to,

i) $\dot{s} = -x_t - \frac{ns_t}{1+t}$

ii) $\dot{L} = \frac{n.L_t}{(1+t)}$

iii) $x_t \geq 0$, $n_t \geq 0$

iv) $s(0) = s_0$ $\quad$ *s(T) free* $\quad$ (s_0 *given)*

v) $\lim_{t \to \infty}$, $\mu L = 0$, Transversality condition for population.

Now the Hamiltonian function would be

$$H = L_t\left\{\frac{[(x_t - x_{\min}).n_t^{\phi}]^{1-\theta} - 1}{(1-\theta)}\right\} - c_0 e^{-\gamma.t} x_t L_t + \varsigma.s_t L_t - \mu_1\left(x_t + \frac{n_t s_t}{1+t}\right)L_t + \mu_2\left(\frac{n_t.L_t}{(1+t)}\right)$$

……….. (4.19)

The first order condition of the Hamiltonian function with respect to *X* variable would result,

$$\mu_1 = (x_t - x_{\min})^{-1} - c_0 e^{-\gamma.t} \qquad \text{…………… (4.20)}$$

Rearranging Eq (4.20), we get

$$x_t = x_{\min} + \frac{1}{\mu_1 + c_0 e^{-\gamma.t}} \qquad \text{………… (4.20a)}$$

Again the first order condition for a state variable, s_t, we get the general solution of μ_1 as

$$\mu_1 = \mu_{1(0)} e^{\left(\rho + \frac{n_t L_t}{1+t}\right).t} + \frac{\varsigma.L_t}{\rho + \frac{n_t L_t}{1+t}} \qquad \text{…………… (4.21)}$$

Substituting the value of μ_1 from Eq (4.21), Eq (4.20a) would be

$$x_t^* = x_{\min} + \frac{1}{\mu_{1(0)} e^{\left(\rho + \frac{n_t L_t}{1+t}\right).t} + \frac{\varsigma.L_t}{\rho + \frac{n_t L_t}{1+t}} + c_0 e^{-\gamma.t}} \qquad \text{……… (4.22)}$$

Eq (4.22) gives an optimal mine phosphate extraction path with endogenous population growth under a quasi-arithmetic progression of population growth.

The quasi-arithmetic progression also gives the same pattern of per capita mine extraction and consumption as of an exponential case, but from a lower level due to the consideration of time factor, *(1+t)*. For a finite terminal time *(T)* though it is far enough, per capita extraction and consumption would be higher than the latter case. It can be said that the terminal time of mine exhaustion will be prolonged in arithmetic progression if the extraction rate of exponential case is maintained. Thus, mine exhaustion won't follow the time path as explained in the literatures that worked with an exponential population growth path. The exhaustibility of mine stock limits the population growth and does not allow to follow the exponential paths that Asheim *et al.* (2007) explained. The quasi-arithmetic progression thus prolongs the life of phosphorus stock and provides a situation not to be worried as predicted in literatures with exponential case.

For the second control variable, n_t, and the state variable, L_t,

The first order condition $\partial H / \partial n = 0$ gives,

$$\mu_2 = (x_t - x_{\min})^{-1} s_t - c_0 e^{-\gamma.t} s_t - \phi.n_t^{-1}(1+t) \quad \text{...... (4.23)}$$

Eq (4.23) can be expressed for n_t as

$$n_t = \frac{\phi(1+t)}{(x_t - x_{\min})^{-1} s_t - c_0 e^{-\gamma.t} s_t - \mu_2} \quad \text{............ (4.24)}$$

The first order condition $-\partial H / \partial L = \dot{\mu}_2 - \mu_2.\rho$ yields the general solution for μ_2 as

$$\mu_2 = \mu_{2(0)} e^{\left(\rho - \frac{n_t}{1+t}\right).t} + \left[\frac{\varsigma.s_t + \frac{c_0 e^{-\gamma.t} n_t s_t}{1+t} - \beta.n_t.s_t - \dot{s}_t (x_t - x_{\min})^{-1}}{\left(\rho - \frac{n_t}{1+t}\right)} \right] \quad \text{........ (4.25)}$$

When the optimization problem satisfies the transversality condition of population - $\lim_{t \to \infty}$, $\mu L = 0$, the $\dot{\mu}_2 = 0$ for all t, and μ_2 always equals its steady-state value,

$$\mu_2 = \frac{\varsigma.s_t + \frac{c_0 e^{-\gamma.t} n_t s_t}{1+t} + \dot{s}_t (x_t - x_{\min})^{-1}}{\left(\rho - \frac{n_t}{1+t}\right)} \quad \text{.......... (4.26)}$$

Solving Eq (4.24) with the above value of μ_2, we get

$$n_t = \frac{\phi\rho(1+t)}{\phi + s_t \left\{ \rho\left[(x_t - x_{\min})^{-1} - c_0 e^{-\gamma.t} \right] - \varsigma \right\} + (x_t - x_{\min})^{-1}.x_t} \quad \text{..... (4.27)}$$

In a quasi-arithmetic case, the population growth rate was slowed down by the presence of the time factor. As explained earlier, the resource availability has exerted a direct influence on population growth and limited the population growth rate. The marginal utility has a negative relation with population growth rate, and instrumented a relation with the exhaustibility of phosphorus resources.

4.6 Conclusion

In this chapter, we considered a child fertility choice to maximize utility obtained from the consumption of mine phosphate. An overlapping generation model was considered to endogenize the population growth rate. It was assumed that the parents have a diminishing marginal utility toward the presence of number of children. In our model, parent in each generation choose consumption of mine phosphate and child fertility rate to maximize their own utility.

From the analysis, it is found that there is a strong positive relation between population growth rate and per capita phosphate consumption. Higher the marginal utility of children, higher will be the per capita phosphate extraction. A higher discount rate deters investment in population and tends to lower population growth rate and lower per capita resource extraction.

Chapter 5: Recycling and Phosphate Extraction

Since the 18th century, the classical economists were expressing their concerns over the challenges of the economy threatened by the exhaustibility of essential natural resources. Malthus (1958) gave a pessimistic view on the situation generated by a geometrically increasing population with a limited availability of non-renewable resources. By the mid 20th century, the fear of resource scarcity was voiced by Meadows et al. in 1972 in their publication of *Limits to Growth*. But in recent years, it has become a big debate whether fears of resource scarcity are actually legit, and arguments of it being not as much alarming as suggested by earlier economists have come to the fore. In the past, it was signaled that resource prices was rocketed geometrically due to the continuous increasing population size and the depletion of low-cost deposits over time (Tilton, 1999). But the recent scientists argued that the cost-reducing effects of new technology and innovations may be more than the offset effect of an upward pressure on costs. Thus it seems that the earlier prediction of soaring resource price was not entirely true.

Burgeoning population has put a lot of pressure to intensify agricultural yields and it demands more fertilizer to replenish the nutrients removed by crops. In earlier times, food consumption was near the production site, and the animal and human wastes were returned to the agricultural land. The societal change in the developmental process has broken the natural recycling loop for fertilizer nutrients, and over 80 percent of phosphate currently mined is used in production of fertilizer products (Tilton, 1999). This dwindling effect of resource mine accentuates to go for recycling the industrial and municipal disposal. The industrial and municipal wastes are generally harmful, or will become harmful to the environment and living beings when the stock is getting larger. Until and unless the waste residual is completely recycled, it reappears as a harmful residual in the environment and leaves negative effects on social welfare. Recycling is a way to reduce environmental cost and produce recycled materials that acts as a substitute to mine phosphate. Many environmentalists, scientists, and other well-informed and thoughtful individuals have been exercising to draw attention of the politicians, bureaucrats and policy makers on recycling and production of substitutes for production inputs. Many policies in the area of environment are focused on recycling and reusing of exhaustible resources. The increasing demand of phosphorus to produce food for burgeoning population together with increasing environmental cost of waste disposal is pushing the people to go for recycling. Recycling is supposed to be a way to support life of increasing population at least as good as at the present level of affluence. In case of phosphate mineral, the recycling provides substitutes and helps to make the scarcity of phosphorus less serious than the case of no-recycling.

According to Carlsen (1972), recycling is an important basis for making a comparison between economic system and ecosystem. In an ecosystem, scavenging organisms and processes convert wastes and organic detritus of the living sphere into the soil nutrients, and plants dissipated as it passes through the food chain. In an economic system, a recycling process brings back the industrial and municipal wastes and disposals in use for production of useful goods and services. In case of exhaustible raw materials, a complete recycling with no losses of materials is crucial for a steady flow of production goods in the long-run. Carlsen (1972) argued that the marginal cost of a percentage point increase in recycling activity exceeds the marginal cost of one percentage point reduction in virgin mine extraction because of the technological advancement in resource extraction and exploration of new reserves in the recent times. But the stock effect of non-renewable resources pushed to shift the economic activity towards recycling process. Furthermore, the increasing spatial competition over land among different economic activities such as waste disposal, agriculture production, human settlement and others induces to move for adopting the recycling process.

Phosphorus is an essential mineral that is usually found in nature combined with oxygen as phosphate. The phosphorus cycle shown in figure 8 below is similar to several other mineral nutrient cycles in that phosphorus exists in soils and minerals, living organisms, and water. The phosphorus cycle explains that plants uptake phosphate from soils, utilized by animals that consume plants, and returned to soils as organic residues decay in soils. The process of mineralization converts the organic phosphates into inorganic phosphates. Many phosphate compounds are not easily soluble in water and therefore, exist in solid form in natural systems.

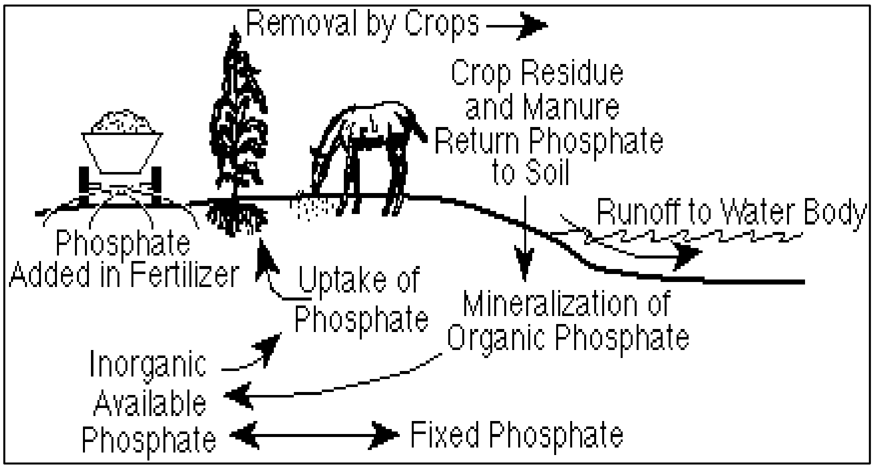


Fig 10. The Phosphorus Cycle.

Source: Busman et al. (2002)

In an ecosystem, phosphorus is a non-renewable resource in the sense that its rate of consumption by humans or other users is faster than the environment's capacity to replenish it. However, one need not worry as much about its complete exhaustion given the fact that it is a non-destroyable and may be regenerated from one form to another form and hence, reused. A matter of present concern is rather on how sustainable increasing rate of consumption from a fixed stock is. A very slow pace of natural mineralization almost makes it impossible to replenish consumed minerals and recognizes the phosphate stock as an exhaustible resource. In this analysis with an incorporation of recycling as a process of regenerating an easily available phosphorus form to the plants, we considered all different forms of phosphorus in the ecosystem as a given stock size. Dasgupta (1979) pointed out that it is not possible to recycle the entire quantity of phosphorus used in consumption and there will be a leakage at every round of recycling into easily available forms to the plants. It is assumed that the recycling process supplies recycled phosphate and decreases the pressure on depletion of exhaustible stock from excess extraction. Recycling augments phosphorus supplies in the market and enables scarce resources to be saved. André *et al.* (2006) mentioned that the recycling possibilities and its effectiveness depend on the composition of waste materials that are used as inputs in the recycling process. It was argued that recycling may increase effective availability of resources and allow for a more intense use for higher output levels in short run. But in the long run, it could not provide a compensation for the exhaustibility of non-renewable resources. Lusky (1976) also argued that recycling induces lesser consumption of original goods with an increased production and consumption of the recycled good. It was suggested to the government authority to formulate a policy of taxing on consumption of original goods and of subsidizing the production of recycled good that would help to minimize the rate of depletion of exhaustible resource stock.

We will develop a simple dynamic model to analyze resource substitution when a recycling technology is present, and the dynamic consequences of recycling. For comparing purposes, we will look on the dynamic consequences on resource use with and without recycling. We assume that the recycling process produces a quantity of output, *R(X)*, from the disposal comes from the use of mine phosphate, *X*. We considered the recycling produce and process as defined by Schulze (1974), Hoel (1978), Dasgupta (1979) and André *et al.* (2006). It was defined as

(i) $0 \leq R(X) \leq X$, and

(ii) $0 \leq R_X \leq 1$

where, R_X indicates the partial derivative of recycling function with respect to mine phosphate, *X*. The definitions explained the non-possibility of full recycling and only a fraction of waste residuals can be recycled to produce recycled materials. The first definition elucidates that it is not possible to recover more resource than what is being employed in production process, and the second definition tells that the recovered quantity increases with *X* but the increment in *R* will be smaller than that in *X*. In our study, *X* is considered as an amount of phosphorus resource consumed in the production process and it is a loss of quantity from the stock that would be easily available for consumption.

Now, net extraction rate of phosphate *(Z)* can be defined as

$Z = X - R(X)$ …………... (5.1)

Eq (5.1) gives a measurement of an instantaneous effective reduction of phosphorus stock taking into a consideration the quantity extracted for consumption in production process and the quantity recovered from recycling

We then defined the equation of motion of natural stock of non-resource as

$\dot{S} = -X_t + R(X)$ …………… (5.2)

In case of no recycling, that is, $R(X) = 0$, the above equation becomes $\dot{S} = -X_t$ and it was the case of our benchmark model. From the above discussion, it can be seen that the production function of recycled output depends on composition of waste disposal and a content of phosphorus, X. The function is linear as defined below,

$R(X) = \kappa.X$

where, κ as a unit of fraction that is eventually recycled to form an input into the resource-using processes and it is $0 < \kappa < 1$.

The resource constraint can be defined as

$$\int_0^\infty (1-\kappa)X_t \leq S_t \text{ or } \int_0^\infty X_t \leq S_t(1-\kappa) \qquad \ldots\ldots\ldots\ldots\ldots (5.3)$$

In the above relation, a factor (1 - κ) is a fractional unit that is lost in recycling process as it cannot be used in further recycling process that may be among the several reasons due to a spatial dispersion in a small quantity in a wide spread area.

The equation of motion of a stock now can be written as,

$$\dot{S} = -(1-\kappa)X_t \qquad \ldots\ldots\ldots\ldots\ldots (5.4)$$

From Equation (5.4), it can be explained that the effective resource base is increased by a factor $1/(1-\kappa)$, and the resource base could be increased by effective recycling policies.

Taking the time derivative of stock size of resource per capita, $s_t = S_t/L_t$, we will get

$$\dot{s} = \frac{\dot{S}}{L} - s\frac{\dot{L}}{L} \qquad \ldots\ldots\ldots\ldots\ldots (5.5)$$

From Eq (5.4) and Eq (3.1), Eq (5.5) becomes

$$\dot{s} = -(1-\kappa).x_t - s_t.n_t \qquad \ldots\ldots\ldots\ldots\ldots (5.6)$$

5.1 Social Optimization Problem

In this study, we are developing a model with an extension of the traditional theory of exhaustion where we also take recycling into consideration. We considered that recycling creates the possibility of supplanting extraction. In this thesis, we exclude the possibility of exploration of new reserves and existence of multiple grades of phosphate mines, but we are rather strictly confined within the presence of a competitive market structure.

In the optimization problem, we maximize the net social welfare benefit with respect to the given constraints. Schulze (1974) and Hoel (1978) presented the utility function obtained from both extracted resource and recycled products but it costs the disutility generated by the piled-up residuals stock and the cost of recycling. $U(X_t + Y_t)$ is the gross utility of a society obtained from the consumption of extracted resource (X_t) and recycled products or substitutes (Y_t) at a time t. Mathematically, it is measured as

$$U(X_t + Y_t) = \int_0^S P(S_t).dS$$

where, S_t is the set of $X_t + Y_t$, and $P(S)$, i.e., $P(X_t+Y_t)$ is the inverse demand curve as a function of the rate of extraction, X_t, plus amount recovered, Y_t.

We assume the properties of inverse demand function as $P'<0$ and $P = U'$ indicating a Pareto optimum in terms of market structure. In our model, we ignored the disutility created by the stock of residual wastes. We directly consider the effect of recycling on mine stock size as in Equation (5.6). This is to make a calculation of our analysis simple. We assumed a constant marginal recycling cost, R_0, and the cost of recycling Y units material is R_0Y measured in the same units as the utility of extraction plus recycling measured.

5.2 The Model

We developed each successive models with an inclusion of new factor upon the most recent one. The optimization problem with recycling was developed upon the model of endogenous population growth. To endogenize the population growth and define an optimal mine resource extraction path, we considered each variable in terms of per capita. Thus, in this section, we worked upon the optimization problem of Eq (4.19) with an inclusion of recycling factor. We compared the results under two scenarios of population growth paths – exponential and quasi-arithmetic progression. In first section of this chapter, we worked with an exponential growth path.

5.2.1 Phosphate Extraction from an Ecosystem

5.2.1.1 Phosphate Extraction under Exponential Population Growth

In this section, we worked in the optimization problem of Eq (4.19) in extension with recycling component. We assumed that the recycled products are perfect substitutes of mined phosphorus. In a competitive resource market, it is to be a Pareto optimum condition for maximizing the present value of the social welfare function,

$$W = \int_0^\infty [U(x_t, y_t) - C_x x_t - C_y y_t] e^{-\rho.t}.dt \qquad \text{............ (5.7)}$$

and it is subject to the constraints,

i) $\dot{s} = -(1-\kappa).x_t - s_t.n$

ii) $\dot{L}_t = n_t L_t$

iii) $x_t \geq 0$, $n_t \geq 0$, $y_t \geq 0$.

iv) $s(0) = s_0$, $s(T)$ *free* and s_0 *given.*

v) $\lim_{t \to \infty}$, $\mu L = 0$, the transversality condition for population.

where, C_x and C_y are a unit cost of extraction and recycling, respectively.

Here, constraint *(i)* explained the changes in a resource stock size in an ecosystem by a fraction of flow of per capita resource, $(1-\kappa)\ x$, that was lost in the recycling process. It was also exacerbated by the population growth. Schulze (1974) and Hoel (1978) accounted waste residuals into consideration as a stock variable in their model because their study was focused on optimizing between the size of recycling and of waste disposal. Conversely, we did not define the waste disposal as second stock variable in our optimization problem as they did. The reason is that our study is only concerned with analyzing the extraction of resource and comparing the optimal control paths under different situations rather than optimizing the size of recycling for maximum benefit. Schulze and Hoel's study was focused on allocating the optimal size of recycling and waste disposal.

Highfill *et al.* (2001) argued that a higher income in a community induced more consumption and also ask for more recycling. It was reported that the per unit cost of recycling increases as the amount of recycling increases while holding the time factor as a constant, and the logic behind this assumption was that the recycling of bottles and newspaper would be less expensive than the that of bottles, newspaper and refrigerators. It was also highlighted that per unit recycling cost is non-increasing over time holding the amount of recycling constant. Thus, the total recycling cost was assumed to be a convex function of recycling input. Seyhan *et al.* (2009) also argued in a similar line of Highfill *et al.* (2001) and considered the recycling cost as below,

$$C^R(X,Y,t) = R_0 e^{-\rho.t}\frac{Y_t}{X_t} \qquad \text{……….. (5.8)}$$

where, R_0 is the initial recycling cost that decreases over a time at a rate of ρ, and C^R is the total recycling cost. The recycling cost increases with the additional quantity of recycled produces and decreases with mine extraction.

Expressing Eq (5.8) in terms of per capita, we get

$$c^r(x,y,t) = r_0 e^{-\rho.t}\frac{y_t}{x_t} \qquad \text{………… (5.9)}$$

Now the objective function of our optimization problem in Eq (5.7) would be defined as

$$\underset{x}{Max.}\;\; U = \int_0^{\infty}\left[L_t\left\{\frac{[(x_t + y_t - x_{\min}).n_t^{\phi}]^{1-\theta} - 1}{(1-\theta)}\right\} - c_0 e^{-\gamma.t}x_t L_t + \varsigma.s_t L_t - r_0 e^{-\rho.t}\frac{y_t}{x_t}L_t\right].e^{-\rho.t}dt$$

$$\text{……………… (5.10)}$$

and subject to the same constraints as explained in Eq (5.7).

The Hamiltonian function to optimize Eq (5.10) is written as

$$H^c = L_t\left\{\frac{[(x_t + y_t - x_{\min}).n_t^{\phi}]^{1-\theta} - 1}{(1-\theta)}\right\} - c_0 e^{-\gamma.t}x_t L_t + \varsigma.s_t L_t - r_0 e^{-\rho.t}\frac{y_t}{x_t}L_t - \mu_1.\{(1-\kappa).x_t + n_t.s_t\}.L_t + \mu_2.n_t.L_t$$

$$\text{……………… (5.11)}$$

The first order conditions of the above the Hamiltonian function as we wrote in Eq (2.9a) – (2.9d) optimized the problem of optimization.

The first order condition of the Hamiltonian function with respect to x yields

$$\mu_1 = \frac{(x_t + y_t - x_m)^{-1} - c_0 e^{-\gamma.t} + r_0 e^{-\rho.t} y_t x_t^{-2}}{1-\kappa} \qquad \text{……….. (5.12)}$$

The First order condition, $-\partial H/\partial S = \dot{\mu}_1 - \rho.\mu_1$, yields

$$\mu_1^* = \mu_{1(0)}.e^{(\rho+n.L)t} + \frac{\varsigma.L}{\rho + n.L} \qquad \text{………… (5.13)}$$

Maximizing the Hamiltonian function with respect produces the condition

$$(x_t + y_t - x_m)^{-1} = r_0 e^{-\rho.t}x_t^{-1} \qquad \text{………. (5.14)}$$

Eq (5.14) holds the marginal utility (MU) equal to marginal cost (MC) for recycling and satisfied the condition of Pareto optimality in a perfect competitive market.

Solving Eq (5.14) for *y*, we have

$$y_t = \frac{x_t\left(1-r_0e^{-\rho.t}\right)+x_m r_0 e^{-\rho.t}}{r_0 e^{-\rho.t}} \qquad \text{……….. (5.14a)}$$

From Eq (5.12) and (5.14a), x_t would be,

$$x_t = \frac{1+r_0e^{-\rho.t}x_t^{-1}x_m}{(1-\kappa).\mu_1 + c_0e^{-\gamma.t}} \qquad \text{……….. (5.15)}$$

Solving Eq (5.15) for *x*, we get

$$\left[\mu_1(1-\kappa)+c_0e^{-\gamma.t}\right]x_t^2 + (-1)x_t + (-r_0e^{-\rho.t}x_m) = 0 \qquad \text{………… (5.15a)}$$

The general solution of above quadratic equation would be[2] ,

$$x_t = \frac{1\pm\sqrt{1+4\mu_1 r_0 e^{-\rho.t}x_m - 4\mu_1 r_0 e^{-\rho.t}\kappa.x_m + 4r_0e^{-\rho.t}x_m c_0 e^{-\gamma.t}}}{2(1-\kappa).\mu_1 + 2c_0e^{-\gamma.t}} \qquad \text{………. (5.16)}$$

In Eq (5.16), the quantity of resource extraction won't have negative sign because it is strictly assigned the positive values for resource extraction. Thus, Eq (5.16) becomes

$$x_t = \frac{1+\sqrt{1+4.r_0e^{-\rho.t}x_m\left[\mu_{1(0)}.e^{(\rho+n.L).t}(1-\kappa)+\dfrac{\varsigma.L}{\rho+n.L}(1-\kappa)+c_0e^{-\gamma.t}\right]}}{2\left[\mu_{1(0)}.e^{(\rho+n.L).t}(1-\kappa)+\dfrac{\varsigma.L}{\rho+n.L}(1-\kappa)+c_0e^{-\gamma.t}\right]} \qquad \text{……. (5.17)}$$

Eq (5.17) can be written as,

$$x_t = \frac{1}{2.\Lambda} + \frac{\sqrt{1+4.r_0e^{-\rho.t}x_m\Lambda}}{2.\Lambda} \qquad \text{………… (5.17a)}$$

where, $\Lambda = \left[\mu_{1(0)}.e^{(\rho+n.L).t}(1-\kappa)+\dfrac{\varsigma.L}{\rho+n.L}(1-\kappa)+c_0e^{-\gamma.t}\right]$

Eq (5.17a) gives an optimal resource extraction path of mine phosphate. The above path shows that per capita consumption of mine phosphorus is lower than the case of no recycling. While looking on phosphate extraction path over time, it rises in the beginning until to reach a peak due to high initial recycling cost and low extraction cost. As marginal extraction cost rises due to the stock effect in mine and recycling cost decreases, it starts to fall continuously over time. It is also revealed that the rate of phosphate extraction from mine decreases as recycling increases, and this effect is obvious because of the perfect substitutability between them.

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[1]*The general solution of a quadratic equation, $ax^2 + bx + c = 0$, would be*

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

5.2.1.2 Phosphate Extraction under Quasi-Arithmetic Population Growth

Under a quasi-arithmetic population growth, the optimal extraction of mine extraction follows a path as below,

$$x_t = \frac{1}{2\Delta} + \frac{\sqrt{1 + 4.r_0 e^{-\rho.t} x_m \Delta}}{2\Delta} \quad \text{.........} (5.18)$$

where, $\Delta = \left[\mu_{1(0)}.e^{\left(\rho + \frac{n.L}{1+t}\right).t}(1-\kappa) + \frac{\varsigma.L}{\rho + \frac{n.L}{1+t}}(1-\kappa) + c_0 e^{-\gamma.t} \right]$

The above extraction path moves in the same directional path as it does in case of exponential population growth, but it goes from a lower level because of a slow growth of population size.

5.2.2 Mine Extraction under a complete recycling

For an analysis of mine extraction under a complete recycling process, we set *κ = 1* to reveal a zero leakage of phosphorus content in recycling process. Eq (5.17) now becomes,

$$x_t = \frac{1 + \sqrt{1 + 4.r_0.c_0.e^{-(\rho+\gamma).t} x_m}}{2c_0 e^{-\gamma.t}}$$

and it is written as,

$$x_t = \frac{1}{2C} + \frac{\sqrt{1 + 4.r_0.e^{-\rho.t} x_m C}}{2C} \quad \text{.........} (5.19)$$

where,

$C = c_0 e^{-\gamma.t}$

Even in the case of complete recycling of industrial and municipal waste for phosphorus production, there would be an extraction of mine phosphate whatever may be the quantity, and the law of motion of stock would be $\dot{s} = -x_t - s_t.n$. It is because the recycling process alone cannot satisfy the demand of phosphorus for food production. It is revealed that we cannot completely stop the reduction of mine stock even with a case of no loss of phosphorus in the recycling process, that is, in complete recycling.

5.2.3 Phosphate Extraction in case of mine stock

In section 5.2.1, phosphorus is described as a non-destroyable resource in the ecosystem. It is assumed that the consumed phosphorus goes back to the ecosystem through human excreta and animal manure, and enters into the phosphorus cycle. We considered the whole ecosystem as a source of phosphorus. Dasgupta (1979) also defined the relation between ecosystem and economy, and considered the ecosystem as a resource stock. However, it is still a non-renewable resource in the sense that the natural process of reformation of phosphate is quite slow and almost sluggish.

In this section, we take into account of phosphate mine as a sole source of phosphorus, and its consumption does not add back to the stock size. Seyhan *et al.* (2009) concentrated on depletion of mine phosphate, but this does not reveal any relationship between recycling activity and extraction of mine phosphate. Obviously, mine phosphate is a non-renewable resource in nature since stock size decreases once it is extracted.

In each case, we discussed the extraction process and stock movement under both exponential and quasi-arithmetic progression of population growth.

5.2.3.1 Exponential Population Growth

In case of an Exponential population growth path, the mine phosphate extraction path would be as mentioned below.

$$x_t = \frac{1}{2\Omega} + \frac{\sqrt{1 + 4.r_0 e^{-\rho.t} x_m \Omega}}{2\Omega} \quad \text{……….} \ (5.20)$$

where, $\Omega = \left[\mu_{1(0)}.e^{(\rho+n.L).t} + \frac{\varsigma.L}{\rho + n.L} + c_0 e^{-\gamma.t} \right]$

Comparing Eq (5.20) and (4.21), it is shown that mine extraction would be lower in case of recycling. It is because of supply of substitutes of mine phosphate. In Eq (5.20), phosphate extraction from mine increases as the recycling cost gets higher, but it does not show a direct correlation of recycling rate and mine extraction. In the case of Eq (5.17a), it is nicely revealed a negative correlation of per capita mine extraction and recycling rate.

5.2.3.2 Quasi-Arithmetic Progression of Population Growth

The mine extraction path under a quasi-arithmetic progression of population growth would be

$$x_t = \frac{1}{2\Theta} + \frac{\sqrt{1 + 4.r_0 e^{-\rho.t} x_m \Theta}}{2\Theta} \qquad \text{……. (5.21)}$$

where, $\Theta = \left[\mu_{1(0)}.e^{\left(\rho + \frac{n.L}{1+t}\right).t} + \frac{\varsigma.L}{\rho + \frac{n.L}{1+t}} + c_0 e^{-\gamma.t} \right]$

Eq (5.21) follows the same directional path of phosphate extraction as of exponential case in Eq (5.20), but the difference is only the level of extraction. In the latter case, the mine extraction path is flatter due to slower growth of population.

5.3 Recycling Path

Recycling of municipal and industrial wastes produces phosphorus and provides substitutes of mine phosphate. A recycling path depends on recycling cost and mine extraction, and holds a functional relation as shown below:

$$y_t = \frac{x_t(1 - r_0 e^{-\rho.t})}{r_0 e^{-\rho.t}} + x_{\min} \qquad \text{………. (5.22)}$$

From Eq (5.22) and (5.17), it is illustrated that a decreasing recycling cost encourages to go for higher production of recycled amount, and a higher recycling rate calls for a lesser mine extraction and prolongs the life of a phosphate mine.

5.4 Conclusion

Phosphorus is a non-destroyable resource and thus we considered phosphorus content in the ecosystem as its stock. A naturally content phosphorus in soil is not sufficient to produce food and supply of mine phosphate is inevitable to feed growing population. Of the extracted mine phosphate, only a fraction will be lost in the production and consumption process and it is not available to recycle back. Other than fraction quantity, all consumed phosphorus will be available for re-consumption through recycling of waste and sewage.

From the analysis, it was found that per capita phosphate extraction from mine would be lower than the case of no recycling. The extraction increases in the beginning until to rich a peak due to a high initial recycling cost and low phosphate extraction cost from mine. After the initial rise, it starts to fall continuously over time because of higher mine extraction cost due to the stock effect and the lower recycling cost. In case of phosphate mines as stock of phosphorus, its consumption causes to deplete mine continuously. Obviously, recycling reduces the consumption of mine phosphate. But in our findings, the optimal mine extraction path does not reveal any relation between recycling rate and mine extraction. Thus, it can be concluded that the stock size would get increase because of recycling and the entire ecosystem can be considered as the stock of phosphorus rather than only phosphate mine.

Chapter 6: Further Direction

News of oil crisis and finding its alternatives is common, but phosphate crisis is less known to the public. Phosphate has no substitute and can't be artificially synthesized or manufactured. Its demand has been rising to produce food for growing population but stocks are only enough for a century and available for a couple of generations. This thesis research worked out on examining phosphate extraction path and its perturbation with changing population and supply of substitutes. It was observed that the phosphate extraction decreases over time as population size increases. In a finite time horizon, supply of substitutes exaggerates phosphate extraction over time. We argued that people find phosphorus less scarce due to the production of substitutes. But in infinite time horizon, people think it is an indispensable resource for the future generations and thus, consume less phosphate at each time. This study was influenced from supply side and was focused on increasing life-time of phosphate stock through the generation of substitutes.

It may be productive to think from the demand side. The developing countries are experiencing high population growth rate and asking more fertilizer to meet their high food requirements. In literatures, it is reported that 18 millions tonnes of phosphate is used in artificial fertilizer and 20 to 30 million tones are drained into the oceans mainly through erosion from fertile soils. Every year more phosphate is eroded into the ocean than it is produced by all the mines combined. It may be possible to reduce phosphate demand while minimizing phosphorus loss if the developing countries checked their population growth and adopted better farm management practices. Obviously, it is an urgent need to think about the possibility of recycling, but at the same time, better soil management and cropping systems especially in the developing countries, will help to reduce phosphate demand and contribute to lessen phosphate crisis in the future.

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